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Asymptotic rigidity of Hadamard 2-spaces

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Abstract. We classify locally compact, geodesically complete, 2-dimensional Hadamard spaces whose Tits ideal boundaries have the minimal diameter π. Furthermore, we classify the universal covering spaces of certain 2-dimensional nonpositively curved spaces, which is an extension of the result obtained in the polyhedral case by W. Ballmann, M. Brin, and S. Barré.

§1. Introduction.

In Riemannian geometry, ever since the work of [**BGS**], many results regarding the asymptotic rigidity of Hadamard manifolds have been obtained. In general, the problem is that of obtaining the rigid properties of the global structures of certain Hadamard manifolds from information about their ideal boundaries.

Since Alexandrov's original study, the various properties of general metric spaces with nonpositive curvature have been investigated (cf. [**Al**], [**ABN**], [**Ba1**], [**Ba2**]). In particular, nonpositively curved 2-polyhedra have been studied in detail (cf. [**BaBr**], [**Bar**]). In [**BaBr**], W. Ballmann and M. Brin obtain the following result:

THEOREM 1 ([**BaBr**], Theorem 6.5). *Let (X, Γ) be a compact 2-dimensional orbihedron without boundary and of nonpositive curvature. Assume that all links of X have diameter π, that all faces of X are Euclidean triangles, and that all edges are geodesics. Then X is either the product of two trees or a thick Euclidean building of dimension 2 of type A_2, B_2, or G_2.*

Also, this result has been obtained later and independently by S. Barré in [**Bar**]. According to [**BaBr**], this result is also related to the (unpublished) result of B. Kleiner (1995): if every geodesic of an n-dimensional complete, simply connected space of X of nonpositive curvature is contained in an n-flat, then X is a Euclidean building or a product of Euclidean buildings.

The main purpose of this paper is to study the asymptotic rigidity of more general Hadamard spaces. In such general metric spaces, we must be careful to consider the possibility of bifurcation of geodesics essentially different from those in the polyhedron case. Furthermore, it was known by B. Kleiner that there is an example of a locally compact, geodesically complete Hadamard space which admits no triangulation. We will construct such an example in the next section (Example 2.7). This construction yields precise information concerning the ideal boundary.

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A *nonpositively curved space* is a complete geodesic space with nonpositive curvature in the sense of Alexandrov. A *Hadamard space* is a simply connected nonpositively curved space. A locally compact *nonpositively curved* 2-*space* (respectively *Hadamard* 2-*space*) is a locally compact nonpositively curved space (respectively Hadamard space) of $\dim_H X = 2$, where $\dim_H$ is the Hausdorff dimension. We say that a nonpositively curved space X is *geodesically complete* if an arbitrary geodesic in X can be extended to a line. The asymptotic classes of rays in a Hadamard space X define the ideal boundary $X(\infty)$ with the Tits metric Td, as in the smooth Riemannian case.

We consider the class of locally compact, geodesically complete Hadamard 2-spaces whose ideal boundaries have the minimal diameter π with respect to the Tits metric.

Example 1. The following are examples whose ideal boundaries possess the minimal diameter π.

(a) The product of two trees with interior metrics.

(b) A thick Euclidean building of dimension 2 of type A_2, B_2, or G_2. (see [**Bro1**, Chapter 4], [**Bro2**].)

On the other hand, there are many examples in which the ideal boundary has a diameter greater than π. In Example 2.7, we construct an example of a locally compact, geodesically complete Hadamard 2-space whose ideal boundary has diameter $\pi + \varepsilon$ for any $\varepsilon > 0$.

One of the main results of this paper is the following:

Theorem A. *Let X be a locally compact, geodesically complete Hadamard 2-space such that the diameter of $(X(\infty), Td)$ is equal to π. Then X is isometric to either the product of two trees, the Euclidean cone over $(X(\infty), Td)$, or a thick Euclidean building of dimension 2 of type A_2, B_2, or G_2.*

We now introduce a concept corresponding to the rank 2 condition that appears in [**Ba2**] and [**Bar**]. We say that a locally compact, geodesically complete nonpositively curved 2-space X satisfies the *Local Flat Condition*—or briefly the *LFC*—if X satisfies the following:

(LFC) For any $x \in X$ there is a positive number $s = s(x) > 0$ such that for any unit speed geodesic $\sigma_x : \boldsymbol{R} \to X$ with $\sigma_x(0) = x$ there is a totally geodesic isometric imbedding $\varphi : D^+(s) \to X$ satisfying $\varphi(t_1, 0) = \sigma_x(t_1)$ for $t_1 \in (-s, s)$.

Here $D^+(s) := \{(t_1, t_2) | t_2 \geq 0, t_1^2 + t_2^2 < s^2\}$ is a local half disk with the standard flat metric.

Example 2. In the following, each X satisfies the LFC. Let $\tilde{X}$ denote the universal covering space of X.

(a) We consider two flat rectangles X_1 and X_2 isometric to each other. Let $e_i, i = 1, 2$, be sides of X_i such that e_1 and e_2 have the same length. We denote by X_i' the torus constructed by gluing the sides of X_i in the usual way. We define $X := X_1' \bigcup_{e_1 = e_2} X_2'$, identifying e_1 and e_2 isometrically. Then $\tilde{X}$ is isometric to the product of a tree and a line.

(b) Let X be the product of two $S^1 \vee S^1$, where $S^1 \vee S^1$ is the one point union of two circles with an interior metric. Then $\tilde{X}$ is the product of two trees.

The other main result of this paper is an extension of Theorem 1:

Theorem B. *Let X be a locally compact, geodesically complete nonpositively curved 2-space satisfying the LFC. Then the universal covering space $\tilde{X}$ of X is isometric to either the product of two trees, the Euclidean cone over $(\tilde{X}(\infty), Td)$, or a thick Euclidean building of dimension 2 of type A_2, B_2, or G_2.*

The basic ideas of the proof of the main results are as follows: Let X be as in Theorem A. Then X satisfies the LFC. First, for any $x \in X$, we construct a local flat (whole) disk around x containing a given geodesic through x. Then, a neighborhood of x is the union of local flat disks. In order to understand how these flat disks meet each other, we observe the following assertion $(*)$, which is a key to the proof of Theorem A.

$(*)$ For any $x \in X$ and each direction at x there is a positive number $r > 0$ such that the geodesics emanating from x directed by that direction coincide within distance r from x. Moreover, such a positive number $r > 0$ can be chosen independently of the choice of the directions at x.

The existence of local flat disks and $(*)$ enable us to obtain a global flat plane containing a given line. Also, we observe that, for any singular point x and an arbitrary geodesic σ_x passing through x which is directed by two vertices of the space of directions at x, every point on σ_x is also a singular point. By using the local properties of X, we can determine the global structure of X. In this way, we obtain Theorem A. Also, we obtain that the diameter of $(\tilde{X}(\infty), Td)$ is equal to π. Then, by applying Theorem A, we conclude Theorem B. Several ideas in our approach are inspired by arguments in [**BaBr**] and [**Bar**].

This paper is organized as follows: After presenting the basic concepts and properties of Hadamard spaces in Section 2, we discuss splitting theorems for certain Hadamard spaces, which are known in the smooth Riemannian case ([**BGS**], [**Oh**]) in Section 3. In Section 4, we discuss the local properties of nonpositively curved 2-spaces with the LFC. In Section 5, we construct a flat plane in a Hadamard 2-space with the LFC. We prove Theorems A and B in Section 6.

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§2. Preliminaries.

Throughout this paper we use Landau's symbol in the following sense: For a real-valued function $f : [0, \infty) \to \boldsymbol{R}$, we write $f(t) = o(t)$ if $\lim_{t \to 0} f(t)/t = 0$. Also, in a metric space (X, d), we denote by $B(x, r)$ (respectively $\bar{B}(x, r)$) the open (respectively closed) metric ball centered at $x \in X$ with radius $r > 0$.

In this section we discuss some concepts and basic facts (mainly stated in [**ABN**], [**Ba2**], [**Ot**], and [**OT**]) which will be used in the following sections.

2.1. Metric spaces with curvature bounded above.

Let (X, d) be a metric space. For an interval I, a curve $\sigma : I \to X$ is a *geodesic* if there is a $c \geq 0$, called the *speed of* σ, such that any $t \in I$ has a neighborhood $J \subset I$ with $d(\sigma(t_1), \sigma(t_2)) = c|t_1 - t_2|$ for all $t_1, t_2 \in J$. If this equality holds for all $t_1, t_2 \in I$, then σ

is called a *minimizing geodesic*. Furthermore, for a minimizing geodesic σ, if $I = \boldsymbol{R}$, then σ is called a *line*, and if $I = [0, \infty)$, then σ is called a *ray*. In this paper, we assume that all geodesics have unit speed $c = 1$. X is a *geodesic space* if for any two points in X there is a minimizing geodesic joining them.

From now on, we assume that X is a complete geodesic space. A (*geodesic*) *triangle* $\triangle$ in X consists of three geodesics in X whose endpoints match in the usual way. For some $\kappa \in \boldsymbol{R}$, we denote by $M^2(\kappa)$ the model surface of constant Gaussian curvature κ. A *comparison triangle* $\overline{\triangle}$ *for* $\triangle \subset X$ is a triangle in $M^2(\kappa)$ whose sides have the same lengths as those of $\triangle$. We say that a triangle $\triangle$ in X is $CAT(\kappa)$ if the lengths of the sides of $\triangle$ satisfy the triangle inequality, and if the perimeter of $\triangle$ is less than $2\pi/\sqrt{\kappa}$ in the case $\kappa > 0$, and if $d(x, y) \leq |\bar{x}, \bar{y}|_\kappa$ for every two points x, y in the sides of $\triangle$ and the corresponding points $\bar{x}, \bar{y}$ in $\overline{\triangle}$, where $|-, -|_\kappa$ is the standard metric on $M^2(\kappa)$. A subset $U \subset X$ is *convex* if each geodesic joining y and z is contained in U for all $y, z \in U$. A convex open subset $U \subset X$ is a $CAT(\kappa)$-*domain* if for any $x, y \in U$ there is a minimizing geodesic $\sigma_{xy} : [0, d(x, y)] \to U$ joining x and y, and if all triangles in U are $CAT(\kappa)$.

We say that X has *curvature bounded above by* κ if there is a $\kappa \in \boldsymbol{R}$ so that every point $x \in X$ has an open neighborhood U_x such that any triangle composed of three points in U_x is $CAT(\kappa)$. Note that if X has curvature bounded above by κ, then for any $x \in X$ there is a $CAT(\kappa)$-domain U of x; in particular, for all $y, z \in U$ there is a unique minimizing geodesic σ_{yz} in U joining y and z, and σ_{yz} depends continuously on y and z in U. In U, we denote by $\triangle(x, y, z)$ the triangle with vertices $x, y, z \in U$.

Let $U \subset X$ be a $CAT(\kappa)$-domain, and let $\sigma_1, \sigma_2 : [0, \varepsilon] \to U$ be geodesics in U emanating from the same point $x = \sigma_i(0) \in U$ for $\varepsilon > 0$. The *angle* $\angle_x(\sigma_1, \sigma_2)$ (*or* $\angle_x(\sigma_1(s_0), \sigma_2(t_0))$ for some $s_0, t_0 \in (0, \varepsilon]$) *at* x *between* σ_1 *and* σ_2 (respectively $\sigma_1(s_0)$ *and* $\sigma_2(t_0)$) is defined as $\lim_{s,t \to 0} \bar{\theta}_\kappa(\bar{\sigma}_1(s), \bar{\sigma}_2(t))$, where $\bar{\theta}_\kappa(\bar{\sigma}_1(s), \bar{\sigma}_2(t))$ is the corresponding angle at $\bar{x}$ in the comparison triangle in $M^2(\kappa)$ for $\triangle(x, \sigma_1(s), \sigma_2(t))$. We have the triangle inequality of angles and the formula

$$\cos(\angle_x(\sigma_1, \sigma_2)) = \lim_{s,t \to 0} \frac{s^2 + t^2 - d^2(\sigma_1(s), \sigma_2(t))}{2st}.$$

Also, we have the claim concerning the (semi)continuity of angles:

$$\limsup_{n \to \infty} \angle_{x_n}(y_n, z_n) \leq \angle_x(y, z) = \lim_{n \to \infty} \angle_x(y_n, z_n)$$

for $x_n \to x$, $y_n \to y$, and $z_n \to z$ as $n \to \infty$ with $x \in U$ and $y, z \in U \backslash \{x\}$. Note that for a triangle $\triangle \subset U$ the angle at an arbitrary vertex of $\triangle$ is not greater than the corresponding angle in $\overline{\triangle}$.

From this point on, we assume that X is locally compact and geodesically complete. For $x, y, z \in U$, two minimizing geodesics σ_{xy} and σ_{xz} extending from x to y and from x to z, respectively, are called *equivalent* if and only if $\angle_x(\sigma_{xy}, \sigma_{xz}) = 0$. Then we denote by v_{xy} (or briefly v_y) the equivalence class representing σ_{xy}. Define $\Sigma_x := \{v_{xy} | y \in X \backslash \{x\}\}$. Then $(\Sigma_x, \angle_x)$ is a metric space. The *space of directions at* $x \in X$ is the interior metric space (Σ_x, ρ_x), where ρ_x is the interior metric induced from $\angle_x$. In general, (Σ_x, ρ_x) is not connected. The *tangent cone at* $x \in X$ is the Euclidean cone over

(Σ_x, ρ_x) and is denoted by $(\Sigma_x, \rho_x) \times [0, \infty)/_\sim$. Note that each connected component of (Σ_x, ρ_x) is a compact, geodesically complete geodesic space with curvature bounded above by 1, and that the tangent cone at x is a locally compact, geodesically complete geodesic space with curvature bounded above by 0. We also remark that, for any $x \in U$, there is a Lipschitz map

$$X \supset U \ni y \mapsto (v_{xy}, d(x, y)) \in \Sigma_x \times [0, \infty)/_\sim$$

(where its image of x is the vertex of the cone).

Lemma 2.1. *If $\angle_x(\sigma_{xu}, \sigma_{xv}) < \pi$, where σ_{xu} and σ_{xv} are geodesics extending from x directed by $u, v \in \Sigma_x$, then $\angle_x(\sigma_{xu}, \sigma_{xv}) = \rho_x(u, v)$. Moreover, if $\rho_x(u, v) < \pi$, then there is a unique minimizing geodesic in Σ_x joining u and v; in other words, Σ_x has an injectivity radius π.*

For a point $x \in X$ and a positive number $\delta > 0$, we say that $y \in X$ is a *δ-branch point of x* if the diameter of the subset of Σ_y whose points are distance π from $v_{yx} \in \Sigma_y$ with respect to $\angle_y$ is not smaller than δ. The following two lemmas are very useful in the investigation of the local structure of X. For the sake of readers' convenience, we give the proofs for them.

Lemma 2.2 ([**OT**]). *Fix $x \in X$ and $\delta > 0$. No sequence (y_n) of δ-branch points of x converges to x.*

Proof. For the simplicity, we discuss the argument below in a $CAT(0)$-domain. Suppose that there is a sequence (y_n) of δ-branch points of x with $y_n \to x$ as $n \to \infty$. Set $t_n := d(x, y_n)$. Since X is locally compact, if necessary, taking a suitable subsequence, we can take z_n and w_n such that

(1) z_n and w_n converge to z_0 and w_0, respectively.
(2) $d(x, z_n) = d(x, y_n) + d(y_n, z_n) = r$ and $d(x, w_n) = d(x, y_n) + d(y_n, w_n) = r$ for some $r > 0$.
(3) $\angle_{y_n}(z_n, w_n) > \delta/2$.

For any $\varepsilon > 0$, consider z_n and w_n satisfying $d(z_n, z_0), d(w_n, w_0) < \varepsilon$ for large n. Let z_0' (respectively w_0') be the point on the geodesic joining x and z_0 (respectively joining x and w_0) with $d(x, z_0') = t_n$ (respectively $d(x, w_0') = t_n$). Now, $\angle_x(z_0', w_0') \geq \delta/2$. By comparison geometry, we obtain the following inequalities: $d(z_0', w_0')/t_n \geq \sqrt{2(1 - \cos(\delta/2))}$ and $d(y_n, z_0'), d(y_n, w_0') < t_n\varepsilon/r$. These inequalities yield a contradiction. $\square$

Lemma 2.3 ([**OT**]). *Let U be a $CAT(\kappa)$-domain. Choose a point z lying on the minimizing geodesic joining x and y such that z is not a δ-branch point of x. Let $w \in U$ be a point with $0 < \angle_z(x, w) < \pi$. Then we have*

$$\pi \leq \angle_z(x, w) + \angle_z(w, y) < \pi + \delta.$$

Proof. Let (w_n) be a sequence of points on the geodesic joining z and w with $w_n \to z$ as $n \to \infty$. And let y_n be a point such that w_n lies on the geodesic extending from x to y_n and $d(x, y_n) = d(x, y)$. We assume that y_n converges to some point y' for a suitable subsequence. Now, $\lim_{n\to\infty}(\angle_z(x, w) + \angle_z(w, y_n)) \leq \pi$. Since z is not a

δ-branch point of x, $\angle_z(w, y) < \angle_z(w, y') + \delta$. Therefore, we obtain $\angle_z(x, w) + \angle_z(w, y) < \pi + \delta$. $\square$

The following theorem, given in [**Ot**] and [**OT**], is important for observing the structure of a metric space with curvature bounded above.

THEOREM 2.4 ([**Ot**], Theorem 3.2, and [**OT**]). *Let X be a locally compact, geodesically complete geodesic space with curvature bounded above. Then for any relatively compact open set O of X there is an $n \in \mathbf{N}$ such that $0 < \mathscr{H}^n(O) < \infty$, where $\mathscr{H}^n(-)$ is the n-dimensional Hausdorff measure.*

If X is a locally compact, geodesically complete geodesic space of $\dim_H X = 2$ with curvature bounded above, then we define

$$S_X := \{x \in X | \Sigma_x \text{ is isometric to neither } S^1(1) \text{ nor } S^0(1)\}.$$

We call the points in S_X *singular points* and the points in $X \backslash S_X$ *regular points*. We note that no regular point is a δ-branch point of an arbitrary $x \in X$ for any $\delta > 0$. Also, note that $\dim_H \Sigma_x \leq 1$ for any $x \in X$ in this case.

2.2. Hadamard spaces and their ideal boundaries.

Let (X, d) be a Hadamard space; that is, a simply connected nonpositively curved space. We have the Hadamard-Cartan theorem as in the smooth Riemannian case.

THEOREM 2.5 (Hadamard-Cartan theorem) ([**Gr**]). *Assume that X is a Hadamard space. Then for any $x, y \in X$ there is a unique minimizing geodesic $\sigma_{xy} : [0, d(x, y)] \to X$ joining x and y.*

Theorem 2.5 and the curvature condition reveal many properties of X.

Let $\triangle$ be a triangle in X. Then the sum of the three angles of $\triangle$ is not greater than π. This sum is identically π if and only if $\triangle$ is flat; that is, if and only if $\triangle$ bounds a convex region in X isometric to the triangular region bounded by $\overline{\triangle}$ in the flat plane. Also, let $\sigma_1, \sigma_2 : I \to X$ be two geodesics. Then $d(\sigma_1(t), \sigma_2(t))$ is convex in $t \in I$. From the convexity of the distance function on X, we can define the projection map pr_C onto a closed convex subset $C \subset X$ by requiring $pr_C(x)$ to be the closest point to x in C. This is a Lipschitz map with Lipschitz constant 1.

Two rays $\sigma_1, \sigma_2 : [0, \infty) \to X$ are called *asymptotic* if $d(\sigma_1(t), \sigma_2(t))$ is uniformly bounded in t. This yields an equivalence relation on the set of rays in X. We denote by $X(\infty)$ the set of all asymptotic equivalence classes of rays in X. If σ is a ray belonging to $\xi \in X(\infty)$, we write $\sigma(\infty) = \xi$ and say that σ is a ray from $\sigma(0)$ to ξ. Note that for any $x \in X$ and $\xi \in X(\infty)$ there is a unique ray $\sigma_{x\xi} : [0, \infty) \to X$ from x to ξ.

Next we define the interior metric on $X(\infty)$. Define $\angle(\xi, \eta) := \sup_{x \in X} \angle_x(\xi, \eta)$ for $\xi, \eta \in X(\infty)$. Then $\angle$ is a metric on $X(\infty)$. The *Tits metric* on $X(\infty)$, denoted by $Td(-,-)$, is the interior metric induced from $\angle$. Of course, $Td(\xi, \eta) \geq \angle(\xi, \eta)$ for any $\xi, \eta \in X(\infty)$.

The following statement is often useful for considering the geometry of $X(\infty)$.

THEOREM 2.6 ([**Ba2**], Theorem II.4.11). *Assume that X is a locally compact Hadamard space. Then each connected component is a complete geodesic space with curvature bounded above by* 1. *Furthermore, for $\xi, \eta \in X(\infty)$ the following hold*:

(i) *If there is no geodesic in X joining ξ and η, then $Td(\xi,\eta) = \angle(\xi,\eta) \le \pi$.*

(ii) *If $\angle(\xi,\eta) < \pi$, then there is no geodesic in X joining ξ and η and there is a unique minimizing geodesic in $(X(\infty), Td)$ joining ξ and η*; *in particular, any triangle in $(X(\infty), Td)$ of perimeter $< 2\pi$ is $CAT(1)$.*

(iii) *If there is a geodesic σ in X joining ξ and η, then $Td(\xi,\eta) \ge \pi$, and equality holds if and only if σ bounds a flat half plane.*

EXAMPLE 2.7. We now construct an example of a Hadamard space which admits no triangulation. This construction is essentially based on an idea of B. Kleiner.

For given $\varepsilon \in (0,\pi)$, we prepare a sequence (α_n) of positive numbers with $\sum_{n=1}^{\infty}(\pi - \alpha_n) < \varepsilon$. Let (l_n) be a sequence of positive numbers with $\sum_{n=1}^{\infty} l_n < \infty$.

We first construct a region C in $\boldsymbol{R}^2$. Set $x_0 := (0,0)$. Let $\triangle_1 := \triangle_1(x_0, p_1, x_1) \subset \{(x,y) | x \ge 0, y \ge 0\}$ be the unique equilateral triangle satisfying the following:

(1) The side joining x_0 and x_1 lies on the x-axis.

(2) $|x_0, p_1| = |p_1, x_1| = l_1$.

(3) $\angle_{p_1}(x_0, x_1) = \alpha_1$.

We denote by c_1 the broken line segment joining x_0 and x_1 through p_1. Similarly, let $\triangle_2 := \triangle_2(x_1, p_2, x_2) \subset \{(x,y) | x \ge 0, y \ge 0\}$ be that equilateral triangle such that the side joining x_1 and x_2 lies on the x-axis, $|x_1, p_2| = |p_2, x_2| = l_2$, and $\angle_{p_2}(x_1, x_2) = \alpha_2$. We denote by c_2 the broken line segment joining x_1 and x_2 through p_2. By continuing this procedure inductively, we obtain the sequence of equilateral triangles $\triangle_n := \triangle_n(x_{n-1}, p_n, x_n) \subset \{(x,y) | x \ge 0, y \ge 0\}$ such that the side joining x_{n-1} and x_n lies on the x-axis, $|x_{n-1}, p_n| = |p_n, x_n| = l_n$, and $\angle_{p_n}(x_{n-1}, x_n) = \alpha_n$. We denote by c_n the broken line segment joining x_{n-1} and x_n through p_n. Define $x_* := \lim_{n\to\infty} x_n$. In this way, we obtain a curve c as the concatenation $\bigcup_{n=1}^{\infty} c_n$ extending from x_0 to x_*. Let c' be the curve of the reflection of c centered on the x-axis. We denote by C the region surrounded by c and c'. And let σ be the ray from x_* to $(+\infty, 0)$ and γ the ray from x_0 to $(-\infty, 0)$.

Next, we construct a region $\hat{R}$ in a second $\boldsymbol{R}^2$, distinct from that considered above. First we set $\hat{x}_0 := (0,0)$ and $\hat{p}_1 := (0, l_1)$. Then, let $\hat{p}_2$ be the point situated at a distance $l_1 + l_2$ from $\hat{p}_1$ such that the two components of the vector $\overrightarrow{\hat{p}_1\hat{p}_2}$ are positive numbers, and that the angle $\hat{\alpha}_1$ between $\overrightarrow{\hat{p}_1\hat{p}_2}$ and the direction parallel to the positive y-axis at $\hat{p}_1$ is equal to $\pi - \alpha_1$. We denote by $\hat{x}_1$ the point on the line segment joining $\hat{p}_1$ and $\hat{p}_2$ situated at a distance l_1 from $\hat{p}_1$. Let $\hat{p}_3$ be the point situated at a distance $l_2 + l_3$ from $\hat{p}_2$ such that the two components of the vector $\overrightarrow{\hat{p}_2\hat{p}_3}$ are positive numbers, and that the angle $\hat{\alpha}_2$ between $\overrightarrow{\hat{p}_2\hat{p}_3}$ and the direction parallel to the positive y-axis at $\hat{p}_2$ is equal to $\sum_{n=1}^{2}(\pi - \alpha_n)$. We denote by $\hat{x}_2$ the point on the line segment joining $\hat{p}_2$ and $\hat{p}_3$ situated at a distance l_2 from $\hat{p}_2$. By continuing this procedure inductively, we obtain the points $\hat{x}_0, \hat{p}_1, \hat{x}_1, \hat{p}_2, \hat{x}_2, \ldots, \hat{p}_n, \hat{x}_n, \ldots$. Now, let $\hat{c}_n$ be the broken line segment joining $\hat{x}_{n-1}$ and $\hat{x}_n$ through $\hat{p}_n$. Note that $\hat{\alpha}_m = \sum_{n=1}^{m}(\pi - \alpha_n)$ and $|\hat{x}_{m-1}, \hat{p}_m| = |\hat{p}_m, \hat{x}_m| = l_m$. Then define $\hat{x}_* := \lim_{n\to\infty} \hat{x}_n$. In this way, we have the curve $\hat{c}$ as the concatenation $\bigcup_{n=1}^{\infty} \hat{c}_n$ extending from $\hat{x}_0$ to $\hat{x}_*$. Next, let $\hat{\gamma}$ be the ray from $\hat{x}_0$ to

$(0, -\infty)$ and $\hat{\sigma}$ the ray in $\{(x, y) | x > 0, y > 0\}$ from $\hat{x}_*$ such that the angle at $\hat{x}_*$ between $\hat{\sigma}$ and the direction parallel to the positive y-axis is equal to $\sum_{n=1}^{\infty}(\pi - \alpha_n)$. We denote by $\hat{R}$ the region spanned by the concatenation $\hat{\gamma} \cup \hat{c} \cup \hat{\sigma}$ containing $\{(0, y) | y \geq 0\}$.

We prepare such $C_0 := \gamma([0, \infty)) \cup C \cup \sigma([0, \infty))$ and $\hat{R}_i$, $1 \leq i \leq 4$, spanned by $\hat{\gamma}_i$, $\hat{c}_i$, and $\hat{\sigma}_i$. Now, we obtain the quotient space X of the disjoint union of $C_0, \hat{R}_1, \hat{R}_2, \hat{R}_3$, and $\hat{R}_4$, where the identification is made by the relations $\hat{c}_1 = c = \hat{c}_2$, $\hat{c}_3 = c' = \hat{c}_4$, $\gamma = \hat{\gamma}_i$, and $\sigma = \hat{\sigma}_i$, $1 \leq i \leq 4$. Then (X, d) is a Hadamard space, where d is the interior metric induced from the Euclidean structure of C and R_i. Of course, there is no triangulation around x_*. Furthermore, note that the diameter of $X(\infty)$ is equal to $\pi + \varepsilon$.

2.3. Graphs and Euclidean buildings.

In this paper, a *graph* G is a 1-simplicial complex with an interior metric $|-,-|_G$. We denote by $V(G)$ the vertex set of G and by $E(G)$ the edge set of G.

Next we introduce the following elementary concept for a complete geodesic space (X, d) with curvature bounded above by κ.

Definition 2.8 (degenerate triangle). We say that a triangle $\triangle = \triangle(x, y, z) \subset X$ of perimeter $< 2\pi/\sqrt{\kappa}$ is *degenerate* if $\triangle = \triangle(x, y, z)$ satisfies the following: there is a point w on the geodesic joining y and z such that $d(x, y) = d(x, w) + d(w, y)$ and $d(x, z) = d(x, w) + d(w, z)$.

This definition is independent of the configuration of x, y, and z.

Lemma 2.9. *Let X be a locally compact, geodesically complete geodesic space of $\dim_H X = 1$ with curvature bounded above. Then X has the structure of a graph.*

Proof. We discuss the argument below in a $CAT(\kappa)$-domain U. First, note that $\dim_H \Sigma_x = 0$ for any $x \in X$. Hence, $\angle_x(y, z) = 0$ or π for arbitrary distinct points $x, y, z \in X$.

To prove that any triangle $\triangle(x, y, z)$ of perimeter $< 2\pi/\sqrt{\kappa}$ is degenerate, it suffices to consider the case in which $\angle_x(y, z) = \angle_y(z, x) = \angle_z(x, y) = 0$ for some $x, y, z \in U$. Let $\hat{y}$ be the closest point to y on the geodesic joining x and z. Note that $\hat{y} \neq x$ and $\hat{y} \neq z$. Then $\angle_{\hat{y}}(x, y) = \angle_{\hat{y}}(y, z) = \pi$. This implies that $\triangle(x, y, z)$ is degenerate, in particular, $\hat{y}$ is a π-branch point of x, y, and z.

Here, we denote by $V(X)$ the set of all π-branch points in X. By Lemma 2.2, for any $y \in V(X)$, $B(y, r)$ contains no π-branch point of y for some $r > 0$. This implies that X has the structure of a graph possessing the vertex set $V(X)$. □

Corollary 2.10. *Let X be a locally compact, geodesically complete geodesic space with curvature bounded above such that, for given $x \in X$, $\dim_H O_x = 2$ for any neighborhood O_x of x. Then each connected component of Σ_x has the structure of a graph.*

The following lemma appears in [**BaBr**] as a characterization of graphs with certain properties.

Lemma 2.11 ([**BaBr**], Lemma 6.1). *Let $G = (V(G), E(G))$ be a connected graph such that the valencies of all vertices are at least* 3, *$(G, |-,-|_G)$ has diameter π, and the injectivity radius of $(G, |-,-|_G)$ is equal to π (that is, for any $\xi, \eta \in G$ with $|\xi, \eta|_G < \pi$*

there is a unique minimizing geodesic in G joining ξ and η). Then G satisfies the following:

(i) *Every geodesic in G of length $\leq \pi$ is contained in a closed geodesic of length 2π.*

(ii) *If $\xi \in V(G)$, then $\eta \in V(G)$ for any $\eta \in G$ with $|\xi, \eta|_G = \pi$.*

(iii) *There is a positive integer $k(G) \geq 1$ such that every edge has the same length $\pi/k(G)$.*

(iv) *If $\xi, \eta \notin V(G)$ with $|\xi, \eta|_G = \pi$, then there is a unique closed geodesic of length 2π containing ξ and η.*

(v) *If $\xi, \eta \in V(G)$ with $|\xi, \eta|_G = \pi$ and $e_1, e_2 \in E(G)$ are adjacent to η, then there is a unique closed geodesic of length 2π containing ξ, η, e_1, and e_2.*

Note that, if $k(G) \leq 2$, then G is a complete bipartive graph (if necessary, adding suitable vertices).

We briefly mention the concept of Euclidean buildings. We refer to [**Bro1**] and [**Bro2**] as references treating Euclidean buildings and their fundamental structures from a geometric point of view. A (*Tits*) *building* is a simplicial complex $\mathscr{B}$ which is the union of subcomplexes (called *apartments*) such that the following hold:

(B0) Each apartment is a Coxeter complex.

(B1) For any two simplices A and A' in $\mathscr{B}$ there is an apartment containing both of them.

(B2) For any two simplices A, and A' in $\mathscr{B}$ and apartments F and F' containing both A and A' there is an isomorphism $F \to F'$ fixing A and A' pointwise.

Note that two apartments are isomorphic; in particular, all apartments have the same dimension. A *chamber* is a simplex of maximal dimension. Indeed we can replaced (B2) by the following:

(B2′) if arbitrary apartments F, and F' have a common chamber C, then there is an isomorphism $F \to F'$ fixing the intersection of F and F' pointwise.

A chamber complex is *thick* if each simplex of codimension 1 is adjacent to at least three chambers.

A building $\mathscr{E}$ is called a *Euclidean building* if $\mathscr{E}$ is a building and its apartments are Euclidean Coxeter complexes. A Euclidean building $\mathscr{E}$ has a canonical piecewise smooth metric $d_{\mathscr{E}}$ associated with its Euclidean structure; in particular, $(\mathscr{E}, d_{\mathscr{E}})$ is a Hadamard space. Note that a subset F of $\mathscr{E}$ is an apartment if and only if F is convex and isometric to $\boldsymbol{R}^2$ with the standard flat metric. A Euclidean building of dimension 2 has *type* A_2 (respectively B_2 and G_2) if the chambers have angles $(\pi/3, \pi/3, \pi/3)$ (respectively $(\pi/2, \pi/4, \pi/4)$ and $(\pi/2, \pi/3, \pi/6)$).

§3. Splitting theorems for Hadamard spaces.

As shown in [**BGS**] and [**Oh**], splitting theorems hold for Hadamard manifolds satisfying certain conditions. In this section, we will extend these results to more general Hadamard spaces. Throughout this section, we assume that (X, d) is a locally compact, geodesically complete Hadamard space.

We say that a closed convex subset $A \subset X$ is *geodesically complete* if any geodesic in A can be extended to a line in A.

DEFINITION 3.1. We say that two lines $\sigma_1, \sigma_2 : \boldsymbol{R} \to X$ are *parallel* if $d(\sigma_1(t), \sigma_2(t))$ is uniformly bounded in $t \in \boldsymbol{R}$, and that two closed, convex, geodesically complete subsets $A_1, A_2 \subset X$ are *parallel* if $Hd(A_1, A_2) < \infty$, where $Hd(-,-)$ is the Hausdorff distance in X.

Note that, if two lines σ_1 and σ_2 are parallel, then σ_1 and σ_2 bound a flat strip ([**Ba2**]).

The next lemma has been proved in the smooth Riemannian case ([**BGS**]). Since we can prove the next lemma by using the same argument as that appearing in [**BGS**], we omit the proof.

LEMMA 3.2 ([**BGS**]). *Let $Y_1, Y_2 \subset X$ be closed, convex, geodesically complete subsets which are parallel. Set $a := Hd(Y_1, Y_2) < +\infty$. Then there is an isometric imbedding $\varphi : Y_1 \times [0, a] \to X$ such that $\varphi(Y_1 \times \{0\}) = Y_1$ and $\varphi(Y_1 \times \{a\}) = Y_2$; in particular, $\varphi(Y_1 \times [0, a])$ is convex.*

The next lemma has also been proved for Hadamard manifolds in [**BGS**]. When we prove our lemma, we need to modify the argument in [**BGS**].

LEMMA 3.3. *Let $Y \subset X$ be a closed, convex, geodesically complete subset and P_Y the union of all closed, convex, geodesically complete subsets parallel to Y. Then P_Y is isometric to $Y \times N$, where N is a closed, convex subset of X.*

PROOF. Set $P_Y := \bigcup_{\lambda \in \Lambda} Y_\lambda$, where Y_λ is a closed, convex, geodesically complete subset parallel to Y. Note that P_Y is closed and convex by Lemma 3.2.

Consider $x \in Y_x \subset P_Y$ and $Y_1, Y_2 \subset P_Y$. Let pr_i be the projection map from P_Y onto Y_i. For some $x' \in Y_x$ there are lines σ in Y_x through x and x' and σ_i in Y_i through $pr_i(x)$ and $pr_i(x')$. Note that σ, σ_1, and σ_2 are parallel to each other. Set $\sigma(0) = x$ and $\sigma_i(0) = pr_i(x)$. Since

$$d(\sigma_2(\pm t), pr_1(x)) - t \le d\left(\sigma_2(\pm t), \sigma\left(\pm \frac{t}{2}\right)\right) - \frac{t}{2} + d\left(\sigma\left(\pm \frac{t}{2}\right), pr_1(x)\right) - \frac{t}{2},$$

and σ and σ_i bound a flat strip, we obtain

$$\limsup_{t \to \infty} d(\sigma_2(t), pr_1(x)) - t \le 0 \quad \text{and} \quad \limsup_{t \to \infty} d(\sigma_2(-t), pr_1(x)) - t \le 0.$$

Hence in the inequality

$$0 \le d(\sigma_2(-t), pr_1(x)) + d(pr_1(x), \sigma_2(t)) - d(\sigma_2(-t), \sigma_2(t)),$$

the right-hand side tends to 0 as $t \to \infty$. Since σ_1 and σ_2 bound a flat strip and since this argument is independent of the choice of $x' \in Y_x$, it follows that $pr_2(x)$ is the closest point to $pr_1(x)$. Therefore $pr_2(x') = pr_2 \circ pr_1(x')$, and similarly we have $pr_1(x') = pr_1 \circ pr_2(x')$ for any $x' \in Y_x$.

For given $x \in Y$, define $N_x := \{pr_\lambda(x) | \lambda \in \Lambda\}$, where pr_λ is the projection map onto Y_λ. We define the map Φ from $Y \times N_x$ to P_Y as $\Phi(y, pr_\lambda(x)) = pr_\lambda(y)$. The surjectivity of Φ is clear from Lemma 3.2. Also, it follows from the above argument that Φ is an isometry. Moreover, because P_Y is closed and convex, it is easily seen that N_x is also closed and convex. This completes the proof of Lemma 3.3. $\square$

In the smooth case, the next proposition has been proved in [**Oh**].

Proposition 3.4. *Assume that X is a locally compact, geodesically complete Hadamard space. Then X is isometric to a product $X' \times \boldsymbol{R}$ if and only if X satisfies the following*:

(i) *The diameter of $(X(\infty), Td)$ is equal to π.*

(ii) *There exists $\xi \in X(\infty)$ such that there is a unique point $\xi' \in X(\infty)$ satisfying $Td(\xi, \xi') = \pi$.*

Proof. It suffices to apply arguments similar to those appearing in [**Oh**] by using Theorem 2.6 and Lemma 3.3. □

The next proposition has also been proved in the smooth Riemannian case ([**BGS**]). In our case, we must be careful to consider the possibility of bifurcation of geodesics.

Proposition 3.5. *Assume that X is a locally compact, geodesically complete Hadamard space. Then X is isometric to a product $X_1 \times X_2$ with $X_1(\infty) = A_1$ and $X_2(\infty) = A_2$ if and only if $A_1, A_2 \subset X(\infty)$ satisfy the following*:

(i) *If $\xi_i \in A_i$, $i = 1, 2$, then $Td(\xi_1, \xi_2) = \pi/2$.*

(ii) *For any $\xi \in X(\infty)$ there are $\xi_1 \in A_1$ and $\xi_2 \in A_2$ such that ξ lies on the minimizing geodesic in $(X(\infty), Td)$ joining ξ_1 and ξ_2.*

Proof. It is clear that if X is isometric to $X_1 \times X_2$ with $X_i(\infty) = A_i$, then A_i satisfies (i) and (ii).

Assume that there are $A_1, A_2 \subset X(\infty)$ satisfying (i) and (ii). In this case, the diameter of $X(\infty)$ is equal to π. For $x \in X$, define

$$F_i(x) := \{\sigma_{x\xi_i}(\boldsymbol{R}) \mid \xi_i \in A_i\},$$

where $\sigma_{x\xi_i}$ is a line through x with $\sigma_{x\xi_i}(\infty) = \xi_i$. Note that $\sigma_{x\xi_i}(-\infty) \in A_i$ for any $\xi_i \in A_i$.

First we show that $F_i(x)$ is convex. For $y, z \in F_i(x)$, let σ_{xy} (respectively σ_{xz}) be a ray from x through y (respectively z) with $\eta := \sigma_{xy}(\infty) \in A_i$ (respectively $\zeta := \sigma_{xz}(\infty) \in A_i$). Next, let $w \in X$ be an arbitrary point on the geodesic joining y and z. Suppose that $w \notin F_i(x)$; that is, for any ray σ_{xw} from x through w, $\sigma_{xw}(\infty) \notin A_i$. Then by (ii), there are $\xi_i \in A_i$ and $\xi_j \in A_j$ such that $\sigma_{xw}(\infty)$ lies on the geodesic in $X(\infty)$ joining ξ_i and ξ_j. Define $\xi_j' := \sigma_{x\xi_j}(-\infty) \in A_j$, where $\sigma_{x\xi_j}$ is a line through x with $\sigma_{x\xi_j}(\infty) = \xi_j$. It then follows from (i) and Theorem 2.6 that there is a flat half plane spanned by $\sigma_{x\xi_j}$ and σ_{xy} (respectively σ_{xz}). Let $b_{x\xi_j} : X \to \boldsymbol{R}$ (respectively $b_{x\xi_j'}$) be the Busemann function based on the ray from x to ξ_j (respectively ξ_j'). Then

$$C := {b_{x\xi_j}}^{-1}((-\infty, 0]) \cap {b_{x\xi_j'}}^{-1}((-\infty, 0])$$

is convex; in particular, $w \in C$, since $y, z \in C$. On the other hand, $w \notin C$ by the assumption regarding w. This is a contradiction, and hence $F_i(x)$ must be convex.

Next we show that $F_i(x)$ is closed. Consider a sequence $(x_n) \subset F_i(x)$ with $x_0 := \lim_{n\to\infty} x_n$. Define $\xi_n := \sigma_{xx_n}(\infty) \in A_i$, where σ_{xx_n} is a ray from x through x_n with $\sigma_{xx_n}(\infty) \in A_i$. Then, suppose that $x_0 \notin F_i(x)$; that is, for any ray σ_{xx_0} from x through

x_0, $\xi_0 := \sigma_{xx_0}(\infty) \notin A_i$. Now we may assume that $\xi_0 \notin A_j$. By (ii), there are $\eta_i \in A_i$ and $\eta_j \in A_j$ such that ξ_0 lies on the geodesic joining η_i and η_j. Define $\eta_j' := \sigma_{x\eta_j}(-\infty)$, where $\sigma_{x\eta_j}$ is a line through x with $\sigma_{x\eta_j}(\infty) = \eta_j$. Then

$$\angle_x(\eta_j, \xi_0) = \lim_{n\to\infty} \angle_x(\eta_j, x_n) = \frac{\pi}{2} \quad \text{and} \quad \angle_x(\eta_j', \xi_0) = \lim_{n\to\infty} \angle_x(\eta_j', x_n) = \frac{\pi}{2}.$$

These yield $Td(\xi_0, \eta_i) = 0$, and hence we have a contradiction. Thus we obtain that $F_i(x)$ is a closed, convex, geodesically complete subset.

We can prove that $F_i(x)$ is parallel to $F_i(y)$ for arbitrary $x, y \in X$ by applying the same argument as that appearing in the proof of Lemma 3.3. Hence, for any $x \in X$, we have $X = P_{F_i(x)}$, where $P_{F_i(x)}$ is the set of all closed, convex, geodesically complete subsets parallel to $F_i(x)$. Also, clearly we have $F_j(x) = \{pr_{F_i(y)}(x) \mid y \in X\}$, since

$$Td(\xi_i, \xi_j) = \angle_x(\xi_i, \xi_j) = \frac{\pi}{2}$$

for any $\xi_i \in A_i$ and $\xi_j \in A_j$, where $pr_{F_i(y)}$ is the projection map onto $F_i(y)$. Therefore we obtain $X = F_1(x) \times F_2(x)$. Now, the result is clear from the definition of $F_i(x)$. This completes the proof of Proposition 3.5. □

§4. The local structure of certain Hadamard spaces.

Throughout this section, we assume that X is a locally compact, geodesically complete nonpositively curved 2-space with the Local Flat Condition. The argument below is given in the context of a $CAT(0)$-domain. Recall the definition of the Local Flat Condition in Section 1. We denote by $lH(x, s; \sigma_x)$ the image of the isometric imbeddings $\varphi(D^+(s)) \subset X$ in the definition of LFC. In this case, for any geodesic σ_x through an arbitrary point $x \in X$, there is a local flat half disk $lH(x, s; \sigma_x)$ spanned by $\sigma_x(-s, s)$. Hence, Σ_x is a connected finite graph with diameter π and injectivity radius π, and hence it is a circle of length 2π or a connected finite graph with the properties given in Lemma 2.11 for any $x \in X$.

In this section, we prove the following proposition corresponding to the assertion $(*)$ in Section 1, which plays a key role in investigating of the local structure of X.

Proposition 4.1. *Let X be a locally compact, geodesically complete nonpositively curved 2-space with the Local Flat Condition. Then, for any $x \in X$, there is a positive number $r = r(x) > 0$ such that for any $u \in \Sigma_x$ there is a unique point $y = y(x, u, r) \in \partial \bar{B}(x, r)$ such that $u = v_{xy}$.*

Remark and Definition 4.2. Proposition 4.1 implies that for any $x \in X$ and some $r > 0$ we can define the map

$$\Sigma_x \times [0, r]/_\sim \ni (u, t) \mapsto \sigma_{xu}(t) \in \bar{B}(x, r) \subset X,$$

where σ_{xu} is a geodesic from x directed by u. We call such a metric ball $\bar{B}(x, r)$ in X a *regular neighborhood of x*.

4.1. Existence of a local flat disk.

Now, for any $x \in X$, $B(x, s)$ is the union of local flat half disks for some $s > 0$. We first observe how these local flat half disks meet each other.

Lemma 4.3. *For* $x \in X$, *let* $\omega_x \subset \Sigma_x$ *be a closed geodesic of length* 2π, *let* $\tau_{x,1}, \tau_{x,2} : [0,\pi] \to \omega_x$ *be geodesics such that* $\tau_{x,1}((0,\pi)) \cap \tau_{x,2}((0,\pi)) \neq \emptyset$, *and let* γ_i, $i = 1,2$, *be a geodesic with* $\gamma_i(0) = x$ *directed by* $\tau_{x,i}(0)$ *and* $\tau_{x,i}(\pi)$ *at* x. *Assume that there is a local flat half disk* $lH_i = lH_i(x,s;\gamma_i) \subset X$ *such that* $\tau_{x,i}$ *corresponds to* lH_i. *Then for any direction* $u \in \tau_{x,1}((0,\pi)) \cap \tau_{x,2}((0,\pi))$ *there is a positive number* $r = r(x,u;\sigma_1,\sigma_2) > 0$ *such that*

$$\sigma_1(r) = \sigma_2(r)$$

for the geodesic $\sigma_i : [0,s) \to lH_i$, $i = 1,2$, *directed by* u.

Proof. Suppose this claim is not true; that is, suppose that $\sigma_1(t) \neq \sigma_2(t)$ for any $t \in (0,s)$. Note that $\angle_x(\sigma_1,\sigma_2) = 0$.

For small $\theta \in (0,\pi/2)$, we denote by $\sigma_{i,\theta}, \sigma'_{i,\theta} : [0,s) \to lH_i$ the two geodesics extending from x satisfying the following:

(1) $\angle_x(\sigma_{i,\theta},\sigma_i) = \angle_x(\sigma'_{i,\theta},\sigma_i) = \theta$.

(2) $\angle_x(\sigma_{i,\theta},\sigma'_{i,\theta}) = 2\theta$.

(3) $\angle_x(\sigma_{1,\theta},\sigma_{2,\theta}) = \angle_x(\sigma'_{1,\theta},\sigma'_{2,\theta}) = 0$.

Also, we denote by $\triangle_i(\theta,t) \subset lH_i$ the triangular flat region spanned by x, $\sigma_{i,\theta}(t/\cos\theta)$, and $\sigma'_{i,\theta}(t/\cos\theta)$. Then, by replacing u by another $\hat{u} \in \tau_{x,1}((0,\pi)) \cap \tau_{x,2}((0,\pi)) \setminus \{u\}$ very close to u, we may assume the following:

(4.3.1) There is a positive number $\theta_0 > 0$ such that, for any $t > 0$, the relations $\sigma_{1,\theta}(t) \neq \sigma_{2,\theta}(t)$ and $\sigma'_{1,\theta}(t) \neq \sigma'_{2,\theta}(t)$ hold for any $\theta \in (0,\theta_0]$.

This follows from the convexity of $lH_1 \cap lH_2$.

We write $y_i(t) := \sigma_i(t) \in lH_i$, $t > 0$. Then, let $\hat{y}_1(t) \in lH_2$ be the closest point to $y_1(t)$. Note that, for any small $t > 0$, $\hat{y}_1(t) \in S_X$ and $\hat{y}_1(t) \neq x$. For a sufficiently small number $\delta > 0$, by Lemma 2.2 there is a positive number $t_0 = t_0(x,\delta) > 0$ such that $B(x,t_0)$ does not contain a δ-branch point of x. Now, consider sufficiently small $t \in (0,t_0)$. Next, let $w_2(t) \in lH_2$ be a point such that $\hat{y}_1(t)$ lies on the geodesic extending from x to $w_2(t)$. Note that $\pi/2 \leq \angle_{\hat{y}_1(t)}(x, y_1(t)) < \pi$. Applying Lemma 2.3, we obtain

$$\angle_{\hat{y}_1(t)}(x, y_1(t)) + \angle_{\hat{y}_1(t)}(y_1(t), w_2(t)) < \pi + \delta,$$

and hence we have

$$\frac{\pi}{2} \leq \angle_{\hat{y}_1(t)}(x, y_1(t)), \quad \angle_{\hat{y}_1(t)}(y_1(t), w_2(t)) < \frac{\pi}{2} + \delta. \tag{4.3.2}$$

Next, define $p_i(t) := \sigma_{i,\theta}(t/\cos\theta)$ and $q_i(t) := \sigma'_{i,\theta}(t/\cos\theta)$ for $\theta > 0$ satisfying (4.3.1). Now $d(y_1(t), y_2(t)) = o(t)$, since the angle between $y_1(t)$ and $y_2(t)$ at x is equal to 0. Hence $d(y_2(t), \hat{y}_1(t)) = o(t)$, since $d(y_1(t), \hat{y}_1(t)) \leq d(y_1(t), y_2(t))$. This implies that $\hat{y}_1(t)$ lies in the interior of $\triangle_2(\theta,t)$ for sufficiently small $t > 0$. Next, let $p'_2(t) \in lH_2$ (respectively $q'_2(t) \in lH_2$) be the intersection point of $\sigma_{2,\theta}([0,s))$ (respectively $\sigma'_{2,\theta}([0,s))$) and the geodesic on lH_2 which is perpendicular to the geodesic extending from x to $w_2(t)$ at $\hat{y}_1(t)$. It follows from (4.3.2) and the structure of $\Sigma_{\hat{y}_1(t)}$ that

$$\pi - \delta \leq \angle_{\hat{y}_1(t)}(y_1(t), p'_2(t)), \quad \angle_{\hat{y}_1(t)}(y_1(t), q'_2(t)) \leq \pi. \tag{4.3.3}$$

Now, $d(p_2(t), p_2'(t)) = o(t)$ by the properties of Euclidean geometry. Let $\hat{p}_2(t) \in lH_1$ be the closest point to $p_2(t)$. Then $d(p_1(t), \hat{p}_2(t)) = o(t)$, since $d(p_1(t), p_2(t)) = o(t)$ and $d(p_2(t), \hat{p}_2(t)) \le d(p_1(t), p_2(t))$. Let $\hat{p}_2'(t) \in lH_1$ be the closest point to $p_2'(t)$. Since

$$d(p_1(t), \hat{p}_2'(t)) \le d(p_1(t), \hat{p}_2(t)) + d(\hat{p}_2(t), \hat{p}_2'(t)) \le d(p_1(t), \hat{p}_2(t)) + d(p_2(t), p_2'(t)),$$

we have $d(p_1(t), \hat{p}_2'(t)) = o(t)$.

We will now find an upper bound for $\angle_{y_1(t)}(p_1(t), \hat{y}_1(t))$. For an arbitrary $\varepsilon > 0$, we have $\angle_{y_1(t)}(p_1(t), \hat{p}_2'(t)) < \varepsilon$ and $|\angle_{\hat{p}_2'(t)}(y_1(t), x) - \angle_{p_1(t)}(y_1(t), x)| < \varepsilon$ for sufficiently small $t > 0$ by the above argument. Therefore

$$\pi/2 - \theta - \varepsilon \le \angle_{\hat{p}_2'(t)}(y_1(t), x) \le \pi/2 - \theta + \varepsilon.$$

Analogous to (4.3.3), we have $\angle_{\hat{p}_2'(t)}(a_1(t), p_2'(t)) \ge \pi - \delta$, where $a_1(t)$ is the point on σ_1 such that the geodesic extending from $\hat{p}_2'(t)$ to $a_1(t)$ is perpendicular to the geodesic from x to $\hat{p}_2'(t)$. Now

$$\angle_{\hat{p}_2'(t)}(y_1(t), a_1(t)) = \frac{\pi}{2} - \angle_{\hat{p}_2'(t)}(y_1(t), x) \le \theta + \varepsilon.$$

Applying the triangle inequality to angles at $\hat{p}_2'(t)$ and the above relations, we obtain

$$\angle_{\hat{p}_2'(t)}(y_1(t), p_2'(t)) \ge \pi - \delta - \theta - \varepsilon.$$

Hence, considering the quadrangle consisting of $y_1(t), \hat{y}_1(t), p_2'(t)$, and $\hat{p}_2'(t)$, we have $\angle_{y_1(t)}(\hat{p}_2'(t), \hat{y}_1(t)) \le 2\delta + \theta + \varepsilon$. Thus by applying the triangle inequality to angles at $y_1(t)$ we obtain

$$\angle_{y_1(t)}(p_1(t), \hat{y}_1(t)) < 2\delta + \theta + 2\varepsilon.$$

Following an argument similar to that given above, for $\angle_{y_1(t)}(q_1(t), \hat{y}_1(t))$, we find the same upper bound $2\delta + \theta + 2\varepsilon$. Hence we obtain

$$\pi \le \angle_{y_1(t)}(p_1(t), \hat{y}_1(t)) + \angle_{y_1(t)}(\hat{y}_1(t), q_1(t)) \le 4\delta + 2\theta + 4\varepsilon.$$

Since we may consider δ, θ, and ε to be small, this is a contradiction, proving Lemma 4.3. □

LEMMA 4.4. *For any $x \in X$, let $u \in \Sigma_x$ be an arbitrary direction at x. Then there are no three points y_i, z_i, and w_i distinct from x such that*

(1) *u is the direction to y_i, z_i, and w_i at x.*
(2) *y_i converges to x, and z_i and w_i converge to z_0 and w_0, respectively.*
(3) *$d(x, z_i) = d(x, y_i) + d(y_i, z_i) = s$ and $d(x, w_i) = d(x, y_i) + d(y_i, w_i) = s$ for some $s > 0$.*
(4) *$\angle_{y_i}(z_i, w_i) > 0$.*

PROOF. Suppose this claim is not true. We then note that $y_i \in S_X$ by the structure of Σ_{y_i}. Also, note that there is a positive number $t_0 = t_0(x, \delta) > 0$ such that $B(x, t_0)$ contains no δ-branch point of x by Lemma 2.2. In this case, $k(\Sigma_{y_i}) \to \infty$ as $i \to \infty$, since $\lim_{i\to\infty} \angle_{y_i}(z_i, w_i) = 0$.

Let $\omega_{y_i} \subset (\Sigma_{y_i}, \rho_{y_i})$ be a closed geodesic of length 2π containing $v_{y_i x}, v_{y_i z_i} \in \Sigma_{y_i}$.

Since $k(\Sigma_{y_i}) \to \infty$ as $i \to \infty$, for sufficiently large i there are $v_i, v_i' \in \omega_{y_i}$ with $v_i, v_i' \in V(\Sigma_{y_i})$ such that $\rho_{y_i}(v_i, v_i') = \pi$, and both $\rho_{y_i}(v_i, v_{y_i x})$ and $\rho_{y_i}(v_i', v_{y_i x})$ are very near $\pi/2$. Then there is a geodesic $\tau_{y_i} : [0, \pi] \to \Sigma_{y_i}$ joining v_i and v_i' such that $\tau_{y_i}([0, \pi]) \cap \omega_{y_i} = \{v_i, v_i'\}$. Hence we can find a direction $u_i \in \Sigma_{y_i}$ for which $\rho_{y_i}(u_i, v_{y_i x})$ is equal to π, and $\rho_{y_i}(u_i, v_{y_i z_i})$ is very near π. This implies that, for some $\delta' > 0$, y_i is a δ'-branch point of x for sufficiently large i. This is a contradiction to Lemma 2.2, which completes the proof of Lemma 4.4. □

The following lemma tells us that, for any $x \in X$, there is a local flat (whole) disk around x containing a given geodesic through x.

Lemma 4.5. *For any $x \in X$ and any geodesic σ_x with $\sigma_x(0) = x$ directed by $u_0, u_0' \in \Sigma_x$ at x there is a positive number $s = s(x, \sigma_x) > 0$ such that for any closed geodesic $\omega_x \subset \Sigma_x$ of length 2π containing u_0 and u_0' there is a totally geodesic isometric imbedding $\varphi : D(s) \to X$ satisfying*

(i) *$\varphi(t_1, 0) = \sigma_x(t_1)$ for $t_1 \in (-s, s)$, and*

(ii) *ω_x corresponds to $\varphi(D(s))$,*

where $D(s) := \{(t_1, t_2) | t_1^2 + t_2^2 < s^2\}$ with the standard flat metric.

Proof. Take a local flat half disk $lH_0 = lH_0(x, s_0; \sigma_x)$ spanned by $\sigma_x((-s_0, s_0))$. Let $\tau_x : [0, \pi] \to (\Sigma_x, \rho_x)$ be the geodesic corresponding to lH_0. For a given closed geodesic ω_x of length 2π in Σ_x containing u_0 and u_0', we will construct a local flat (whole) disk with radius $s(\omega_x) > 0$ centered at x.

First we assume that $u_0, u_0' \in V(\Sigma_x)$ and that $\tau_x((0, \pi)) \cap \omega_x = \emptyset$. Let $\tau_x' : [0, \pi] \to \omega_x$ be a geodesic joining u_0 and u_0' with $\tau_x([0, \pi]) \cap \tau_x'([0, \pi]) = \{u_0, u_0'\}$, and let $\omega_x' \subset \Sigma_x$ be the closed geodesic with $\omega_x' = \tau_x([0, \pi]) \cup \tau_x'([0, \pi])$.

Now, we will verify the following claim:

(4.5.1) For ω_x', there is a local flat disk with radius $s(\omega_x') > 0$ centered at x such that ω_x' corresponds to this local flat disk.

For a point $v_1 \in \tau_x((0, \pi))$ with $v_1 \notin V(\Sigma_x)$, let $v_1' \in \tau_x'((0, \pi))$ be the point with $\rho_x(v_1, v_1') = \pi$. Then there is a geodesic σ_1 with $\sigma_1(0) = x$ which is directed by v_1 and v_1' at x such that $\sigma_1((-s_0, 0]) \subset lH_0$. Since X satisfies the LFC, there is a local flat half disk $lH_1 = lH_1(x, s_0; \sigma_1)$ spanned by $\sigma_1((-s_0, s_0))$. Let $\tau_{x,1} : [0, \pi] \to \Sigma_x$ be the geodesic joining v_1 and v_1' corresponding to lH_1. Note that $\tau_{x,1}([0, \pi]) \subset \omega_x'$ and $\tau_x((0, \pi)) \cap \tau_{x,1}((0, \pi)) \neq \emptyset$ by the choice of v_1.

In this case, without loss of generality, we may assume that $u_0 \in \tau_{x,1}([0, \pi])$, and set $u_1 := u_0$. For a sequence $(u_{1,i})$ with $u_{1,i} \to u_1$ and $u_{1,i} \in \tau_x((0, \pi)) \cap \tau_{x,1}((0, \pi))$, there is a positive number $r_{1,i} = r_{1,i}(x, u_{1,i}) > 0$ such that the geodesics in lH_0 and lH_1 extending from x directed by $u_{1,i}$ coincide over a distance $r_{1,i}$, as seen by Lemma 4.3. Let $\bar{r}_{1,i} > 0$ be the supremum of $r_{1,i}$ satisfying the above. We show the following:

(4.5.2) For u_1 there is a positive number $r_1 = r_1(x, u_1) > 0$ such that the geodesics in lH_0 and lH_1 extending from x directed by u_1 coincide over a distance r_1.

Suppose that $\bar{r}_{1,i}$ tends to 0 as $i \to \infty$. Then for $u_{1,i}$, if necessary, taking a suitable subsequence, we obtain three points y_i, z_i, and w_i distinct from x satisfying the condition listed in the statement in Lemma 4.4. Hence, by Lemma 4.4, we obtain a contradiction, which completes the proof of (4.5.2).

Let $\bar{r}_1 > 0$ be the supremum satisfying (4.5.2). In this way, we obtain, for some $\theta_1 \in (\pi, 2\pi)$ and some $s_1 \in (0, \min\{s_0, \bar{r}_1\})$, a region $lR_1 = lR_1(x, s_1, \theta_1; \sigma_x) \subset X$ corresponding to the subarc of ω'_x of length θ_1 such that lR_1 is the union of two local flat half disks $lH_0(x, s_1; \sigma_x)$ and $lH_1(x, s_1; \sigma_1)$. In particular, lR_1 contains $\sigma_x((-s_1, s_1))$ and is isometric to the sector in the flat plane with radius s_1 and possessing inner angle θ_1 with respect to the interior metric in themselves.

Next, for a point $v_2 \in \tau_x((0,\pi)) \cap \tau_{x,1}((0,\pi))$ with $v_2 \notin V(\Sigma_x)$, let $v'_2 \in \tau'_x((0,\pi))$ be the point with $\rho_x(v_2, v'_2) = \pi$. Then there is a geodesic σ_2 with $\sigma_2(0) = x$ which is directed by v_2 and v'_2 at x such that $\sigma_2((-s_1, 0]) \subset lR_1$. Since X satisfies the LFC, there is a local flat half disk $lH_2 = lH_2(x, s_0; \sigma_2)$. In this way, continuing this procedure inductively, for some θ_n with $\theta_n \nearrow 2\pi$, we obtain u_n, $\bar{r}_n$, and a region $lR_n(x, s_n, \theta_n; \sigma_x) \subset X$ with the same property as in the case of $n = 1$. Note that (s_n) is non-increasing.

Suppose that we can not construct a local flat disk using finitely many procedures and that s_n tends to 0; that is, $\bar{r}_n$ tends to 0. Then for u_n—if necessary, taking a suitable subsequence—we obtain three points y_n, z_n, and w_n distinct from x satisfying the condition listed in Lemma 4.4. Hence, by Lemma 4.4, we obtain a contradiction. Thus, we have completed the proof of the claim (4.5.1).

Similarly, for ω_x we obtain a local flat disk with radius $s(\omega_x) > 0$ centered at x containing $\sigma_x((-s(\omega_x), s(\omega_x)))$ such that ω_x corresponds to this local flat disk. Because there are at most finitely many closed geodesics in Σ_x containing u_0 and u'_0, we obtain Lemma 4.5.

If u_0 and u'_0 are not vertices of Σ_x, there is a unique closed geodesic $\omega_x \subset \Sigma_x$ containing u_0 and u'_0. Similarly to the case in which u_0 and u'_0 are vertices, we obtain a local flat disk in the statement in Lemma 4.5. This completes the proof of Lemma 4.5. □

Indeed we can consider the condition given in Section 1 to be the LFC in view of Lemma 4.5. We denote $\varphi(D(s))$ (respectively $\varphi(\bar{D}(s))$) in the statement of Lemma 4.5 by $lF(x, s; \sigma) \subset X$ (respectively $l\bar{F}(x, s; \sigma)$).

4.2. Proof of Proposition 4.1.

Lemma 4.6. *For any $x \in X$ and $u \in \Sigma_x$, let σ_1 and σ_2 be arbitrary geodesics emanating from x directed by u. Then there is a positive number $r = r(x, u; \sigma_1, \sigma_2) > 0$ such that*

$$\sigma_1(r) = \sigma_2(r).$$

Proof. We now note that $\angle_x(\sigma_1, \sigma_2) = 0$. For a $u' \in (\Sigma_x, \rho_x)$ with $\rho_x(u, u') = \pi$, let $\omega_x \subset \Sigma_x$ be a closed geodesic of length 2π containing u and u'. Then we may consider that both σ_1 and σ_2 are directed by u and u'. It follows from Lemma 4.5 that there are two local flat disks $lF_1(x, s; \sigma_1)$ and $lF_2(x, s; \sigma_2)$. Then, by Lemma 4.3, there is an $r = r(x, u; \sigma_1, \sigma_2) > 0$ such that $\sigma_1(r) = \sigma_2(r)$, which proves Lemma 4.6. □

Lemma 4.7. *For any $x \in X$ and each $u \in \Sigma_x$ there is a positive number $r = r(x, u) > 0$ such that $\sigma_1(t) = \sigma_2(t)$ for any $t \in [0, r]$ and for arbitrary geodesics σ_1, σ_2 in X emanating from x directed by u.*

Proof. Suppose this claim is not true; that is, suppose that there is a point $x \in X$ and a direction $u \in \Sigma_x$ satisfying the following: for any $t_n \searrow 0$ there are two geodesics $\sigma_{n,1}$ and $\sigma_{n,2}$ emanating from x directed by u such that $\sigma_{n,1}(t_n) \neq \sigma_{n,2}(t_n)$. Now, define $r_n = r_n(u; \sigma_{n,1}, \sigma_{n,2}) := \sup\{t > 0 | \sigma_{n,1}(t) = \sigma_{n,2}(t)\}$. Note that $r_n > 0$ by Lemma 4.6, and that r_n tends to 0 as $n \to \infty$. Furthermore, we obtain three points $y_n := \sigma_{n,1}(r_n) = \sigma_{n,2}(r_n)$, $z_n := \sigma_{n,1}(r_1)$, and $w_n := \sigma_{n,2}(r_1)$. Now, $y_n \in S_X$ and $\angle_{y_n}(z_n, w_n) > 0$. Then we obtain a contradiction to Lemma 4.4, and hence we have proved Lemma 4.7. □

Proof of Proposition 4.1. Define $r(u) := \sup\{t > 0 | (**)\}$, where $(**)$ represents the condition that $\sigma_{u,1}(t) = \sigma_{u,2}(t)$ for arbitrary geodesics $\sigma_{u,1}$ and $\sigma_{u,2}$ emanating from x directed by u. Note that $r(u) > 0$ by Lemma 4.7. Now, since Σ_x is compact, it suffices to prove that the function $r(u)$ is lower semicontinuous. We do this by considering a sequence $(u_i) \subset \Sigma_x$ with $u_i \to u_0$. Note that in this case all rays σ_{xu_0} from x directed by u_0 coincide over a distance $r(u_0)$.

Suppose that there is a positive number $\varepsilon > 0$ such that $r(u_i) < r(u_0) - \varepsilon$ for any i. Then there is a point $y_i \in X$ with $u_i = v_{xy_i}$ satisfying $r(u_i) = d(x, y_i)$. Let $z_i, w_i \in X$ be the points for which $u_i = v_{xz_i} = v_{xw_i}$ satisfying the following:

(1) $\angle_{y_i}(z_i, w_i) > 0$.

(2) The distance from each point to x is equal to $r(u_0)$.

(3) y_i lies on the geodesics joining x and these points.

Note that $y_i \in S_X$. Then by the uniqueness of $\sigma_{xu_0}|_{[0, r(u_0)]}$ and the choice of y_i we may assume that there are points y_0, z_0, and w_0 on the geodesic σ_{xu_0} with $y_i \to y_0, z_i \to z_0$, and $w_i \to w_0$ such that $d(x, y_0) < r(u_0) - \varepsilon$ and $d(x, z_0) = d(x, w_0) = r(u_0)$ (if necessary, taking a suitable subsequence). Then, note that

$$\lim_{i \to \infty} \angle_{y_i}(z_i, w_i) = \angle_{y_0}(z_0, w_0) = 0.$$

This implies that $k(\Sigma_{y_i}) \to \infty$ as $i \to \infty$. Now, applying the similar argument appearing in the proof of Lemma 4.4, we obtain that y_i is a δ-branch point of y_0 for some $\delta > 0$, which contradicts Lemma 2.2. We have completed the proof of Proposition 4.1. □

§5. Existence of a flat plane in certain Hadamard spaces.

Throughout this section, we assume that X is a locally compact, geodesically complete nonpositively curved 2-spaces with the Local Flat Condition. Note that since X is locally simply connected, X naturally has the universal covering space $\tilde{X}$. Then $\tilde{X}$ is a locally compact, geodesically complete Hadamard 2-space satisfying the LFC, because $\tilde{X}$ is locally isometric to X.

The next proposition provides a basic construction of a global flat plane in the universal covering space $\tilde{X}$ spanned by an arbitrary geodesic from the Local Flat Condition. This proposition has been proved for some nonpositively curved 2-polyhedra ([**BaBr**], [**Bar**]).

Proposition 5.1. *Let X be a locally compact, geodesically complete nonpositively curved 2-space satisfying the LFC, and let $\tilde{X}$ be the universal covering space of X. For* *any $x \in \tilde{X}$, let $\sigma = \sigma_x : \boldsymbol{R} \to \tilde{X}$ be an arbitrary line with $\sigma(0) = x$ directed by $u, u' \in \Sigma_x$ at x, and let $\omega_x \subset \Sigma_x$ be a closed geodesic of length 2π containing u and u'. Then there is a flat plane $F = F(\sigma, \omega_x) \subset \tilde{X}$ such that $\sigma(\boldsymbol{R}) \subset F$ and such that ω_x corresponds to F.*

Proof. Fix a positive number $b > 0$. We will first construct a flat strip of width $2b > 0$ containing $\sigma([-b, b])$ so that ω_x corresponds to this flat strip.

Let $A \subset [0, \infty)$ be the set of nonnegative numbers $a \geq 0$ satisfying the condition that there is a flat $(2b \times 2a)$-rectangle R spanned by $x_{a,b}, x_{a,-b}, x_{-a,-b}$, and $x_{-a,b}$ such that

(1) $\sigma([-b, b]) \subset R$.
(2) $\sigma(\pm b) \in R$ is the midpoint between $x_{a,\pm b}$ and $x_{-a,\pm b}$.
(3) The geodesic joining $x_{\pm a,-b}$ and $x_{\pm a,b}$ is parallel to $\sigma([-b, b])$.
(4) ω_x corresponds to R at the center $x = \sigma(0)$ of R.

Then A is closed in $[0, \infty)$, since $\tilde{X}$ is locally compact. Hence it suffices to show that A is open.

Assume that there is a flat $(2b \times 2a)$-rectangle R spanned by $x_{a,b}, x_{a,-b}, x_{-a,-b}$, and $x_{-a,b}$ for $a \in A$. Let σ_a and σ_{-a} be lines with $x_{\pm a,-b} = \sigma_{\pm a}(-b)$ and $x_{\pm a,b} = \sigma_{\pm a}(b)$. Then, for $c \in [-b, b]$, let $\gamma_c : [-a, a] \to R$ be the geodesic with $\gamma_c(\pm a) = \sigma_{\pm a}(c)$. Note that $x = \gamma_0(0)$, where $\gamma_0 : [-a, a] \to R$ is the geodesic with $\gamma_0(\pm a) = \sigma_{\pm a}(0)$.

We will extend the flat $(2b \times 2a)$-rectangle R beyond σ_a in the following manner.

Write $x_0 := \sigma_a(0) = \gamma_0(a)$. Let $\tau_0 : [0, \pi] \to \Sigma_{x_0}$ be the geodesic corresponding to R, and let $\omega_0 \subset \Sigma_{x_0}$ be a closed geodesic of length 2π containing $\tau_0([0, \pi])$. Then by Proposition 4.1 and Lemma 4.5 there exists a positive number $s_0 = s_0(x_0, \sigma_a)$ such that there is a totally geodesic isometric imbedding $\varphi_0 : \bar{D}(s_0) \to \tilde{X}$ (where $\bar{D}(s_0) := \{(t_1, t_2) | t_1^2 + t_2^2 \leq s_0^2\}$) which satisfies the following:

(1) $\varphi_0(t_1, 0) = \sigma_a(t_1)$ for $t_1 \in [-s_0, s_0]$.
(2) $\varphi_0(0, -t_2) = \gamma_0(a - t_2)$ for $-t_2 \in [-s_0, 0]$.
(3) $\varphi_0(\bar{D}(s_0))$ is contained in a regular neighborhood of x_0.
(4) $\varphi_0(\bar{D}(s_0)) \cap R = \varphi_0(\{(t_1, t_2) | t_2 \leq 0, t_1^2 + t_2^2 \leq s_0^2\})$.

We next define $l\bar{F}_0 = l\bar{F}_0(x_0, s_0; \sigma_a, R) := \varphi_0(\bar{D}(s_0))$ and $l\bar{H}_0 = l\bar{H}_0(x_0, s_0; \sigma_a, R) := \varphi_0(\{(t_1, t_2) | t_2 \geq 0, t_1^2 + t_2^2 \leq s_0^2\})$. Note that $R \cap l\bar{H}_0 = \sigma_a([-s_0, s_0])$.

We extend R beyond σ_a toward $x_{a,b} = \sigma_a(b) = \gamma_b(a)$.

Now, write $x_1 := \sigma_a(s_0)$. Let $\tau_1 : [0, \pi] \to \Sigma_{x_1}$ be the geodesic corresponding to R and $e_1 \in E(\Sigma_{x_1})$ the edge corresponding to the subset of $l\bar{H}_0$ and which satisfies $v_{x_1 x_0} \in e_1$. Then there is a unique closed geodesic $\omega_1 \subset \Sigma_{x_1}$ of length 2π containing e_1 and $\tau_1([0, \pi])$. It follows from Proposition 4.1 and Lemma 4.5 that there is a positive number $s_1 := s_1(x_1, \sigma_a, R, l\bar{H}_0)$ such that there is a totally geodesic isometric imbedding $\varphi_1 : \bar{D}(s_1) \to \tilde{X}$ for which we have:

(1) $\varphi_1(t_1, 0) = \sigma_a(s_0 + t_1)$ for $t_1 \in [-s_1, s_1]$.
(2) $\varphi_1(0, -t_2) = \gamma_{s_0}(a - t_2)$ for $-t_2 \in [-s_1, 0]$.
(3) $\varphi_1(\bar{D}(s_1))$ is contained in a regular neighborhood of x_1.
(4) $\varphi_1(\bar{D}(s_1)) \cap R = \varphi_1(\{(t_1, t_2) | t_2 \leq 0, t_1^2 + t_2^2 \leq s_1^2\})$.
(5) There is a point $z_1 \in l\bar{H}_0 \cap l\bar{H}_1$ such that $d(x_1, z_1) \leq s_1$, $z_1 \notin \sigma_a([-b, b])$, and the direction to z_1 at x_1 is contained in e_1.

Here we define $l\bar{F}_1 = l\bar{F}_1(x_1, s_1; \sigma_a, R, l\bar{H}_0) := \varphi_1(\bar{D}(s_1))$ and $l\bar{H}_1 = l\bar{H}_1(x_1, s_1; \sigma_a, R, l\bar{H}_0) := \varphi_1(\{(t_1, t_2) | t_2 \geq 0, t_1^2 + t_2^2 \leq s_1^2\})$. Then there is a point $w_1 \in \sigma_a([-b, b])$ such that

$\triangle(x_1, w_1, z_1) \subset l\overline{H}_0 \cap l\overline{H}_1$ bounds a flat triangular region. This implies that R is extended by a flat rectangle around $\sigma_a((-s_0, s_0 + s_1))$. Next, write $x_2 := \sigma_a(s_0 + s_1)$. By continuing this procedure inductively, we obtain $l\overline{H}_n, x_n, s_n$, and so on.

Now let us show that we can take (s_n) in the above manner such that there is a finite positive number $N \in \boldsymbol{N}$ for which $b \le \sum_{n=0}^{N} s_n$. Suppose that for any (s_n) chosen in the above manner, $\sum_{n=0}^{N} s_n < b$ for any N. Let $\bar{s}_*$ be the supremum of $\sum_{n=0}^{\infty} s_n$ such that (s_n) satisfies the above condition, and define $p_* := \sigma_a(s_*)$. Let τ_* be the geodesic in Σ_{p_*} corresponding to R. Then there are at most finitely many geodesics $\tau_{*,i} : [0, \pi] \to \Sigma_{p_*}$ such that τ_* and $\tau_{*,i}$ compose of a closed geodesic of length 2π. Let $\omega_{*,i} \subset \Sigma_{p_*}$ be the closed geodesic of length 2π composed by τ_* and $\tau_{*,i}$. It then follows from Proposition 4.1 and Lemma 4.5 that there is a positive number $r_{*,i} = r_{*,i}(p_*, \sigma_a, \omega_{*,i}) > 0$ such that $\bar{B}(p_*, r_{*,i})$ is a regular neighborhood and contains the local flat disk with radius $r_{*,i}$ centered at p_* corresponding to $\omega_{*,i}$. Next, define $r_* := \min_i r_{*,i}$. Now for an arbitrary $\varepsilon > 0$ there is a sequence (s_n) which can be chosen in the above manner such that $\bar{s}_* < s_* + \varepsilon$, where $s_* := \sum_{n=0}^{\infty} s_n$. Then there is a sufficiently large number N such that $s_* - \sum_{n=0}^{N} s_n < \varepsilon$. Let $x_N \in \sigma_a([-b, b])$ be the point corresponding to the above N. Hence we may assume that $l\overline{H}_N \subset \bar{B}(p_*, r_*)$. Then there is a point y_N in the interior of $l\overline{H}_N$ such that $0 < \angle_{p_*}(x_N, y_N) < \pi/k(\Sigma_{p_*})$. This implies that we can extend R by a flat rectangle beyond p_* in the above manner. This is a contradiction of the definition of p_*.

Considering the above construction, we can extend R beyond σ_a toward $x_{a,-b} = \sigma_a(-b) = \gamma_{-b}(a)$. Also, we can extend R beyond σ_{-a}. Thus A is open in $[0, \infty)$.

In this way, we obtain a flat strip of width $2b > 0$ containing $\sigma([-b, b])$ so that ω_x corresponds to this flat strip at x for any $b > 0$. Hence, we obtain a flat plane containing $\sigma(\boldsymbol{R})$ so that ω_x corresponds to this flat plane. This completes the proof of Proposition 5.1. □

The following lemma helps us to determine the global structure of $\tilde{X}$.

LEMMA 5.2. *For a singular point $x \in S_{\tilde{X}}$, let σ_x be a line through x directed by two vertices $u, u' \in V(\Sigma_x)$. Then we have the following*:

(i) *$\sigma_x(t) \in S_{\tilde{X}}$ for any $t \in \boldsymbol{R}$.*

(ii) *σ_x is directed by two vertices of $\Sigma_{\sigma_x(t)}$ at $\sigma_x(t)$ for any $t \in \boldsymbol{R}$.*

PROOF. By the structure of Σ_x there are (at least three) geodesics $\tau_{x,1}, \tau_{x,2}, \tau_{x,3} : [0, \pi] \to \Sigma_x$ joining u and u'. Hence it follows from Proposition 5.1 that there is a flat half plane $H_i = H_i(\sigma_x, \tau_{x,i})$, $1 \le i \le 3$, spanned by σ_x such that $\tau_{x,i}$ corresponds to H_i. In particular, there are flat planes $H_1 \bigcup_{\sigma_x(\boldsymbol{R})} H_2$, $H_2 \bigcup_{\sigma_x(\boldsymbol{R})} H_3$, and $H_3 \bigcup_{\sigma_x(\boldsymbol{R})} H_1$. Note that $\bigcap_{1 \le i \le 3} H_i = \sigma_x(\boldsymbol{R})$. Then for any $t \in \boldsymbol{R}$ there is a geodesic $\tau_{\sigma_x(t),i} : [0, \pi] \to \Sigma_{\sigma_x(t)}$, $1 \le i \le 3$, corresponding to H_i. This completes the proof of (i) and (ii). □

Hereafter, we call a geodesic $\sigma : I \to \tilde{X}$ for which $\sigma(t) \in S_{\tilde{X}}$ for any $t \in I$ a *singular segment* in $\tilde{X}$.

§6. A classification of certain Hadamard spaces.

In this section we prove the following main theorem:

Theorem 6.1. *Let X be a locally compact, geodesically complete Hadamard 2-space such that the diameter of $(X(\infty), Td)$ is equal to π. Then X is isometric to either*
(1) *the product of two trees,*
(2) *the Euclidean cone over $(X(\infty), Td)$, or*
(3) *a thick Euclidean building of dimension 2 of type A_2, B_2, or G_2.*

If X is as in Theorem 6.1, then X satisfies the Local Flat Condition. Hence, we can apply the propositions obtained in the previous sections to X.

Assume that for any $x \in X$, x is a regular point. In this case, it is not difficult to show that X is isometric to $\boldsymbol{R}^2$ with the standard flat metric. We will consider the case of $S_X \neq \varnothing$.

Theorem 6.2. *Let X be as in Theorem 6.1. Assume that for any $x \in S_X$, $k(\Sigma_x) = 1$. Then X is the product of a tree and a line.*

Proof. For $x \in S_X$ with $k(\Sigma_x) = 1$, let σ_x be a line through x directed by the two vertices in Σ_x. By Proposition 5.1, there is a flat plane $F = F(\sigma_x) \subset X$ satisfying $\sigma_x(\boldsymbol{R}) \subset F$. For $y \notin F$, there is the unique point $p = p(y, F) \in F$ closest to y. In fact, $p \in S_X$ with $k(\Sigma_p) = 1$. Let $\omega_{p,F} \subset \Sigma_p$ be the closed geodesic of length 2π corresponding to F. Then there is a (unique) line σ_p in F through p directed by the two vertices in $\omega_{p,F}$. Note that for any $t \in \boldsymbol{R}$, $\sigma_p(t)$ and $\sigma_x(t)$ are singular points with $k(\Sigma_{\sigma_p(t)}), k(\Sigma_{\sigma_x(t)}) = 1$, as seen from Lemma 5.2. Therefore σ_p is parallel to σ_x: if this were not the case, there would be a (unique) intersection point $x' \in S_X$ of $\sigma_p(\boldsymbol{R})$ and $\sigma_x(\boldsymbol{R})$ with $k(\Sigma_{x'}) \geq 2$.

Now, there is a line $\sigma_p^{\perp}$ for which $\sigma_p^{\perp}(0) = p$, $\sigma_p^{\perp}(d(p, y)) = y$, and $\sigma_p^{\perp}((-\infty, 0]) \subset F$. By Proposition 5.1, there is a flat plane $F' = F'(\sigma_p^{\perp})$ such that $\sigma_p^{\perp}(\boldsymbol{R}) \subset F'$. Let $\omega_{p,F'} \subset \Sigma_p$ be the closed geodesic of length 2π corresponding to F'. Then there is a line σ_p' in F' through p directed by the two vertices in $\omega_{p,F'}$. In particular, for any $t \in \boldsymbol{R}$, $\sigma_p'(t)$ is a singular point with $k(\Sigma_{\sigma_p'(t)}) = 1$. Indeed we conclude that $\sigma_p = \sigma_p'$ by Proposition 4.1 and by the assumption in the statement of the present theorem. Hence we can find the line σ_y in F' through y parallel to σ_p. Of course, σ_y is parallel to σ_x. Thus we find that $X = P_{\sigma_x(\boldsymbol{R})}$, where $P_{\sigma_x(\boldsymbol{R})}$ is the set of all the images of the lines parallel to σ_x. Hence, applying Lemma 3.3, we obtain $X = T \times \boldsymbol{R}$. Moreover, T has the structure of a tree by Lemma 2.9. $\square$

Theorem 6.3. *Let X be as in Theorem 6.1. Assume that for any $z \in S_X$, $k(\Sigma_z) \leq 2$ and that there is a singular point $x \in S_X$ with $k(\Sigma_x) = 2$. Then X is the product of two trees.*

Proof. Let $V_1(x) := V_1(\Sigma_x)$ and $V_2(x) := V_2(\Sigma_x)$ be the vertex subsets of (Σ_x, ρ_x) such that $V_1(x) \cup V_2(x) = V(\Sigma_x)$, $V_1(x) \cap V_2(x) = \varnothing$, and $\rho_x(v_1, v_2) = \pi/2$ for any $v_i \in V_i(x)$, $i = 1, 2$. Then $\rho_x(u_i, v_i) = \pi$ for any $u_i, v_i \in V_i(x)$. We denote by $T_i(x) \subset X$ the set of the images of all rays from x directed by the directions in $V_i(x)$.

We first show that $T_i(x)$ is convex, and in fact that it has the structure of a tree. In order to prove this, for arbitrary $y_i, z_i, w_i \in T_i(x)$, we show that $\triangle(y_i, z_i, w_i) \subset T_i(x)$ is degenerate.

Assume that each v_{xy_i}, v_{xz_i}, and v_{xw_i} represents a distinct direction. Then all three angles at x consisted of y_i, z_i, and w_i are equal to π. Now, assume that v_{xy_i}, v_{xz_i} are the same and v_{xw_i} is distinct. Then there are two possible cases. In the one case, y_i, z_i, and w_i lie on the same geodesic through x. In the other case, y_i and z_i do not lie on the same geodesic through x. Then there is a point $x' \in X$ such that x' lies on the two geodesics extending from x to y_i and from x to z_i with $k(\Sigma_{x'}) = 2$, as follows from Proposition 4.1 and Lemma 5.2 and the structure of $\Sigma_{x'}$. In particular, each angle at x' consisting of y_i, z_i, and w_i is equal to π. Next, assume that v_{xy_i}, v_{xz_i}, and v_{xw_i} are identical. Then, following arguments similarly to those given above, we find that $\triangle(y_i, z_i, w_i)$ is degenerate. Thus in each case $\triangle(y_i, z_i, w_i)$ is degenerate, and hence $\triangle(y_i, z_i, w_i) \subset T_i(x)$. Therefore $T_i(x)$ is convex, and therefore has the structure of a tree.

Next, consider $y \in X \backslash \{x\}$. By Proposition 5.1, there is a flat plane $F = F(\sigma_{xy}) \subset X$ containing $\sigma_{xy}(\boldsymbol{R})$, where σ_{xy} is a line through x and y. Let $\omega_{x,F} \subset \Sigma_x$ be the closed geodesic of length 2π corresponding to F. Then there are singular segments $\sigma_1, \sigma_2 : \boldsymbol{R} \to F$ such that σ_1 and σ_2 are perpendicular to each other at x and directed by the four vertices in $\omega_{x,F}$ marked correspondingly by $V_1(x)$ and $V_2(x)$. Let $p_i \in F$ be the foot point on $\sigma_i(\boldsymbol{R})$ to y, and let v_{p_i} be the direction to p_i at y. Now, by the assumption in the statement of the present theorem we can define, if necessary, adding suitable vertices, the vertex subsets $V_1(y)$ and $V_2(y)$ of (Σ_y, ρ_y) as $V_1(y) := V_1(\Sigma_y)$, $V_2(y) := V_2(\Sigma_y)$ so that $V_1(y) \cup V_2(y) = V(\Sigma_y), V_1(y) \cap V_2(y) = \emptyset$, and $\rho_y(v_1, v_2) = \pi/2$ for any $v_i \in V_i(y)$, and $v_{p_1} \in V_2(y), v_{p_2} \in V_1(y)$. Thus we can define $T_i(y)$ similarly to $T_i(x)$.

We now verify the compatibility of the definition of $T_i(y)$:

(6.3.1) $v_y \in V_i(y') \subset \Sigma_{y'}$ for any $y' \in T_i(y)$, where v_y is the direction from y' to y in $\Sigma_{y'}$.

Let $F' = F(\sigma_{xy'}), p_i'$, and $V_i(y')$ be defined for y' in analogy to F, p_i, and $V_i(y)$ defined for y.

Let $q_i \in T_i(x)$ (respectively $q_i' \in T_i(x)$) be the point closest to y (respectively y'). Suppose that $p_i \neq q_i$. Let $\widehat{\sigma_i}$ be a line in $T_i(x)$ through p_i and q_i. By the structure of Σ_{p_i} and the convexity of $T_i(x)$, we have $\angle_{p_i}(y, q_i) = \pi/2$. Also, $\angle_{q_i}(x, y) = \pi/2$ by the assumption in the statement of the present theorem, because we can not find a closed geodesic of length $>2\pi$ in Σ_{q_i}. Furthermore, $\angle_{q_i}(y, p_i) \geq \pi/2$, by the choice of q_i. These yield a contradiction to the convexity of the function $t \mapsto d(y, \widehat{\sigma_i}(t))$; that is, the above function in t would have local minima at no fewer than two points. Therefore $p_i = q_i$ (respectively $p_i' = q_i'$). This implies that p_i is also the closest point to y on $T_i(x)$ (respectively, p_i' is the closest point to y').

Now, in order to verify (6.3.1) it suffices to show $\angle_{y'}(y, p_j') = 0$ or π. First, note that $\angle_y(y', p_j) = 0$ or π by the choice of y'. Assume that $\angle_y(y', p_j) = \pi$. Then, because y lies on the geodesic extending from y' to p_j, we have $p_j = p_j'$, as follows from an argument similar to that above. This implies that y lies on the geodesic in F' extending from y' to p_j', and hence $\angle_{y'}(y, p_j') = 0$.

Next, assume that $\angle_y(y', p_j) = 0$. Now, three cases are possible. The first case is that in which y' lies on a line through y and p_j. Then $p_j = p_j'$, and hence $\angle_{y'}(y, p_j') = 0$ or π. We note that, if y' does not lies on a line through y and p_j, then there is a point $y_0 \in S_X$ such that y_0 lies on the line through y and p_j and that $\angle_{y_0}(y', p_j) > 0$, by Proposition 4.1. Since $k(\Sigma_{y_0}) = 1$ or 2, we have $\angle_{y_0}(y', p_j) = \pi$ by considering the singular segments on F. The second case is that in which y' does not lie on a line through y and p_j and that y' lies on the geodesic extending from y to p_j'. Then $\angle_{y'}(y, p_j') = \pi$. The third case is that in which y' does not lie on a line through y and p_j and that y' does not lie on the geodesic extending from y to p_j'. In this case we also have $\angle_{y'}(y, p_j') = 0$. Hence in each case $\angle_{y'}(y, p_j') = 0$ or π. Thus we have verified the compatibility of the definition of $T_i(y)$.

Considering the structure of $T_i(y)$, similarly to the case for $T_i(x)$, we find that $T_i(y)$ is convex, and thus it has the structure of a tree, by the compatibility of the definition of $T_i(y)$. Furthermore, by the above arguments, the quadrangle spanned by y, y', p_i', and p_i is a flat rectangle. Hence $d(-, T_i(x)) \leq d(x, y)$ on $T_i(y)$, and similarly we have $d(-, T_i(y)) \leq d(x, y)$ on $T_i(x)$. This implies that $T_i(y)$ is parallel to $T_i(x)$ for any $y \in X$. Applying Lemma 3.3, we find that $X = T_1(x) \times N$, where $N = \{pr_{T_1(y)}(x) | y \in X\}$ and $pr_{T_1(y)}$ is the projection map onto $T_1(y)$. Now, it follows from the definition of $T_1(x)$ and $T_2(x)$ that $N = T_2(x)$. Thus we conclude that X is the product of two trees. $\square$

THEOREM 6.4. *Let X be as in Theorem 6.1. Assume that there is only one singular point $x \in S_X$ with $k(\Sigma_x) \geq 2$. Then X is the Euclidean cone over $(X(\infty), Td)$.*

PROOF. Suppose that there is a direction $u \in \Sigma_x$ such that there are two distinct rays σ_{xu} and σ_{xu}' from x directed by u. Then by Proposition 4.1 there is a point $x_0 (\neq x)$ that lies on σ_{xu} and σ_{xu}' which satisfies $\angle_{x_0}(\sigma_{xu}(\infty), \sigma_{xu}'(\infty)) > 0$. This implies that $x_0 \in S_X$. By assumption, $k(\Sigma_{x_0}) = 1$. Now, by Proposition 5.1, there is a flat plane $F(\sigma_{xu})$ containing $\sigma_{xu}(\boldsymbol{R})$. Let $\sigma_{x_0} : \boldsymbol{R} \to F(\sigma_{xu})$ be the singular segment directed by the two vertices of Σ_{x_0}. Considering the structure of Σ_{x_0}, we find that x does not lie on σ_{x_0}. This implies that we can find a singular point with $k(-) \geq 2$ distinct from x. This contradicts our assumption. Hence for any $u \in \Sigma_x$ there is a unique ray σ_{xu} from x directed by u.

By the above argument, we can define the map φ_x from the tangent cone at x $\Sigma_x \times [0, \infty)/_\sim$ to X as $\varphi_x(u, t) = \sigma_{xu}(t)$. The surjectivity of φ_x is clear. Now, consider $(u, s), (v, t) \in \Sigma_x \times [0, \infty)/_\sim$. Since there is a closed geodesic $\omega_x \subset \Sigma_x$ of length 2π containing u and v, there is a flat plane $F = F(\sigma_{xu}, \omega_x)$ containing $\sigma_{xu}(\boldsymbol{R})$ such that ω_x corresponds to F by Lemma 5.2, where σ_{xu} is a line through x directed by u. Now, by the uniqueness of rays, for the ray σ_{xv} from x directed by v, we have $\sigma_{xv}([0, \infty)) \subset F$. Hence it follows from the geometry on F that φ_x is an isometry.

Furthermore, we can also define the isometry Φ_x from Σ_x to $X(\infty)$ as $\Phi_x(u) = \sigma_{xu}(\infty)$ by the above arguments. Therefore we have shown that X is the Euclidean cone over $X(\infty)$. $\square$

THEOREM 6.5. *Let X be as in Theorem 6.1. Assume that there are points $x, y \in S_X$ with $k(\Sigma_x) \geq 3$ and $k(\Sigma_y) \geq 2$. Then X is a thick Euclidean building of dimension 2 of type A_2, B_2, or G_2.*

PROOF. Recall Proposition 4.1 implies that for any $w \in S_X$ with $k(\Sigma_w) \geq 2$ there is a positive number $r = r(w) > 0$ such that $\bar{B}(w, r)$ contains no singular point with $k(-) \geq 2$. Hence it follows from Proposition 4.1 and the local compactness of X that $\bar{B}(x, r')$ contains at most finitely many singular points with $k(-) \geq 2$ for any $r' > 0$. Choose $y \in S_X$ with $k(\Sigma_y) \geq 2$ satisfying $d(x, y) = \min\{d(x, w) | w \in S_X \backslash \{x\}, k(\Sigma_w) \geq 2\} > 0$. Then by Proposition 5.1, there is a flat plane $F = F(\sigma_{xy})$ containing $\sigma_{xy}(\boldsymbol{R})$, where σ_{xy} is a line through x and y.

Let $\omega_{x,F} \subset \Sigma_x$ (respectively $\omega_{y,F} \subset \Sigma_y$) be the closed geodesic of length 2π corresponding to F, and let $v_1, v_2, \ldots, v_{2k(\Sigma_x)} \in \omega_{x,F}$ be the vertices of Σ_x ordered by the rotation manner. Then there is a singular segment $\sigma_i : [0, \infty) \to F$, $1 \leq i \leq 2k(\Sigma_x)$, emanating from x directed by v_i. Let $R_i(x) \subset F$ be the flat sector spanned by two singular segments $\sigma_i([0, \infty))$ and $\sigma_{i+1}([0, \infty))$ (if $i = 2k(\Sigma_x)$, then $R_i(x)$ is spanned by $\sigma_{2k(\Sigma_x)}([0, \infty))$ and $\sigma_1([0, \infty))$).

We now show that there is a singular segment σ_i such that $y \in \sigma_i([0, \infty))$. Suppose that there is a flat sector $R_i(x)$ such that the interior of $R_i(x)$ contains y. Then there are two points $p_i \in \sigma_i([0, \infty))$ and $p_{i+1} \in \sigma_{i+1}([0, \infty))$ such that v_{yp_i} and $v_{yp_{i+1}}$ are adjacent vertices in $\omega_{y,F}$; that is, $\angle_y(p_i, p_{i+1}) = \pi/k(\Sigma_y)$, and that v_{yx} is contained in the edge joining v_{yp_i} and $v_{yp_{i+1}}$. Note that $p_i, p_{i+1} \in S_X$ with $k(\Sigma_{p_i}) \geq 2$ and $k(\Sigma_{p_{i+1}}) \geq 2$. Consider the flat quadrangle in $R_i(x)$ spanned by x, p_i, p_{i+1}, and y. Then since $\angle_x(p_i, p_{i+1})$ is not greater than $\pi/3$, either $d(x, p_i)$ or $d(x, p_{i+1})$ is smaller than $d(x, y)$. This contradicts the choice of y.

Now there is a point $z \in \sigma_{i+1}([0, \infty))$ such that $\angle_y(x, z) = \pi/k(\Sigma_y)$. This implies that $\sigma_{yz} : \boldsymbol{R} \to F$ is a singular segment through y and z. In particular, $z \in S_X$ with $k(\Sigma_z) \geq 2$. Then the interior of the flat triangular region spanned by $\triangle(x, y, z)$ does not contain singular points: if it did, we could find a singular point $y' \in S_X$ with $k(\Sigma_{y'}) \geq 2$ and $d(x, y') < d(x, y)$, since $\angle_x(y, z) \leq \pi/3$ and $\angle_y(x, z), \angle_z(x, y) \leq \pi/2$. Moreover, by the Gauss-Bonnet formula, the flat triangular region spanned by $\triangle(x, y, z)$ is isometric to that of type A_2, B_2, or G_2. Thus we verify that F has a triangulation of type A_2, B_2, or G_2 which satisfies the following:

(1) Every 2-simplex is isometric to the flat triangular region spanned by $\triangle(x, y, z)$.
(2) No interior of any 2-simplex contains a singular point.
(3) Every point in every 1-simplex is singular.
(4) Every interior point of every 1-simplex has type of $k(-) = 1$.

For $q \notin F$, let $p = p(q, F) \in F$ be the closest point to q. Also, let $\omega_{p,F} \subset (\Sigma_p, \rho_p)$ be the closed geodesic of length 2π corresponding to F, and let $v_q \in \Sigma_p$ be the direction to q. Then by the structure of Σ_p there is a point $v_q' \in \omega_{p,F}$ such that $\rho_p(v_q, v_q') = \pi$. Let σ_{pq} be a line directed by v_q and v_q' at p with $\sigma_{pq}(0) = p$, $\sigma_{pq}(d(p, q)) = q$, and $\sigma_{pq}((-\infty, 0]) \subset F$. Then, applying Proposition 5.1 to σ_{pq} and Proposition 4.1 to p, we obtain a flat plane $F' = F'(\sigma_{pq}) \subset X$ such that F' contains a 2-simplex of such a triangulation of F. Hence F' has a triangulation of the same type as F.

Therefore it follows from the above arguments that X has a triangulation whose 2-simplices are all isometric. We next verify the axioms for thick Euclidean buildings of dimension 2 (see Subsection 2.3). First, in order to verify (B1), consider arbitrary 2-simplices $\triangle_1$ and $\triangle_2$. Let x_i, $i = 1, 2$, be an interior point of $\triangle_i$, and let $\sigma_{x_1x_2}$ be a line through x_1 and x_2. It then follows from Propositions 4.1 and 5.1 that there is a flat plane $F(\sigma_{x_1x_2})$ containing $\sigma_{x_1x_2}(\boldsymbol{R})$ such that $\triangle_1, \triangle_2 \subset F(\sigma_{x_1x_2})$. This implies that X satisfies (B1). Now, it is also clear that X satisfies (B2′) and the thickness condition by the above arguments. Thus X is a thick Euclidean building of dimension 2. This completes the proof of Theorem 6.5. □

We have thus completed the proof of Theorem 6.1. □

With the preceding preparation we obtain the following main theorem as a corollary of Theorem 6.1, which is a natural extension of Theorem 1 stated in Section 1.

THEOREM 6.6. *Assume that X is a locally compact, geodesically complete nonpositively curved* 2*-space satisfying the LFC. Then $\tilde{X}$ is isometric to either the product of two trees, the Euclidean cone over $(\tilde{X}(\infty), Td)$, or a thick Euclidean building of dimension* 2 *of type A_2, B_2, or G_2.*

PROOF. Proposition 5.1 implies that $\tilde{X}$ is a locally compact, geodesically complete Hadamard 2-space such that the diameter of $(\tilde{X}(\infty), Td)$ is equal to π. Applying Theorem 6.1, we obtain Theorem 6.6. □

Furthermore we assume that X is compact. Then, because a fundamental domain of the deck transformation group on $\tilde{X}$ has a finite diameter, we obtain the following corollary:

COROLLARY 6.7. *Let X be a compact, geodesically complete nonpositively curved* 2*-space satisfying the LFC. Then $\tilde{X}$ is isometric to either the product of two trees or a thick Euclidean building of dimension* 2 *of type A_2, B_2,* or *G_2.*

REMARK 6.8. Let X be as in Theorem 6.1. Then, $(X(\infty), Td)$ also has the structure of a graph with the properties listed in Lemma 2.11. Conversely, it follows from Propositions 3.4 and 3.5 that, if $k(X(\infty)) \leq 2$, then X is the product of two trees.

Recently Bernhard Leeb has obtained the following result, closely related to Theorem 6.1: Let X be a locally compact Hadamard space with extendible geodesic segments and assume that $\partial_{Tits}X$ is a connected thick irreducible spherical building. Then X is a Riemannian symmetric space or a Euclidean building. ([**Le**]) His result is discussed in the case of general dimension. Assume that X is 2-dimensional in his result. Then, actually, the results here intersect with his result in [**Le**] in the case $k(X(\infty)) \geq 3$. However our approach, which is obtained independently, is more elementary than his approach in [**Le**].

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