A note on a conjecture of Xiao

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(Received Feb. 1, 1999)

Abstract. We prove that the image of the relative dualizing sheaf of a fibration from a smooth projective surface onto a smooth projective curve is ample under some extra conditions.

When \( f : S \to B \) is a surjective morphism of a complex, smooth surface \( S \) onto a complex, smooth, genus \( b \) curve \( B \), such that the fibre \( F \) of \( f \) has genus \( g \), it is well known that \( f_*\omega_{S/B} = \mathcal{E} \) is a locally free sheaf of rank \( g \) and degree \( d = \mathcal{M}(S) - (b - 1)(g - 1) \) and that \( f \) is not an holomorphic fibre bundle if and only if \( d > 0 \). In this case the slope, \( \lambda(f) = \{ K_S^2 - 8(b - 1)(g - 1) \}/d \), is a natural invariant associated by Xiao to \( f \) (cf. [7]). In [7, Conjecture 2] he conjectured that \( \mathcal{E} \) has no locally free quotient of degree zero (i.e., \( \mathcal{E} \) is ample) if \( \lambda(f) < 4 \). We give a partial affirmative answer to this conjecture:

**Theorem 1.** Let \( f : S \to B \) be a relatively minimal fibration with general fibre \( F \). Let \( b = g(B) \) and assume that \( g = g(F) \geq 2 \) and that \( f \) is not locally trivial.

If \( \lambda(f) < 4 \) then \( \mathcal{E} = f_*\omega_{S/B} \) is ample provided one of the following conditions hold.

(i) \( F \) is non hyperelliptic.

(ii) \( b \leq 1 \).

(iii) \( g(F) \leq 3 \).

**Proof.** (i) If \( q(S) > b \) the result follows from [7, Corollary 2.1]. Now assume \( q(S) = b \). By Fujita’s decomposition theorem (see [3], [4] and also [5] for a proof)

\[
\mathcal{E} = \mathcal{A} \oplus \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_r
\]

where \( h^0(B, (\mathcal{A} \oplus \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_r)^+) = 0 \), \( \mathcal{A} \) is an ample sheaf and \( \mathcal{F}_i \) are non trivial stable degree zero sheaves. Then we only must prove that \( \mathcal{F}_i = 0 \). If \( F \) is not hyperelliptic and rank \( (\mathcal{F}_i) \geq 2 \) the claim is the content of [7, Proposition 3.1]. 

2000 Mathematics Subject Classification. Primary 14H10, Secondary 14J29.

Key Words and Phrases. Fibration, Relative dualizing sheaf.

* Partially supported by CICYT PS93-0790 and HCM project n.ERBCHRXCT-940557.

** Partially supported by HCM project n.ERBCHRXCT-940557.
rank(\mathcal{F}_i) = 1 we can use [2, §4.2] or [1, Theorem 3.4] to conclude that \mathcal{F}_i is torsion in Pic^0(B). Hence it induces an étale base change:

\[
\begin{array}{ccc}
\hat{S} & \longrightarrow & S \\
\downarrow f & & \downarrow f \\
\hat{B} & \longrightarrow & B
\end{array}
\]

By flatness \( \hat{f}_* \omega_{\hat{S}/B} = \sigma^*(f_* \omega_{S/B}) \). Since \( \sigma \) is étale \( \lambda(f) = \lambda(\hat{f}) \) and \( \sigma^*(\mathcal{F}_i) = \mathcal{O}_B \) is a direct summand of \( \hat{f}_* \omega_{\hat{S}/B} \). In particular by [3] \( q(\hat{S}) > \hat{b} = g(\hat{B}) \) hence \( \lambda(\hat{f}) \geq 4 \) by [7, Theorem 3.3]: a contradiction.

(ii) If \( b = 0 \) the claim is trivial. If \( b = 1 \), any stable degree zero sheaf has rank one, then as in (i) we conclude.

(iii) If \( g = 2 \) and \( \mathcal{E} \neq \mathcal{A} \), then \( \mathcal{E} = \mathcal{A} \oplus \mathcal{L} \) where \( \mathcal{L} \) torsion and we are done. The only non trivial case if \( g = 3 \) is \( \mathcal{E} = \mathcal{A} \oplus \mathcal{F} \) where \( \mathcal{A} \) an ample line bundle and \( \mathcal{F} \) a stable, degree zero, rank two vector bundle. Then \( K^2_{\hat{S}/B} \geq (2g - 2) \deg \mathcal{A} = 4d \) and we are done by [7, Theorem 2].

Theorem 3.3 of [7] Xiao says that if \( q(S) > b \) and \( \lambda(f) = 4 \) then \( \mathcal{E} = \mathcal{F} \oplus \mathcal{O}_B \), where \( \mathcal{F} \) is a semistable sheaf. We have the following improvement:

**Theorem 2.** Let \( f : S \to B \) be a relatively minimal non locally trivial fibration. If \( \lambda(f) = 4 \) then \( \mathcal{E} = f_* \omega_{S/B} \) has at most one degree zero, rank one quotient \( \mathcal{L} \). Moreover, in this case \( \mathcal{E} = \mathcal{A} \oplus \mathcal{L} \) with \( \mathcal{A} \) semistable and \( \mathcal{L} \) torsion.

**Proof.** As in the previous theorem the torsion subsheaf \( \mathcal{L} \) becomes the trivial one after an étale base change; thus

\[
\hat{f}_* \omega_{\hat{S}/B} = \mathcal{A} \oplus \mathcal{O}_B, \quad \mathcal{A} = \sigma^* \mathcal{A}.
\]

By [7, Theorem 3.3], \( \mathcal{A} \) is semistable. Then \( \mathcal{A} \) is also semistable by [6, Proposition 3.2].

**References**


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