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Double group construction of quantum groups in the von Neumann algebra framework

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Abstract. We introduce a notion of double group construction within the category of quasi Woronowicz algebras which are regarded as quantum groups in the von Neumann algebra framework. We show that the quantum double in this setting is always unimodular. The Kac-Takesaki operator of the double group is explicitly described. It is also proven that the dual of the quantum double has a quasitriangular structure.

Introduction.

In [**Dr**], Drinfeld devised a remarkable ingenious method, called the double group construction, which generates a quasitriangular Hopf algebra out of any finite-dimensional Hopf algebra. This method was used to find solutions to the quantum Yang-Baxter equation in statistical mechanics. It was Podleś and Woronowicz [**PW**] who employed this method from the viewpoint of operator algebras in order to define a quantum deformation of Lorentz group. Later, Baaj and Skandalis [**BS**] introduced a notion of a Kac system, using (regular and irreducible) multiplicative unitaries. They showed that one can equally define the quantum double of a Kac system, and that the framework of Kac systems is stable under the construction of the quantum double. Afterwards, Nakagami [**N**] discussed the double group construction for Woronowicz algebras. The category of Woronowicz algebras can be naturally regarded as a "subcategory" of Kac systems. In [**N**], Nakagami was able to define the quantum double of a *compact* Woronowicz algebra, and to show that the double group is again a (noncompact, unimodular) Woronowicz algebra. It is, however, not so transparent how Nakagami's double construction is related to Baaj-Skandalis'.

The purpose of this note is to define (construct) the quantum double for a *general* (quasi) Woronowicz algebra, and to prove that the category of (quasi) Woronowicz algebras is stable under this construction. We exhibit an explicit relationship between our double group construction and Baaj-Skandalis'. We also examine the dual of the quantum double, which was left untouched in [**N**]. Roughly speaking, our main result asserts that, if a Kac system "comes from" a (quasi) Woronowicz algebra, then the Kac system obtained from the double group construction in the sense of [**BS**] admits a natural Haar measure.

The organization of this note is as follows. Section 1 is concerned with notation which will be used in the sections that follow. We also briefly recall fundamental

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facts on quasi Woronowicz algebras. The reason why we deal with quasi Woronowicz algebras here, not with Woronowicz algebras is also discussed. In Section 2, following [**BS**], we equip the tensor product of a given quasi Woronowicz algebra $\boldsymbol{W}$ and its dual $\hat{\boldsymbol{W}}$ with a structure of a coinvolutive Hopf-von Neumann algebra. This construction turns out to be the same as the one used in [**N**]. In Section 3, we prove that the coinvolutive Hopf-von Neumann algebra constructed in the previous section does admit a Haar measure. We call the resulting quasi Woronowicz algebra $D(\boldsymbol{W})$ the quantum double of $\boldsymbol{W}$. The quantum double of a (quasi) Woronowicz algebra is always unimodular even if the original (quasi) Woronowicz algebra is not unimodular. The argument of this section more or less explains why Nakagami's case [**N**] (i.e., the case of compact Woronowicz algebras) was relatively easy to handle. The last Section is devoted to analysis on the dual of the quantum double. We give an explicit relation between the Kac-Takesaki operator of $D(\boldsymbol{W})$ and the mutiplicative unitary V in [**BS**, Section 8]. This fully answers the problem raised in [**N**, Section 2].

1. Notation.

In this section, we give a quick review on quasi Woronowicz algebras, introducing notation that will be used in our later discussion. Quasi Woronowicz algebras are almost like Woronowicz algebras introduced in [**MN**]. It is not too much to say that what is true for Woronowicz algebras is equally true for quasi Woronowicz algebras. Thus, for the general theory of quasi Woronowicz algebras, we may refer readers to [**MN**] and [**N**] (also see [**Y**]). Our notation will be mainly adopted from these literatures.

Given a von Neumann algebra $\mathscr{M}$ and a faithful normal semifinite weight ψ on $\mathscr{M}$, we introduce subsets $\mathfrak{n}_\psi$, $\mathfrak{m}_\psi$ and $\mathfrak{m}_\psi^+$ of $\mathscr{M}$ by

$$\mathfrak{n}_\psi = \{x \in \mathscr{M} : \psi(x^*x) < \infty\}, \quad \mathfrak{m}_\psi = \mathfrak{n}_\psi^*\mathfrak{n}_\psi, \quad \mathfrak{m}_\psi^+ = \mathfrak{m}_\psi \cap \mathscr{M}_+.$$

We denote by π_ψ the standard (GNS) representation associated with ψ. Its representation space is denoted by $\mathfrak{H}_\psi$. We use the symbol Λ_ψ for the canonical embedding of $\mathfrak{n}_\psi$ into $\mathfrak{H}_\psi$. Let $\mathfrak{a}_\psi = \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*$ and set $\mathfrak{A}_\psi = \Lambda_\psi(\mathfrak{a}_\psi)$, which is the full left Hilbert algebra associated with ψ. For a left bounded vector $\xi \in \mathfrak{H}$ with respect to the left Hilbert algebra $\mathfrak{A}_\psi$, we write $\pi_\ell(\xi)$ for the left multiplication operator corresponding to ξ. For a right bounded vector η, we use $\pi_r(\eta)$ for the corresponding right multiplication operator. The modular automorphism group of ψ is denoted by σ^ψ.

A *coinvolutive Hopf-von Neumann algebra* is a triple $(\mathscr{M}, \delta, R)$ in which:

(1) $\mathscr{M}$ is a von Neumann algebra;

(2) δ is an injective normal $*$-homomorphism, called a *coproduct* (or a *comultiplication*), from $\mathscr{M}$ into $\mathscr{M} \bar{\otimes} \mathscr{M}$ with the coassociativity condition: $(\delta \otimes id_{\mathscr{M}}) \circ \delta = (id_{\mathscr{M}} \otimes \delta) \circ \delta$;

(3) R is a $*$-antiautomorphism of $\mathscr{M}$, called a *coinvolution* or a *unitary antipode*, such that $R^2 = id_{\mathscr{M}}$ and $\sigma \circ (R \otimes R) \circ \delta = \delta \circ R$, where σ is the usual flip.

A *quasi Woronowicz algebra* is a family $\boldsymbol{W} = (\mathscr{M}, \delta, R, \tau, h)$ in which:

(1) $(\mathscr{M}, \delta, R)$ is a coinvolutive Hopf-von Neumann algebra;

(2) τ is a continuous one-parameter automorphism group of $\mathscr{M}$, called the *deformation automorphism*, which commutes with the coproduct δ and the antipode R;

(3) h is a τ-invariant faithful normal semifinite weight on $\mathscr{M}$, called the *Haar measure* of $\boldsymbol{W}$, satisfying the following conditions:

(a) *Quasi left invariance*: For any ϕ in $\mathscr{M}_*^+$, we have $(\phi \otimes h) \circ \delta(x) = h(x)\phi(1)$ for all $x \in \mathfrak{m}_h^+$;

(b) *Strong left invariance*: For any $x, y \in \mathfrak{n}_h$ and $\phi \in \mathscr{M}_*$ which is analytic with respect to the adjoint action of the deformation automorphism τ on $\mathscr{M}_*$, the following equality holds:

$$(\phi \otimes h)((1 \otimes y^*)\delta(x)) = (\phi \circ \tau_{-i/2} \circ R \otimes h)(\delta(y^*)(1 \otimes x)).$$

(c) *Commutativity*: $h \circ \sigma_t^{h \circ R} = h$ for all $t \in \boldsymbol{R}$ (or, equivalently, $h \circ R \circ \sigma_t^h = h \circ R$).

We say that a quasi Woronowicz algebra $\boldsymbol{W} = (\mathscr{M}, \delta, R, \tau, h)$ is *unimodular* (resp. *compact*) if $h = h \circ R$ (resp. h is bounded).

Remark that only difference between a Woronowicz algebra and a quasi Woronowicz algebra is the requirement that the weight h is left invariant or quasi left invariant. In other words, in the definition of a Woronowicz algebra, one requires that h should satisfy $(\phi \otimes h) \circ \delta(x) = h(x)\phi(1)$ for all $\phi \in \mathscr{M}_*^+$ and *all* $x \in \mathscr{M}_+$. At the present stage, the author does not know whether left invariance and quasi left invariance are distinct notions. Let us briefly tell the reason why we work with quasi Woronowicz algebras rather than with Woronowicz algebras in this note. In the paper [**MN**], there is a crucial gap at the end of the proof of Proposition 3.8. Because of this gap, we do <u>not</u> yet know that the dual Woronowicz algebra in the sense of [**MN**] is really a Woronowicz algebra. One can, however, easily see that the dual <u>is</u> a quasi Woronowicz algebra. Moreover, most of the argument in [**MN**] goes through perfectly without any change even if we start with a quasi Woronowicz algebra, not with a Woronowicz algebra. (There are some points in which we really have to be careful, but those points are irrelevant to our discussion that follows). This is why we stick to working with quasi Woronowicz algebras. Besides, as shown in [**Y**], every matched pair of (locally compact) groups gives rise to a quasi Woronowicz algebra. Hence there are plenty of examples of quasi Woronowicz algebras.

Throughout the remainder of this note, we fix a quasi Woronowicz algebra $\boldsymbol{W} = (\mathscr{M}, \delta, R, \tau, h)$. Identifying $\mathscr{M}$ with $\pi_h(\mathscr{M})$, we always think of $\mathscr{M}$ as represented on the Hilbert space $\mathfrak{H} := \mathfrak{H}_h$. We denote by Δ and J the modular operator and the modular conjugation of h, respectively. By the commutativity of h, there exists a non-singular positive self-adjoint operator Q on $\mathfrak{H}$ affiliated with the centralizer $\mathscr{M}_h = \{x \in \mathscr{M} : \sigma_t^h(x) = x \ \ (t \in \boldsymbol{R})\}$ of h such that the Connes' Radon Nikodym derivative $(D(h \circ R) : Dh)_t$ satisfies $(D(h \circ R) : Dh)_t = Q^{it}$ for $t \in \boldsymbol{R}$. In the notation in [**MN**], we have $Q = \rho^{-1}$. For any positive self-adjoint operator K and $\varepsilon > 0$, set $K_\varepsilon := K(1 + \varepsilon K)^{-1}$. With this notation, it follows from [**PT**, Theorem 5.12] that we have

$$h \circ R(x) = \lim_{\varepsilon \downarrow 0} h(Q_\varepsilon^{1/2} x Q_\varepsilon^{1/2}). \quad (x \in \mathscr{M}_+)$$

In this case, following the notation in [**PT**], we write $h \circ R = h(Q\,\cdot)$. Since h is τ-invariant, $\Lambda_h(x) \mapsto \Lambda_h(\tau_t(x))$ $(x \in \mathfrak{n}_h, t \in \boldsymbol{R})$ defines a one-parameter unitary group on

$\mathfrak{H}$. We write H for the *analytic generator* (see [**SZ**, p243] for this terminology) of this one-parameter unitary group: $H^{it}\Lambda_h(x) := \Lambda_h(\tau_t(x))$. An element $\phi \in \mathscr{M}_*$ is said to be $L^2(h)$-bounded if

$$\sup\{|\phi(x^*)| : h(x^*x) \leq 1\} < \infty.$$

We denote by $\hat{\eta}(\phi)$ the unique vector in $\mathfrak{H}$ such that $\phi(x^*) = (\hat{\eta}(\phi)|\Lambda_h(x))$ for $x \in \mathfrak{n}_h$. For $\phi, \psi \in \mathscr{M}_*$, define an element $\phi * \psi$ in $\mathscr{M}_*$ by

$$(\phi * \psi)(x) := (\phi \otimes \psi)(\delta(x)) \quad (x \in \mathscr{M}).$$

This operation $*$ turns $\mathscr{M}_*$ into a Banach algebra. Let $(\mathscr{M}_*)_\tau^\infty$ be the set of analytic elements in $\mathscr{M}_*$ with respect to the action $\phi \mapsto \phi \circ \tau_t$ of the deformation automorphism on $\mathscr{M}_*$. For $\phi \in (\mathscr{M}_*)_\tau^\infty$, put $\phi^\sharp := \phi^* \circ \tau_{-i/2} \circ R$. This defines an involution on the subalgebra $(\mathscr{M}_*)_\tau^\infty$. Thanks to quasi left invariance, the equation

$$W\Lambda_{h\otimes h}(x \otimes y) = \Lambda_{h\otimes h}(\delta(y)(x \otimes 1)) \quad (x, y \in \mathfrak{n}_h)$$

defines an isometry (in fact, a unitary) on $\mathfrak{H} \otimes \mathfrak{H}$. This unitary W is called the *Kac-Takesaki operator* of $\boldsymbol{W}$ and satisfies

$$W_{12}W_{23} = W_{23}W_{13}W_{12}, \quad \delta(x) = W(1 \otimes x)W^* \quad (x \in \mathscr{M}).$$

With W, the equation

$$\hat{\pi}(\phi) := (\phi \otimes id)(W^*) \quad (\phi \in \mathscr{M}_*)$$

defines a homomorphism (resp. $*$-homomorphism) of $\mathscr{M}_*$ (resp. $(\mathscr{M}_*)_\tau^\infty$) into the set $\mathscr{B}(\mathfrak{H})$ of all bounded operators on $\mathfrak{H}$. The mapping $\hat{\pi}$ is called the *Fourier representation* of $\boldsymbol{W}$. Let $\hat{\mathscr{M}}$ stand for the von Neumann algebra generated by $\hat{\pi}(\phi)$ $(\phi \in \mathscr{M}_*)$. By [**BS**, Proposition 3.5], $\hat{\mathscr{M}}$ is the σ-strong* closure of the subalgebra $\hat{\pi}(\mathscr{M}_*)$ (or the $*$-subalgebra $\hat{\pi}((\mathscr{M}_*)_\tau^\infty)$). It is possible to equip $\hat{\mathscr{M}}$ with a quasi Woronowicz algebra structure as follows:

$$\text{coproduct:} \quad \hat{\delta}(y) := \hat{W}(1 \otimes y)\hat{W}^* \quad (y \in \hat{\mathscr{M}})$$

$$\text{unitary antipode:} \quad \hat{R}(y) := Jy^*J$$

$$\text{deformation automorphism:} \quad \hat{\tau}_t := \operatorname{Ad} H^{it}$$

$$\text{Haar measure:} \quad \hat{h}(x) := \begin{cases} \|\xi\|^2, & \text{if } x^{1/2} = \hat{\pi}_\ell(\xi) \text{ for } \xi \in \hat{\mathfrak{A}}'', \\ \infty, & \text{otherwise,} \end{cases}$$

where $\hat{W} = \Sigma W^* \Sigma$ and Σ is the flip on $\mathfrak{H} \otimes \mathfrak{H}$. $\hat{\mathfrak{A}}$ is a left Hilbert algebra obtained as the image of some suitable $*$-subalgebra in $(\mathscr{M}_*)_\tau^\infty$ under the map $\hat{\eta}$. In particular, we have

$$\hat{h}(\hat{\pi}(\omega)^*\hat{\pi}(\phi)) = (\hat{\eta}(\phi)\,|\,\hat{\eta}(\omega))$$

for $L^2(h)$-bounded functionals ϕ, ω. We denote this quasi Woronowicz algebra by $\hat{\boldsymbol{W}}$ and call it the quasi Woronowicz algebra dual to $\boldsymbol{W}$. The Kac-Takesaki operator of

$\hat{\boldsymbol{W}}$ is $\hat{W}$. The Fourier representation of $\hat{\boldsymbol{W}}$ is denoted by $\hat{\hat{\pi}}$. The linear mapping $\mathscr{F}$ defined by

$$\mathscr{F}\Lambda_{\hat{h}}(\hat{\pi}(\phi)) := \hat{\eta}(\phi) \quad (\phi : L^2(h)\text{-bounded})$$

extends to a unitary, still denoted by $\mathscr{F}$, from $\mathfrak{H}_{\hat{h}}$ onto $\mathfrak{H}$. We call this unitary the *Fourier transform*. Note that $\{\hat{\mathscr{M}}, \mathfrak{H}\}$ is a standard representation. Thus we regard $\hat{\Delta} := \Delta_{\hat{h}}$ and $\hat{J} := J_{\hat{h}}$ as acting on the Hilbert space $\mathfrak{H}$. We have $R(x) = \hat{J}x^*\hat{J}$ $(x \in \mathscr{M})$. The analytic generator of the Radon Nikodym derivative $(D(\hat{h} \circ \hat{R}) : D\hat{h})_t$ is denoted by $\hat{Q} : (D(\hat{h} \circ \hat{R}) : D\hat{h})_t = \hat{Q}^{it}$.

Finally, for a linear operator T on a Hilbert space, let $\mathfrak{D}(T)$ designate the domain of T.

2. Hopf-von Neumann algebraic structure on $\mathscr{M} \bar{\otimes} \hat{\mathscr{M}}$.

In this section, we shall equip the tensor product $\mathscr{N} := \mathscr{M} \bar{\otimes} \hat{\mathscr{M}}$ with a Hopf-von Neumann algebraic structure. The method for this is exactly the same as the one set out in Section 2 of [**N**]. But, here, we will reconsider it more carefully along the line of argument given in [**BS**, Section 8].

Let $\boldsymbol{X} = (\hat{W}')^*$, where $\hat{W}'$ stands for the Kac-Takesaki operator associated with the commutant of the dual of the given quasi Woronowicz algebra $\boldsymbol{W}$. Then set

$$\boldsymbol{Y}_0 := \Sigma \boldsymbol{X}^* \Sigma, \quad Z_0 := \Sigma \boldsymbol{X}(u \otimes u)\boldsymbol{X}^*(u \otimes u)\Sigma.$$

Here u is the self-adjoint unitary given by $u = J\hat{J} = \hat{J}J$. Then, by [**BS**, Théorème 8.17], the family $\{(\mathfrak{H}, \boldsymbol{X}, u), (\mathfrak{H}, \boldsymbol{Y}_0, u), Z_0\}$ forms a matched pair of Kac systems. Hence, by [**BS**, Proposition 8.14], if we set $V_0 := (Z_0)_{12}^* \boldsymbol{X}_{13}(Z_0)_{12}(\boldsymbol{Y}_0)_{24}$, then the map δ_τ given by

$$\delta_\tau(X) := V_0(X \otimes 1)V_0^* \quad (X \in S_{\boldsymbol{X}}'' \bar{\otimes} S_{\boldsymbol{Y}_0}'')$$

defines a coproduct on the von Neumann algebra $S_{\boldsymbol{X}}'' \bar{\otimes} S_{\boldsymbol{Y}_0}''$. In our notation, we have

$$S_{\boldsymbol{X}}'' = \boldsymbol{W}, \quad S_{\boldsymbol{Y}_0}'' = \hat{\boldsymbol{W}}'^{\sigma}.$$

Since we want to work with $\mathscr{N} := \mathscr{M} \bar{\otimes} \hat{\mathscr{M}}$ rather than $\mathscr{M} \bar{\otimes} \hat{\mathscr{M}}'$, we modify the above construction in the following way. First we note that the map $\mathrm{Ad}\, u$ gives a quasi Woronowicz algebra isomorphism from $\hat{\boldsymbol{W}}$ onto $\hat{\boldsymbol{W}}'^{\sigma}$ (cf. [**N**, Section 4]). So, through the isomorphism $id_{\mathscr{M}} \otimes \mathrm{Ad}\, u$, everything that is true for the above construction can be translated in terms of our setting $\mathscr{N} = \mathscr{M} \bar{\otimes} \hat{\mathscr{M}}$. Thus we put

$$\boldsymbol{Y} := (u \otimes u)\boldsymbol{Y}_0(u \otimes u), \quad Z := (1 \otimes u)Z_0(1 \otimes u).$$

Then the family $\{(\mathfrak{H}, \boldsymbol{X}, u), (\mathfrak{H}, \boldsymbol{Y}, u), Z\}$ forms a matched pair. Hence the map γ given by $\gamma := \sigma \circ \mathrm{Ad}\, Z$ defines an "inversion" on $\mathscr{M}$ and $\hat{\mathscr{M}}$ (in the sense of [**BS**]). Namely, γ is an isomorphism from $\mathscr{M} \bar{\otimes} \hat{\mathscr{M}}$ onto $\hat{\mathscr{M}} \bar{\otimes} \mathscr{M}$ satisfying

$$(2.1) \qquad \begin{cases} (\gamma \otimes id_{\mathscr{M}}) \circ (id_{\mathscr{M}} \otimes \gamma) \circ (\delta \otimes id_{\hat{\mathscr{M}}}) = (id_{\hat{\mathscr{M}}} \otimes \delta) \circ \gamma, \\ (id_{\hat{\mathscr{M}}} \otimes \gamma) \circ (\gamma \otimes id_{\hat{\mathscr{M}}}) \circ (id_{\mathscr{M}} \otimes \hat{\delta}) = (\hat{\delta} \otimes id_{\mathscr{M}}) \circ \gamma. \end{cases}$$

Then the map

$$\delta^{\mathcal{N}} := (id_{\mathcal{M}} \otimes \gamma \otimes id_{\hat{\mathcal{M}}}) \circ (\delta \otimes \hat{\delta})$$

defines a coproduct on $\mathcal{N}$. Moreover, with $V = Z_{12}^* \boldsymbol{X}_{13} Z_{12} \boldsymbol{Y}_{24}$, we have

$$\delta^{\mathcal{N}}(x) = V(x \otimes 1_{\mathfrak{H} \otimes \mathfrak{H}})V^* \quad (x \in \mathcal{N}).$$

It can be verified that $Z = \hat{\boldsymbol{X}}\tilde{\boldsymbol{X}}^*$ with the notation in [**BS**, Section 6]. Since $\tilde{\boldsymbol{X}} = (u \otimes u)W^*(u \otimes u)$ belongs to $\mathcal{M}' \bar{\otimes} \hat{\mathcal{M}}'$, the map γ actually equals $\sigma \circ \operatorname{Ad} \hat{\boldsymbol{X}}$. Since $\hat{\boldsymbol{X}} = W^*$, the inversion γ coincides with σ_W introduced in [**N**, Section 2]. Therefore, our coproduct $\delta^{\mathcal{N}}$ is the same as Nakagami's.

Theorem 2.2. *Retain the notation established above. Let*

$$R^{\mathcal{N}} := (R \otimes \hat{R}) \circ \operatorname{Ad} W^*,$$

$$\tau_t^{\mathcal{N}} := \tau_t \otimes \hat{\tau}_t.$$

Then $(\mathcal{N}, \delta^{\mathcal{N}}, R^{\mathcal{N}})$ *is a coinvolutive Hopf-von Neumann algebra. Each* $\tau_t^{\mathcal{N}}$ *is a coinvolutive Hopf-von Neumann algebra automorphism, i.e., it satisfies*

$$(\tau_t^{\mathcal{N}} \otimes \tau_t^{\mathcal{N}}) \circ \delta^{\mathcal{N}} = \delta^{\mathcal{N}} \circ \tau_t^{\mathcal{N}},$$

$$R^{\mathcal{N}} \circ \tau_t^{\mathcal{N}} = \tau_t^{\mathcal{N}} \circ R^{\mathcal{N}}$$

for any $t \in \boldsymbol{R}$.

Proof. This follows from a combination of Lemma 5 and Lemma 6 of [**N**]. □

3. Haar measure for $(\mathcal{N}, \delta^{\mathcal{N}}, R^{\mathcal{N}}, \tau^{\mathcal{N}})$.

The purpose of this section is to construct a Haar measure for the coinvolutive Hopf-von Neumann algebra $(\mathcal{N}, \delta^{\mathcal{N}}, R^{\mathcal{N}})$ with the deformation automorphism $\tau^{\mathcal{N}}$ defined in the previous section.

Let $h^{\mathcal{N}}$ be the faithful normal semifinite weight on $\mathcal{N} = \mathcal{M} \otimes \hat{\mathcal{M}}$ defined by

$$h^{\mathcal{N}} := h \otimes \hat{h} \circ \hat{R}.$$

We shall show that $h^{\mathcal{N}}$ is the desired Haar measure.

For the next lemma, note that, since the Kac-Takesaki operator W is a unitary in $\mathcal{N}$, the function $t \in \boldsymbol{R} \mapsto W^* \sigma_t^{h \circ R \otimes \hat{h}}(W)$ is a unitary 1-cocycle (coboundary) for the weight $h \circ R \otimes \hat{h}$.

Lemma 3.1. *Let* $u_t := W^* \sigma_t^{h \circ R \otimes \hat{h}}(W)$ *be the* 1*-cocycle mentioned above. Then we have*

$$u_t = W^*(Q^{it} \otimes \hat{Q}^{-it})W.$$

Moreover, the faithful normal semifinite weight on $\mathcal{N}$ *determined by this cocycle is*

$(h \circ R \otimes \hat{R}) \circ \mathrm{Ad}\, W$. *Namely, we have*

$$u_t = (D((h \circ R \otimes \hat{h}) \circ \mathrm{Ad}\, W) : D(h \circ R \otimes \hat{h}))_t$$

for any $t \in \boldsymbol{R}$.

Proof. Since $(D(h \circ R) : Dh)_t = Q^{it}$, it follows from [**N**, Lemma 7] that

$$\sigma_t^{h \circ R \otimes \hat{h}}(W) = (Q^{it} \otimes 1)\sigma_t^{h \otimes \hat{h}}(W)(Q^{-it} \otimes 1) = (Q^{it} \otimes \hat{Q}^{-it})W.$$

This proves the first assertion. Let $\Phi = (h \circ R \otimes \hat{h}) \circ \mathrm{Ad}\, W$. By [**N**, Lemma 7] again, we easily find that $\sigma_t^{\Phi} = \mathrm{Ad}\, u_t \circ \sigma_t^{h \circ R \otimes \hat{h}}$. Now let X be in $\mathfrak{n}_{\Phi} \cap \mathfrak{n}^*_{h \circ R \otimes \hat{h}}$ and Y be in $\mathfrak{n}^*_{\Phi} \cap \mathfrak{n}_{h \circ R \otimes \hat{h}}$. Since $\mathfrak{n}_{\Phi} = \mathfrak{n}_{h \circ R \otimes \hat{h}} W$, it follows that both XW^* and WY belong to $\mathfrak{n}_{h \circ R \otimes \hat{h}} \cap \mathfrak{n}^*_{h \circ R \otimes \hat{h}}$. Hence, by the KMS condition for the weight $h \circ R \otimes \hat{h}$, there exists a bounded continuous function F defined on the strip $\boldsymbol{D} = \{z \in \boldsymbol{C} : 0 \leq \mathrm{Im}\, z \leq 1\}$, which is analytic in the interior, such that

$$F(t) = (h \circ R \otimes \hat{h})(\sigma_t^{h \circ R \otimes \hat{h}}(WY)XW^*) \quad (t \in \boldsymbol{R}),$$

$$F(t+i) = (h \circ R \otimes \hat{h})(XW^* \sigma_t^{h \circ R \otimes \hat{h}}(WY)).$$

These identities can be transformed into

$$F(t) = \Phi(u_t \sigma_t^{h \circ R \otimes \hat{h}}(Y)X), \quad F(t+i) = (h \circ R \otimes \hat{h})(Xu_t \sigma_t^{h \circ R \otimes \hat{h}}(Y)).$$

Therefore, by uniqueness of the Randon-Nikodym derivative, we conclude that the 1-cocycle u_t must equal the Radon-Nikodym derivative $(D\Phi : D(h \circ R \otimes \hat{h}))_t$. □

Lemma 3.2. *Let* $v_t := W^* \sigma_t^{h^{\mathcal{N}}}(W)$ *be the unitary* 1-*cocycle for the weight* $h^{\mathcal{N}}$. *Then the faithful normal semifinite weight on* $\mathcal{N}$ *determined uniquely by this cocycle is* $h \circ R \otimes \hat{h}$. *Thus we have*

$$(D(h \circ R \otimes \hat{h}) : Dh^{\mathcal{N}})_t = W^* \sigma_t^{h^{\mathcal{N}}}(W)$$

for any $t \in \boldsymbol{R}$.

Proof. Since $(D(h \circ R) : Dh)_t = Q^{it}$ and $(D(\hat{h} \circ \hat{R}) : D\hat{h})_t = \hat{Q}^{it}$, one has

$$(D(h \circ R \otimes \hat{h}) : Dh^{\mathcal{N}})_t = (D(h \circ R) : Dh)_t \otimes (D\hat{h} : D(\hat{h} \circ \hat{R}))_t = Q^{it} \otimes \hat{Q}^{-it}.$$

Hence it suffices to show that $v_t = Q^{it} \otimes \hat{Q}^{-it}$. But this follows from a direct computation with the help of [**N**, Lemma 7]. □

Theorem 3.3. *The weight* $h^{\mathcal{N}}$ *is* $R^{\mathcal{N}}$-*invariant, i.e.*,

$$h^{\mathcal{N}} = h^{\mathcal{N}} \circ R^{\mathcal{N}}.$$

Proof. Let $\Phi = ((h \circ R) \otimes \hat{h}) \circ \mathrm{Ad}\, W$. As we saw in the proof of the preceding lemma, we have

$$(D(h \circ R \otimes \hat{h}) : Dh^{\mathcal{N}})_t = Q^{it} \otimes \hat{Q}^{-it}.$$

Hence it follows from Lemma 3.1 that

$$W(D\Phi : D(h\circ R\otimes \hat{h}))_t W^* = (D(h\circ R\otimes \hat{h}) : Dh^{\mathcal{N}})_t.$$

On the other hand, one has

$$\begin{aligned} W(D\Phi : D(h\circ R\otimes \hat{h}))_t W^* &= \mathrm{Ad}\, W((D\Phi : D(h\circ R\otimes \hat{h}))_t) \\ &= (D\Phi\circ \mathrm{Ad}\, W^* : D(h\circ R\otimes \hat{h})\circ \mathrm{Ad}\, W^*)_t \\ &= (D(h\circ R\otimes \hat{h}) : D(h^{\mathcal{N}}\circ R^{\mathcal{N}}))_t \\ &= (D(h\circ R\otimes \hat{h}) : Dh^{\mathcal{N}})_t (Dh^{\mathcal{N}} : D(h^{\mathcal{N}}\circ R^{\mathcal{N}}))_t. \end{aligned}$$

Therefore, we conclude that $(Dh^{\mathcal{N}} : D(h^{\mathcal{N}}\circ R^{\mathcal{N}}))_t \equiv 1$, which implies that $h^{\mathcal{N}} = h^{\mathcal{N}}\circ R^{\mathcal{N}}$. □

In what follows, we set $\Psi = (h\otimes \hat{h})\circ \mathrm{Ad}\, W^*$.

Proposition 3.4. *The weight $h^{\mathcal{N}}$ is σ^{Ψ}-invariant. The Radon Nikodym derivative $(Dh^{\mathcal{N}} : D\Psi)$ of $h^{\mathcal{N}}$ with respect to Ψ is given by*

$$(Dh^{\mathcal{N}} : D\Psi)_t = W(Q^{it}\otimes 1)W^*.$$

In particular, we have $h^{\mathcal{N}} = \Psi(P\,\cdot)$, where P is the nonsingular positive self-adjoint operator defined by $P := W(Q\otimes 1)W^*$.

Proof. Let P be the operator defined as above. First we claim that P is affiliated with the centralizer of the weight Ψ. Indeed, since $P^{it} = W(Q^{it}\otimes 1)W^*$ and $\sigma_t^{\Psi} = \mathrm{Ad}\, W\circ \sigma_t^{h\otimes\hat{h}}\circ \mathrm{Ad}\, W^*$, we clearly obtain $\sigma_s^{\Psi}(P^{it}) = P^{it}$ for any $s, t\in \boldsymbol{R}$.

Next note that we have

$$\sigma_t^{\Psi} = \mathrm{Ad}\, W\circ \sigma_t^{h\otimes\hat{h}}\circ \mathrm{Ad}\, W^* = \mathrm{Ad}\, W\sigma_t^{h\otimes\hat{h}}(W^*)\circ \sigma_t^{h\otimes\hat{h}}.$$

Thus, by [**N**, Lemma 7], we find that

$$\sigma_t^{\Psi} = \mathrm{Ad}\, W(Q^{-it}\otimes 1)W^*(1\otimes \hat{Q}^{it})\circ \sigma_t^{h\otimes\hat{h}} = \mathrm{Ad}\, W(Q^{-it}\otimes 1)W^*\circ \sigma_t^{h^{\mathcal{N}}}. \tag{3.4.1}$$

This shows that $\sigma_t^{h^{\mathcal{N}}} = \mathrm{Ad}\, P^{it}\circ \sigma_t^{\Psi}$. In particular, one has $\Psi\circ \sigma^{h^{\mathcal{N}}} = \Psi$, which in turn implies that $h^{\mathcal{N}}$ is σ^{Ψ}-invariant. Suppose that X is an arbitrary element in $\mathfrak{n}_{h^{\mathcal{N}}}$. For any positive self-adjoint operator K and $\varepsilon > 0$, set $K_\varepsilon := K(1+\varepsilon K)^{-1}$. With this notation, we have

$$\begin{aligned} h^{\mathcal{N}}(X^*X) &= h^{\mathcal{N}}\circ R^{\mathcal{N}}(X^*X) \quad \text{by Theorem 3.3} \\ &= (h\circ R\otimes \hat{h})(W^*X^*XW) \\ &= \lim_{\varepsilon\downarrow 0}(h\otimes \hat{h})((Q_\varepsilon^{1/2}\otimes 1)W^*X^*XW(Q_\varepsilon^{1/2}\otimes 1)) \\ &= \lim_{\varepsilon\downarrow 0}(h\otimes \hat{h})(W^*P_\varepsilon^{1/2}X^*XP_\varepsilon^{1/2}W) \\ &= \lim_{\varepsilon\downarrow 0}\Psi(P_\varepsilon^{1/2}X^*XP_\varepsilon^{1/2}) = \Psi(PX^*X). \end{aligned}$$

Hence $h^{\mathscr{N}}$ equals $\Psi(P\cdot)$ on $\mathfrak{m}_{h^{\mathscr{N}}}$. It follows from [**PT**, Proposition 5.9] that $h^{\mathscr{N}} = \Psi(P\cdot)$. □

Lemma 3.5. *Let $\mathscr{A}$ be a von Neumann algebra and ψ a faithful normal semifinite weight on $\mathscr{A}$. Suppose that X is in $\mathfrak{n}_{\psi\otimes h}$. Then we have*

$$(\psi\otimes\omega\otimes h)((id_{\mathscr{A}}\otimes\delta)(X^*X)) = \omega(1)(\psi\otimes h)(X^*X)$$

for any $\omega\in\mathscr{M}_*^+$.

Proof. First we consider the case where ω is of the form $\omega = \omega_{\Lambda_h(x)}$ for some $x\in\mathfrak{n}_h$. Since $\omega(y) = h(x^*yx)$ for any $y\in\mathscr{M}_+$, it follows that

$$\omega(y) = h(x^*yx)$$

for any y in the extended positive part $\mathrm{Ext}_+(\mathscr{M})$ in the sense of [**H1**]. Thus we have

$$\begin{aligned}\omega((\psi\otimes id\otimes h)&((id_{\mathscr{A}}\otimes\delta)(X^*X)))\\ &= h(x^*(\psi\otimes id\otimes h)((id_{\mathscr{A}}\otimes\delta)(X^*X))x)\\ &= (\psi\otimes h\otimes h)((1\otimes x^*\otimes 1)(id_{\mathscr{A}}\otimes\delta)(X^*X)(1\otimes x\otimes 1)).\end{aligned}$$

Namely,

$$\begin{aligned}(3.5.1)\qquad (\psi\otimes\omega\otimes h)&((id_{\mathscr{A}}\otimes\delta)(X^*X))\\ &= (\psi\otimes h\otimes h)((1\otimes x^*\otimes 1)(id_{\mathscr{A}}\otimes\delta)(X^*X)(1\otimes x\otimes 1)).\end{aligned}$$

Now, by the density theorem for left Hilbert algebra in [**H**], there exists a sequence $\{X_n\}$ in the algebraic tensor product $\mathfrak{n}_\psi\otimes\mathfrak{n}_h$ such that

$$\|X_n\|\le\|X\|,\quad \lim_{n\to\infty}\Lambda_{\psi\otimes h}(X_n) = \Lambda_{\psi\otimes h}(X).$$

In particular, X_n strongly converges to X. With this $\{X_n\}$, it is readily checked that one has

$$(3.5.2)\qquad (1\otimes W)(\Sigma\otimes 1)\Lambda_{h\otimes\psi\otimes h}(x\otimes X_n) = \Lambda_{\psi\otimes h\otimes h}((id_{\mathscr{A}}\otimes\delta)(X_n)(1\otimes x\otimes 1)).$$

For any $\eta\in(\mathfrak{A}_\psi\otimes\mathfrak{A}_h\otimes\mathfrak{A}_h)' = (\mathfrak{A}_{\psi\otimes h\otimes h})'$, we have

$$\begin{aligned}\lim_{n\to\infty}\pi_r(\eta)\Lambda_{\psi\otimes h\otimes h}((id_{\mathscr{A}}\otimes\delta)(X_n)(1\otimes x\otimes 1)) &= \lim_{n\to\infty}(id_{\mathscr{A}}\otimes\delta)(X_n)(1\otimes x\otimes 1)\eta\\ &= (id_{\mathscr{A}}\otimes\delta)(X)(1\otimes x\otimes 1)\eta.\end{aligned}$$

On the other hand, by (3.5.2), one has

$$\begin{aligned}\lim_{n\to\infty}\pi_r(\eta)\Lambda_{\psi\otimes h\otimes h}&((id_{\mathscr{A}}\otimes\delta)(X_n)(1\otimes x\otimes 1))\\ &= \lim_{n\to\infty}\pi_r(\eta)(1\otimes W)(\Sigma\otimes 1)\Lambda_{h\otimes\psi\otimes h}(x\otimes X_n)\\ &= \lim_{n\to\infty}\pi_r(\eta)(1\otimes W)(\Sigma\otimes 1)(\Lambda_h(x)\otimes\Lambda_{\psi\otimes h}(X_n))\\ &= \pi_r(\eta)(1\otimes W)(\Sigma\otimes 1)(\Lambda_h(x)\otimes\Lambda_{\psi\otimes h}(X)).\end{aligned}$$

This shows that the vector $(1 \otimes W)(\Sigma \otimes 1)(\Lambda_{h \otimes \psi \otimes h}(x \otimes X))$ is left bounded with respect to the left Hilbert algebra $\mathfrak{A}_{\psi \otimes h \otimes h}$, and

$$\pi_\ell((1 \otimes W)(\Sigma \otimes 1)(\Lambda_{h \otimes \psi \otimes h}(x \otimes X))) = (id_{\mathscr{A}} \otimes \delta)(X)(1 \otimes x \otimes 1).$$

Particularly, $(id_{\mathscr{A}} \otimes \delta)(X)(1 \otimes x \otimes 1)$ belongs to $\mathfrak{n}_{\psi \otimes h \otimes h}$. So we obtain

$$(1 \otimes W)(\Sigma \otimes 1)\Lambda_{h \otimes \psi \otimes h}(x \otimes X) = \Lambda_{\psi \otimes h \otimes h}((id_{\mathscr{A}} \otimes \delta)(X)(1 \otimes x \otimes 1)). \tag{3.5.3}$$

From (3.5.1), (3.5.2), (3.5.3) and quasi left invariance of h, it follows that

$$\begin{aligned}(\psi \otimes \omega \otimes h)((id_{\mathscr{A}} \otimes \delta)(X^*X)) &= \|\Lambda_{\psi \otimes h \otimes h}((id_{\mathscr{A}} \otimes \delta)(X)(1 \otimes x \otimes 1))\|^2 \\ &= \|\Lambda_{h \otimes \psi \otimes h}(x \otimes X)\|^2 \\ &= \lim_{n \to \infty} \|\Lambda_{h \otimes \psi \otimes h}(x \otimes X_n)\|^2 \\ &= \lim_{n \to \infty} \|\Lambda_{\psi \otimes h \otimes h}((id_{\mathscr{A}} \otimes \delta)(X_n)(1 \otimes x \otimes 1))\|^2 \\ &= \lim_{n \to \infty} (\psi \otimes \omega \otimes h)((id_{\mathscr{A}} \otimes \delta)(X_n^* X_n)) \\ &= \lim_{n \to \infty} \omega(1)(\psi \otimes h)(X_n^* X_n^*) \\ &= \omega(1) \lim_{n \to \infty} \|\Lambda_{\psi \otimes h}(X_n)\|^2 \\ &= \omega(1)\|\Lambda_{\psi \otimes h}(X)\|^2 = \omega(1)(\psi \otimes h)(X^*X).\end{aligned}$$

To show that the assertion of this lemma is true for any ω in $\mathscr{M}_*^+$, we proceed as follows (cf. [**S**, Lemma II.8]). Regard $m = (\psi \otimes id \otimes h)((id_{\mathscr{A}} \otimes \delta)(X^*X))$ as an element of $\mathrm{Ext}_+(\mathscr{M})$. By [**H1**, Lemma 1.4], there exist a closed subspace $\mathfrak{K}$ of $\mathfrak{H}$ and a densely defined positive self-adjoint operator A on $\mathfrak{K}$ such that

$$m(\omega_\xi) = \begin{cases} \|A^{1/2}\xi\|^2, & (\xi \in \mathfrak{D}(A^{1/2})); \\ \infty, & \text{otherwise.} \end{cases}$$

From the preceding paragraph, $\mathfrak{D}(A^{1/2})$ contains $\Lambda_h(\mathfrak{n}_h)$, so that $\mathfrak{K}$ equals $\mathfrak{H}$. If $\xi \in \Lambda_h(\mathfrak{n}_h)$, then

$$\|A^{1/2}\xi\|^2 = \|(\psi \otimes h)(X^*X)^{1/2}\xi\|^2.$$

From the uniqueness of polar decomposition, we find that $A^{1/2} = (\psi \otimes h)(X^*X)^{1/2} \cdot 1$. Since $\{\mathscr{M}, \mathfrak{H}\}$ is a standard representation, it follows that

$$m(\omega) = \omega(1)(\psi \otimes h)(X^*X)$$

for any $\omega \in \mathscr{M}_*^+$. This proves the assertion. In particular, $m = (\psi \otimes id \otimes h) \cdot ((id_{\mathscr{A}} \otimes \delta)(X^*X))$ actually belongs to $\mathscr{M}$. □

COROLLARY 3.6. *Let $\mathscr{B}$ be a von Neumann algebra and ϕ a faithful normal semifinite weight on $\mathscr{B}$. Suppose that X is in $\mathfrak{n}_{id_{\mathscr{B}} \otimes \phi \otimes h}$. Then we have*

$$(\theta \otimes \phi \otimes \omega \otimes h)((id_{\mathscr{B}} \otimes id_{\mathscr{B}} \otimes \delta)(X^*X)) = \omega(1)(\theta \otimes \phi \otimes h)(X^*X)$$

for any $\theta \in \mathscr{B}_*^+$ *and* $\omega \in \mathscr{M}_*^+$.

Proof. Let us take an $\omega \in \mathscr{M}_*^+$ and fix it. First we treat the case where $\theta \in \mathscr{B}_*^+$ is of the form $\theta = \omega_{\Lambda_\phi(x)}$ for some $x \in \mathfrak{n}_\phi$. Then, as in the previous lemma, one can show that

$$\begin{aligned}(\theta \otimes \phi \otimes \omega \otimes h)&((id \otimes id \otimes \delta)(X^*X)) \\ &= (\phi \otimes \phi \otimes \omega \otimes h)((id \otimes id \otimes \delta)((x^* \otimes 1 \otimes 1)X^*X(x \otimes 1 \otimes 1))).\end{aligned}$$

Applying Lemma 3.5 to our situation in which $\mathscr{A} = \mathscr{B} \bar{\otimes} \mathscr{B}$ and $\psi = \phi \otimes \phi$, we obtain

$$\begin{aligned}(\theta \otimes \phi \otimes \omega \otimes h)&((id \otimes id \otimes \delta)(X^*X)) \\ &= \omega(1)(\phi \otimes \phi \otimes h)((x^* \otimes 1 \otimes 1)X^*X(x \otimes 1 \otimes 1)) \\ &= \omega(1)(\theta \otimes \phi \otimes h)(X^*X).\end{aligned}$$

To prove that the above identity holds for a general $\theta \in \mathscr{B}_*^+$, one may just follow the argument set out in the last paragraph of the proof of the preceding lemma. The details are left to readers. □

Theorem 3.7. *The weight* $h^{\mathscr{N}}$ *is quasi left invariant.*

Proof. Let X be any element in the algebraic tensor product $\mathfrak{n}_h \otimes \mathfrak{n}_{\hat{h} \circ \hat{R}} \subseteq \mathfrak{n}_{h^{\mathscr{N}}}$. First we claim that the identity

$$(\psi \otimes h^{\mathscr{N}})(\delta^{\mathscr{N}}(X^*X)) = h^{\mathscr{N}}(X^*X)\psi(1)$$

holds for any $\psi \in \mathscr{N}_*^+$. For simplicity, we assume that X has the form $X = a \otimes x$, where $a \in \mathfrak{n}_h$ and $x \in \mathfrak{n}_{\hat{h} \circ \hat{R}}$. We assume also for the moment that ψ is of the form $\psi = \theta \otimes \omega$ for some $\theta \in \mathscr{M}_*^+$ and $\omega \in \hat{\mathscr{M}}_*^+$. Then, by Proposition 3.4, we have

$$\begin{aligned}(\psi \otimes h^{\mathscr{N}})&(\delta^{\mathscr{N}}(X^*X)) \\ &= \lim_{\varepsilon \downarrow 0}(\theta \otimes \omega \otimes \Psi)((P_\varepsilon^{1/2})_{34}\delta^{\mathscr{N}}(a^*a \otimes x^*x)(P_\varepsilon^{1/2})_{34}) \\ &= \lim_{\varepsilon \downarrow 0}(\theta \otimes \omega \otimes h \otimes \hat{h})(W_{34}^*(P_\varepsilon^{1/2})_{34}\delta^{\mathscr{N}}(a^*a \otimes x^*x)(P_\varepsilon^{1/2})_{34}W_{34}) \\ &= \lim_{\varepsilon \downarrow 0}(\theta \otimes \omega \otimes h \otimes \hat{h})((1 \otimes 1 \otimes Q_\varepsilon^{1/2} \otimes 1)W_{34}^*\delta^{\mathscr{N}}(a^*a \otimes x^*x)W_{34}(1 \otimes 1 \otimes Q_\varepsilon^{1/2} \otimes 1)) \\ &= (\theta \otimes \omega \otimes h \circ R \otimes \hat{h})(W_{34}^*\delta^{\mathscr{N}}(a^*a \otimes x^*x)W_{34}).\end{aligned}$$

Here we used the notation introduced in the proof of Proposition 3.4. Since

$$\delta^{\mathscr{N}} = \mathrm{Ad}\,\Sigma_{23}W_{23}^* \circ (\delta \otimes \hat{\delta}),$$

$$W_{13}^*W_{12}^* = (id_{\mathscr{M}} \otimes \hat{\delta})(W^*),$$

we find that

$$\begin{aligned}\operatorname{Ad} W_{34}^* \circ \delta^{\mathcal{N}}(a^*a \otimes x^*x) &= \operatorname{Ad} W_{34}^* \Sigma_{23} W_{23}^* \circ (id_{\mathcal{M}} \otimes id_{\mathcal{M}} \otimes \hat{\delta})(\delta(a^*a) \otimes x^*x) \\ &= \operatorname{Ad} \Sigma_{23} W_{24}^* W_{23}^* \circ (id_{\mathcal{M}} \otimes id_{\mathcal{M}} \otimes \hat{\delta})(\delta(a^*a) \otimes x^*x) \\ &= \operatorname{Ad} \Sigma_{23} \circ (id_{\mathcal{M}} \otimes id_{\mathcal{M}} \otimes \hat{\delta})(W_{23}^*(\delta(a^*a) \otimes x^*x) W_{23}).\end{aligned}$$

From this, it follows that

$$(\psi \otimes h^{\mathcal{N}})(\delta^{\mathcal{N}}(X^*X)) = (\theta \otimes h \circ R \otimes \omega \otimes \hat{h})((id_{\mathcal{M}} \otimes id_{\mathcal{M}} \otimes \hat{\delta})(W_{23}^*(\delta(a^*a) \otimes x^*x) W_{23})).$$

Meanwhile, by Theorem 3.3, we obtain

$$\begin{aligned}&(id_{\mathcal{M}} \otimes h \circ R \otimes \hat{h})(W_{23}^*(\delta(a^*a) \otimes x^*x) W_{23}) \\ &\quad = (id_{\mathcal{M}} \otimes (h \circ R \otimes \hat{h}) \circ \operatorname{Ad} W^*)(\delta(a^*a) \otimes x^*x) \\ &\quad = (id_{\mathcal{M}} \otimes h^{\mathcal{N}} \circ R^{\mathcal{N}})(\delta(a^*a) \otimes x^*x) \\ &\quad = (id_{\mathcal{M}} \otimes h^{\mathcal{N}})(\delta(a^*a) \otimes x^*x) \\ &\quad = (id_{\mathcal{M}} \otimes h)(\delta(a^*a))\hat{h} \circ \hat{R}(x^*x) \\ &\quad = h(a^*a)\hat{h} \circ \hat{R}(x^*x) \cdot 1.\end{aligned}$$

This implies that the element $(\delta(a) \otimes x) W_{23}$ belongs to $\mathfrak{n}_{id_{\mathcal{M}} \otimes h \circ R \otimes \hat{h}}$. It results from Corollary 3.6 that

$$\begin{aligned}&(\theta \otimes h \circ R \otimes \omega \otimes \hat{h})((id_{\mathcal{M}} \otimes id_{\mathcal{M}} \otimes \hat{\delta})(W_{23}^*(\delta(a^*a) \otimes x^*x) W_{23})) \\ &\quad = \omega(1)(\theta \otimes h \circ R \otimes \hat{h})(W_{23}^*(\delta(a^*a) \otimes x^*x) W_{23}) \\ &\quad = \omega(1)\theta(1)h(a^*a)\hat{h} \circ \hat{R}(x^*x) \\ &\quad = \omega(1)\theta(1)h^{\mathcal{N}}(X^*X) = \psi(1)h^{\mathcal{N}}(X^*X).\end{aligned}$$

Therefore our claim is valid when ψ has the form described above. But the argument set out in the last paragraph of the proof of Lemma 3.5 guarantees that this claim is still valid even for a general $\psi \in \mathcal{N}_*^+$.

Now we let X be an arbitrary element in $\mathfrak{n}_{h^{\mathcal{N}}}$. Then, by the density theorem for left Hilbert algebras mentioned before, there exists a sequence $\{X_n\}$ in $\mathfrak{n}_h \otimes \mathfrak{n}_{\hat{h} \circ \hat{R}}$ such that

$$\|X_n\| \leq \|X\|, \quad \lim_{n \to \infty} \Lambda_{h^{\mathcal{N}}}(X_n) = \Lambda_{h^{\mathcal{N}}}(X).$$

In particular, X_n strongly converges to X. In the meantime, from the preceding paragraph, it is easy to see that, for $A, B \in \mathfrak{n}_h \otimes \mathfrak{n}_{\hat{h} \circ \hat{R}}$, we have

$$\begin{aligned}(h^{\mathcal{N}} \otimes h^{\mathcal{N}})((A^* \otimes 1)\delta^{\mathcal{N}}(B^*B)(A \otimes 1)) &= (\omega_{\Lambda_{h^{\mathcal{N}}}(A)} \otimes h^{\mathcal{N}})(\delta^{\mathcal{N}}(B^*B)) \\ &= h^{\mathcal{N}}(A^*A)h^{\mathcal{N}}(B^*B).\end{aligned}$$

Thus the equation

$$W^{\mathcal{N}}(\Lambda_{h^{\mathcal{N}}}(A)\otimes\Lambda_{h^{\mathcal{N}}}(B))=\Lambda_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}(\delta^{\mathcal{N}}(B)(A\otimes 1_{\mathcal{N}}))\quad (A,B\in\mathfrak{n}_h\otimes\mathfrak{n}_{\hat{h}\circ\hat{R}})$$

defines an isometry $W^{\mathcal{N}}$ on $\mathfrak{H}_{h^{\mathcal{N}}}\otimes\mathfrak{H}_{h^{\mathcal{N}}}$. For any $A\in\mathfrak{n}_h\otimes\mathfrak{n}_{\hat{h}\circ\hat{R}}$, one has

$$\begin{aligned}\lim_{n\to\infty}\Lambda_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}(\delta^{\mathcal{N}}(X_n)(A\otimes 1_{\mathcal{N}}))&=\lim_{n\to\infty}W^{\mathcal{N}}(\Lambda_{h^{\mathcal{N}}}(A)\otimes\Lambda_{h^{\mathcal{N}}}(X_n))\\&=W^{\mathcal{N}}(\Lambda_{h^{\mathcal{N}}}(A)\otimes\Lambda_{h^{\mathcal{N}}}(X)).\end{aligned}$$

Hence, for any $\eta_1,\eta_2\in\mathfrak{A}'_{h^{\mathcal{N}}}$, we have

$$\begin{aligned}&(\pi_r(\eta_1)\otimes\pi_r(\eta_2))W^{\mathcal{N}}(\Lambda_{h^{\mathcal{N}}}(A)\otimes\Lambda_{h^{\mathcal{N}}}(X))\\&\quad=\lim_{n\to\infty}(\pi_r(\eta_1)\otimes\pi_r(\eta_2))\Lambda_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}(\delta^{\mathcal{N}}(X_n)(A\otimes 1_{\mathcal{N}}))\\&\quad=\lim_{n\to\infty}\delta^{\mathcal{N}}(X_n)(A\otimes 1_{\mathcal{N}})(\eta_1\otimes\eta_2)\\&\quad=\delta^{\mathcal{N}}(X)(A\otimes 1_{\mathcal{N}})(\eta_1\otimes\eta_2).\end{aligned}$$

This proves that the vector $W^{\mathcal{N}}(\Lambda_{h^{\mathcal{N}}}(A)\otimes\Lambda_{h^{\mathcal{N}}}(X))$ is left bounded with respect to the left Hilbert algebra $\mathfrak{A}_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}$, and we have

$$\pi_\ell(W^{\mathcal{N}}(\Lambda_{h^{\mathcal{N}}}(A)\otimes\Lambda_{h^{\mathcal{N}}}(X)))=\delta^{\mathcal{N}}(X)(A\otimes 1).$$

In particular, $\delta^{\mathcal{N}}(X)(A\otimes 1)$ belongs to $\mathfrak{n}_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}$, and one has

$$W^{\mathcal{N}}(\Lambda_{h^{\mathcal{N}}}(A)\otimes\Lambda_{h^{\mathcal{N}}}(X))=\Lambda_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}(\delta^{\mathcal{N}}(X)(A\otimes 1)).$$

From this, it follows that

$$\begin{aligned}h^{\mathcal{N}}(A^*A)h^{\mathcal{N}}(X^*X)&=\|\Lambda_{h^{\mathcal{N}}}(A)\otimes\Lambda_{h^{\mathcal{N}}}(X))\|^2\\&=\|W^{\mathcal{N}}(\Lambda_{h^{\mathcal{N}}}(A)\otimes\Lambda_{h^{\mathcal{N}}}(X))\|^2\\&=\|\Lambda_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}(\delta^{\mathcal{N}}(X)(A\otimes 1))\|^2\\&=(h^{\mathcal{N}}\otimes h^{\mathcal{N}})((A^*\otimes 1)\delta^{\mathcal{N}}(X^*X)(A\otimes 1))\\&=(\omega_{\Lambda_{h^{\mathcal{N}}}(A)}\otimes h^{\mathcal{N}})(\delta^{\mathcal{N}}(X^*X))\end{aligned}$$

Therefore, the identity

$$(\psi\otimes h^{\mathcal{N}})(\delta^{\mathcal{N}}(X^*X))=h^{\mathcal{N}}(X^*X)\psi(1)$$

holds for any $\psi\in\mathcal{N}_*^+$ of the form $\psi=\omega_{\Lambda_{h^{\mathcal{N}}}(A)}$ for some $A\in\mathfrak{n}_h\otimes\mathfrak{n}_{\hat{h}\circ\hat{R}}$. But, again, the argument set out in the last paragraph of the proof of Lemma 3.5 enables us to conclude that this identity is still true for <u>*all*</u> $\psi\in\mathcal{N}_*^+$. This completes the proof. □

THEOREM 3.8. *The weight $h^{\mathcal{N}}$ is strongly left invariant.*

PROOF. We closely follow the proof of Lemma 10 in [**N**]. But there are minor misprints in that proof, so that we proceed, correcting them.

Let X,Y be in $\mathfrak{n}_{h^{\mathcal{N}}}$. We denote the set of entire analytic elements in $\mathcal{M}_*$ with resepct to the adjoint action of the deformation automorphism $\{\tau_t\}$ by $(\mathcal{M}_*)_\tau^\infty$. The set

$(\hat{\mathscr{M}}_*)_{\hat{\tau}}^{\infty}$ is defined in the same way. We first take an element ϕ from the algebraic tensor product $(\mathscr{M}_*)_{\tau}^{\infty} \otimes (\hat{\mathscr{M}}_*)_{\hat{\tau}}^{\infty}$. For simplicity, we assume that ϕ has the form $\theta \otimes \omega$. Put $\tilde{\theta} := \theta \circ \tau_{-i/2} \circ R$ and $\tilde{\omega} := \omega \circ \hat{\tau}_{-i/2} \circ \hat{R}$. Then we have

$$\begin{aligned}
&(\phi \circ \tau_{-i/2}^{\mathscr{N}} \circ R^{\mathscr{N}} \otimes h^{\mathscr{N}})(\delta^{\mathscr{N}}(Y^*)(1_{\mathscr{N}} \otimes X)) \\
&\quad = (\tilde{\theta} \otimes \tilde{\omega} \otimes h \otimes \hat{h} \circ \hat{R}) \circ \operatorname{Ad} W_{12}^*(\operatorname{Ad} \Sigma_{23} W_{23}^* \circ (\delta \otimes \hat{\delta})(Y^*) \cdot (1 \otimes 1 \otimes X)) \\
&\quad = (\tilde{\theta} \otimes \tilde{\omega} \otimes h \otimes \hat{h} \circ \hat{R}) \circ \operatorname{Ad} \Sigma_{23}(\operatorname{Ad} W_{13}^* W_{23}^* \circ (\delta \otimes \hat{\delta})(Y^*) \cdot \operatorname{Ad} \Sigma_{23}(1 \otimes 1 \otimes X)).
\end{aligned}$$

Since $W_{13}^* W_{23}^* = (\delta \otimes id_{\hat{\mathscr{M}}})(W^*)$, it follows that

$$\begin{aligned}
(3.8.1) \qquad &(\phi \circ \tau_{-i/2}^{\mathscr{N}} \circ R^{\mathscr{N}} \otimes h^{\mathscr{N}})(\delta^{\mathscr{N}}(Y^*)(1_{\mathscr{N}} \otimes X)) \\
&\quad = (\tilde{\theta} \otimes h \otimes \tilde{\omega} \otimes \hat{h} \circ \hat{R})((\delta \otimes id_{\hat{\mathscr{M}}} \otimes id_{\hat{\mathscr{M}}}) \circ \operatorname{Ad} W_{12}^* \circ (id_{\mathscr{M}} \otimes \hat{\delta})(Y^*) \\
&\qquad \times \operatorname{Ad} \Sigma_{23}(1 \otimes 1 \otimes X)).
\end{aligned}$$

Now, from Theorem 3.3, we find that

$$(h \circ R \otimes \hat{h})(W^* Y^* Y W) = (h \circ R \otimes \hat{h}) \circ \operatorname{Ad} W^*(Y^* Y) = h^{\mathscr{N}}(Y^* Y) < \infty.$$

Hence, from Lemma 3.5 and Theorem 3.3 again, it results that

$$\begin{aligned}
&(h \otimes \tilde{\omega} \otimes \hat{h} \circ \hat{R})(W_{12}^*(id_{\mathscr{M}} \otimes \hat{\delta})(Y^* Y) W_{12}) \\
&\quad = (h \otimes \tilde{\omega} \otimes \hat{h} \circ \hat{R})(W_{13}^* W_{12}^*(id_{\mathscr{M}} \otimes \hat{\delta})(Y^* Y) W_{12} W_{13}) \\
&\quad = (h \otimes \tilde{\omega} \otimes \hat{h} \circ \hat{R})((id_{\mathscr{M}} \otimes \hat{\delta})(W^* Y^* Y W)) \\
&\quad = \tilde{\omega}(1)(h \circ R \otimes \hat{R})(W^* Y^* Y W) \\
&\quad = \tilde{\omega}(1) h^{\mathscr{N}}(Y^* Y) < \infty.
\end{aligned}$$

This shows that $W_{12}^*(id_{\mathscr{M}} \otimes \hat{\delta})(Y) W_{12}$ belongs to $\mathfrak{n}_{h \otimes \tilde{\omega} \otimes \hat{h} \circ \hat{R}}$. Thus we may use the strong left invariance of the weight h to deduce that the right-hand side of (3.8.1) equals

$$\begin{aligned}
(3.8.2) \qquad &(\theta \otimes h \otimes \tilde{\omega} \otimes \hat{h} \circ \hat{R})(\operatorname{Ad} W_{23}^* \circ (id_{\mathscr{M}} \otimes id_{\mathscr{M}} \otimes \hat{\delta})(Y_{23}^*) \\
&\qquad \times (\delta \otimes id_{\hat{\mathscr{M}}} \otimes id_{\hat{\mathscr{M}}})(\operatorname{Ad} \Sigma_{12}(1 \otimes X))).
\end{aligned}$$

Let $A = \operatorname{Ad} W_{23}^* \circ (id_{\mathscr{M}} \otimes id_{\mathscr{M}} \otimes \hat{\delta})(Y_{23}^*) \cdot (\delta \otimes id_{\hat{\mathscr{M}}} \otimes id_{\hat{\mathscr{M}}})(\operatorname{Ad} \Sigma_{12}(1 \otimes X))$. Using Proposition 3.4, we further compute (3.8.2) as follows.

$$\begin{aligned}
(3.8.2) &= \lim_{\varepsilon \downarrow 0} (\theta \otimes h \otimes \tilde{\omega} \otimes \hat{h})(W_{24}^* (P_\varepsilon^{1/2})_{24} A (P_\varepsilon^{1/2})_{24} W_{24}) \\
&= \lim_{\varepsilon \downarrow 0} (\theta \otimes h \otimes \tilde{\omega} \otimes \hat{h})((1 \otimes Q_\varepsilon^{1/2} \otimes 1 \otimes 1) W_{24}^* A W_{24} (1 \otimes Q_\varepsilon^{1/2} \otimes 1 \otimes 1)) \\
&= \lim_{\varepsilon \downarrow 0} (\theta \otimes h \otimes \tilde{\omega} \otimes \hat{h})((id_{\mathscr{M}} \otimes id_{\mathscr{M}} \otimes \hat{\delta})((1 \otimes Q_\varepsilon^{1/2} \otimes 1) W_{23}^* Y_{23}^* W_{23}) \\
&\qquad \times \operatorname{Ad} W_{24}^* \circ (\delta \otimes id_{\hat{\mathscr{M}}} \otimes id_{\hat{\mathscr{M}}})(\operatorname{Ad} \Sigma_{12}(1 \otimes X))(1 \otimes Q_\varepsilon^{1/2} \otimes 1 \otimes 1)).
\end{aligned}$$

Let $X(\varepsilon) := W_{24}^*(\delta \otimes id_{\hat{\mathcal{M}}} \otimes id_{\hat{\mathcal{M}}})(\mathrm{Ad}\,\Sigma_{12}(1 \otimes X))W_{24}(1 \otimes Q_\varepsilon^{1/2} \otimes 1 \otimes 1)$ and $Y(\varepsilon) := (Q_\varepsilon^{1/2} \otimes 1)W^*YW$. Then, by Proposition 3.4, we have

$$\begin{aligned}(\theta \otimes h \otimes \tilde{\omega} \otimes \hat{h})&(X(\varepsilon)^*X(\varepsilon))\\ &= (\theta \otimes \Psi_{24} \otimes \tilde{\omega})((P_\varepsilon^{1/2})_{24}(\delta \otimes id_{\hat{\mathcal{M}}} \otimes id_{\hat{\mathcal{M}}})(\mathrm{Ad}\,\Sigma_{12}(1 \otimes X))(P_\varepsilon^{1/2})_{24})\\ &\leq (\theta \otimes h_{24}^{\mathcal{N}} \otimes \tilde{\omega})((\delta \otimes id_{\hat{\mathcal{M}}} \otimes id_{\hat{\mathcal{M}}})(\mathrm{Ad}\,\Sigma_{12}(1 \otimes X))\\ &= \theta(1)(h \otimes \tilde{\omega} \otimes \hat{h} \circ \hat{R})(\mathrm{Ad}\,\Sigma_{12}(1 \otimes X))\\ &= \theta(1)\tilde{\omega}(1)h^{\mathcal{N}}(X^*X) < \infty.\end{aligned}$$

Moreover, one has

$$\begin{aligned}(h \otimes \hat{h})(Y(\varepsilon)^*Y(\varepsilon)) &= \Psi(P_\varepsilon^{1/2}Y^*YP_\varepsilon^{1/2})\\ &\leq \Psi(PY^*Y) = h^{\mathcal{N}}(Y^*Y) < \infty.\end{aligned}$$

Hence we may use the strong left invariance of the weight $\hat{h}$ to conclude that (3.8.2) equals

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}(\theta \otimes h \otimes \omega \otimes \hat{h})&((1 \otimes Q_\varepsilon^{1/2} \otimes 1 \otimes 1)W_{24}^*Y_{24}^*W_{24}\\ &\times (id_{\mathcal{M}} \otimes id_{\mathcal{M}} \otimes \hat{\delta})(\mathrm{Ad}\,W_{23}^* \circ (\delta \otimes id_{\hat{\mathcal{M}}})(X)(1 \otimes Q_\varepsilon^{1/2} \otimes 1))).\end{aligned}$$

Since $(id_{\mathcal{M}} \otimes id_{\mathcal{M}} \otimes \hat{\delta})(W_{23}^*) = W_{24}^*W_{23}^*$, we can continue to calculate the above limit as follows.

$$\begin{aligned}\text{the limit} &= \lim_{\varepsilon\downarrow 0}(\theta \otimes h \otimes \omega \otimes \hat{h})(W_{24}^*(P_\varepsilon^{1/2})_{24}Y_{24}^* \cdot \mathrm{Ad}\,W_{23}^* \circ (\delta \otimes \hat{\delta})(X)(P_\varepsilon^{1/2})_{24}W_{24})\\ &= (\theta \otimes h \otimes \omega \otimes \hat{h} \circ \hat{R})(Y_{24}^* \cdot \mathrm{Ad}\,W_{23}^* \circ (\delta \otimes \hat{\delta})(X))\\ &= (\theta \otimes \omega \otimes h^{\mathcal{N}})(\Sigma_{23}Y_{24}^*\Sigma_{23} \cdot \mathrm{Ad}\,\Sigma_{23}W_{23}^* \circ (\delta \otimes \hat{\delta})(X))\\ &= (\phi \otimes h^{\mathcal{N}})((1_{\mathcal{N}} \otimes Y^*)\delta^{\mathcal{N}}(X)).\end{aligned}$$

Therefore, strong left invariance holds for any normal functional in $(\mathcal{M}_*)_\tau^\infty \otimes (\hat{\mathcal{M}}_*)_{\hat{\tau}}^\infty$.

Now we take any analytic element ϕ in $(\mathcal{N}_*)_{\tau^{\mathcal{N}}}^\infty$. Then we choose a sequence $\{\phi_n\}$ in $(\mathcal{M}_*)_\tau^\infty \otimes (\hat{\mathcal{M}}_*)_{\hat{\tau}}^\infty$ that converges to ϕ in norm. We define entire functions f_n and f which is bounded on the strip $\boldsymbol{D} = \{z \in \boldsymbol{C} : 0 \leq \mathrm{Im}\,z \leq 1/2\}$ as follows.

$$f_n(z) := (\phi_n \circ \tau_{z-i/2}^{\mathcal{N}} \circ R^{\mathcal{N}} \otimes h^{\mathcal{N}})(\delta^{\mathcal{N}}(Y^*)(1_{\mathcal{N}} \otimes X)), \quad (z \in \boldsymbol{C}),$$

$$f(z) := (\phi \circ \tau_{z-i/2}^{\mathcal{N}} \circ R^{\mathcal{N}} \otimes h^{\mathcal{N}})(\delta^{\mathcal{N}}(Y^*)(1_{\mathcal{N}} \otimes X)).$$

Indeed, from the preceding paragraph, we have

$$f_n(t) = (\phi_n \otimes h^{\mathcal{N}})((1 \otimes \tau_t^{\mathcal{N}}(Y^*))\delta^{\mathcal{N}}(\tau_t^{\mathcal{N}}(X))), \quad (t \in \boldsymbol{R})$$

$$f_n\left(t + \frac{i}{2}\right) = (\phi_n \circ R^{\mathcal{N}} \otimes h^{\mathcal{N}})(\delta^{\mathcal{N}}(\tau_t^{\mathcal{N}}(Y^*))(1 \otimes \tau_t^{\mathcal{N}}(X))).$$

(Note: the above equalities are stated wrong in [**N**]). From the proof of [**N**, Lemma 10], the functions $\{f_n\}$ converges uniformly to a bounded continuous function g on the strip $\boldsymbol{D}$, which is analytic in the interior and satisfies

$$g(t) = (\phi \otimes h^{\mathcal{N}})((1 \otimes \tau_t^{\mathcal{N}}(Y^*))\delta^{\mathcal{N}}(\tau_t^{\mathcal{N}}(X))),$$
$$g\left(t + \frac{i}{2}\right) = (\phi \circ R^{\mathcal{N}} \otimes h^{\mathcal{N}})(\delta^{\mathcal{N}}(\tau_t^{\mathcal{N}}(Y^*))(1 \otimes \tau_t^{\mathcal{N}}(X))).$$

Hence f equals g on the line $\boldsymbol{R} + i/2$, which implies that f must coincide with g on the strip $\boldsymbol{D}$. In particular, we have $f(0) = g(0)$. This means that the identity

$$(\phi \circ \tau_{-i/2}^{\mathcal{N}} \circ R^{\mathcal{N}} \otimes h^{\mathcal{N}})(\delta^{\mathcal{N}}(Y^*)(1_{\mathcal{N}} \otimes X)) = (\phi \otimes h^{\mathcal{N}})((1 \otimes Y^*)\delta^{\mathcal{N}}(X)).$$

holds. This completes the proof. □

We summarize the results obtained in this section in the theorem that follows.

THEOREM 3.9. *The system* $(\mathcal{N}, \delta^{\mathcal{N}}, R^{\mathcal{N}}, \tau^{\mathcal{N}}, h^{\mathcal{N}})$ *is a unimodular quasi Woronowicz algebra.*

DEFINITION 3.10. We call the unimodular quasi Woronowicz algebra constructed above the *quantum double* (*group*) of the given quasi Woronowicz algebra $\boldsymbol{W}$, and denote it by $D(\boldsymbol{W})$. The construction is referred to as the *double group construction*.

The next proposition states that the operation of taking the double group construction is closed in the category of Kac algebras.

COROLLARY 3.11. *If a quasi Woronowicz algebra* $\boldsymbol{W}$ *is a Kac algebra, then so is the quantum double* $D(\boldsymbol{W})$.

PROOF. We retain the notation introduced so far. Note first that a quasi Woronowicz algebra $\boldsymbol{W}$ is a Kac algebra if and only if $\sigma^h = \sigma^{h \circ R}$ and the deformation automorphism is trivial. By Theorem 3.3, we certainly have $\sigma^{h^{\mathcal{N}}} = \sigma^{h^{\mathcal{N}} \circ R^{\mathcal{N}}}$. If $\boldsymbol{W}$ is a Kac algebra, then $H = 1$, so that $\tau^{\mathcal{N}}$ is trivial. Hence the quantum double $D(\boldsymbol{W})$ is a Kac algebra. □

4. The dual of $D(W)$.

In this section, we shall study the dual of the quantum double $D(\boldsymbol{W})$ and clarify how the original algebra $\boldsymbol{W}$ or $\hat{\boldsymbol{W}}$ etc. are related to it.

Our immediate aim is to describe the Kac-Takesaki operator $W^{\mathcal{N}}$ of the double group $D(\boldsymbol{W})$ in more detail in terms of W or $\hat{W}$. Since the Haar measure of $D(\boldsymbol{W})$ is $h^{\mathcal{N}} = h \otimes h \circ R$, the algebra $\mathcal{N} = \mathcal{M} \bar{\otimes} \hat{\mathcal{M}}$ should be represented on the Hilbert space $\mathfrak{H}_{h^{\mathcal{N}}} = \mathfrak{H} \otimes \mathfrak{H}_{\hat{h} \circ \hat{R}}$ from the viewpoint of theory of (quasi) Woronowicz algebras. It, however, seems more natural (convenient) for our purpose mentioned above to work with the weight $h \otimes \hat{h}$ and represent $\mathcal{N}$ on $\mathfrak{H} \otimes \mathfrak{H}_{\hat{h}} \cong \mathfrak{H} \otimes \mathfrak{H}$, even though we know that $\mathfrak{H}_{h^{\mathcal{N}}}$ is *canonically* isomorphic to $\mathfrak{H} \otimes \mathfrak{H}_{\hat{h}}$. Thus we first study this canonical isomorphism.

Following [**N**], we define a one-paramenter transformation L_t on $\mathscr{M}$ by

$$L_t(x) := xQ^{-it} \quad (x \in \mathscr{M}).$$

As noted in [**N**], all the one-parameter transformations σ^h, $\sigma^{h\circ R}$, τ, L are mutually commuting. Now we denote by $\mathfrak{a}_0$ the set of all entire analytic elements $x \in \mathfrak{a}_h$ with respect to these four transformations satisfying $\sigma_z^h(x)$, $\sigma_z^{h\circ R}(x)$, $\tau_z(x)$, $L_z(x) \in \mathfrak{a}_h$ for all $z \in \boldsymbol{C}$. This set $\mathfrak{a}_0$ is a σ-weakly dense $*$-subalgebra of $\mathfrak{a}_h$. Remark that $\mathfrak{A}_0 := \Lambda_h(\mathfrak{a}_0)$ is a left Hilbert algebra dense in $\mathfrak{H}$ such that $\mathfrak{A}_0'' = \mathfrak{A}_h$.

Lemma 4.1. *Let $x \in \mathfrak{a}_0$. Then we have*

$$JQ^{i\bar{z}}J\Lambda_h(x) = \Lambda_h(L_z(x))$$

for any $z \in \boldsymbol{C}$.

Proof. Since $x \in \mathfrak{a}_0$, the function $t \in \boldsymbol{R} \mapsto \Lambda_h(L_t(x))$ has an analytic extension to the whole plane $\boldsymbol{C}$. In the meantime, since Q^{it} lies in the centralizer $\mathscr{M}_h$ of the weight h, we find that

$$\Lambda_h(L_t(x)) = \Lambda_h(xQ^{-it}) = JQ^{it}J\Lambda_h(x).$$

Hence, from [**PT**, Lemma 3.2], it follows that the vector $\Lambda_h(x)$ belongs to $\bigcap_{z\in\boldsymbol{C}} \mathfrak{D}((JQJ)^z)$. Therefore, we obtain

$$\Lambda_h(L_z(x)) = (JQJ)^{-iz}\Lambda_h(x) = JQ^{i\bar{z}}J\Lambda_h(x).$$

This completes the proof. □

Lemma 4.2. *Let $\mathfrak{P}$ be the self-dual closed convex cone in $\mathfrak{H}$ associated with the standard representation $\{\mathscr{M}, \mathfrak{H}, J = J_h\}$. Then we have*

$$\mathfrak{P} = \overline{\{xJ\Lambda_h(x) : x \in \mathfrak{a}_o\}}.$$

Proof. By definition (see [**H**]), $\mathfrak{P} = \overline{\{xJ\Lambda_h(x) : x \in \mathfrak{a}_h\}}$. Our assertion now follows from the density theorem for left Hilbert algebras. □

Proposition 4.3. *The unitary conjugation $\hat{J}$ on $\mathfrak{H}$ is the unique canonical implementation of the antipode R in the sense of Haagerup* ([**H**]).

Proof. By [**MN**, Lemma 3.9], $\hat{J}$ implements the antipode $R : R(x) = \hat{J}x^*\hat{J}$ $(x \in \mathscr{M})$. From [**MN**, Corollary 3.6.2], $\hat{J}$ commutes with J. Hence, in order to prove this proposition, it suffices by [**H**, Theorem 2.3] to show that $\hat{J}$ leaves the self-dual closed convex cone $\mathfrak{P}$ in Lemma 4.2 invariant. So let $x \in \mathfrak{a}_0$. Then, by Lemma 4.1 and [**MN**, Corollary 3.6.2], we have

$$\begin{aligned}\hat{J}xJ\Lambda_h(x) &= R(x^*)\hat{J}J\Lambda_h(x) = R(x^*)J\Lambda_h(L_{i/2}(R(x^*))) \\ &= R(x^*)Q^{1/2}J\Lambda_h(R(x^*)).\end{aligned}$$

Let $Q^{1/2} = \int_0^\infty \lambda\, de(\lambda)$ be the spectral decomposition of $Q^{1/2}$. Put $y_n = \int_{1/n}^n \lambda\, de(\lambda)$. Note that each y_n belongs to the centralizer of the weight h. Thus we have

$$R(x^*)y_nJ\Lambda_h(R(x^*)) = R(x^*)y_n^{1/2}J(Jy_n^{1/2}J)\Lambda_h(R(x^*))$$
$$= R(x^*)y_n^{1/2}J\Lambda_h(R(x^*)y_n^{1/2})$$

Since $\mathfrak{P} = \overline{\{xJ\Lambda_h(x) : x \in \mathfrak{n}_h\}}$ and $R(x^*)y_n^{1/2}$ is in $\mathfrak{n}_h$, it follows that the vector $R(x^*)y_nJ\Lambda_h(R(x^*))$ lies in $\mathfrak{P}$. Since

$$R(x^*)Q^{1/2}J\Lambda_h(R(x^*)) = \lim_{n\to\infty} R(x^*)y_nJ\Lambda_h(R(x^*)),$$

we find that $\hat{J}xJ\Lambda_h(x)$ still belongs to $\mathfrak{P}$, $\mathfrak{P}$ being closed. From Lemma 4.2, we conclude that $\hat{J}$ leaves $\mathfrak{P}$ invariant. □

COROLLARY 4.4. *Let $\pi_{h\circ R}$ be the standard representation of $\mathcal{M}$ on the Hilbert space $\mathfrak{H}_{h\circ R}$ constructed from the weight $h\circ R$. Then the unique canonical implementation $U_{h\circ R,h} : \mathfrak{H}_{h\circ R} \to \mathfrak{H}$ of $\pi_{h\circ R}$ is given by*

$$U_{h\circ R,h}\Lambda_{h\circ R}(x) = \hat{J}\Lambda_h(R(x)^*) \quad (x \in \mathfrak{n}_{h\circ R}).$$

PROOF. First note that $\mathfrak{n}_{h\circ R} = R(\mathfrak{n}_h^*)$. Thus the operator $U_{h\circ R,h}$ defined above is clearly a unitary transformation from $\mathfrak{H}_{h\circ R}$ onto $\mathfrak{H}$. It can be easily verified that $\pi_{h\circ R}(x) = U_{h\circ R,h}^* x U_{h\circ R,h}$ for any $x \in \mathcal{M}$.

Let $x \in \mathfrak{a}_{h\circ R}$. We denote the S-operator, the modular operator and the modular conjugation associated with $h\circ R$ by $S_{h\circ R}$, $\Delta_{h\circ R}$ and $J_{h\circ R}$, respectively. Then we have

$$U_{h\circ R,h}S_{h\circ R}\Lambda_{h\circ R}(x) = \hat{J}\Lambda_h(R(x)) = \hat{J}S\Lambda_h(R(x)^*) = \hat{J}S\hat{J}U_{h\circ R,h}\Lambda_{h\circ R}(x).$$

Since $\Lambda_{h\circ R}(\mathfrak{a}_{h\circ R})$ (resp. $\Lambda_h(\mathfrak{a}_h)$) is a core for $S_{h\circ R}$ (resp. S) and

$$\mathfrak{D}(\hat{J}S\hat{J}U_{h\circ R,h}) = U_{h\circ R,h}^*\hat{J}\mathfrak{D}(S), \quad U_{h\circ R,h}^*\hat{J}\Lambda_h(\mathfrak{a}_h) = \Lambda_{h\circ R}(\mathfrak{a}_{h\circ R}),$$

it follows that $U_{h\circ R,h}S_{h\circ R} = \hat{J}S\hat{J}U_{h\circ R,h}$. From the uniqueness of polar decomposition, it results that

$$U_{h\circ R,h}J_{h\circ R} = JU_{h\circ R,h}, \quad \Delta_{h\circ R} = U_{h\circ R,h}^*\hat{J}\Delta\hat{J}U_{h\circ R,h}.$$

Finally, if $x \in \mathfrak{n}_{h\circ R}$, then, by the preceding paragraphs, we have

$$U_{h\circ R,h}\pi_{h\circ R}(x)J_{h\circ R}\Lambda_{h\circ R}(x) = xU_{h\circ R,h}J_{h\circ R}\Lambda_{h\circ R}(x)$$
$$= xJU_{h\circ R,h}\Lambda_{h\circ R}(x)$$
$$= xJ\hat{J}\Lambda_h(R(x)^*)$$
$$= \hat{J}R(x)^*J\Lambda_h(R(x)^*).$$

This shows that $U_{h\circ R,h}\{yJ_{h\circ R}\Lambda_{h\circ R}(y) : y \in \mathfrak{n}_{h\circ R}\} = \hat{J}\{xJ\Lambda_h(x) : x \in \mathfrak{n}_h\}$. From Proposition 4.3, we conclude that the unitary $U_{h\circ R,h}$ carries the self-dual closed convex cone $\mathfrak{P}_{h\circ R}$ asscoiated with the standard representation of $h\circ R$ onto the self-dual cone $\mathfrak{P}$ introduced in Lemma 4.2. Therefore, by [**H**, Theorem 2.3], $U_{h\circ R,h}$ is the unique canonical implementation of the isomorphism $\pi_{h\circ R}$. □

COROLLARY 4.5. *The unitary* $U_{h\circ R,h}$ *in the preceding corollary is characterized by*

$$U_{h\circ R,h}\Lambda_{h\circ R}(a) = JQ^{1/2}J\Lambda_h(a) \quad (a \in \mathfrak{n}_h \cap \mathfrak{n}_{h\circ R})$$

PROOF. First we assert that $\Lambda_{h\circ R}(\mathfrak{n}_h \cap \mathfrak{n}_{h\circ R})$ is dense in $\mathfrak{H}_{h\circ R}$. Indeed, take any $x \in \mathfrak{n}_{h\circ R}$. Let $Q^{-1} = \int_0^\infty \lambda\, de(\lambda)$ be the spectral decomposition of Q^{-1}. Set $e_n := \int_{1/n}^n de(\lambda)$. Since e_n belongs to the centralizer of $h\circ R$, we have

$$h\circ R((xe_n)^*(xe_n)) = \|\Lambda_{h\circ R}(xe_n)\|^2 \leq \|\Lambda_{h\circ R}(x)\|^2 < \infty.$$

Thus xe_n is in $\mathfrak{n}_{h\circ R}$ for any $n \geq 1$. Meanwhile, we have

$$\begin{aligned} h((xe_n)^*(xe_n)) &= \lim_{\varepsilon\downarrow 0} h\circ R(Q_\varepsilon^{-1/2}e_n x^* x e_n Q_\varepsilon^{-1/2}) \\ &= \lim_{\varepsilon\downarrow 0} \|\Lambda_{h\circ R}(xe_n Q_\varepsilon^{-1/2})\|^2 \\ &= \lim_{\varepsilon\downarrow 0} \|J_{h\circ R} e_n Q_\varepsilon^{-1/2} J_{h\circ R}\Lambda_{h\circ R}(x)\|^2 \\ &= \|J_{h\circ R} y_n^{1/2} J_{h\circ R}\Lambda_{h\circ R}(x)\|^2 \\ &\leq \|y_n\|\|\Lambda_{h\circ R}(x)\|^2 < \infty, \end{aligned}$$

where $y_n := Q^{-1}e_n$. This shows that xe_n lies in $\mathfrak{n}_h$. Since $\lim_{n\to\infty}\|\Lambda_{h\circ R}(xe_n) - \Lambda_{h\circ R}(x)\| = 0$, it results that $\Lambda_{h\circ R}(\mathfrak{n}_h \cap \mathfrak{n}_{h\circ R})$ is dense in $\mathfrak{H}_{h\circ R}$, as asserted.

Let $a \in \mathfrak{n}_h \cap \mathfrak{n}_{h\circ R}$. One has

$$\lim_{\varepsilon\downarrow 0}\|JQ_\varepsilon^{1/2}J\Lambda_h(a)\|^2 = \lim_{\varepsilon\downarrow 0}\|\Lambda_h(aQ_\varepsilon^{1/2})\|^2 = \lim_{\varepsilon\downarrow 0} h(Q_\varepsilon^{1/2}a^*aQ_\varepsilon^{1/2}) = h\circ R(a^*a) < \infty.$$

Hence, as shown in [**PT**, Lemma 7.9], $\Lambda_h(a)$ belongs to the domain $\mathfrak{D}(JQ^{1/2}J)$ of the positive operator $JQ^{1/2}J$. Therefore, the mapping T given by

$$T\Lambda_{h\circ R}(a) = JQ^{1/2}J\Lambda_h(a) \quad (a \in \mathfrak{n}_h \cap \mathfrak{n}_{h\circ R})$$

is well-defined and can be extended to an isometry from $\mathfrak{H}_{h\circ R}$ into $\mathfrak{H}$. Now we suppose that a lies in $\mathfrak{a}_0$. Then, by Lemma 4.1, one finds that $T\Lambda_{h\circ R}(a) = \Lambda_h(L_{i/2}(a))$. In the meantime, from [**MN**, Corollary 3.6.2 (iii)], it follows that

$$U_{h\circ R,h}\Lambda_{h\circ R}(a) = \hat{J}\Lambda_h(R(x)^*) = \Lambda_h(L_{i/2}(a)).$$

Since $\Lambda_{h\circ R}(\mathfrak{a}_0)$ is dense in $\mathfrak{H}_{h\circ R}$, $U_{h\circ R,h}$ equals T. □

The next corollary is irrelevant to the subject under discussion. We, however, think it worth mentioning.

COROLLARY 4.6. *Let* $\boldsymbol{W}^\circ = (\mathscr{M}, \sigma\circ\delta, R, \tau^{-1}, h\circ R)$ *be the quasi Woronowicz algebra co-opposite to* $\boldsymbol{W}$. (*see* [**N**, Section 4]). *Under the identification of* $\mathfrak{H}_{h\circ R}$ *with* $\mathfrak{H}$ *through the unitary* $U_{h\circ R,h}$ *in the preceding corollary, the Kac-Takesaki operator* W° *of* $\boldsymbol{W}^\circ$ *is given by*

$$W^\circ = (\hat{J}\otimes\hat{J})W(\hat{J}\otimes\hat{J}).$$

(*Compare this result with the assertion* (i) *of* [**MN**, Proposition 4.3]).

Proof. One needs to show that

$$(U_{h\circ R,h}\otimes U_{h\circ R,h})W^{\circ}(U_{h\circ R,h}\otimes U_{h\circ R,h})^* = (\hat{J}\otimes\hat{J})W(\hat{J}\otimes\hat{J}).$$

But this follows from a direct computation. So we leave the verification to readers. □

Lemma 4.7. *Let* $x, y \in \mathscr{M}$ *be in* $\mathfrak{n}_h$, $\mathfrak{n}_h \cap \mathfrak{n}_{h\circ R}$, *respectively. Then we have*

$$\Lambda_{h\circ R\otimes h}(\delta(x)(y\otimes 1)) = (U_{h\circ R,h}^*\otimes 1)W(U_{h\circ R,h}\otimes 1)\Lambda_{h\circ R\otimes h}(y\otimes x).$$

Proof. The weight h being quasi left invariant, $\delta(x)(y\otimes 1)$ is in fact in $\mathfrak{n}_{h\circ R\otimes h}$. Let $z_i \in \mathfrak{n}_h \cap \mathfrak{n}_{h\circ R}$ $(i=1,2)$. By Corollary 4.5, we have

$$\begin{aligned}
&(\Lambda_{h\circ R\otimes h}(\delta(x)(y\otimes 1))|\Lambda_{h\circ R\otimes h}(z_1\otimes z_2))\\
&\quad = (h\circ R\otimes h)((z_1^*\otimes z_2^*)\delta(x)(y\otimes 1))\\
&\quad = \lim_{\varepsilon\downarrow 0}(h\otimes h)((Q_\varepsilon^{1/2}z_1^*\otimes z_2^*)\delta(x)(yQ_\varepsilon^{1/2}\otimes 1))\\
&\quad = \lim_{\varepsilon\downarrow 0}(W(JQ_\varepsilon^{1/2}J\otimes 1)\Lambda_{h\otimes h}(y\otimes x)\,|\,(JQ_\varepsilon^{1/2}J\otimes 1)\Lambda_{h\otimes h}(z_1\otimes z_2))\\
&\quad = (W(U_{h\circ R,h}\otimes 1)\Lambda_{h\circ R\otimes h}(y\otimes x)\,|\,(U_{h\circ R,h}\otimes 1)\Lambda_{h\circ R\otimes h}(z_1\otimes z_2)).
\end{aligned}$$

Thus we are done. □

Lemma 4.8. *Let* $\mathscr{B}$ *be a von Neumann algebra and* ψ *a faithful normal semifinite weight on* $\mathscr{B}$. *Suppose that* X, b *and* m *are in* $\mathscr{B}\bar{\otimes}\mathscr{M}, \mathscr{B}, \mathscr{M}$, *respectively, so that* $X(b\otimes 1)\in\mathfrak{n}_{\psi\otimes h}$ *and* $m\in\mathfrak{n}_{h\circ R}$. *Then* $(id_{\mathscr{B}}\otimes\delta)(X(b\otimes 1))(1\otimes m\otimes 1) = (id_{\mathscr{B}}\otimes\delta)(X)\cdot(b\otimes m\otimes 1)$ *belongs to* $\mathfrak{n}_{\psi\otimes h\circ R\otimes h}$, *and we have*

$$\begin{aligned}
&\Lambda_{\psi\otimes h\circ R\otimes h}((id_{\mathscr{B}}\otimes\delta)(X)(b\otimes m\otimes 1))\\
&\quad = (1\otimes U_{h\circ R,h}^*\otimes 1)W_{23}(1\otimes U_{h\circ R,h}\otimes 1)\Lambda_{\psi\otimes h\circ R\otimes h}(X_{13}(b\otimes m\otimes 1)).
\end{aligned}$$

Proof. The claim that $(id_{\mathscr{B}}\otimes\delta)(X)(b\otimes m\otimes 1)$ belongs to $\mathfrak{n}_{\psi\otimes h\circ R\otimes h}$ follows from Lemma 3.5. The identity above can be verified by the method analogous to the proof of Lemma 3.5. Indeed, since $X(b\otimes 1)$ lies in $\mathfrak{n}_{\psi\otimes h}$, it follows from the density theorem for left Hilbert algebras that there exists a sequence $\{Y_n\}$ in the algebraic tensor product $\mathfrak{n}_\psi\otimes\mathfrak{n}_h$, converging strongly to $X(b\otimes 1)$, such that $\Lambda_{\psi\otimes h}(X(b\otimes 1)) = \lim_{n\to\infty}\Lambda_{\psi\otimes h}(Y_n)$. From Lemma 4.7, we easily find that

$$\begin{aligned}
&\Lambda_{\psi\otimes h\circ R\otimes h}((id_{\mathscr{B}}\otimes\delta)(Y_n)(1\otimes m\otimes 1))\\
&\quad = (1\otimes U_{h\circ R,h}^*\otimes 1)W_{23}(1\otimes U_{h\circ R,h}\otimes 1)\Lambda_{\psi\otimes h\circ R\otimes h}((Y_n)_{13}(1\otimes m\otimes 1)).
\end{aligned}$$

The right-hand side of the above identity converges to the vector

$$(1\otimes U_{h\circ R,h}^*\otimes 1)W_{23}(1\otimes U_{h\circ R,h}\otimes 1)\Lambda_{\psi\otimes h\circ R\otimes h}(X_{13}(b\otimes m\otimes 1)).$$

From this, with $\eta\in\mathfrak{A}'_{\psi\otimes h\circ R\otimes h}$, one has

$$\pi_r(\eta)\Lambda_{\psi\otimes h\circ R\otimes h}((id_{\mathscr{B}}\otimes\delta)(X)(b\otimes m\otimes 1))$$
$$= (id_{\mathscr{B}}\otimes\delta)(X)(b\otimes m\otimes 1)\eta$$
$$= \lim_{n\to\infty}(id_{\mathscr{B}}\otimes\delta)(Y_n)(1\otimes m\otimes 1)\eta$$
$$= \lim_{n\to\infty}\pi_r(\eta)\Lambda_{\psi\otimes h\circ R\otimes h}((id_{\mathscr{B}}\otimes\delta)(Y_n)(1\otimes m\otimes 1))$$
$$= \pi_r(\eta)(1\otimes U^*_{h\circ R,h}\otimes 1)W_{23}(1\otimes U_{h\circ R,h}\otimes 1)\Lambda_{\psi\otimes h\circ R\otimes h}(X_{13}(b\otimes m\otimes 1)).$$

This completes the proof. □

Lemma 4.9. *The set* $W^*\mathfrak{n}_{h^{\mathscr{N}}}W$ *is contained in* $\mathfrak{n}_{h\circ R\otimes\hat{h}}$, *and there exists a unitary* T_W *from* $\mathfrak{H}_{h^{\mathscr{N}}}$ *onto* $\mathfrak{H}_{h\circ R\otimes\hat{h}}$ *given by*

$$T_W\Lambda_{h^{\mathscr{N}}}(X) = \Lambda_{h\circ R\otimes\hat{h}}(W^*XW)\quad (X\in\mathfrak{n}_{h^{\mathscr{N}}}).$$

In particular, $\mathrm{Ad}\,T_W(\pi_{h^{\mathscr{N}}}(X)) = \pi_{h\circ R\otimes\hat{h}}(W^*XW)$.

Proof. By Theorem 3.3, one has

$$(h\circ R\otimes\hat{h})\circ\mathrm{Ad}\,W^* = h^{\mathscr{N}}\circ R^{\mathscr{N}} = h^{\mathscr{N}}.$$

This shows that $\mathrm{Ad}\,W^*$ is a weight-preserving isomorphism. Hence it follows from a general theory that the canonical implementation T_W of $\mathrm{Ad}\,W^*$ is characterized (defined) by the equation in the statement of this lemma. □

Before we state the next lemma, let us recall the unitary Z introduced in Section 2. It is the unitary given by $Z = \hat{X}\tilde{X} = W^*(u\otimes u)W(u\otimes u)$.

Lemma 4.10. *Let* $\mathscr{F}$ *be the Fourier transform and* T_W *the unitary in the preceding lemma. Then the mapping* $Z(1\otimes\mathscr{F})(1\otimes U_{\hat{h}\circ\hat{R},\hat{h}})T^*_W:\mathfrak{H}_{h\circ R\otimes\hat{h}}\to\mathfrak{H}\otimes\mathfrak{H}$ *is the canonical implementation of the standard representation* $\pi_{h\circ R\otimes\hat{h}}$. *In particular, we have*

$$\mathrm{Ad}(1\otimes\mathscr{F}U_{\hat{h}\circ\hat{R},\hat{h}})T^*_W\circ\pi_{h\circ R\otimes\hat{h}}(X) = \mathrm{Ad}\,Z^*(X)\quad (X\in\mathscr{N}).$$

Proof. By the definition of T_W, it is easy to see that the unitary $(1\otimes\mathscr{F})(1\otimes U_{\hat{h}\circ\hat{R},\hat{h}})T^*_W:\mathfrak{H}_{h\circ R\otimes\hat{h}}\to\mathfrak{H}\otimes\mathfrak{H}$ is the canonical implementation of the isomorphism $\pi_{h\circ R\otimes\hat{h}}(X)\mapsto\mathrm{Ad}\,W(X)$. Thus it suffices to show that Z^* is the canonical implementation of $\mathrm{Ad}\,W$. Since W is a unitary in $\mathscr{N}$, the canonical implementation of $\mathrm{Ad}\,W$ is given by $W(J\otimes\hat{J})W(J\otimes\hat{J})$. Since $W^* = (\hat{J}\otimes J)W(\hat{J}\otimes J)$, it is exactly the unitary Z^*. □

Theorem 4.11. *Let* $W^{\mathscr{N}}$ *be the Kac-Takesaki operator of the quantum double* $D(\boldsymbol{W})$ *and* $\mathscr{F}$ *the Fourier transform. Then we have*

$$(1\otimes\mathscr{F}U_{\hat{h}\circ\hat{R},\hat{h}}\otimes 1\otimes\mathscr{F}U_{\hat{h}\circ\hat{R},\hat{h}})W^{\mathscr{N}}(1\otimes U^*_{\hat{h}\circ\hat{R},\hat{h}}\mathscr{F}^*\otimes 1\otimes U^*_{\hat{h}\circ\hat{R},\hat{h}}\mathscr{F}^*) = Z^*_{34}\hat{W}_{24}Z_{34}W_{13}.$$

Proof. Let a,b,c,d be in $\mathfrak{a}_0$ and y,z,p,q be in $\hat{\mathfrak{a}}_0$. Then, by Proposition 3.4, we have

$$\begin{aligned}
&(W^{\mathcal{N}}\Lambda_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}(a\otimes y\otimes b\otimes z)\,|\,\Lambda_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}(c\otimes p\otimes d\otimes q))\\
&\quad=(h^{\mathcal{N}}\otimes h^{\mathcal{N}})((c\otimes p\otimes d\otimes q)^*\delta^{\mathcal{N}}(b\otimes z)(a\otimes y\otimes 1\otimes 1))\\
&\quad=\lim_{\varepsilon\downarrow 0}(\Psi\otimes\Psi)((P_\varepsilon^{1/2}\otimes P_\varepsilon^{1/2})(c\otimes p\otimes d\otimes q)^*\delta^{\mathcal{N}}\\
&\qquad\times(b\otimes z)(a\otimes y\otimes 1\otimes 1)(P_\varepsilon^{1/2}\otimes P_\varepsilon^{1/2}))\\
&\quad=\lim_{\varepsilon\downarrow 0}(h\otimes\hat{h}\otimes h\otimes\hat{h})((Q_\varepsilon^{1/2}\otimes 1\otimes Q_\varepsilon^{1/2}\otimes 1)W_{12}^*W_{34}^*\\
&\qquad\times(c\otimes p\otimes d\otimes q)^*\delta^{\mathcal{N}}(b\otimes z)(a\otimes y\otimes 1\otimes 1)\\
&\qquad\times W_{12}W_{34}(Q_\varepsilon^{1/2}\otimes 1\otimes Q_\varepsilon^{1/2}\otimes 1)).
\end{aligned}$$

As we showed in the proof of Theorem 3.7, we have

$$\mathrm{Ad}\,W_{34}^*\circ\delta^{\mathcal{N}}(b\otimes z)=\mathrm{Ad}\,\Sigma_{23}\circ(id_{\mathcal{M}}\otimes id_{\mathcal{M}}\otimes\hat{\delta})(W_{23}^*(\delta(b)\otimes z)W_{23}).$$

From this and unimodularity of $h^{\mathcal{N}}$, it results that

$$\begin{aligned}
&(W^{\mathcal{N}}\Lambda_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}(a\otimes y\otimes b\otimes z)\,|\,\Lambda_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}(c\otimes p\otimes d\otimes q))\\
&\quad=\lim_{\varepsilon\downarrow 0}(h\otimes h\otimes\hat{h}\otimes\hat{h})((Q_\varepsilon^{1/2}\otimes Q_\varepsilon^{1/2}\otimes 1\otimes 1)W_{13}^*W_{24}^*(c\otimes d\otimes p\otimes q)^*W_{24}\\
&\qquad\times(id_{\mathcal{M}}\otimes id_{\mathcal{M}}\otimes\hat{\delta})(W_{23}^*(\delta(b)\otimes z)W_{23})(a\otimes 1\otimes y\otimes 1)W_{13}(Q_\varepsilon^{1/2}\otimes Q_\varepsilon^{1/2}\otimes 1\otimes 1))\\
&\quad=(h\circ R\otimes h\circ R\otimes\hat{h}\otimes\hat{h})(W_{13}^*W_{24}^*(c\otimes d\otimes p\otimes q)^*W_{24}\\
&\qquad\times(id_{\mathcal{M}}\otimes id_{\mathcal{M}}\otimes\hat{\delta})(W_{23}^*(\delta(b)\otimes z)W_{23})\times(a\otimes 1\otimes y\otimes 1)W_{13})\\
&\quad=(h\otimes h\circ R\otimes\hat{h}\circ\hat{R}\otimes\hat{h})(W_{24}^*(c\otimes d\otimes p\otimes q)^*\\
&\qquad\times W_{24}(id_{\mathcal{M}}\otimes id_{\mathcal{M}}\otimes\hat{\delta})(W_{23}^*(\delta(b)\otimes z)W_{23})\times(a\otimes 1\otimes y\otimes 1))\\
&\quad=(\Lambda_{h\otimes h\circ R\otimes\hat{h}\circ\hat{R}\otimes\hat{h}}(\mathcal{X})|\Lambda_{h\otimes h\circ R\otimes\hat{h}\circ\hat{R}\otimes\hat{h}}(W_{24}^*(c\otimes d\otimes p\otimes q)W_{24})),
\end{aligned}$$

where $\mathcal{X}=(id_{\mathcal{M}}\otimes id_{\mathcal{M}}\otimes\hat{\delta})(W_{23}^*(\delta(b)\otimes z)W_{23})(a\otimes 1\otimes y\otimes 1)$. With the notation in the preceding lemma, we obtain

$$\Lambda_{h\otimes h\circ R\otimes\hat{h}\circ\hat{R}\otimes\hat{h}}(W_{24}^*(c\otimes d\otimes p\otimes q)W_{24})=(T_W)_{24}\Sigma_{23}\Lambda_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}(c\otimes p\otimes d\otimes q).$$

From this, it follows that

$$W^{\mathcal{N}}\Lambda_{h^{\mathcal{N}}\otimes h^{\mathcal{N}}}(a\otimes y\otimes b\otimes z)=\Sigma_{23}(T_W)_{24}^*\Lambda_{h\otimes h\circ R\otimes\hat{h}\circ\hat{R}\otimes\hat{h}}(\mathcal{X}).\tag{4.11.1}$$

By Lemma 4.8, we have

$$\begin{aligned}
&\Lambda_{h\otimes h\circ R\otimes\hat{h}\circ\hat{R}\otimes\hat{h}}(\mathcal{X})\\
&\quad=(1\otimes 1\otimes U_{\hat{h}\circ\hat{R},\hat{h}}^*\otimes 1)\hat{W}_{34}(1\otimes 1\otimes U_{\hat{h}\circ\hat{R},\hat{h}}\otimes 1)\Lambda_{h\otimes h\circ R\otimes\hat{h}\circ\hat{R}\otimes\hat{h}}\\
&\qquad\times(W_{24}^*(\delta(b)\otimes 1\otimes z)W_{24}(a\otimes 1\otimes y\otimes 1))
\end{aligned}$$

$$= (1 \otimes 1 \otimes U^*_{\hat{h} \circ \hat{R}, \hat{h}} \otimes 1)\hat{W}_{34}(1 \otimes 1 \otimes U_{\hat{h} \circ \hat{R}, \hat{h}} \otimes 1)$$

$$\times (T_W)_{24} \Lambda_{h \otimes h \otimes \hat{h} \circ \hat{R} \otimes \hat{h} \circ \hat{R}}((\delta(b)(a \otimes 1) \otimes y \otimes z)$$

$$= (1 \otimes 1 \otimes U^*_{\hat{h} \circ \hat{R}, \hat{h}} \otimes 1)\hat{W}_{34}(1 \otimes 1 \otimes U_{\hat{h} \circ \hat{R}, \hat{h}} \otimes 1)$$

$$\times (T_W)_{24} W_{12} \Sigma_{23} \Lambda_{h^{\mathcal{N}} \otimes h^{\mathcal{N}}}(a \otimes y \otimes b \otimes z).$$

From this and (4.11.1), it follows that

$$W^{\mathcal{N}} = \Sigma_{23}(T_W)^*_{24}(1 \otimes 1 \otimes U^*_{\hat{h} \circ \hat{R}, \hat{h}} \otimes 1)\hat{W}_{34}(1 \otimes 1 \otimes U_{\hat{h} \circ \hat{R}, \hat{h}} \otimes 1)(T_W)_{24} W_{12} \Sigma_{23}$$

$$= (1 \otimes U^*_{\hat{h} \circ \hat{R}, \hat{h}} \otimes T^*_W)\hat{W}_{24}(1 \otimes U_{\hat{h} \circ \hat{R}, \hat{h}} \otimes T_W) W_{13}.$$

Hence we obtain

$$(1 \otimes \mathscr{F} U_{\hat{h} \circ \hat{R}, \hat{h}} \otimes 1 \otimes \mathscr{F} U_{\hat{h} \circ \hat{R}, \hat{h}}) W^{\mathcal{N}} (1 \otimes U^*_{\hat{h} \circ \hat{R}, \hat{h}} \mathscr{F}^* \otimes 1 \otimes U^*_{\hat{h} \circ \hat{R}, \hat{h}} \mathscr{F}^*)$$

$$= (1 \otimes \mathscr{F} \otimes (1 \otimes \mathscr{F})(1 \otimes U_{\hat{h} \circ \hat{R}, \hat{h}}) T^*_W)\hat{W}_{24}$$

$$\times (1 \otimes \mathscr{F}^* \otimes T_W(1 \otimes U^*_{\hat{h} \circ \hat{R}, \hat{h}})(1 \otimes \mathscr{F}^*)) W_{13}.$$

The assertion of this theorem now follows from Lemma 4.10 by identifying $\hat{W}$ with $(\mathscr{F} \otimes \mathscr{F})^* \hat{W} (\mathscr{F} \otimes \mathscr{F})$. □

From Theorem 4.11, we may and do identify the Kac-Takesaki operator $W^{\mathcal{N}}$ of the quantum double $D(\boldsymbol{W})$ with the unitary $Z^*_{34} \hat{W}_{24} Z_{34} W_{13}$. Hence we will think that both $D(\boldsymbol{W})$ and its dual $\widehat{D(\boldsymbol{W})}$ "live" on the Hilbert space $\mathfrak{H} \otimes \mathfrak{H}$.

Remark. Theorem 4.11 fully answers the problem raised in Section 2 of [**N**, Page 532]. In other words, Theorem 4.11 gives an explicit relation between the Kac-Takesaki operators for a *general* quasi Woronowicz algebra $\boldsymbol{W}$ and its quantum double $D(\boldsymbol{W})$.

Corollary 4.12. *The quasi Woronowicz algebra $\hat{\mathcal{N}}$ dual to $\mathcal{N}$ is generated by $\hat{\mathcal{M}} \otimes \boldsymbol{C}$ and $Z^*(\boldsymbol{C} \otimes \mathcal{M})Z$. Indeed, it is the σ-strong* closure of the linear span of the set*

$$\{(y \otimes 1)Z^*(1 \otimes a)Z : a \in \mathcal{M}, y \in \hat{\mathcal{M}}\}.$$

Proof. Let θ, ω be in $\mathscr{B}(\mathfrak{H})_*$. We denote the Fourier repesentation of $D(\boldsymbol{W})$ by $\hat{\pi}_{\mathcal{N}}$. By definition, $\hat{\mathcal{N}}$ is the σ-strong* closure of the subalgebra $\hat{\pi}_{\mathcal{N}}(\mathcal{N}_*)$. By [**MN**, Lemma 2.10], we have

$$\hat{\pi}_{\mathcal{N}}(\theta \otimes \omega) = (\theta \otimes \omega \otimes id \otimes id)((W^{\mathcal{N}})^*)$$

$$= (\theta \otimes \omega \otimes id \otimes id)(W^*_{13} Z^*_{34} \hat{W}^*_{24} Z_{34})$$

$$= (\hat{\pi}(\theta) \otimes 1)Z^*(1 \otimes \hat{\hat{\pi}}(\omega))Z.$$

From this, the assertion of this corollary immediately follows. □

Proposition 4.13. *The modular conjugation $\hat{J}_{\mathcal{N}}$ associated with the dual quasi Woronowicz algebra $\widehat{D(\boldsymbol{W})}$ is $(\hat{J} \otimes J)Z = Z^*(\hat{J} \otimes J)$.*

Proof. By Proposition 4.3, it suffices to show that the conjugation $(\hat{J} \otimes J)Z$ is the canonical implementation of the antipode $R^{\mathcal{N}}$. By definition, $R^{\mathcal{N}} = (R \otimes \hat{R}) \circ \operatorname{Ad} W^*$. From Proposition 4.3, $\hat{J} \otimes J$ is the canonical implementation of $R \otimes \hat{R}$. As we noted in the proof of Lemma 4.10, the unitary Z is the canonical implementation of $\operatorname{Ad} W^*$. Thus we are done. □

Corollary 4.14. *The commutant $\hat{\mathcal{N}}'$ of the dual $\hat{\mathcal{N}}$ is generated by $Z^*(\hat{\mathcal{M}}' \otimes \boldsymbol{C})Z$ and $\boldsymbol{C} \otimes \mathcal{M}'$. Indeed, it is the σ-strong* closure of the linear span of the set*

$$\{Z^*(z \otimes 1)Z(1 \otimes b) : b \in \mathcal{M}', z \in \hat{\mathcal{M}}'\}.$$

Proof. The assertion easily follows from a combination of Corollary 4.12 and Proposition 4.13. □

In what follows, we set $\Sigma_{34}^{12} := \Sigma_{13}\Sigma_{24}$. In other words, Σ_{34}^{12} is the unitary on $\mathfrak{H} \otimes \mathfrak{H} \otimes \mathfrak{H} \otimes \mathfrak{H}$ given by $\Sigma_{34}^{12}(\xi \otimes \eta) = \eta \otimes \xi$ for $\xi, \eta \in \mathfrak{H} \otimes \mathfrak{H}$.

Theorem 4.15. *With the unitary V defined in Section 2, we have*

$$(\hat{J}_{\mathcal{N}} \otimes \hat{J}_{\mathcal{N}})\Sigma_{34}^{12} V \Sigma_{34}^{12}(\hat{J}_{\mathcal{N}} \otimes \hat{J}_{\mathcal{N}}) = W^{\mathcal{N}}.$$

Therefore, V is the adjoint of the Kac-Takesaki operator $W(\widehat{D(\boldsymbol{W})}')$ of the commutant of the dual quasi Woronowicz algebra.

Proof. We employ the notation introduced in Section 2. By Proposition 4.13, we have

$$\Sigma_{34}^{12}(\hat{J}_{\mathcal{N}} \otimes \hat{J}_{\mathcal{N}})V(\hat{J}_{\mathcal{N}} \otimes \hat{J}_{\mathcal{N}})\Sigma_{34}^{12} = \Sigma_{34}^{12}(\hat{J} \otimes J \otimes \hat{J} \otimes J)Z_{12}Z_{34}VZ_{12}^*Z_{34}^*(\hat{J} \otimes J \otimes \hat{J} \otimes J)\Sigma_{34}^{12}.$$

By the remark following [**BS**, Proposition 8.10], one has $Z_{12}Z_{34}VZ_{12}^*Z_{34}^* = Z_{12}\boldsymbol{Y}_{24}Z_{12}^*\boldsymbol{X}_{13}$. Thus we obtain

$$\begin{aligned}
&\Sigma_{34}^{12}(\hat{J}_{\mathcal{N}} \otimes \hat{J}_{\mathcal{N}})V(\hat{J}_{\mathcal{N}} \otimes \hat{J}_{\mathcal{N}})\Sigma_{34}^{12} \\
&\quad = \Sigma_{34}^{12}(\hat{J} \otimes J \otimes \hat{J} \otimes J)Z_{12}\boldsymbol{Y}_{24}Z_{12}^*\boldsymbol{X}_{13}(\hat{J} \otimes J \otimes \hat{J} \otimes J)\Sigma_{34}^{12} \\
&\quad = (\hat{J} \otimes J \otimes \hat{J} \otimes J)Z_{34}(\Sigma_{34}^{12}\boldsymbol{Y}_{24}\Sigma_{34}^{12})Z_{34}^*(\Sigma_{34}^{12}\boldsymbol{X}_{13}\Sigma_{34}^{12})(\hat{J} \otimes J \otimes \hat{J} \otimes J).
\end{aligned}$$

Since $(\hat{J} \otimes J)Z = Z^*(\hat{J} \otimes J)$, we have

$$(\hat{J} \otimes J \otimes \hat{J} \otimes J)Z_{34} = Z_{34}^*(\hat{J} \otimes J \otimes \hat{J} \otimes J).$$

Moreover, since

$$\boldsymbol{X} = (\hat{J} \otimes \hat{J})\Sigma W^* \Sigma(\hat{J} \otimes \hat{J}),$$

$$\boldsymbol{Y} = (J \otimes J)W^*(J \otimes J),$$

we find that

$$\Sigma_{34}^{12} \boldsymbol{X}_{13} \Sigma_{34}^{12} = (\hat{J} \otimes 1 \otimes \hat{J} \otimes 1) W_{13} (\hat{J} \otimes 1 \otimes \hat{J} \otimes 1),$$

$$\Sigma_{34}^{12} \boldsymbol{Y}_{24} \Sigma_{34}^{12} = (1 \otimes J \otimes 1 \otimes J) \hat{W}_{24} (1 \otimes J \otimes 1 \otimes J).$$

From these identities, it follows that

$$(\hat{J} \otimes J \otimes \hat{J} \otimes J) Z_{34} (\Sigma_{34}^{12} \boldsymbol{Y}_{24} \Sigma_{34}^{12}) Z_{34}^{*} (\Sigma_{34}^{12} \boldsymbol{X}_{13} \Sigma_{34}^{12}) (\hat{J} \otimes J \otimes \hat{J} \otimes J)$$
$$= Z_{34}^{*} \hat{W}_{24} Z_{34} W_{13} = W^{\mathcal{N}}.$$

The last part of this theorem results from Proposition 4.13. □

As a direct consequence of Theorem 4.15, we immediately obtain the proposition that follows. It is merely a rephrase of a part of Proposition 8.14 and Proposition 8.19 in [**BS**].

Proposition 4.16. (1) *For any* $z \in \hat{\mathcal{M}}'$ *and* $b \in \mathcal{M}'$, *put* $\pi(z) := Z^{*}(z \otimes 1)Z$, $\pi'(b) := 1 \otimes b$. *Then* $\pi : \hat{\mathcal{M}}' \to \hat{\mathcal{N}}'$ *and* $\pi' : \mathcal{M}' \to \hat{\mathcal{N}}'$ *are Hopf-von Neumann algebra morphisms, i.e., we have*

$$(\pi \otimes \pi) \circ \hat{\delta}'(z) = (\widehat{\delta^{\mathcal{N}}})' \circ \pi(z),$$

$$(\pi' \otimes \pi') \circ \delta'(b) = (\widehat{\delta^{\mathcal{N}}})' \circ \pi'(b).$$

(2) *Set* $\mathcal{R} := Z_{12}^{*} \boldsymbol{X}_{14} Z_{12}$. *Then* $\mathcal{R}$ *is a unitary in* $\hat{\mathcal{N}}'$. *One also has*

$$(id \otimes (\widehat{\delta^{\mathcal{N}}})')(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \quad ((\widehat{\delta^{\mathcal{N}}})' \otimes id)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}.$$

For any $x \in \hat{\mathcal{N}}'$, *we have* $\sigma \circ (\widehat{\delta^{\mathcal{N}}})'(x) = \mathcal{R}(\widehat{\delta^{\mathcal{N}}})'(x)\mathcal{R}^{*}$. *Moreover, it satisfies the quantum Yang-Baxter equation*: $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$.

We now summarize the quasi Woronowicz algebraic structure of $\widehat{D(\boldsymbol{W})}$. The Kac-Takesaki operator $\widehat{W^{\mathcal{N}}}$ of $\widehat{D(\boldsymbol{W})}$ is, by definition, $\Sigma_{34}^{12}(W^{\mathcal{N}})^{*}\Sigma_{34}^{12}$. From Theorem 4.11, it follows that

$$\text{(QD1)} \qquad \widehat{W^{\mathcal{N}}} = \hat{W}_{13} Z_{12}^{*} W_{24} Z_{12}.$$

We denote the modular operator and the modular conjugation of $D(\boldsymbol{W})$ on $\mathfrak{H} \otimes \mathfrak{H}$ by $\Delta_{\mathcal{N}}$ and $J_{\mathcal{N}}$, respectively. Since the Haar measure of $D(\boldsymbol{W})$ is $h \otimes \hat{h} \circ \hat{R}$ and we need to represent everything on $\mathfrak{H} \otimes \mathfrak{H}$, we have (cf. Corollary 4.4)

$$\Delta_{\mathcal{N}} = \Delta \otimes J\hat{\Delta}J, \quad J_{\mathcal{N}} = J \otimes \hat{J}.$$

The antipode $\widehat{R^{\mathcal{N}}}$ of $\widehat{D(\boldsymbol{W})}$ is then given by

$$\text{(QD2)} \qquad \widehat{R^{\mathcal{N}}}(X) = J_{\mathcal{N}} X^{*} J_{\mathcal{N}}, \quad (X \in \hat{\mathcal{N}}).$$

By the definition of $\tau_t^{\mathcal{N}}$, it is easy to see that the canonical implementation of $\tau_t^{\mathcal{N}}$ on $\mathfrak{H} \otimes \mathfrak{H}$ is $H^{it} \otimes H^{it}$. Set $H_{\mathcal{N}} := H \otimes H$. By [**MN**, Proposition 3.6], the deformation automorphism $\{\widehat{\tau_t^{\mathcal{N}}}\}$ of the dual $\widehat{D(\boldsymbol{W})}$ is given by

$$\widehat{\tau_t^{\mathcal{N}}} = \operatorname{Ad} H_{\mathcal{N}}^{it} = \operatorname{Ad}(H^{it} \otimes H^{it}). \tag{QD3}$$

Let $\hat{\Delta}_{\mathcal{N}}$ stand for the modular operator of $\widehat{D(\boldsymbol{W})}$ on $\mathfrak{H} \otimes \mathfrak{H}$. From [**MN**, Lemma 2.14] and Theorem 3.3, it follows that $\hat{\Delta}_{\mathcal{N}}$ is nothing but $H_{\mathcal{N}} = H \otimes H$. Thus one has $\sigma_t^{\widehat{h^{\mathcal{N}}}} = \widehat{\tau_t^{\mathcal{N}}}$. The modular conjugation $\hat{J}_{\mathcal{N}}$ of $\widehat{D(\boldsymbol{W})}$ has already been given in Proposition 4.13. We denote by $\hat{Q}_{\mathcal{N}}$ the analytic generator of the Radon Nikodym derivative $(D(\widehat{h^{\mathcal{N}}} \circ \widehat{R^{\mathcal{N}}}) : D\widehat{h^{\mathcal{N}}})_t$, which is a one-parameter unitary group. Namely, $(D(\widehat{h^{\mathcal{N}}} \circ \widehat{R^{\mathcal{N}}}) : D\widehat{h^{\mathcal{N}}})_t = \hat{Q}_{\mathcal{N}}^{it}$. By [**MN**, Lemma 2.14] again, we have $\Delta_{\mathcal{N}}^{it} = \hat{J}_{\mathcal{N}} \hat{Q}_{\mathcal{N}}^{-it} \hat{J}_{\mathcal{N}} H_{\mathcal{N}}^{it}$. Hence we obtain

$$\hat{Q}_{\mathcal{N}}^{it} = Z^*(\hat{J}\Delta^{-it}\hat{J}H^{it} \otimes \hat{\Delta}^{it}H^{it})Z. \tag{QD4}$$

We define two homomorphisms $\alpha : \mathcal{M} \to \hat{\mathcal{N}}$ and $\beta : \hat{\mathcal{M}} \to \hat{\mathcal{N}}$ by

$$\alpha(a) := Z^*(1 \otimes a)Z, \quad \beta(y) := y \otimes 1 \quad (a \in \mathcal{M}, y \in \hat{\mathcal{M}}).$$

By virtue of Corollary 4.12, $\hat{\mathcal{N}}$ is the σ-strong* closure of the linear span of elements of the form $\beta(y)\alpha(a)$ $(a \in \mathcal{M}, y \in \hat{\mathcal{M}})$. Recall that the unitary Z is the canonical implementation of the inner automorphism $\operatorname{Ad} W^*$ of $\mathcal{N}$. So Z commutes with $J_{\mathcal{N}}$. From this, one easily finds that both α and β "commute" with the antipode $\hat{R}^{\mathcal{N}}$, i.e.,

$$\widehat{R^{\mathcal{N}}} \circ \alpha = \alpha \circ R, \quad \widehat{R^{\mathcal{N}}} \circ \beta = \beta \circ \hat{R}. \tag{QD5}$$

By [**MN**, Corollary 3.6.1], $H_{\mathcal{N}}$ commutes with W. We also have $[H_{\mathcal{N}}, u \otimes u] = 0$, where $u = J\hat{J}$. Consequently, Z commutes with $H_{\mathcal{N}}$. It is now plain to see that

$$\widehat{\tau_t^{\mathcal{N}}} \circ \alpha = \alpha \circ \tau_t, \quad \widehat{\tau_t^{\mathcal{N}}} \circ \beta = \beta \circ \hat{\tau}_t. \tag{QD6}$$

With the notation in Proposition 4.16, it can be verified that

$$\alpha(a) = \hat{J}_{\mathcal{N}} \pi'(JaJ) \hat{J}_{\mathcal{N}}, \quad \beta(y) = \hat{J}_{\mathcal{N}} \pi(\hat{J}y\hat{J}) \hat{J}_{\mathcal{N}}.$$

From this and the definition of the coproduct $(\widehat{\delta^{\mathcal{N}}})'$, it follows that

$$(\alpha \otimes \alpha) \circ \delta = \widehat{\delta^{\mathcal{N}}} \circ \alpha, \quad (\beta \otimes \beta) \circ \hat{\delta} = \widehat{\delta^{\mathcal{N}}} \circ \beta. \tag{QD7}$$

Therefore we have shown that the maps $\alpha : \mathcal{M} \to \hat{\mathcal{N}}$ and $\beta : \hat{\mathcal{M}} \to \hat{\mathcal{N}}$ defined above are coinvolutive Hopf-von Neumann algebra morphisms.

Finally, we examine the dual Haar measure $\widehat{h^{\mathcal{N}}}$. As before, we denote the Fourier representation of $D(\boldsymbol{W})$ by $\hat{\pi}_{\mathcal{N}}$. Let Θ, Φ be in $\mathcal{N}_*$ which are $L^2(h^{\mathcal{N}})$-bounded. We use the notation $\hat{\eta}_{\mathcal{N}}(\Theta)$ etc., instead of $\hat{\eta}(\Theta)$, for the corresponding L^2-vector in $\mathfrak{H} \otimes \mathfrak{H}$. Then, by definition, one has

$$\widehat{h^{\mathcal{N}}}(\hat{\pi}_{\mathcal{N}}(\Theta)^* \hat{\pi}_{\mathcal{N}}(\Phi)) = (\hat{\eta}_{\mathcal{N}}(\Phi) \,|\, \hat{\eta}_{\mathcal{N}}(\Theta)). \tag{HM1}$$

If Θ and Φ are of the form $\Theta = \theta \otimes \omega$, $\Phi = \phi \otimes \psi$ for some $\theta, \phi, \omega, \psi \in \mathcal{B}(\mathfrak{H})_*$, then, as shown in the proof of Corollary 4.12, we have

$$\hat{\pi}_{\mathcal{N}}(\Theta) = \beta(\hat{\pi}(\theta))\alpha(\hat{\hat{\pi}}(\omega)), \quad \hat{\pi}_{\mathcal{N}}(\Phi) = \beta(\hat{\pi}(\phi))\alpha(\hat{\hat{\pi}}(\psi)).$$

Moreover, if θ, ϕ are $L^2(h)$-bounded and if ω, ψ are $L^2(\hat{h} \circ \hat{R})$-bounded, then $\theta \otimes \omega$ and $\phi \otimes \psi$ are certainly $L^2(h^{\mathcal{N}})$-bounded, and we easily find that

$$\hat{\eta}_{\mathcal{N}}(\Theta) = \hat{\eta}(\theta) \otimes \hat{\eta}_{\hat{h} \circ \hat{R}}(\omega), \quad \hat{\eta}_{\mathcal{N}}(\Phi) = \hat{\eta}(\phi) \otimes \hat{\eta}_{\hat{h} \circ \hat{R}}(\psi),$$

where $\hat{\eta}_{\hat{h} \circ \hat{R}}(\omega)$ stands for the vector in $\mathfrak{H}$ corresponding to the $L^2(\hat{h} \circ \hat{R})$-bounded element ω. Under these circumstances, (HM1) can be written as

$$\widehat{h^{\mathcal{N}}}(\alpha(\hat{\hat{\pi}}(\omega))^*\beta(\hat{\pi}(\theta)^*\hat{\pi}(\phi))\alpha(\hat{\hat{\pi}}(\psi))) = (\hat{\eta}(\phi) \,|\, \hat{\eta}(\theta))(\hat{\eta}_{\hat{h} \circ \hat{R}}(\psi) \,|\, \hat{\eta}_{\hat{h} \circ \hat{R}}(\omega)).$$

In the meantime, we have

$$(\hat{\eta}(\phi) \,|\, \hat{\eta}(\theta)) = \hat{h}(\hat{\pi}(\theta)^*\hat{\pi}(\phi)), \quad (\hat{\eta}_{\hat{h} \circ \hat{R}}(\psi) | \hat{\eta}_{\hat{h} \circ \hat{R}}(\omega)) = h'(\hat{\pi}_{\hat{h} \circ \hat{R}}(\omega)^*\hat{\pi}_{\hat{h} \circ \hat{R}}(\psi)),$$

where h' is the Haar measure for the commutant $\boldsymbol{W}'$ of $\boldsymbol{W}$, and $\hat{\pi}_{\hat{h} \circ \hat{R}}$ is the Fourier representation of $(\hat{\boldsymbol{W}})^{\circ}$. (Remark that the dual of $(\hat{\boldsymbol{W}})^{\circ}$ is $\boldsymbol{W}'$). Therefore, we get

$$\text{(HM2)} \qquad \widehat{h^{\mathcal{N}}}(\alpha(\hat{\hat{\pi}}(\omega))^*\beta(\hat{\pi}(\theta)^*\hat{\pi}(\phi))\alpha(\hat{\hat{\pi}}(\psi))) = h'(\hat{\pi}_{\hat{h} \circ \hat{R}}(\omega)^*\hat{\pi}_{\hat{h} \circ \hat{R}}(\psi))\hat{h}(\hat{\pi}(\theta)^*\hat{\pi}(\phi)).$$

In some sense, this identity characterizes the Haar measure $\widehat{h^{\mathcal{N}}}$ as we see in the next proposition.

Remark. In the above discussion, if ω and ψ are, in addition, analytic with respect to the deformation automorphism $\hat{\tau}$, then one can easily verify that, for example,

$$\hat{\pi}_{\hat{h} \circ \hat{R}}(\omega) = J\hat{\hat{\pi}}(\omega \circ \hat{\tau}_{-i/2})^*J = J\tau_{i/2}(\hat{\hat{\pi}}(\omega))^*J.$$

Thus we have

$$\widehat{h^{\mathcal{N}}}(\alpha(\hat{\hat{\pi}}(\omega))^*\beta(\hat{\pi}(\theta)^*\hat{\pi}(\phi))\alpha(\hat{\hat{\pi}}(\psi))) = h(\tau_{i/2}(\hat{\hat{\pi}}(\psi))\tau_{i/2}(\hat{\hat{\pi}}(\omega))^*)\hat{h}(\hat{\pi}(\theta)^*\hat{\pi}(\phi)).$$

Proposition 4.17. *If the original quasi Woronowicz algebra* $\boldsymbol{W}$ *is compact, then there exists a unique faithful normal conditional expectation* E *from* $\hat{\mathcal{N}}$ *onto* $\beta(\hat{\mathcal{M}})$ *such that*

$$\widehat{h^{\mathcal{N}}} = \hat{h} \circ \beta^{-1} \circ E.$$

Proof. If $\boldsymbol{W}$ is compact, then $Q = 1$. From [**MN**, Lemma 2.14], it then follows that $\sigma^{\hat{h}} = \hat{\tau}$. Since $\sigma^{\widehat{h^{\mathcal{N}}}} = \widehat{\tau^{\mathcal{N}}}$, it results from (QD6) that $\sigma^{\widehat{h^{\mathcal{N}}}} \circ \beta = \beta \circ \sigma^{\hat{h}}$. This implies that, identifying $\beta(\hat{\mathcal{M}})$ with $\hat{\mathcal{M}}$, the modular automorpshim $\sigma^{\widehat{h^{\mathcal{N}}}}$ leaves $\hat{\mathcal{M}}$ globally invariant and that its restriction to $\hat{\mathcal{M}}$ equals $\sigma^{\hat{h}}$. Moreover, from (HM2), we find that the restriction of $\widehat{h^{\mathcal{N}}}$ to $\beta(\hat{\mathcal{M}}) = \hat{\mathcal{M}}$ is semifinite. Therefore, by [**H2**] and [**T**], there exists a unique faithful normal conditional expectation E from $\hat{\mathcal{N}}$ onto $\beta(\hat{\mathcal{M}})$ with the desired property. □

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