J. Math. Soc. Japan
Vol. 52, No. 4, 2000

Cohomology of discrete groups and their finite subgroups

Dedicated to the memory of Professor Katsuo Kawakubo

By Toshiyuki Akita

(Received Oct. 13, 1998)
(Revised Jun. 3, 1999)

Abstract. We investigate the cohomology of a group having finite virtual cohomological dimension in terms of the contributions from finite subgroups. As a result, we prove a variant of Quillen's F-isomorphism theorem which remains valid for an arbitrary commutative ring of coefficients and for suitable families of finite subgroups.

1. Introduction.

A discrete group Γ is said to have *finite virtual cohomological dimension* (written $\operatorname{vcd}\Gamma < \infty$) if there is a subgroup $\Gamma' \subseteq \Gamma$ of finite index such that Γ' has finite cohomological dimension. Some of examples of this type of groups are finite groups, arithmetic groups [**8**], mapping class groups [**4**], outer automorphism groups of free groups [**3**], and Coxeter groups [**7**]. One of the fundamental problems in topology as well as in group cohomology is to understand the cohomology of these groups having finite virtual cohomological dimension.

This paper is an attempt to understand the cohomology of a group Γ with $\operatorname{vcd}\Gamma < \infty$ in terms of the contributions from finite subgroups. In his famous paper [**5**], Quillen showed that the natural map

$$H^*(\Gamma, \boldsymbol{F}_p) \to \varprojlim_{E} H^*(E, \boldsymbol{F}_p)$$

is a uniform F-isomorphism, where $\boldsymbol{F}_p$ is the field of p elements (p a prime) and E runs all the elementary abelian p-subgroups of Γ [**5**, Theorem 14.1]. He also remarked that, for a finite group G, the kernel and the cokernel of the natural map

$$H^*(G, \boldsymbol{Z}) \to \varprojlim_{A} H^*(A, \boldsymbol{Z})$$

consist of nilpotent elements, where $\boldsymbol{Z}$ is the ring of integers and A runs all the abelian subgroups of G [**5**, p. 599].

In this paper, we will prove a variant of the last mentioned result of Quillen which holds for groups having finite virtual cohomological dimension. To be more precise, for an *arbitrary* commutative ring k with identity and an arbitrary family $\mathfrak{F}$ of finite subgroups of Γ such that (i) for every finite subgroup H in Γ, there is $K \in \mathfrak{F}$ such that

---

2000 *Mathematics Subject Classification.* Primary 55R40; Secondary 20J05, 55R35.

Key Words and Phrases. Group cohomology, Farrell-Tate cohomology, discrete group.

The author is partially supported by Grand-in-Aid for Encouragement of Young Scientists (No. 09740072), the Ministry of Education, Science, Sports and Culture.

$H \subseteq K$, and (ii) $\mathfrak{F}$ is closed under conjugation and taking intersections, we will show that the kernel and the cokernel of the natural map

$$H^*(\Gamma, k) \to \varprojlim_{H \in \mathfrak{F}} H^*(H, k)$$

consist of nilpotent elements (see Theorem 1 for detail). Our result illustrates the importance of the contributions from finite subgroups in understanding the cohomology of a group Γ with $\operatorname{vcd} \Gamma < \infty$, not only with coefficients in $\boldsymbol{F}_p$ but also with coefficients in an arbitrary commutative ring, particularly in $\boldsymbol{Z}$.

In trying to explain the cohomology of a group Γ with $\operatorname{vcd} \Gamma < \infty$ in terms of its finite subgroups, one could also consider the *Farrell-Tate cohomology* $\hat{H}^*(\Gamma, k)$ of Γ (or the *Farrell cohomology* in the literature), which is expected to have a better behavior than the ordinary cohomology in relation to finite subgroups. We will also prove the analogue of our main result for the Farrell-Tate cohomology (see Theorem 2 for detail). We refer to [**2**, Chapter X] for the general theory of the Farrell-Tate cohomology.

The rest of this paper is organized as follows: In §2, we state our main results in full detail. In §3, we consider the Leray spectral sequences associated to a contractible, proper Γ-simplicial complex, which are necessary to prove our results. The proofs of Theorems 1 and 2 will be given in §4. As byproducts of Theorems 1 and 2, we will prove in §5 that if $u \in H^*(\Gamma, k)$ (*resp.* $u \in \hat{H}^*(\Gamma, k)$) restricts to zero for every maximal finite subgroup of Γ, then u is nilpotent (Theorem 4).

This study grew out of our earlier paper [**1**], in which the analogues of Theorems 1 and 2 for Coxeter groups were proved. Indeed, the main result of [**1**] follows from Theorems 1 and 2. Some of the arguments in §4 are refinements of those used in [**1**].

Notation. Let Γ be a group with $\operatorname{vcd} \Gamma < \infty$ and H a subgroup of Γ. For a subgroup K of H, we denote by res^H_K the restriction map $H^*(H, k) \to H^*(K, k)$ or $\hat{H}^*(H, k) \to \hat{H}^*(K, k)$. For $u \in H^*(H, k)$ (*resp.* $u \in \hat{H}^*(H, k)$) and $\gamma \in \Gamma$, we denote by γu the image of u under the isomorphism $H^*(H, k) \to H^*(\gamma H \gamma^{-1}, k)$ (*resp.* $\hat{H}^*(H, k) \to \hat{H}^*(\gamma H \gamma^{-1}, k)$) induced by conjugation $\gamma H \gamma^{-1} \to H$ by γ^{-1}.

2. Statement of main results.

Let Γ be a group having finite virtual cohomological dimension. A family $\mathfrak{F}$ of finite subgroups of Γ is said to be *admissible* if the following three conditions are satisfied:

1. $\gamma H \gamma^{-1} \in \mathfrak{F}$ for all $\gamma \in \Gamma$, $H \in \mathfrak{F}$.
2. If $H_1, H_2 \in \mathfrak{F}$ then $H_1 \cap H_2 \in \mathfrak{F}$.
3. For every finite subgroup H in Γ, there exists $K \in \mathfrak{F}$ such that $H \subseteq K$.

In particular, the family of all finite subgroups is admissible. Let k be an arbitrary commutative ring with identity. Let $\mathfrak{H}^*(\Gamma, k)$ (*resp.* $\hat{\mathfrak{H}}^*(\Gamma, k)$) be the subring of $\prod_{H \in \mathfrak{F}} H^*(H, k)$ (*resp.* $\prod_{H \in \mathfrak{F}} \hat{H}^*(H, k)$) consisting of those families $(u_H)_{H \in \mathfrak{F}}$ satisfying the following two conditions:

1. $\gamma u_H = u_{\gamma H \gamma^{-1}}$ for all $\gamma \in \Gamma$, $H \in \mathfrak{F}$.
2. If $K \subseteq H$ then $\operatorname{res}^H_K u_H = u_K$.

The subring $\mathfrak{H}^*(\Gamma,k)$ is identical to the inverse limit of $H^*(H,k)$ which appeared in the introduction. Let $\rho : H^*(\Gamma,k) \to \mathfrak{H}^*(\Gamma,k)$ (resp. $\hat{\rho} : \hat{H}^*(\Gamma,k) \to \hat{\mathfrak{H}}^*(\Gamma,k)$) be the canonical ring homomorphism induced by restriction maps $H^*(\Gamma,k) \to H^*(H,k)$ (*resp.* $\hat{H}^*(\Gamma,k) \to \hat{H}^*(H,k)$). Now we can state our main results.

Theorem 1. *Let Γ be a group with* $\operatorname{vcd}\Gamma < \infty$*, $\mathfrak{F}$ an admissible family of finite subgroups of Γ, and k a commutative ring with identity. Then the kernel and the cokernel of the canonical ring homomorphism $\rho : H^*(\Gamma,k) \to \mathfrak{H}^*(\Gamma,k)$ consist of nilpotent elements.*

We also have an analogue of Theorem 1 for the Farrell-Tate cohomology:

Theorem 2. *Let Γ be a group with* $\operatorname{vcd}\Gamma < \infty$*, $\mathfrak{F}$ an admissible family of finite subgroups of Γ, and k a commutative ring with identity. Then the kernel and the cokernel of the canonical ring homomorphism $\hat{\rho} : \hat{H}^*(\Gamma,k) \to \hat{\mathfrak{H}}^*(\Gamma,k)$ consist of nilpotent elements.*

The proof of Theorem 2, for the case $\mathfrak{F}$ is the family of all finite subgroups and k is the field $\boldsymbol{F}_p$ of p elements (p a prime), can be found in [**2**, Proposition X.4.6].

3. Preliminaries.

We begin with some definitions concerning of the notion of a Γ-simplicial complex. An *ordered simplicial complex* is a simplicial complex together with a partial ordering on its vertices, such that the vertices of any simplex are linearly ordered: $v_0 < v_1 < \cdots < v_n$. Let Γ be a group. An *ordered Γ-simplicial complex* is an ordered simplicial complex X together with a simplicial action of Γ on X which preserves an ordering on vertices. If X is an ordered Γ-simplicial complex, then for each simplex σ of X the isotropy subgroup Γ_σ fixes σ pointwise. An ordered Γ-simplicial complex X is *proper* if the isotropy subgroup Γ_σ is finite for every simplex σ of X. The following lemma due to Serre [**7**] occurs as a fundamental ingredient for proving Theorems 1 and 2.

Lemma 1. *Let Γ be a group having finite virtual cohomological dimension. Then there exists a finite dimensional, contractible, proper, ordered Γ-simplicial complex having the following property: For every finite subgroup $H \subset \Gamma$, the fixed point set X^H is non-empty and contractible.*

See also [**2**, Chapter VIII]. Associated with such a Γ-simplicial complex X, we have the Leray spectral sequences

$$E_1^{p,q} = \prod_{\sigma \in \Sigma_p} H^q(\Gamma_\sigma, k) \Rightarrow H^{p+q}(\Gamma, k) \tag{1}$$

for the ordinary cohomology of Γ, and

$$\hat{E}_1^{p,q} = \prod_{\sigma \in \Sigma_p} \hat{H}^q(\Gamma_\sigma, k) \Rightarrow \hat{H}^{p+q}(\Gamma, k) \tag{2}$$

for the Farrell-Tate cohomology of Γ, where Σ_p is a set of representatives for the p-simplices of X modulo Γ, and Γ_σ is the isotropy subgroup of σ [**2**, §VII.7 and §X.4].

Let X_0 be the set of vertices of X. The fiber terms of the spectral sequences (1) and (2) can be characterized as follows:

Lemma 2. *$E_2^{0,*}$ (resp. $\hat{E}_2^{0,*}$) can be identified with the subring of $\prod_{v \in X_0} H^*(\Gamma_v, k)$ (resp. $\prod_{v \in X_0} \hat{H}^*(\Gamma_v, k)$) consisting of those families $(u_v)_{v \in X_0}$ satisfying the following two conditions*:

1. *$\gamma u_v = u_{\gamma v}$ for any $\gamma \in \Gamma$, $v \in X_0$.*
2. *If e is a 1-simplex of X with vertices v_0, v_1, then u_{v_0} and u_{v_1} restrict to the same element of $H^*(\Gamma_e, k)$ (resp. $\hat{H}^*(\Gamma_e, k)$).*

Proof. See [**2**, §X.4]. $\square$

The rest of this section is devoted to prove $\mathfrak{H}^*(\Gamma, k)$ (*resp.* $\hat{\mathfrak{H}}^*(\Gamma, k)$) is isomorphic to $E_2^{0,*}$ (*resp.* $\hat{E}_2^{0,*}$). Let $(u_H)_{H \in \mathfrak{F}}$ be an element of $\mathfrak{H}^*(\Gamma, k)$ or $\hat{\mathfrak{H}}^*(\Gamma, k)$. For each $v \in X_0$, choose $H \in \mathfrak{F}$ such that $\Gamma_v \subseteq H$ and set $w_v = \operatorname{res}^H_{\Gamma_v} u_H$. Then w_v is independent of the choice of $H \in \mathfrak{F}$. Indeed, if $\Gamma_v \subseteq K \in \mathfrak{F}$ is another choice, then $\Gamma_v \subseteq H \cap K \in \mathfrak{F}$ and $\operatorname{res}^H_{\Gamma_v} u_H = \operatorname{res}^{H\cap K}_{\Gamma_v} \circ \operatorname{res}^H_{H \cap K} u_H = \operatorname{res}^{H \cap K}_{\Gamma_v} u_{H \cap K} = \operatorname{res}^{H\cap K}_{\Gamma_v} \circ \operatorname{res}^K_{H\cap K} u_K = \operatorname{res}^K_{\Gamma_v} u_K$ by the definition of $\mathfrak{H}^*(\Gamma, k)$ or $\hat{\mathfrak{H}}^*(\Gamma, k)$.

Lemma 3. *Under the identifications of $E_2^{0,*}$ (resp. $\hat{E}_2^{0,*}$) with the subring of $\prod_{v \in X_0} H^*(\Gamma_v, k)$ (resp. $\prod_{v \in X_0} \hat{H}^*(\Gamma_v, k)$) given in Lemma 2, the family $(w_v)_{v \in X_0}$ defined above belongs to $E_2^{0,*}$ (resp. $\hat{E}_2^{0,*}$).*

Proof. The family $(w_v)_{v \in X_0}$ satisfies the first condition of Lemma 2. Indeed, if $H \in \mathfrak{F}$ with $\Gamma_v \subseteq H$ then $\gamma H \gamma^{-1} \in \mathfrak{F}$ and $\Gamma_{\gamma v} = \gamma \Gamma_v \gamma^{-1} \subseteq \gamma H \gamma^{-1}$, which implies

$$\gamma w_v = \gamma \cdot \operatorname{res}^H_{\Gamma_v} u_H = \operatorname{res}^{\gamma H \gamma^{-1}}_{\Gamma_{\gamma v}} \gamma u_H = \operatorname{res}^{\gamma H \gamma^{-1}}_{\Gamma_{\gamma v}} u_{\gamma H \gamma^{-1}} = w_{\gamma v}.$$

Here the third equality follows from the definition of $\mathfrak{H}^*(\Gamma, k)$ or $\hat{\mathfrak{H}}^*(\Gamma, k)$.

To see $(w_v)_{v \in X_0}$ satisfies the second condition of Lemma 2, for every 1-simplex e of X with vertices v_0, v_1, choose $H_0, H_1 \in \mathfrak{F}$ with $\Gamma_{v_0} \subseteq H_0$, $\Gamma_{v_1} \subseteq H_1$. Then $\Gamma_e \subseteq \Gamma_{v_0} \cap \Gamma_{v_1} \subseteq H_0 \cap H_1 \in \mathfrak{F}$ and the following diagram is commutative:

$$\begin{array}{ccccc}
H^*(H_0, k) & \xrightarrow{\operatorname{res}^{H_0}_{H_0 \cap H_1}} & H^*(H_0 \cap H_1, k) & \xleftarrow{\operatorname{res}^{H_1}_{H_0 \cap H_1}} & H^*(H_1, k) \\
{\scriptstyle \operatorname{res}^{H_0}_{\Gamma_{v_0}}} \downarrow & & {\scriptstyle \operatorname{res}^{H_0 \cap H_1}_{\Gamma_e}} \downarrow & & {\scriptstyle \operatorname{res}^{H_1}_{\Gamma_{v_1}}} \downarrow \\
H^*(\Gamma_{v_0}, k) & \xrightarrow{\operatorname{res}^{\Gamma_{v_0}}_{\Gamma_e}} & H^*(\Gamma_e, k) & \xleftarrow{\operatorname{res}^{\Gamma_{v_1}}_{\Gamma_e}} & H^*(\Gamma_{v_1}, k).
\end{array}$$

This proves $\operatorname{res}^{\Gamma_{v_0}}_{\Gamma_e} w_{v_0} = \operatorname{res}^{\Gamma_{v_1}}_{\Gamma_e} w_{v_1}$, since $w_{v_0} = \operatorname{res}^{H_0}_{\Gamma_v} u_{H_0}$ and $w_{v_1} = \operatorname{res}^{H_1}_{\Gamma_v} u_{H_1}$. The proof for the Farrell-Tate cohomology is similar. $\square$

Now define $\varphi : \mathfrak{H}^*(\Gamma, k) \to E_2^{0,*}$ and $\hat{\varphi} : \hat{\mathfrak{H}}^*(\Gamma, k) \to \hat{E}_2^{0,*}$ to be the ring homomorphisms which assign $(w_v)_{v \in X_0}$ to $(u_H)_{H \in \mathfrak{F}}$.

Lemma 4. *$\varphi : \mathfrak{H}^*(\Gamma, k) \to E_2^{0,*}$ and $\hat{\varphi} : \hat{\mathfrak{H}}^*(\Gamma, k) \to \hat{E}_2^{0,*}$ are isomorphisms.*

Proof. To prove φ is injective, suppose φ assigns to $(u_H)_{H \in \mathfrak{F}}$ and $(u'_H)_{H \in \mathfrak{F}}$ the same element $(w_v)_{v \in X_0}$. Given $H \in \mathfrak{F}$, choose $v \in X_0$ such that $H \subseteq \Gamma_v$. This is possible since $X^H \neq \emptyset$ by Lemma 1. Choose $K \in \mathfrak{F}$ such that $H \subseteq \Gamma_v \subseteq K$, then $\mathrm{res}^K_{\Gamma_v} u_K = w_v = \mathrm{res}^K_{\Gamma_v} u'_K$ by the assumption. It follows from the second condition in the definition of $\mathfrak{H}^*(\Gamma, k)$ that

$$u_H = \mathrm{res}^K_H u_K = \mathrm{res}^{\Gamma_v}_H \circ \mathrm{res}^K_{\Gamma_v} u_K = \mathrm{res}^{\Gamma_v}_H \circ \mathrm{res}^K_{\Gamma_v} u'_K = \mathrm{res}^K_H u'_K = u'_H,$$

which proves that φ is injective.

To prove that φ is surjective, suppose $(w_v)_{v \in X_0}$ satisfies conditions 1 and 2 of Lemma 2. Given $H \in \mathfrak{F}$, choose a vertex v such that $H \subseteq \Gamma_v$ and set $u_H = \mathrm{res}^{\Gamma_v}_H w_v$. Since X^H is connected, the second condition of Lemma 2 shows that u_H is independent of the choice of v. It is easy to check that the resulting family $(u_H)_{H \in \mathfrak{F}}$ is in $\mathfrak{H}^*(\Gamma, k)$ and that its image under φ is $(w_v)_{v \in X_0}$. The proof for $\hat{\varphi}$ is similar. $\square$

4. Proof of theorems.

Using the identifications $\mathfrak{H}^*(\Gamma, k) \cong E_2^{0,*}$ and $\hat{\mathfrak{H}}^*(\Gamma, k) \cong \hat{E}_2^{0,*}$ given in the last section, it is easy to see that ρ is simply the edge homomorphism

$$H^*(\Gamma, k) \twoheadrightarrow E_\infty^{0,*} \hookrightarrow E_2^{0,*} \overset{\varphi}{\cong} \mathfrak{H}^*(\Gamma, k)$$

associated to the spectral sequence (1), and that $\hat{\rho}$ is the edge homomorphism

$$\hat{H}^*(\Gamma, k) \twoheadrightarrow \hat{E}_\infty^{0,*} \hookrightarrow \hat{E}_2^{0,*} \overset{\hat{\varphi}}{\cong} \hat{\mathfrak{H}}^*(\Gamma, k)$$

associated to the spectral sequence (2) (cf. the proof of Proposition X.4.6 and Exercise 1 of §VII.7 in [**2**]). Thus, to prove Theorems 1 and 2, it suffices to show the following two claims: (i) Every element in the kernel of the edge homomorphism $H^*(\Gamma, k) \to E_2^{0,*}$ (*resp.* $\hat{H}^*(\Gamma, k) \to \hat{E}_2^{0,*}$) is nilpotent. (ii) For any $u \in E_2^{*,0}$ (*resp.* $u \in \hat{E}_2^{*,0}$), there is an integer $n \geq 1$ such that $u^n \in E_\infty^{*,0}$ (*resp.* $u^n \in \hat{E}_\infty^{*,0}$). Since $\dim X < \infty$, the first claim is a formal consequence of the multiplicative structure of the spectral sequences (1) and (2) (cf. the proof of [**2**, Proposition X.4.6]).

Let $l = l(\Gamma)$ be the least common multiple of the orders of the finite subgroups of Γ. Such $l(\Gamma)$ exists since Γ is a group having finite virtual cohomological dimension.

Lemma 5. *In the spectral sequences* (1) *and* (2), *$E_r^{p,q}$ is annihilated by $l(\Gamma)$ for any p and $q \neq 0$, and $\hat{E}_r^{p,q}$ is annihilated by $l(\Gamma)$ for any p and q.*

Proof. Since $E_r^{p,q}$ (*resp.* $\hat{E}_r^{p,q}$) is a subquotient of $E_{r-1}^{p,q}$ (*resp.* $\hat{E}_{r-1}^{p,q}$), it suffices to prove the case $r = 1$. For any simplex σ of X the isotropy subgroup Γ_σ is finite since X is proper. Hence $H^q(\Gamma_\sigma, k)$ $(q > 0)$ and $\hat{H}^q(\Gamma, k)$ (all q) are annihilated by the order of Γ_σ. Since $E_1^{p,q}$ (*resp.* $\hat{E}_1^{p,q}$) is a product of $H^q(\Gamma_\sigma, k)$ (*resp.* $\hat{H}^q(\Gamma_\sigma, k)$) with $\sigma \in \Sigma_p$, $E_1^{p,q}$ $(q > 0)$ and $\hat{E}_1^{p,q}$ is annihilated by $l(\Gamma)$. $\square$

Lemma 6. *Suppose $r \geq 2$. For any element $u \in E_r^{0,q}$* (*resp.* $u \in \hat{E}_r^{0,q}$), *there is an integer $n \geq 1$ such that $u^n \in E_{r+1}^{0,nq}$* (*resp.* $u^n \in \hat{E}_{r+1}^{0,nq}$).

Proof. Let d_r and $\hat{d}_r$ be the differentials $d_r^{0,q} : E_r^{0,q} \to E_r^{r,q-r+1}$ and $\hat{d}_r^{0,q} : \hat{E}_r^{0,q} \to \hat{E}_r^{r,q-r+1}$, respectively. Let $u \in \hat{E}_r^{0,q}$. If q is even, then $\hat{d}_r(u^l) = lu^{l-1}\hat{d}_r(u) = 0$, since

$\hat{E}_r^{r,q-r+1}$ is annihilated by $l = l(\Gamma)$. Hence $u^l \in \hat{E}_{r+1}^{0,lq}$. If q is odd, then $\hat{d}_r(u^2) = 0$ as in the proof of [**1**, Lemma 1]. Hence $u^2 \in \hat{E}_{r+1}^{0,2q}$.

Now let $u \in E_r^{0,q}$. When $q \neq r-1$, the proof is similar to the case of $u \in \hat{E}_r^{0,q}$. When $q = r-1$, the image of the differential $d_r^{0,q}$ belongs to $E_r^{r,0}$, which may not be annihilated by $l(\Gamma)$. However, applying to u^2 the assertion for $u \in \hat{E}_r^{0,q}$ with q even, we obtain $u^{2l} \in E_{r+1}^{0,2lq}$. □

By making use of Lemma 6, the proof of the second claim is similar to that of [**1**, Theorem 1]. This completes the proof of Theorems 1 and 2.

5. An application.

The result of Quillen mentioned in the introduction implies the following:

Theorem 3. *Let Γ be a group with* $\mathrm{vcd}\,\Gamma < \infty$. *If $u \in H^*(\Gamma, \boldsymbol{F}_p)$ restricts to zero on every elementary abelian p-subgroup of Γ, then u is nilpotent.*

The alternative proof of Theorem 3 for finite groups can be found in [**6**]. As an application of Theorems 1 and 2, we obtain a variant of Theorem 3 for an arbitrary commutative ring of coefficients:

Theorem 4. *Let Γ be a group with* $\mathrm{vcd}\,\Gamma < \infty$ *and k a commutative ring with identity. If $u \in H^*(\Gamma, k)$ (resp. $u \in \hat{H}^*(\Gamma, k)$) restricts to zero on every maximal finite subgroup of Γ, then u is nilpotent.*

Proof. Suppose that $u \in H^*(\Gamma, k)$ (*resp.* $u \in \hat{H}^*(\Gamma, k)$) satisfies $\mathrm{res}_K^\Gamma u = 0$ for every maximal finite subgroup K. Choose an admissible family $\mathfrak{F}$ of finite subgroups of Γ and let $\rho : H^*(\Gamma, k) \to \mathfrak{H}^*(\Gamma, k)$ (*resp.* $\hat{\rho} : \hat{H}^*(\Gamma, k) \to \hat{\mathfrak{H}}^*(\Gamma, k)$) be the canonical ring homomorphism induced by restrictions as in §2. In view of Theorems 1 and 2, to prove Theorem 4 it suffices to show $\rho u = 0$ (*resp.* $\hat{\rho} u = 0$). A family $\mathfrak{F}$ contains all maximal finite subgroups of Γ by the definition of $\mathfrak{F}$. We have $\mathrm{res}_H^\Gamma u = 0$ for every $H \in \mathfrak{F}$, since there is a maximal finite subgroup $K \in \mathfrak{F}$ with $H \subseteq K$. As the homomorphism ρ (*resp.* $\hat{\rho}$) is induced by res_H^Γ ($H \in \mathfrak{F}$), we have $\rho u = 0$ (resp. $\hat{\rho} u = 0$). □

References

[1] T. Akita, On the cohomology of Coxeter groups and their finite parabolic subgroups II, in Group Representations: Cohomology, Group Actions and Topology, Proc. Symp. in Pure Math., Vol. 63, Amer. Math. Soc., Providence, 1998, 1–5.

[2] K. S. Brown, Cohomology of Groups, Graduate texts in mathematics, Vol. 87, Springer-Verlag, New York-Heidelberg-Berlin, 1982.

[3] M. Culler and K. Vogtman, Moduli of graphs and automorphisms of free groups, Invent. Math., **84** (1986), 91–119.

[4] J. Harer, The virtual cohomological dimension of the mapping class group of an orientable surface, Invent. Math., **84** (1986), 157–176.

[5] D. Quillen, The spectrum of an equivariant cohomology ring, I, II, Ann. Math., **94**, (1971), 549–572 and 573–602.

[6] D. Quillen, B. B. Venkov, Cohomology of finite groups and elementary abelian subgroups, Topology, **11** (1972), 317–318.

[7] J.-P. Serre, Cohomologie de groupes discrets, in Prospects in Mathematics, Ann. of Math. Studies, Vol. 70, Princeton University Press, Princeton, 1971, 77–169.

[8] J.-P. Serre, Arithmetic groups, in Homological group theory, London Math. Soc. Lecture Notes, Vol. 36, Cambridge University Press, Cambridge, 1979, 105–136.

Toshiyuki Akita
Department of Applied Mathematics
Fukuoka University
Fukuoka 814-0180, Japan

Current address:
Division of Mathematics
Graduate School of Science
Hokkaido University
Sapporo 060-0810, Japan
E-mail: akita@math.sci.hokudai.ac.jp