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Lifting to mod 2 Moore spaces

Dedicated to the memory of Professor Katsuo Kawakubo

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Abstract. We try to investigate whether an element of order 2 in homotopy groups of spheres has a lift to an element of homotopy groups of mod 2 Moore spaces or not. A typical element of the affirmative property is Thomeier's element of $\pi_{8n+7}(S^{4n+3})$. And the Whitehead square $[\iota_{2n+1}, \iota_{2n+1}]$ of the identity class ι_{2n+1} of S^{2n+1} is a negative example except for $n = 0$, 1 and 3.

Introduction.

Let ι_n be the identity class of S^n. We denote by $M^n = S^{n-1} \cup_{2\iota_{n-1}} e^n$ the Moore space of type $(\boldsymbol{Z}_2, n-1)$. An element $\beta \in \pi_k(M^n)$ is called a lift of $\alpha \in \pi_k(S^n)$ if $p_*\beta = \alpha$, where $p : M^n \to S^n$ is the collapsing map. If $2\iota_n \circ \alpha = 0$, then by the definition, a coextension of α to M^{n+1} is a lift of a suspension $\Sigma\alpha$. A lift which is not a coextension is called a strict lift. The main purpose of this note is to examine which element of $\pi_k(S^n)$ is strictly lifted to an element of $\pi_k(M^n)$.

To state our result, we need the notations and results of [**22**]. Throughout this note, we deal with 2-primary components. Let $\eta_2 \in \pi_3(S^2)$ be the Hopf map, $\eta_n = \Sigma^{n-2}\eta_2$ and $\eta_n^2 = \eta_n \circ \eta_{n+1}$ for $n \geq 2$. We know that $\pi_3(S^2) = \boldsymbol{Z}\{\eta_2\}$, $\pi_{n+1}(S^n) = \boldsymbol{Z}_2\{\eta_n\}$ for $n \geq 3$ and $\pi_{n+2}(S^n) = \boldsymbol{Z}_2\{\eta_n^2\}$ for $n \geq 2$. Let $\nu_4 \in \pi_7(S^4)$ be the Hopf map and $\nu_n = \Sigma^{n-4}\nu_4$ for $n \geq 4$. We know that $\pi_6(S^3) = \boldsymbol{Z}_4\{\nu'\}$, $\pi_7(S^4) = \boldsymbol{Z}\{\nu_4\} \oplus \boldsymbol{Z}_4\{\Sigma\nu'\}$ and $\pi_{n+3}(S^n) = \boldsymbol{Z}_8\{\nu_n\}$ for $n \geq 5$. We recall the relations $2\nu' = \eta_3^3$ and $\pm[\iota_4, \iota_4] = 2\nu_4 - \Sigma\nu'$. Let $\sigma_8 \in \pi_{15}(S^8)$ be the Hopf map and $\sigma_n = \Sigma^{n-8}\sigma_8$ for $n \geq 8$. We know that $\pi_{12}(S^5) = \boldsymbol{Z}_2\{\sigma'''\}$, $\pi_{13}(S^6) = \boldsymbol{Z}_4\{\sigma''\}$, $\pi_{14}(S^7) = \boldsymbol{Z}_8\{\sigma'\}$, $\pi_{15}(S^8) = \boldsymbol{Z}\{\sigma_8\} \oplus \boldsymbol{Z}_8\{\Sigma\sigma'\}$ and $\pi_{n+7}(S^n) = \boldsymbol{Z}_{16}\{\sigma_n\}$ for $n \geq 9$. We know the relations $\Sigma\sigma''' = 2\sigma''$, $\Sigma\sigma'' = 2\sigma'$ and $\pm[\iota_8, \iota_8] = 2\sigma_8 - \Sigma\sigma'$. We recall the following elements: $\varepsilon_3 \in \pi_{11}(S^3)$; $\nu_4^2 \in \pi_{10}(S^4)$; $\mu_3 \in \pi_{12}(S^3)$; $\bar{\nu}_6 \in \pi_{14}(S^6)$; $\theta' \in \pi_{23}(S^{11})$; $\kappa_7 \in \pi_{21}(S^7)$; $\rho^{IV} \in \pi_{20}(S^5)$; $\omega_{14} \in \pi_{30}(S^{14})$; $\eta^{*\prime} \in \pi_{31}(S^{15})$.

Let $H : \pi_k(S^n) \to \pi_k(S^{2n-1})$ be the Hopf homomorphism. According to [**19**], there exist elements $\delta_{4n} \in \pi_{8n}(S^{4n})$ and $\delta'_{4n-1} \in \pi_{8n-1}(S^{4n-1})$ of order 2 satisfying $\Sigma\delta_{4n} =$

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$[\iota_{4n+1}, \iota_{4n+1}]$, $\Sigma\delta'_{4n-1} = [\iota_{4n}, \eta_{4n}]$, $H(\delta_{4n}) = \eta_{8n-1}$ and $H(\delta'_{4n-1}) = \eta^2_{8n-3}$. These elements can be taken as follows: $\delta'_3 = \nu'\eta_6$, $\delta_4 = \nu_4\eta_7$, $\delta'_7 = \sigma'\eta_{14}$, $\delta_8 = \sigma_8\eta_{15}$, $\delta'_{11} = \theta'$, $\delta_{12} = \theta$, $\delta'_{15} = \eta^{*\prime}$ and $\delta_{16} = \eta^*_{16}$. Our result is stated as follows.

Theorem 1. *The following elements are strictly lifted:* η_3; $\nu'\eta_6$; $4\nu^2_4$; ν^2_5; σ'''; $4\bar{\nu}_6$; $\bar{\nu}_7$; $\sigma'\eta_{14}$; $2[\iota_{10}, \nu_{10}]$; $2\kappa_7$; $8\sigma^2_8$; ρ^{IV}; $4\omega_{14}$; ω_{15}.

Theorem 2. *The following elements are not lifted:* κ_{10}; σ^2_{16}; δ_{4n}, $\delta_{4n}\eta_{8n}$ *for* $n \geq 1$; $[\iota_{2n+1}, \iota_{2n+1}]$ *for* $n \neq 0, 1$ *or* 3; $[\iota_{4n+1}, \eta_{4n+1}]$ *for* $n \geq 1$.

Let $\tilde{\eta}_3 \in \pi_5(M^4)$ be a coextension of η_3 and $\tilde{\eta}_n = \Sigma^{n-3}\tilde{\eta}_3 \in \pi_{n+2}(M^{n+1})$ for $n \geq 3$. There exists a lift α of η_3 which is a generator of $\pi_4(M^3) \cong \boldsymbol{Z}_4$ (Lemma 4.1 of [**14**]). We use the notation $\tilde{\eta}_2 = \alpha$. The following result gives an example of the element of order 8 in homotopy groups of M^n (cf. [**3**]).

Theorem 3. *Let* $n \equiv 3 \bmod 4$ *and* $n \geq 11$. *Then there exists a lift* $\beta_n \in \pi_{2n+1}(M^n)$ *of* δ'_n *such that*

$$2\beta_n \equiv \tilde{\eta}_{n-1}[\iota_{n+1}, \iota_{n+1}] \mod \Sigma\pi_{2n}(M^{n-1}).$$

The assertion of Theorem 3 for $n = 3$, 7 except for the last one was obtained by Wu [**25**]. Our method is to use the composition methods developed by Toda [**22**]. We use effectively the informations of two homotopy fibers of the pinching maps $p : M^n \to S^n$ and $p' : (M^n, S^{n-1}) \to (S^n, *)$ (James [**5**]).

Although Tipple [**20**] omitted the details, some of our result overlaps with that of [**20**].

The authors wish to thank Mahowald for informing the property of the Mahowald element.

1. Fundamental facts and some homotopy groups of $M^n \wedge M^n$.

Let X be a connected finite CW-complex, $\theta : S^{n-1} \to X$ be a mapping and $X^* = X \cup_\theta e^n$ be a complex formed by attaching an n-cell. Let $p' = p'_n : (X^*, X) \to (S^n, *)$ be the collapsing map. Let $\gamma_n \in \pi_n(X^*, X)$ be the characteristic map of the n-cell e^n of X^*. Let CY be the reduced cone of a space Y. For an element $\alpha \in \pi_{k-1}(Y)$, we denote by $\hat{\alpha}' \in \pi_k(CY, Y)$ an element satisfying $\partial\hat{\alpha}' = \alpha$, where $\partial : \pi_k(CY, Y) \to \pi_{k-1}(Y)$ is the connecting isomorphism. For $\alpha \in \pi_{k-1}(S^{n-1})$, we set $\hat{\alpha} = \gamma_n\hat{\alpha}' \in \pi_n(X^*, X)$. We note the following:

$$\partial\hat{\alpha} = \theta \circ \alpha \quad \text{and} \quad p'_*\hat{\alpha} = \Sigma\alpha,$$

where $\partial : \pi_n(X^*, X) \to \pi_{n-1}(X)$ is the boundary map. Let $\Sigma' : \pi_k(X^*, X) \to \pi_{k+1}(\Sigma X^*, \Sigma X)$ be the relative suspension homomorphism ([**21**]). In the case that $X^* = M^n$, we can take $\gamma_{n+k} = (\Sigma')^k \gamma_n$. Let $i_2 : S^1 \hookrightarrow M^2$ be the inclusion and $p_2 : M^2 \to S^2$ the collapsing map. We set $i_n = \Sigma^{n-2} i_2 : S^{n-1} \hookrightarrow M^n$ and $p_n = \Sigma^{n-2} p_2 : M^n \to S^n$.

First of all, we recall the Hilton formula. By (8.16) of Chapter 9 of [**24**], we have

Lemma 1.1 (Hilton). *$2\iota_n \circ \alpha = 2\alpha + [\iota_n, \iota_n] \circ H(\alpha)$ for any element $\alpha \in \pi_k(S^n)$.*

Next we state the homotopy excision theorem (Theorem 2.1 of [**5**]) in the following form.

Lemma 1.2 (James). *Assume that $r \leq 3n - 4$. Then we have the exact sequence*

$$\pi_r(M^n, S^{n-1}) \xrightarrow{p'_{n*}} \pi_r(S^n) \xrightarrow{H_2} \pi_{r-n}(S^{n-1}) \xrightarrow{Q} \pi_{r-1}(M^n, S^{n-1}) \longrightarrow \cdots,$$

where $H_2 = 2\iota_{n-1} \circ \Sigma^{-n} \circ H$ and $Q(\ ) = [\gamma_n, \ ]$.

We denote by ι'_n the identity class of M^n. Let JX be the James reduced product of X ([**6**]). The $(3n-4)$-skeleton of JM^n is the second filtration $J_2 M^n = M^n \cup_\beta C\Sigma(M^{n-1} \wedge M^{n-1})$ of JM^n, where $\beta = [\iota'_n, \iota'_n]$. Then we have the following.

Lemma 1.3. *Let $X = M^n \wedge M^n$ and let $\partial : [C\Sigma^{-1}X, \Sigma^{-1}X; JM^n, M^n] \to [\Sigma^{-1}X, M^n]$ be the connecting map. Then we have $\partial \hat{\imath}_X = [\iota'_n, \iota'_n]$ for $n \geq 3$.*

Let F be the homotopy fiber of the map $p_n : M^n \to S^n$. From the cohomology structure of F, we have $F = S^{n-1} \cup e^{2(n-1)} \cup e^{3(n-1)} \cup \cdots$. Let Y be the $(3n-4)$-skeleton of F. We set $w_n = [\iota_n, \iota_n]$. Then we have the following.

Lemma 1.4. *$Y = S^{n-1} \cup_{2w_{n-1}} e^{2n-2}$ for n odd and $Y = S^{n-1} \vee S^{2n-2}$ for n even.*

Proof. Let $A = S^{n-1}$ and X be a mapping cylinder of a mapping of degree 2 of S^{n-1}. Let $i : S^{n-1} \hookrightarrow X$ be the embedding satisfying $i(x) = [x, 1]$. Then X is homotopy equivalent to S^{n-1}, i is regarded as $2\iota_{n-1}$ and $X \cup CA = M^n$. So the collapsing map $p_n : M^n \to S^n$ has the homotopy fiber $(X, A)_\infty$ ([**4**]). By Corollary 5.8 of [**4**], we have $Y = (X, A)_2 = S^{n-1} \cup_\delta e^{2n-2}$, where $\delta = [\iota_{n-1}, i] = 2[\iota_{n-1}, \iota_{n-1}]$. This completes the proof. □

We recall that $M^n \wedge M^n$ is a mapping cone of a mapping $2\iota'_{2n-1} = i_{2n-1}\eta_{2n-2}p_{2n-1}$ ([**1**], [**23**]). Let $i'_4 : M^4 \hookrightarrow \Sigma(M^2 \wedge M^2)$ be the inclusion. We set $i'_n = \Sigma^{n-4} i'_4 : M^n \hookrightarrow \Sigma^{n-3}(M^2 \wedge M^2)$ for $n \geq 4$. We also set $i''_n = i'_n \circ i_n$. A Toda bracket $\{i'_4, 2\iota'_4, i_4\} \subset \pi_4(\Sigma(M^2 \wedge M^2))$ is well-defined and its representative $\tilde{\imath}_4$ is a coextension of i_4. Since $2\tilde{\imath}_4 \in \{i'_4, 2\iota'_4, i_4\} \circ 2\iota_4 = -i'_4\{2\iota'_4, i_4, 2\iota_3\} \supset -i'_4 i_4 \eta_3 \{p_4, i_4, 2\iota_3\} \ni i''_4 \eta_3 \bmod 0$, we have $2\tilde{\imath}_4 =$

$i_4''\eta_3$. We set $\tilde{\imath}_n = \Sigma^{n-4}\tilde{\imath}_4 \in \pi_n(\Sigma^{n-3}(M^2 \wedge M^2))$ for $n \geq 4$. We note that $\tilde{\imath}_n \in \{i_n', 2\imath_n', i_n\}$. We denote by $p' : \Sigma(M^n \wedge M^n) \to M^{2n+1}$ and $p'' = p'p : \Sigma(M^n \wedge M^n) \to S^{2n+1}$ the collapsing maps. By making use of the cofiber sequence starting with $2\imath_{2n}'$, we have the following.

LEMMA 1.5. *Suppose* $n \geq 3$. *Then*
(i) $\pi_{2n}(\Sigma(M^n \wedge M^n)) = \boldsymbol{Z}_4\{\tilde{\imath}_{2n}\}$;
(ii) $\pi_{2n+1}(\Sigma(M^n \wedge M^n)) = \boldsymbol{Z}_2\{\tilde{\imath}_{2n}\eta_{2n}\} \oplus \boldsymbol{Z}_2\{i_{2n}'\tilde{\eta}_{2n-1}\}$;
(iii) $\pi_{2n+2}(\Sigma(M^n \wedge M^n)) = \boldsymbol{Z}_2\{\tilde{\imath}_{2n}\eta_{2n}^2\} \oplus \boldsymbol{Z}_2\{i_{2n}'\tilde{\eta}_{2n-1}\eta_{2n+1}\} \oplus \boldsymbol{Z}_2\{i_{2n}''\nu_{2n-1}\}$.

We set $\alpha_2 = \tilde{\eta}_3\eta_5$ and $\alpha_n = \Sigma^{n-2}\alpha_2 = \tilde{\eta}_{n+1}\eta_{n+3}$ for $n \geq 2$. We have $2\imath_4' \circ \alpha_2 = i_4\eta_3 p_4\alpha_2 = i_4\eta_3^3 = 2i_4\nu' = 0$. So we can define a coextension $\tilde{\alpha}_2 \in \{i_4', 2\imath_4', \alpha_2\} \subset \pi_7(\Sigma(M^2 \wedge M^2))$ of α_2. We set $\tilde{\alpha}_n = \Sigma^{n-2}\tilde{\alpha}_2$ for $n \geq 2$. Then we show

LEMMA 1.6. (i) $2\tilde{\alpha}_n = i_{n+2}'\tilde{\eta}_{n+1}\eta_{n+3}^2$ *for* $n \geq 2$.
(ii) $\pi_{2n+3}(\Sigma(M^n \wedge M^n)) = \boldsymbol{Z}_4\{\tilde{\alpha}_{2n}\} \oplus \boldsymbol{Z}_2\{\tilde{\imath}_{2n}\nu_{2n}\}$ *for* $n \geq 4$ *and* $\pi_9(\Sigma(M^3 \wedge M^3)) = \boldsymbol{Z}_4\{\tilde{\alpha}_4\} \oplus \boldsymbol{Z}_2\{\tilde{\imath}_6\nu_6\} \oplus \boldsymbol{Z}_2\{i_6''\nu_5\eta_8\}$, *where* $\Sigma(\tilde{\eta}_{n-1} \wedge \tilde{\eta}_{n-1}) \equiv \pm\tilde{\alpha}_{2n} \bmod i_{2n}''\nu_{2n-1}\eta_{2n+2}$ *for* $n \geq 3$.
(iii) $\pi_{2n+4}(\Sigma(M^n \wedge M^n)) = \boldsymbol{Z}_2\{\tilde{\alpha}_{2n}\eta_{2n+3}\}$ *for* $n \geq 4$.

PROOF. First we recall the following ([**12**], [**14**]):

$$\pi_{2n+3}(M^{2n+1}) = \boldsymbol{Z}_2\{\tilde{\eta}_{2n}\eta_{2n+2}\} \oplus \boldsymbol{Z}_2\{i_{2n+1}\nu_{2n}\} (n \geq 3);$$

$$\pi_9(M^6) = \boldsymbol{Z}_2\{\tilde{\eta}_5\eta_7^2\} \oplus \boldsymbol{Z}_2\{i_6\nu_5\eta_8\}; \quad \pi_{2n+3}(M^{2n}) = \boldsymbol{Z}_2\{\tilde{\eta}_{2n-1}\eta_{2n+1}^2\} (n \geq 4).$$

We have $2\tilde{\alpha}_2 \in -i_4'\{2\imath_4', \alpha_2, 2\imath_6\}$. And we have

$$\begin{aligned} \{2\imath_4', \alpha_2, 2\imath_6\} &\subset \{i_4\eta_3, \eta_4^2, 2\imath_6\} \\ &\subset \{i_4, 2\nu', 2\imath_6\} \\ &\supset \{i_4, 2\imath_3, \eta_3^3\} \\ &\ni \tilde{\eta}_3\eta_5^2 \\ &\bmod i_{4*}\pi_7(S^3) + 2\pi_7(M^4) = 0, \end{aligned}$$

by Lemma 2.2 of [**14**]. So we have $2\tilde{\alpha}_2 = i_4'\tilde{\eta}_3\eta_5^2$. This leads to (i).

We have $\pi_{2n+4}(M^{2n+1}) = \boldsymbol{Z}_2\{\tilde{\eta}_{2n}\eta_{2n+2}^2\}$ and $\pi_{2n+4}(M^{2n}) = 0$ for $n \geq 4$. So, by the exact sequence

$$0 \longrightarrow \pi_{2n+4}(M^{2n}) \xrightarrow{i_{2n*}'} \pi_{2n+4}(\Sigma(M^n \wedge M^n)) \xrightarrow{p_*'} \pi_{2n+4}(M^{2n+1}) \longrightarrow 0,$$

we have (iii).

We note that $M^{2n} \vee S^{2n}$ is the 2n-skeleton of $\Sigma(M^n \wedge M^n)$. Let $k_1 : M^{2n} \hookrightarrow M^{2n} \vee S^{2n}$ and $k_2 : S^{2n} \hookrightarrow M^{2n} \vee S^{2n}$ be the inclusions, respectively. Then $\Sigma(M^n \wedge M^n)$ is a mapping cone of $k_1 i_{2n}\eta_{2n-1} + 2k_2$. In the homotopy exact sequence of a pair $(\Sigma(M^n \wedge M^n), M^{2n} \vee S^{2n})$:

$$\pi_{2n+4}(\Sigma(M^n \wedge M^n), M^{2n} \vee S^{2n}) \xrightarrow{\partial} \pi_{2n+3}(M^{2n} \vee S^{2n}) \xrightarrow{i_*} \pi_{2n+3}(\Sigma(M^n \wedge M^n))$$

$$\xrightarrow{j_*} \pi_{2n+3}(\Sigma(M^n \wedge M^n), M^{2n} \vee S^{2n})$$

$$\xrightarrow{\partial} \pi_{2n+2}(M^{2n} \vee S^{2n}),$$

the first group is isomorphic to $\pi_{2n+4}(S^{2n+1})$ and the fourth group is isomorphic to $\pi_{2n+3}(S^{2n+1})$ for $n \geq 3$. We have $\partial\hat{\nu}_{2n} = 2k_2\nu_{2n}$ and $\partial\widehat{\eta^2_{2n}} = k_1 i_{2n}\eta^3_{2n-1} + 2k_2\eta^2_{2n} = 0$. We have $p''_*\tilde{\alpha}_{2n} = \eta^2_{2n+1}$. Hence we have the first half of (ii).

The second half of (ii) is obtained by the parallel argument to the first half.

In the exact sequence

$$\pi_9(M^6) \xrightarrow{i'_{6*}} \pi_9(\Sigma M^3 \wedge M^3) \xrightarrow{j'_*} \pi_9(\Sigma M^3 \wedge M^3, M^6),$$

we have $\pi_9(M^6) = \boldsymbol{Z}_2\{\tilde{\eta}_5\eta^2_7\} \oplus \boldsymbol{Z}_2\{\nu_5\eta_8\}$ and $\pi_9(\Sigma M^3 \wedge M^3, M^6) \cong \pi_9(M^7) = \boldsymbol{Z}_2\{\tilde{\eta}_6\eta_8\} \oplus \boldsymbol{Z}_2\{i_7\nu_6\}$. So the inclusion j' is identified with the collapsing map $p' = \Sigma p_3 \wedge i'_3$ and $p'_*(\Sigma\tilde{\eta}_2 \wedge \tilde{\eta}_2) = \eta_4 \wedge \tilde{\eta}_2 = \tilde{\eta}_6\eta_8 = p'_*\tilde{\alpha}_4$. Therefore we have the relation $\Sigma\tilde{\eta}_2 \wedge \tilde{\eta}_2 - \tilde{\alpha}_4 \in \operatorname{Im} i'_{6*} = \{i'_6\tilde{\eta}_5\eta^2_7, i''_6\nu_5\eta_8\}$. So we have $\Sigma\tilde{\eta}_2 \wedge \tilde{\eta}_2 \equiv \pm\tilde{\alpha}_4 \bmod i''_6\nu_5\eta_8$. This completes the proof. □

2. Some lifts to M^n.

If $\alpha \in \pi_k(S^n)$ has a lift $\beta \in \pi_k(M^n)$ and the orders of α and β are same, then β is called a splitting lift of α. We shall use the notion of lifting for an element of the relative homotopy group. We remark that $\alpha \in \pi_k(S^n)$ for $k \leq 3n-2$ has a lift in $\pi_k(M^n, S^{n-1})$ if and only if $H_2(\alpha) = 0$ by Lemma 1.2. We show

LEMMA 2.1. (i) *Suppose that $\alpha \in \pi_k(S^n)$ is lifted to M^n. Then the composite $\alpha \circ \beta$ is lifted to M^n, where $\beta \in \pi_m(S^k)$. Furthermore, $2\iota_n \circ \alpha = 0$ for $n \geq 3$.*

(ii) *Any nontrivial element of $\pi_k(S^2)$ has no lift.*

(iii) *Any element of $\Sigma\pi_k(S^2)$ has a lift.*

PROOF. The first half of (i) is obviously obtained. Since $p_n = \Sigma p_{n-1}$ for $n \geq 3$ and it is of order 2, we have $2\iota_n \circ \alpha = 2p_n \circ \delta = 0$, where $\delta \in \pi_k(M^n)$ is a lift of α. This leads to the second half of (i).

Since M^2 is the real projective plane, $\pi_k(M^2)$ is isomorphic to $\pi_k(S^2)$ through the double covering map $\gamma : S^2 \to M^2$. Then the fact $p_2\gamma = 0$ implies (ii).

Any element of $\Sigma\pi_k(S^2)$ is represented as $\eta_3 \circ \Sigma\alpha$ for $\alpha \in \pi_k(S^3)$ and η_3 has a lift. So, the first assertion of (i) implies (iii), completing the proof. □

Next we show

LEMMA 2.2. *Let $\alpha \in \pi_k(S^n)$ for $k \leq 3n-3$. Then $\Sigma\alpha$ is lifted to M^{n+1} if and only if there exists an element $\beta \in \pi_{k-n+1}(S^n)$ satisfying $2\iota_n \circ \alpha = 2[\iota_n, \beta]$.*

PROOF. By Lemma 1.2, we have an exact sequence

$$\pi_{k-n+1}(S^n) \xrightarrow{Q} \pi_{k+1}(M^{n+1}, S^n) \xrightarrow{p'_*} \pi_{k+1}(S^{n+1}).$$

Let $\alpha_1 \in \pi_{k+1}(M^{n+1})$ be a lift of $\Sigma\alpha$. By the exact sequence, there exists an element $\beta \in \pi_{k-n+1}(S^n)$ satisfying $j_*\alpha_1 = \hat{\alpha} + [\gamma_{n+1}, \beta]$, where $j : (M^{n+1}, *) \hookrightarrow (M^{n+1}, S^n)$ is the inclusion. We have $\partial[\gamma_{n+1}, \beta] = -2[\iota_n, \beta]$ and $\partial\hat{\alpha} = 2\iota_n \circ \alpha$. Since $\partial(\hat{\alpha} + [\gamma_{n+1}, \beta]) = 2\iota_n \circ \alpha - [2\iota_n, \beta] = 0$, the converse follows. This completes the proof. □

Here we need the information about the Mahowald element $\eta'_i \in \pi^S_{2^i}(S^0)$ for $i \geq 3$ ([**9**]). It satisfies the relation $H(\eta'_i) = \nu$ on S^{2^i-2} and so we have $[\iota_{n-2}, \nu_{n-2}] = 0$ if $n = 2^i - 1$ for $i \geq 3$. We set $\eta'_{i,n} = \eta'_i$ on S^n and $\eta'_{i,n+1} = \Sigma\eta'_{i,n}$. Making use of the EHP-sequence, we have the following two cases:

$$2\eta'_{i,n-1} = [\iota_{n-1}, \nu_{n-1}] \tag{1}$$

and there exists an element $\theta \in \pi_{2n-1}(S^{n-2})$ satisfying

$$2\eta'_{i,n-1} - [\iota_{n-1}, \nu_{n-1}] = \Sigma\theta \neq 0. \tag{2}$$

By Lemma 1.1, $2\iota_{n-1} \circ \eta'_{i,n-1} = 2\eta'_{i,n-1} + [\iota_{n-1}, \nu_{n-1}]$. So, by Lemma 2.2, $\eta'_{i,n}$ has a lift in the case (1) and does not have a lift in the case (2).

According to Mahowald, the order of $\eta'_i \in \pi^S_{2^i}(S^0)$ is 2 for $i = 3, 4, 5$; 4 for $i = 6$ and $4\eta'_i = 0$. We can take $\eta'_{3,6} = \bar{\nu}_6$ and $\eta'_{4,14} = \omega_{14}$. Since $2\bar{\nu}_6 = [\iota_6, \nu_6]$ and $2\omega_{14} = [\iota_{14}, \nu_{14}]$ ([**22**]), $\bar{\nu}_7$ and ω_{15} have lifts, respectively.

We recall the Mahowald-Thomeier result: $[\iota_{2n-1}, \nu_{2n-1}] = 0$ if $n = 2^i - 1$ or $n \equiv 0 \bmod 4$ and $[\iota_{2n-1}, \nu_{2n-1}]$ is of order 2 if otherwise. We show

LEMMA 2.3. (i) *$[\gamma_{2n+1}, \nu_{2n}]$ is of order 8 for $n \geq 3$.*

(ii) *$[\gamma_{2n}, \nu_{2n-1}]$ is of order 2 if $n = 2^i - 1$ or $n \equiv 0 \bmod 4$. $[\gamma_{2n}, \nu_{2n-1}]$ is of order 4 if otherwise.*

Proof. By Lemma 1.2, we have the exact sequence for $n \geq 3$:

$$\pi_{4n+4}(S^{2n+1}) \xrightarrow{H_2} \pi_{2n+3}(S^{2n}) \xrightarrow{Q} \pi_{4n+3}(M^{2n+1}, S^{2n}).$$

Since the order of $[\iota_{2n}, \nu_{2n}]$ is 4 or 8, we have $H\pi_{4n+4}(S^{2n+1}) \subset \{4\nu_{4n+1}\}$. So we have H_2 is trivial and Q is a monomorphism. This leads to (i).

Next we consider the exact sequence

$$\pi_{4n+2}(S^{2n}) \xrightarrow{H_2} \pi_{2n+2}(S^{2n-1}) \xrightarrow{Q} \pi_{4n+1}(M^{2n}, S^{2n-1}).$$

We have $\mathrm{Im}\, H = \boldsymbol{Z}_8\{\nu_{4n-1}\}$ if $n = 2^i - 1$ or $n \equiv 0 \bmod 4$ and $\mathrm{Im}\, H = \boldsymbol{Z}_4\{2\nu_{4n-1}\}$ if otherwise. This leads to (ii) and completes the proof. □

For the element κ_9 or σ_{15}^2, we know that $2\iota_9 \circ \kappa_9 = 2\kappa_9 \neq 0$ and $2\iota_{15} \circ \sigma_{15}^2 = 2\sigma_{15}^2 \neq 0$ ([**22**]). Since $2\iota_{4n} \circ \delta_{4n} = [\iota_{4n}, \eta_{4n}] \neq 0$ and $2\iota_{4n} \circ (\delta_{4n}\eta_{8n}) = [\iota_{4n}, \eta_{4n}^2] \neq 0$ by [**19**], we have the assertion of Theorem 2 except for the last two. We show the following result which completes the proof of Theorem 2.

Lemma 2.4. *There exists an element $\tau_{2n} \in \pi_{4n}(S^{2n})$ satisfying $\Sigma\tau_{2n} = [\iota_{2n+1}, \iota_{2n+1}]$ for $n \geq 4$. τ_{2n} satisfies $2\iota_{2n} \circ \tau_{2n} \neq 0$. Furthermore we have the following.*

(i) *Let n be even. Then $2\iota_{2n} \circ (\tau_{2n}\eta_{4n}) \neq 0$.*

(ii) *Let n be odd and $n \geq 5$. Then there exists an element $\tau' \in \pi_{4n-1}(S^{2n-1})$ satisfying $\Sigma\tau' = \tau_{2n}$ and $2\tau_{2n} = [\iota_{2n}, \eta_{2n}] \neq 0$.*

Proof. Let $\boldsymbol{C}\mathrm{P}^n$ the complex n-dimensional projective space and let $T : S^{2n} \to \Sigma\boldsymbol{C}\mathrm{P}^{n-1} \hookrightarrow SU(n)$ be the characteristic map in the unitary group $U(n)$. According to [**18**], the characteristic map T' in the rotation group $SO(2n+1)$ is obtained from T followed by the inclusion $SU(n) \hookrightarrow SO(2n+1)$. We know that the J-image of T' is just $[\iota_{2n+1}, \iota_{2n+1}]$. Then τ_{2n} is taken as the J-image of T followed by the inclusion $SU(n) \hookrightarrow SO(2n)$. By [**18**], $H(\tau_{2n}) = \eta_{4n-1}$ or 0 according as n even or odd. Since $2\pi_{2n}(SO(2n)) = 0$ for n even ([**7**]), τ_{2n} is of order 2 for n even. Then we have $2\iota_{2n} \circ \tau_{2n} = [\iota_{2n}, \eta_{2n}]$ and $2\iota_{2n} \circ (\tau_{2n}\eta_{4n}) = [\iota_{4n}, \eta_{4n}^2]$ by Lemma 1.1. By [**19**], these are nontrivial. This leads to (i).

Next assume that n is odd and $n \geq 5$. Then we consider the following natural map up to sign between exact sequences:

$$\begin{array}{ccccccc}
\pi_{2n+1}(S^{2n}) & \xrightarrow{\Delta'} & \pi_{2n}(SO(2n)) & \xrightarrow{i_*} & \pi_{2n}(SO(2n+1)) & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle \Sigma^{2n+1}} & & \big\downarrow{\scriptstyle J} & & \big\downarrow{\scriptstyle J} & & \\
\pi_{4n+2}(S^{4n+1}) & \xrightarrow{\Delta} & \pi_{4n}(S^{2n}) & \xrightarrow{\Sigma} & \pi_{4n+1}(S^{2n+1}) & \longrightarrow & 0.
\end{array}$$

We know $\pi_{2n}(SO(2n)) \cong \boldsymbol{Z}_4$ and $\pi_{2n}(SO(2n+1)) \cong \boldsymbol{Z}_2$ ([**7**]). This shows that τ_{2n} is taken as a suspended element and $2\tau_{2n} = [\iota_{2n}, \eta_{2n}] \neq 0$ ([**19**]). This leads to (ii) and completes the proof. $\square$

In the rest of the section, we shall prove Theorem 1 except the last one. First we consider the exact sequence induced from the fibration $p : M^n \to S^n$ for n odd:

$$\pi_n(S^n) \xrightarrow{\Delta} \pi_{n-1}(Y) \xrightarrow{i_*} \pi_{n-1}(M^n) \longrightarrow 0,$$

where $Y = S^{n-1} \cup_{2w_{n-1}} e^{2n-2}$. Since $\pi_{n-1}(M^n) \cong \boldsymbol{Z}_2$, we have $\Delta\iota_n = 2i'$ for the inclusion $i' : S^{n-1} \hookrightarrow Y$.

Here we reprove the existence of a lift of η_3. By Lemma 1.4, we have the exact sequence

$$\pi_4(S^3) \xrightarrow{\Delta} \pi_3(S^2 \cup_{4\eta_2} e^4) \xrightarrow{i_*} \pi_3(M^3).$$

We have $\pi_3(S^2 \cup_{4\eta_2} e^4) = \boldsymbol{Z}_4\{i'\eta_2\}$ and $\Delta\eta_3 = 2i' \circ \eta_2 = 4i'\eta_2 = 0$. This shows the existence of a lift of η_3.

Let $\widehat{\nu'} = \gamma_4 \widehat{\nu'}' \in \pi_7(M^4, S^3)$ be a lift of ν'. We show

Lemma 2.5. *ν_5^2 has a splitting lift.*

Proof. In the homotopy exact sequence of a pair (M^5, S^4):

$$\pi_{11}(S^4) \xrightarrow{i_{5*}} \pi_{11}(M^5) \xrightarrow{j_*} \pi_{11}(M^5, S^4) \xrightarrow{\partial} \pi_{10}(S^4),$$

there exists a lift $\alpha = (\hat{\nu}_4 + [\gamma_5, \iota_4]) \circ \hat{\nu}_7' \in \pi_{11}(M^5, S^4)$ of ν_5^2. We know $\nu'\nu_6 = 0$ ([**22**]), $\partial(\hat{\nu}_4 + [\gamma_5, \iota_4]) = \Sigma\nu'$ and $2(\hat{\nu}_4 + [\gamma_5, \iota_4]) = \Sigma'\widehat{\nu'}$ ([**14**]), where $\Sigma' : \pi_7(M^5, S^4) \to \pi_8(M^6, S^5)$ is the relative suspension ([**22**]). So we have $\partial\alpha = \Sigma\nu' \circ \nu_7 = 0$ and $2\alpha = \Sigma'\widehat{\nu'} \circ \hat{\nu}_7' = \gamma_5 \Sigma' \widehat{\nu'}' \circ \hat{\nu}_7' = 0$. Since $\pi_{11}(S^5) = \boldsymbol{Z}_2\{\nu_5^2\}$ and $\pi_{11}(S^4) \cong \boldsymbol{Z}_{15}$, there exists an element $\beta \in \pi_{11}(M^5)$ of order 2 satisfying $j_*\beta = \alpha$ and $p_{5*}\beta = \nu_5^2$. This completes the proof. $\square$

Next in the exact sequence

$$\pi_{14}(M^6) \xrightarrow{j_*} \pi_{14}(M^6, S^5) \xrightarrow{\partial} \pi_{13}(S^5) \xrightarrow{i_{6*}} \pi_{13}(M^6),$$

$4\bar{\nu}_6$ is lifted to an element of $\pi_{14}(M^6, S^5)$ by Lemma 1.2. Since $\pi_{13}(S^5) = \boldsymbol{Z}_2\{\varepsilon_5\}$ and $i_6\varepsilon_5$ survives stably, $4\bar{\nu}_6$ is lifted. By the parallel argument, $4\omega_{14}$ is lifted

We recall that the existence of a lift θ_1 of σ''' is ensured by [**16**] and [**14**]. We show

Lemma 2.6. *ρ^{IV} has a splitting lift θ_2.*

PROOF. We recall that $\rho^{IV} \in \{\sigma''', 2\iota_{12}, 8\sigma_{12}\}_1$. Since the lift θ_1 of σ''' is of order 2 ([**14**]), we can define an element $\theta_2 \in \{\theta_1, 2\iota_{12}, 8\sigma_{12}\}_1 \subset \pi_{20}(M^5)$. We have $p_5\theta_2 = \rho^{IV}$ and

$$2\{\theta_1, 2\iota_{12}, 8\sigma_{12}\}_1 = \theta_1\Sigma\{2\iota_{11}, 8\sigma_{11}, 2\iota_{18}\} \ni 0 \bmod 0.$$

This completes the proof. $\square$

Since $2\kappa_7 \equiv \bar{\nu}_7\nu_{15}^2 \bmod (\Sigma^2\sigma''')\sigma_{14}$ ([**22**]), $2\kappa_7$ has a lift. We shall show that there exist lifts of $4\nu_4^2$ and $8\sigma_8^2$, respectively.

Let $V_{n,2}$ be the Stiefel manifold of 2-frames in $\boldsymbol{R}^n$. We have a cell strucutre $V_{2n+1,2} = M^{2n} \cup_{\lambda_n} e^{4n-1}$. By [**13**], we have the following.

LEMMA 2.7. *The order of λ_n is 4 for n even and 8 for n odd. Furthermore λ_n satisfies*

$$\Delta(\tilde{\iota}_{4n}) = \pm 2\lambda_n, \quad \Sigma\lambda_n = i_{2n+1}[\iota_{2n}, \iota_{2n}] \quad \textit{and} \quad j_*\lambda_n = \pm[\gamma_{2n}, \iota_{2n-1}].$$

We show

LEMMA 2.8. (i) $\pi_6(M^4) = \boldsymbol{Z}_4\{\lambda_2\} \oplus \boldsymbol{Z}_2\{\tilde{\eta}_3\eta_5\}$ *and* $2\lambda_2 = i_4\nu'$.
(ii) $\pi_{14}(M^8) = \boldsymbol{Z}_4\{\lambda_4\} \oplus \boldsymbol{Z}_2\{\widetilde{\nu_7^2}\}$ *and* $2\lambda_4 = i_8\sigma'$, *where* $\widetilde{\nu_7^2}$ *is a coextension of* ν_7^2.

PROOF. We know (i) by [**14**]. In the exact sequence

$$\pi_{14}(S^7) \xrightarrow{i_{8*}} \pi_{14}(M^8) \xrightarrow{j_*} \pi_{14}(M^8, S^7) \xrightarrow{\partial} \pi_{14}(S^6),$$

we have $\pi_{14}(M^8, S^7) = \boldsymbol{Z}_2\{\widehat{\nu_7^2}\} \oplus \boldsymbol{Z}_2\{[\gamma_8, \iota_7]\}$. So we have $j_*\lambda_4 = [\gamma_8, \iota_7]$ and $2\lambda_4 = i_8\sigma'$. This completes the proof. $\square$

We show

LEMMA 2.9. $4\nu_4^2$ *and* $8\sigma_8^2$ *have lifts, respectively.*

PROOF. We only prove the second assertion. The first one is obtained by the parallel argument. Let F be a homotopy fiber of $p_8 : M^8 \to S^8$. Then, by Lemma 1.4, the 20-skeleton of F is $S^7 \vee S^{14}$. It suffices to prove $8\Delta(\sigma_8^2) = 0$ for the connecting map $\Delta : \pi_{22}(S^8) \to \pi_{21}(F)$. By Lemma 1.4, we have an exact sequence

$$\pi_{15}(S^8) \xrightarrow{\Delta} \pi_{14}(S^7 \vee S^{14}) \xrightarrow{i_*} \pi_{14}(M^8) \xrightarrow{p_*} \pi_{14}(S^8).$$

We have $\Delta(\Sigma\sigma') = 2i'\sigma'$ and $\Delta(\sigma_8) = 2\iota_{14} \pm i'\sigma'$ by Lemma 2.8.(ii). So we have $\Delta(\sigma_8^2) = 2\sigma_{14} \pm i'\sigma'\sigma_{14}$ and $\Delta(8\sigma_8^2) = 0$. This completes the proof. $\square$

3. Existence of a lift of the Thomeier element.

In this section, we construct an element $\delta'_n \in \pi_{2n+1}(S^n)$ for $n \equiv 3 \bmod 4$ satisfying

$$\Sigma\delta'_{4n-1} = [\iota_{4n}, \eta_{4n}]; \quad H(\delta'_{4n-1}) = \eta^2_{8n-3}; \quad 2\delta'_n = 0.$$

We set $\delta'_3 = \nu'\eta_6 = \eta_3\nu_4$ and $\delta'_7 = \sigma'\eta_{14}$. δ'_3 has a lift $\tilde{\eta}_2\nu_4$. Since $\eta_7\sigma_8 = \sigma'\eta_{14} + \bar{\nu}_7 + \varepsilon_7$, δ'_7 has a lift $\tilde{\eta}_6\sigma_8 + \tilde{\bar{\nu}}_6 + \tilde{\varepsilon}_6$, where $\tilde{\bar{\nu}}_6$ is a lift of $\bar{\nu}_7$ and $\tilde{\varepsilon}_6$ is a coextension of ε_6. Let n be odd and $n \geq 11$. Then, by Lemma 2.4, there exists an element $\tau' \in \pi_{2n-3}(S^{n-2})$ satisfying $2\Sigma\tau' = [\iota_{n-1}, \eta_{n-1}]$ and $\Sigma^2\tau' = [\iota_n, \iota_n]$. We show

Lemma 3.1. *Let $n \equiv 3 \bmod 4$ and $n \geq 11$. Then δ'_n is taken as a represetative of the Toda bracket $\{\Sigma^2\tau', 2\iota_{2n-1}, \eta_{2n-1}\}_1$.*

Proof. First we remark that τ' is taken as the J-image of a generator of $\pi_{2n}(SO(2n-1)) \cong \boldsymbol{Z}_8$ ([**7**]) and $\Sigma^2\tau' = Jq'_{2n}$, where $q'_{2n} \in \pi_{2n}(SO(2n+1)) \cong \boldsymbol{Z}_2$ is the composite of the embedding from the real $2n$-dimensional projective space $\boldsymbol{R}\mathrm{P}^{2n}$ to $SO(2n+1)$ and the covering map $q_{2n} : S^{2n} \to \boldsymbol{R}\mathrm{P}^{2n}$. The Toda bracket $\{q'_{2n}, 2\iota_{2n}, \eta_{2n}\}$ is well defined. We take δ'_{2n+1} as a representative of

$$J\{q'_{2n}, 2\iota_{2n}, \eta_{2n}\} \subset \{\Sigma^2\tau', 2\iota_{4n+1}, \eta_{4n+1}\}.$$

We have

$$\begin{aligned} 2\{q'_{2n}, 2\iota_{2n}, \eta_{2n}\} &= q'_{2n} \circ \{2\iota_{2n}, \eta_{2n}, 2\iota_{2n+1}\} \\ &= q'_{2n} \circ \eta^2_{2n} \\ &= 0, \end{aligned}$$

because $\pi_{2n+1}(SO(2n+1)) \cong \boldsymbol{Z}$. So we have $2\delta'_{2n+1} = 0$.

Since $p'q'_{2n} = 0$ for the projection map $p' : SO(2n+1) \to S^{2n}$, we have $p' \circ \{q'_{2n}, 2\iota_{2n}, \eta_{2n}\} = \eta_{2n+1}$. So we have $\Sigma\delta'_{2n+1} = [\iota_{2n+2}, \eta_{2n+2}]$.

Next we note that $\{q'_{2n}, 2\iota_{2n}, \eta_{2n}\} = \{q'_{2n}, 2\iota_{2n}, \eta_{2n}\}_1$. So, by Proposition 2.6 of [**22**], we have

$$H\{\Sigma^2\tau', 2\iota_{4n+1}, \eta_{4n+1}\}_1 = -\Delta^{-1}(2\Sigma\tau') \circ \eta_{4n+2}.$$

Since $\{\Sigma^2\tau', 2\iota_{4n+1}, \eta_{4n+1}\}_1 = \{\Sigma^2\tau', 2\iota_{4n+1}, \eta_{4n+1}\}$, we have $H(\delta'_{2n+1}) = \eta^2_{4n+1}$. This completes the proof. $\square$

Next we consider the fibration $p_n : M^n \to S^n$ for odd n. We recall that $Y = S^{n-1} \cup_{2w_{n-1}} e^{2n-2}$. Let $\tilde{\eta}'_{2n-3} \in \{i', 2w_{n-1}, \eta_{2n-3}\} \subset \pi_{2n-1}(Y)$ be a coextension of η_{2n-3}. By

Lemma 6.6 of [**17**], we know that $[\iota_{n-1}, \nu_{n-1}]$ is of order 4 or 8 according as $n \equiv 3 \bmod 4$ or $n \equiv 1 \bmod 4$. We show

Lemma 3.2. *Let n be odd and $n \geq 7$. Then*

(i) $\pi_{2n-2}(Y) \cong \pi_{2n-2}(S^{n-1})$.

(ii) $\pi_{2n-1}(Y) \cong \boldsymbol{Z}_2\{\tilde{\eta}'_{2n-3}\} \oplus \pi_{2n-1}(S^{n-1})$ *if* $n \equiv 3 \bmod 4$ *and* $\pi_{2n-1}(Y) = \boldsymbol{Z}_4\{\tilde{\eta}'_{2n-3}\} + i'_*\pi_{2n-1}(S^{n-1})$ *if* $n \equiv 1 \bmod 4$

(iii) *There exist independent elements* $i'[\iota_{n-1}, \nu_{n-1}]$ *and* $\tilde{\eta}'_{2n-3}\eta_{2n-1}$ *of order* 2 *in* $\pi_{2n}(Y)$.

(iv) *Let* $b(n) = 2$ *or* 1 *according as* $n \equiv 3 \bmod 4$ *or* $n \equiv 1 \bmod 4$. *Then* $\pi_{2n+1}(Y) \cong \boldsymbol{Z}_{2b(n)} \oplus \pi_{2n+1}(S^{n-1})$, *where the direct summand* $\boldsymbol{Z}_{2b(n)}$ *is generated by a coextension of* $(4/b(n))\nu_{2n-3}$.

Proof. Let $k = 2n-2$. Then, in the exact sequence

$$\pi_{k+1}(Y, S^{n-1}) \xrightarrow{\partial} \pi_k(S^{n-1}) \xrightarrow{i'_*} \pi_k(Y) \xrightarrow{j_*} \pi_k(Y, S^{n-1}) \xrightarrow{\partial} \pi_{k-1}(S^{n-1}),$$

$\pi_{k+1}(Y, S^{n-1}) = \boldsymbol{Z}_2\{\hat{\eta}_{2n-3}\}$ and $\pi_k(Y, S^{n-1}) = \boldsymbol{Z}\{\hat{\iota}_{2n-3}\}$. Since $\partial\hat{\iota}_{2n-3} = 2w_{n-1}$ and $\partial\hat{\eta}_{2n-3} = 0$, i'_* is an isomorphism and we have (i).

We consider the exact sequence

$$\pi_{2n}(Y, S^{n-1}) \xrightarrow{\partial} \pi_{2n-1}(S^{n-1}) \xrightarrow{i'_*} \pi_{2n-1}(Y) \xrightarrow{j_*} \pi_{2n-1}(Y, S^{n-1}) \xrightarrow{\partial} \pi_{2n-2}(S^{n-1}).$$

Since $\pi_{2n-1}(Y, S^{n-1}) \cong \pi_{2n-1}(S^{2n-2})$ and $\pi_{2n}(Y, S^{n-1}) \cong \pi_{2n}(S^{2n-2})$, ∂ are trivial. So j_* is an epimorphism. We have

$$2\tilde{\eta}'_{2n-3} \in i'[\iota_{n-1}, \iota_{n-1}] \circ \{2\iota_{2n-3}, \eta_{2n-3}, 2\iota_{2n-2}\} \ni i'[\iota_{n-1}, \iota_{n-1}] \circ \eta^2_{2n-3} \bmod 0.$$

Since $[\iota_{n-1}, \eta^2_{n-1}] = 0$ or $\neq 0$ according as $n \equiv 3$ or $1 \bmod 4$ ([**19**]), we have (ii).

In the exact sequence

$$\pi_{2n+1}(Y, S^{n-1}) \xrightarrow{\partial} \pi_{2n}(S^{n-1}) \xrightarrow{i'_*} \pi_{2n}(Y) \xrightarrow{j_*} \pi_{2n}(Y, S^{n-1}),$$

we have $\pi_{2n+1}(Y, S^{n-1}) = \boldsymbol{Z}_8\{\hat{\nu}_{2n-3}\}$ and $\pi_{2n}(Y, S^{n-1}) = \boldsymbol{Z}_2\{\hat{\eta}_{2n-3}\hat{\eta}'_{2n-2}\}$. Since $\partial\hat{\nu}_{2n-3} = 2[\iota_{n-1}, \nu_{n-1}]$ and $\partial\hat{\eta}_{2n-3}\hat{\eta}'_{2n-2} = 0$, we have (iii).

In the exact sequence

$$\pi_{2n+2}(Y, S^{n-1}) \xrightarrow{\partial} \pi_{2n+1}(S^{n-1}) \xrightarrow{i'_*} \pi_{2n+1}(Y) \xrightarrow{j_*} \pi_{2n+1}(Y, S^{n-1}),$$

we have $\pi_{2n+2}(Y, S^{n-1}) = 0$. If $[\iota_{n-1}, \nu_{n-1}]$ is of order 4, we can define a coextension of $2\nu_{2n-3}$ by an element of $\{i', 2w_{n-1}, 2\nu_{2n-3}\} \subset \pi_{2n+1}(Y)$. $[\iota_{n-1}, \nu_{n-1}]$ is of order 4 for

$n \equiv 3 \bmod 4$ ([**19**]). If $[\iota_{n-1}, \nu_{n-1}]$ is of order 8, we can define a coextension of η^3_{2n-3} by $\{i', 2w_{n-1}, \eta^3_{2n-3}\}$. To determine the orders of these coextensions, we define a coextension $\tilde{\eta}'_{2n-3}$ of η_{2n-3} by an element of $\{i'[\iota_{n-1}, \iota_{n-1}], 2\iota_{2n-3}, \eta_{2n-3}\} \subset \{i', 2w_{n-1}, \eta_{2n-3}\} \subset \pi_{2n-1}(Y)$. Since $2\nu_{2n-3} \in \{\eta_{2n-3}, 2\iota_{2n-2}, \eta_{2n-2}\} \bmod 4\nu_{2n-3}$, any element of the bracket $\{\tilde{\eta}'_{2n-3}, 2\iota_{2n-1}, \eta_{2n-1}\}$ is taken as a coextension of $2\nu_{2n-3}$ if $n \equiv 3 \bmod 4$. We have $2\{\tilde{\eta}'_{2n-3}, 2\iota_{2n-1}, \eta_{2n-1}\} \ni \tilde{\eta}'_{2n-3} \circ \eta^2_{2n-1} \bmod 0$. Therefore any element of the bracket $\{\tilde{\eta}'_{2n-3}, 2\iota_{2n-1}, \eta_{2n-1}\}$ is of order 4 if $n \equiv 3 \bmod 4$.

If $[\iota_{n-1}, \nu_{n-1}]$ is of order 8, then a coextension of η^3_{2n-3} is taken as $\tilde{\eta}'_{2n-3}\eta^2_{2n-1}$ of order 2. Hence we have a split exact sequence

$$0 \longrightarrow \pi_{2n+1}(S^{n-1}) \xrightarrow{i_*} \pi_{2n+1}(Y) \xrightarrow{p_*} \boldsymbol{Z}_{2b(n)} \longrightarrow 0$$

and we have (iv). This completes the proof. $\square$

We denote by $i'' : Y \hookrightarrow M^n$ the inclusion. We show

LEMMA 3.3. *Let n be odd and $n \geq 7$. Then*

(i) $ji''\tilde{\eta}'_{2n-3} = [\gamma_n, \eta_{n-1}]$. *Furthermore, for a suitable choice of $\tilde{\eta}'_{2n-3}$, $i''\tilde{\eta}'_{2n-3} = [\tilde{\eta}_{n-1}, \iota_n] \neq 0$ and $i''\tilde{\eta}'_{2n-3}\eta^2_{2n-1} \neq 0$.*

(ii) *Let $n \equiv 3 \bmod 4$. Then $\Delta\delta'_n \in i'_*\Sigma\pi_{2n-2}(S^{n-2}) \circ \eta_{2n-1}$.*

PROOF. We consider the natural map between exact sequences for $k = 2n - 1$:

$$\begin{array}{ccccccc}
\pi_k(S^{n-1}) & \xrightarrow{i'_*} & \pi_k(Y) & \xrightarrow{j'_*} & \pi_k(Y, S^{n-1}) & \xrightarrow{\partial} & \cdots \\
\Big\downarrow{=} & & \Big\downarrow{i''_*} & & \Big\downarrow{i'_*} & & \\
\pi_k(S^{n-1}) & \xrightarrow{i_*} & \pi_k(M^n) & \xrightarrow{j_*} & \pi_k(M^n, S^{n-1}) & \xrightarrow{\partial} & \cdots \\
 & & & & \Big\downarrow{p'_*} & & \\
 & & & & \pi_k(M^n, Y), & &
\end{array}$$

where the vertical sequence is the exact one induced from a triple (M^n, Y, S^{n-1}). We have $\pi_{2n-1}(Y, S^{n-1}) = \boldsymbol{Z}_2\{\hat{\eta}_{2n-3}\}$, $\partial\hat{\eta}_{2n-3} = 2w_n \circ \eta_{2n-3} = 0$ and $j'\tilde{\eta}'_{2n-3} = \hat{\eta}_{2n-3}$. Remark that $\pi_{2n-1}(M^n, Y) \cong \pi_{2n-1}(S^n)$ and so $p'_* : \pi_{2n-1}(M^n, S^{n-1}) \to \pi_{2n-1}(M^n, Y)$ is regarded as the map induced from the collapsing $p' : (M^n, S^{n-1}) \to (S^n, *)$. So we have $j_*i''_*\tilde{\eta}'_{2n-3} = [\gamma_n, \eta_{n-1}]$. By [**2**] and [**14**], we have

$$j_*[\tilde{\eta}_{n-1}, \iota_n] = [\gamma_n\hat{\eta}'_{n-1}, \iota_{n-1}] = [\gamma_n, \iota_{n-1}]\hat{\eta}'_{2n-3} = [\gamma_n, \eta_{n-1}]. \tag{3}$$

Therefore there exists an element $\alpha \in \pi_{2n-1}(S^{n-1})$ satisfying $i''_*\tilde{\eta}'_{2n-3} \equiv [\tilde{\eta}_{n-1}, \iota_n] \bmod i_*\alpha$. Hence, by a suitable choice of $\tilde{\eta}'_{2n-3}$, we have the second assertion of (i). Since

$[\gamma_n, \eta_n^3] = 4[\gamma_n, \nu_{n-1}] \neq 0$ by Lemma 2.3.(i), we have the third of (i). This leads to (i).

The assertion of (ii) for $n = 7$ is true since $\delta_7' = \sigma'\eta_{14}$ has a lift. Hereafter we assume that $n \geq 11$. We consider the exact sequence:

$$\pi_{2n+1}(M^n) \xrightarrow{p_{n*}} \pi_{2n+1}(S^n) \xrightarrow{\Delta} \pi_{2n}(Y) \xrightarrow{i_*''} \pi_{2n}(M^n).$$

By use of the EHP-sequence, we have $\pi_{2n+1}(S^n) = \boldsymbol{Z}_2\{\delta_n'\} \oplus \Sigma\pi_{2n}(S^{n-1})$ ([**19**]). By Theorem 5.2 of [**11**] and by Lemma 3.1, we have

$$\begin{aligned}
\Delta\delta_n' &\in \Delta\{\Sigma^2\tau', 2\iota_{2n-1}, \eta_{2n-1}\}_1 \\
&\subset \{\Delta\Sigma^2\tau', 2\iota_{2n-2}, \eta_{2n-2}\} \\
&= \{2i'\Sigma\tau', 2\iota_{2n-2}, \eta_{2n-2}\} \\
&\supset i'\{2\Sigma\tau', 2\iota_{2n-2}, \eta_{2n-2}\} \\
&= i'\{[\iota_{n-1}, \eta_{n-1}], 2\iota_{2n-2}, \eta_{2n-2}\} \\
&\supset i'[\iota_{n-1}, \iota_{n-1}]\{\eta_{2n-3}, 2\iota_{2n-2}, \eta_{2n-2}\} \\
&= 2i'[\iota_{n-1}, \iota_{n-1}]\nu_{2n-3} \\
&= 0 \bmod \pi_{2n-1}(Y) \circ \eta_{2n-1}.
\end{aligned}$$

So we have $\Delta\delta_n' \in \pi_{2n-1}(Y) \circ \eta_{2n-1}$. By Lemma 3.3.(ii), we have $\pi_{2n-1}(Y) \circ \eta_{2n-1} = \{\tilde{\eta}_{2n-3}'\eta_{2n-1}\} + i_*'\pi_{2n-1}(S^{n-1}) \circ \eta_{2n-1}$. Since $[\iota_{n-2}, \eta_{n-2}^2] \neq 0$ ([**19**]), we have $\pi_{2n-1}(S^{n-1}) = \Sigma\pi_{2n-2}(S^{n-2})$. This and (i) lead to (ii) and complete the proof. □

We show

Lemma 3.4. *$i_n[\iota_{n-1}, \nu_{n-1}] \neq 0$ in $\pi_{2n}(M^n)$ if n is odd and $n \geq 5$.*

Proof. Let $n \equiv 1 \bmod 4$. For $n = 5$, we have $[\iota_4, \nu_4] = [\iota_4, \iota_4]\nu_7$. So we have the assertion. Since $[\iota_{n-1}, \eta_{n-1}^2] \neq 0$ for $n \equiv 1 \bmod 4$ and $n \geq 9$ ([**19**]), we have $\pi_{2n+1}(S^n) = \Sigma\pi_{2n}(S^{n-1})$ in this case. Assume that $i_n[\iota_{n-1}, \nu_{n-1}] = 0$. Then we have $i'[\iota_{n-1}, \nu_{n-1}] \in \Delta(\pi_{2n+1}(S^n))$. So there exists an element $\beta \in \pi_{2n}(S^{n-1})$ satisfying $i'[\iota_{n-1}, \nu_{n-1}] = i'(2\iota_{n-1} \circ \beta)$. We know $\pi_{2n+1}(Y, S^{n-1}) = \boldsymbol{Z}_{24}\{\hat{\nu}_{2n-3}\}$ and $\partial\hat{\nu}_{2n-3} = 2[\iota_{n-1}, \nu_{n-1}]$. So, by the homotopy exact sequence of a pair (Y, S^{n-1}), we have the relation $\pm[\iota_{n-1}, \nu_{n-1}] = 2\iota_{n-1} \circ \beta$. Apply the Hopf homomorphism to this relation, we have $\pm 2\nu_{2n-3} = 4H(\beta)$. This is a contadiction and leads to the assertion in the case $n \equiv 1 \bmod 4$.

Next we consider the case $n \equiv 3 \bmod 4$. By [**19**], we have $\pi_{2n+1}(S^n) = \boldsymbol{Z}_2\{\delta_n'\} \oplus \Sigma\pi_{2n}(S^{n-1})$. Suppose that $i_n[\iota_{n-1}, \nu_{n-1}] = 0$. Then we have $i'[\iota_{n-1}, \nu_{n-1}] \in \Delta\pi_{2n+1}(S^n)$.

So, by Lemma 3.3.(ii), we have $i'[\iota_{n-1}, \nu_{n-1}] = i'(\Sigma\alpha \circ \eta_{2n-1} + 2\iota_{n-1} \circ \beta)$, where $\alpha \in \pi_{2n-2}(S^{n-2})$, $\beta \in \pi_{2n}(S^{n-1})$. So, by the parallel argument to the preceding case, we have $\pm[\iota_{n-1}, \nu_{n-1}] = \Sigma\alpha' \circ \eta_{2n-1} + 2\iota_{n-1} \circ \beta$. Apply the Hopf homomorphism to this relation, we have $\pm 2\nu_{2n-3} = 4H(\beta)$. This is a contadiction and completes the proof. $\square$

Here we recall the methods to determine the metastable homotopy groups of spheres ([**22**]). By use of (11.10) and Theorem 11.7 of [**22**], we have the following exact sequence for $i \leq 4m - 5$:

$$\pi_{i+1}(S^m) \xrightarrow{\Sigma^k} \pi_{i+k+1}(S^{m+k}) \xrightarrow{I} \pi_i(\Sigma^{m-1}\mathrm{P}_m^{m+k-1}) \xrightarrow{\Delta'} \pi_i(S^m) \longrightarrow \cdots, \tag{4}$$

where $\mathrm{P}^j = \boldsymbol{R}\mathrm{P}^j$ and $\mathrm{P}_m^{m+k-1} = \mathrm{P}^{m+k-1}/\mathrm{P}^{m-1}$.

Now we show

Lemma 3.5. *δ_n' has a lift if $n \equiv 3 \bmod 4$.*

Proof. It suffices to assume that $n \geq 11$. We apply (4) for $i = 2n+1$, $m = n-1$ and $k = \infty$. Then we have an exact sequence

$$0 \longrightarrow \pi_{n+2}^S(\mathrm{P}_{n-1}) \xrightarrow{\Delta'} \pi_{2n}(S^{n-1}) \xrightarrow{\Sigma^\infty} \pi_{n+1}^S(S^0) \xrightarrow{I} \pi_{n+1}^S(\mathrm{P}_{n-1}),$$

where $\mathrm{P}_{n-1} = \mathrm{P}^\infty/\mathrm{P}^{n-1}$. As is easily seen ([**8**]), $\pi_{n+1}^S(\mathrm{P}_{n-1}) = 0$, $\pi_{n+2}^S(\mathrm{P}_{n-1}) \cong \boldsymbol{Z}_4$ and this group is generated by ν. So the sequence becomes the exact sequence

$$0 \longrightarrow \boldsymbol{Z}_4\{[\iota_{n-1}, \nu_{n-1}]\} \longrightarrow \pi_{2n}(S^{n-1}) \xrightarrow{\Sigma^\infty} \pi_{n+1}^S(S^0) \longrightarrow 0.$$

By Lemma 1.2, there exists a lift $\delta'' \in \pi_{2n+1}(M^n, S^{n-1})$ of δ_n'. Consider the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & \pi_{n+2}(S^{n-1}) \arrow[d, "{[\gamma_n,]}"'] \arrow[dr, "{2[\iota_{n-1},]}"] & \boldsymbol{Z}_4\{[\iota_{n-1}, \nu_{n-1}]\} \arrow[d] & \\
\pi_{2n+1}(M^n) \arrow[r, "j_*"] \arrow[dr, "p_*"'] & \pi_{2n+1}(M^n, S^{n-1}) \arrow[r, "\partial"] \arrow[d, "p'_*"] & \pi_{2n}(S^{n-1}) \arrow[r, "i_*"] \arrow[d, "\Sigma^\infty"'] & \pi_{2n}(M^n) \arrow[d, "\Sigma^\infty"] \\
 & \pi_{2n+1}(S^n) \arrow[d, "H_2"'] & \pi_{n+1}^s(S^0) \arrow[r, "i_*"] \arrow[d] & \pi_{n+1}^s(M^1) \\
 & \pi_{n+1}(S^{n-1}) & 0. &
\end{tikzcd}$$

From the diagram, we see $\Sigma^\infty(\partial\delta'') \equiv 0 \bmod 2\pi_{n+1}^S(S^0)$. Since Σ^∞ is epimorphic, there exists an element $\alpha \in \pi_{2n}(S^{n-1})$ such that $\Sigma^\infty(\partial\delta'') = 2\Sigma^\infty(\alpha) = \Sigma^\infty(2\iota_{n-1} \circ \alpha)$. So $\partial(\delta'' - \hat{\alpha}) = \partial\delta'' - 2\iota_{n-1} \circ \alpha$ belongs to $\operatorname{Ker}\Sigma^\infty$. Therefore $\partial(\delta'' - \hat{\alpha}) = k[\iota_{n-1}, \nu_{n-1}]$ for some $k \in \boldsymbol{Z}$. By Lemma 3.4, we see that k is even. Thus we can choose δ'' as $\partial(\delta'' - \hat{\alpha}) = 0$. Then $p'(\delta'' - \hat{\alpha}) = \delta_n' - \Sigma\alpha$ has a lift and $0 = 2\iota_n \circ (\delta_n' - \Sigma\alpha) = 2\Sigma\alpha$, so $\Sigma^\infty(\partial\delta'') = 0$. We see that $\partial\delta'' = l[\iota_{n-1}, \nu_{n-1}]$ for some $l \in \boldsymbol{Z}$. Again, by Lemma 3.4, we get that l is even. Therefore we can choose δ'' so that $\partial\delta'' = 0$. Thus we have $\Delta(\delta_n') = 0$. This completes the proof. □

4. Proof of Theorem 3 and some results.

First we show Theorem 3.

Theorem 4.1. (i) *$\tilde{\eta}_{n-1}[\iota_{n+1}, \iota_{n+1}]$ is of order 4 if n is odd.*
(ii) *If $n \equiv 3 \bmod 4$ and $n \geq 11$, a lift β_n of δ_n' satisfies a relation*

$$2\beta_n \equiv \tilde{\eta}_{n-1}[\iota_{n+1}, \iota_{n+1}] \bmod \Sigma\pi_{2n}(M^{n-1})$$

and the order of β_n is 8.

Proof. For $n = 3$ and 5, $\tilde{\eta}_{n-1}[\iota_{n+1}, \iota_{n+1}]$ is of order 4 ([**25**]). Hereafter we assume $n \geq 7$.

In the exact sequence

$$\pi_{2n+2}(M^n) \xrightarrow{p_{n*}} \pi_{2n+2}(S^n) \xrightarrow{\Delta} \pi_{2n+1}(Y) \xrightarrow{i''_*} \pi_{2n+1}(M^n),$$

$\pi_{2n+2}(S^n) = \boldsymbol{Z}_2\{\delta_n'\eta_{2n+1}\} \oplus \Sigma\pi_{2n+1}(S^{n-1})$ if $n \equiv 3 \bmod 4$ and $\pi_{2n+2}(S^n) = \Sigma\pi_{2n+1}(S^{n-1})$ if $n \equiv 1 \bmod 4$. So, by Lemma 3.3, $\Delta\pi_{2n+2}(S^n) \subset i'_*\pi_{2n+1}(S^{n-1})$ and i''_* is a monomorphism on the direct summand $\boldsymbol{Z}_{2b(n)}$. By Lemmas 1.6 and 3.3.(i), we have

$$\begin{aligned} i''\tilde{\eta}'_{2n-3} \circ \eta^2_{2n-1} &= [\tilde{\eta}_{n-1}, i_n]\eta^2_{2n-1} \\ &= [\iota_n', i_n]\tilde{\eta}_{2n-3}\eta^2_{2n-1} \\ &= 2[\iota_n', \iota_n'] \circ \Sigma(\tilde{\eta}_{n-2} \wedge \tilde{\eta}_{n-2}) \\ &= 2(\tilde{\eta}_{n-1}[\iota_{n+1}, \iota_{n+1}]). \end{aligned}$$

Therefore $\tilde{\eta}_{n-1}[\iota_{n+1}, \iota_{n+1}]$ is of order 4 if n is odd and $n \geq 7$. This leads to (i).

We consider the commutative diagram $(p = p_n)$:

$$\begin{tikzcd}
\pi_{2n+1}(M^n) \arrow[r, "H"] \arrow[d, "\Sigma p_*"] & \pi_{2n+1}(\Sigma(M^{n-1} \wedge M^{n-1})) \arrow[d, "\Sigma(p \wedge p)_*"] \\
\pi_{2n+1}(S^n) \arrow[r, "H"] & \pi_{2n+1}(S^{2n-1}).
\end{tikzcd}$$

By Lemma 1.6, we have

$$H(\beta_n) \equiv \Sigma(\tilde{\eta}_{n-2} \wedge \tilde{\eta}_{n-2}) \bmod \tilde{\imath}_{2n-2}\nu_{2n-2}. \tag{5}$$

We have

$$H(\tilde{\eta}_{n-1}[\imath_{n+1}, \imath_{n+1}]) = \Sigma(\tilde{\eta}_{n-2} \wedge \tilde{\eta}_{n-2})H([\imath_{n+1}, \imath_{n+1}]) = 2\Sigma(\tilde{\eta}_{n-2} \wedge \tilde{\eta}_{n-2}).$$

Thus we have the equation (ii). Since the identity class $\imath_n'$ is of order 4 for $n \geq 3$, we have $4\Sigma\pi_{2n}(M^{n-1}) = 0$. This implies that $8\beta_n = 0$.

Now suppose that $4\beta_n = 0$. Then the equation (ii) implies that there exists an element $\alpha \in \pi_{2n}(M^n)$ such that

$$2\tilde{\eta}_{n-1}[\imath_{n+1}, \imath_{n+1}] = 2\Sigma\alpha.$$

Thus we have the equation

$$i''\tilde{\eta}_{2n-3}' \circ \eta_{2n-1}^2 = 2\Sigma\alpha.$$

Applying $j_* : \pi_{2n+1}(M^n) \to \pi_{2n+1}(M^n, S^{n-1})$ to this equation, we get

$$j_*i''\tilde{\eta}_{2n-3}' \circ \eta_{2n-1}^2 = 4[\gamma_n, \nu_{n-1}] \neq 0.$$

On the other hand, since

$$2\Sigma\alpha = 2\imath_n' \circ \Sigma\alpha = i_n \circ \eta_{n-1} \circ p_n \circ \Sigma\alpha \in \pi_{2n+1}(S^{n-1}),$$

we have $j_*(2\Sigma\alpha) = 0$. This is a contradiction. This completes the proof. □

Remark. Wu [**25**] obtained the groups $\pi_{2n+1}(M^n)$ for $n = 3$, 5 and 7. For $n = 3$, a lift $\tilde{\eta}_2\nu_4 \in \pi_7(M^3)$ of $\eta_3\nu_4$ is of order 8. We have relations $2\tilde{\eta}_2\nu_4 = \pm\tilde{\eta}_2[\imath_4, \imath_4] + \tilde{\eta}_2\Sigma\nu'$ and $4\tilde{\eta}_2\nu_4 = i_3\eta_2^2\nu_4 + \tilde{\eta}_2\Sigma\nu'$ in $\pi_7(M^3)$ ([**15**]).

We know $\pi_{15}(S^6) = \boldsymbol{Z}_2\{\nu_6^3\} \oplus \boldsymbol{Z}_2\{\mu_6\} \oplus \boldsymbol{Z}_2\{\eta_6\varepsilon_7\}$ and $\Sigma^\infty : \pi_{15}(S^6) \to \pi_9^S(S^0)$ is an isomorphism ([**22**]). We denote by $\tilde{\bar{\nu}}_6$ a lift of $\bar{\nu}_7$ and by $\tilde{\varepsilon}_6$ a lift of ε_7. Note that $2\tilde{\bar{\nu}}_6 = i_7\nu_6^3$ and $2\tilde{\varepsilon}_6 = i_7\eta_6\varepsilon_7$. We show

Proposition 4.2. *$\tilde{\eta}_6\Sigma\sigma' = 2(\tilde{\eta}_6[\imath_8, \imath_8] + \tilde{\bar{\nu}}_6 + \tilde{\varepsilon}_6)$, $2(\tilde{\eta}_6\sigma_8 + \tilde{\bar{\nu}}_6 + \tilde{\varepsilon}_6) = \pm\tilde{\eta}_6[\imath_8, \imath_8]$ and $4\tilde{\eta}_6\sigma_8 = \tilde{\eta}_6\Sigma\sigma' + 2(\tilde{\bar{\nu}}_6 + \tilde{\varepsilon}_6) = 2\tilde{\eta}_6[\imath_8, \imath_8] \neq 0$.*

Proof. By use of the fact $\eta_6\sigma' = 4\bar{\nu}_6 = 2[\iota_6, \nu_6]$, we have

$$\begin{aligned}
\tilde{\eta}_6 \Sigma\sigma' &\in \{i_7, 2\iota_6, \eta_6\}\Sigma\sigma' \\
&\subset \{i_7, 2\iota_6, \eta_6\sigma'\} \\
&= \{i_7, 2\iota_6, 2[\iota_6, \nu_6]\} \\
&\subset \{i_7, 2\iota_6 \circ [\iota_6, \iota_6], 2\nu_{11}\} \\
&= \{i_7, 4[\iota_6, \iota_6], 2\nu_{11}\} \\
&\supset i''\{i', 2[\iota_6, \iota_6], 4\nu_{11}\} \\
&\ni i''\tilde{\eta}'_{11}\eta_{13}^2 = 2\tilde{\eta}_6[\iota_8, \iota_8] \\
&\quad \bmod i_{7*}\pi_{15}(S^6) + \pi_{12}(M^7) \circ 2\nu_{12}.
\end{aligned}$$

In the exact sequence

$$\pi_{13}(S^7) \xrightarrow{\Delta} \pi_{12}(Y) \xrightarrow{i''_*} \pi_{12}(M^7) \xrightarrow{p_{7*}} \pi_{12}(S^7) = 0,$$

we have $\operatorname{Im}\Delta = 0$ since $\Delta(\nu_7^2) = 2i'\nu_6^2 = 0$. So, by Lemma 3.2.(i), we have $\pi_{12}(M^7) = \boldsymbol{Z}_2\{i_7\nu_6^2\}$. Hence we have

$$\tilde{\eta}_6\Sigma\sigma' \equiv 2\tilde{\eta}_6[\iota_8, \iota_8] \bmod i_{7*}\pi_{15}(S^6).$$

Since $\eta^2\sigma = \nu^3 + \eta\varepsilon$ in $\pi_9^S(S^0)$ ([**22**]), we have a relation

$$2\tilde{\eta}\sigma = 2\tilde{\bar{\nu}} + 2\tilde{\varepsilon}.$$

We have $\Sigma^\infty(\tilde{\eta}_6\Sigma\sigma') = 2\tilde{\eta}\sigma = i\eta^2\sigma$ in $\pi_{10}^S(M^2)$. Hence we have the first relation.

Since $\pm[\iota_8, \iota_8] = 2\sigma_8 - \Sigma\sigma'$, we have $\pm\tilde{\eta}_6[\iota_8, \iota_8] = (2\tilde{\eta}_6\sigma_8 - \tilde{\eta}_6\Sigma\sigma')$. This leads to the second relation.

The last relation is a direct consequence of the first two relations and Theorem 4.1.(i) for $n = 7$. This completes the proof. □

Remark. Since $\eta_{2n}[\iota_{2n+1}, \iota_{2n+1}] = [\eta_{2n}, \eta_{2n}] = [\iota_{2n}, \eta_{2n}^2] \neq 0$ for n even ([**19**]), $\tilde{\eta}_{2n-1}[\iota_{2n+1}, \iota_{2n+1}]$ is of order 2 for n even.

We have to show

Lemma 4.3. (i) $\pi_{23}(M^{11}) = \boldsymbol{Z}_8\{\beta_2\} \oplus \boldsymbol{Z}_2\{i_{11}\sigma_{10}\nu_{17}^2\}$, *where* β_2 *is a lift of* θ'.
(ii) $\pi_{22}(M^{10}) = \boldsymbol{Z}_2\{\alpha\} \oplus \boldsymbol{Z}_2\{i_{10}\sigma_9\nu_{16}^2\}$, *where* α *is a lift of* $2[\iota_{10}, \nu_{10}]$ *and* $\Sigma\alpha = 4\beta_2$.

PROOF. First we know the following ([22]):

$$\pi_{24}(S^{11}) = \boldsymbol{Z}_2\{\theta'\eta_{23}\} \oplus \boldsymbol{Z}_2\{\sigma_{11}\nu_{18}^2\}; \quad \pi_{23}(S^{11}) = \boldsymbol{Z}_2\{\theta'\}$$

and $\pi_{n+13}(S^n) = \boldsymbol{Z}_2\{\sigma_n\nu_{n+7}^2\}$ for $n = 9$ and 10. In the exact sequence

$$\pi_{24}(S^{11}) \xrightarrow{\varDelta} \pi_{23}(Y) \xrightarrow{i''_*} \pi_{23}(M^{11}) \xrightarrow{p_{11*}} \pi_{23}(S^{11}),$$

we have $\varDelta(\theta'\eta_{23}) = 0$, $\varDelta(\sigma_{11}\nu_{18}^2) = 2i'\sigma_{10}\nu_{17}^2 = 0$ and $\varDelta(\theta') = 0$ by Lemma 3.5. By Lemma 3.2.(iv), we have $\pi_{23}(Y) \cong \boldsymbol{Z}_4 \oplus \pi_{23}(S^{10})$. Hence, by Theorem 4.1, we have (i).

We know $\pi_{21}(S^9) = 0$ and $\pi_{22}(S^{10}) = \boldsymbol{Z}_4\{[\iota_{10}, \nu_{10}]\}$ ([22]). In the exact sequence

$$\pi_{23}(M^{10}, S^9) \xrightarrow{\partial} \pi_{22}(S^9) \xrightarrow{i_{10*}} \pi_{22}(M^{10}) \xrightarrow{j_*} \pi_{22}(M^{10}, S^9) \longrightarrow 0,$$

we have $\pi_{22}(M^{10}, S^9) \cong \pi_{22}(S^{10})$ and $\pi_{23}(M^{10}, S^9) \cong \pi_{23}(S^{10})$ by use of Lemma 1.2. So there exists a lift α of $2[\iota_{10}, \nu_{10}]$ and $\pi_{22}(M^{10})$ is generated by α and $i_{10}\sigma_9\nu_{16}^2$. The assumption $2\alpha = i_{10}\sigma_9\nu_{16}^2$ induces a relation $2\Sigma\alpha = i_{11}\sigma_{10}\nu_{17}^2$. This contradicts the result of (i), obtaining the group $\pi_{22}(M^{10})$.

Next we consider the exact sequence

$$\pi_{24}(\Sigma(M^{10} \wedge M^{10})) \xrightarrow{\varDelta} \pi_{22}(M^{10}) \xrightarrow{\Sigma} \pi_{23}(M^{11}).$$

By use of the fact $[\iota_{11}, \iota_{11}] = \sigma_{11}\nu_{18}$, by Lemmas 1.3 and 1.6.(iii), we have $\varDelta(\Sigma(\tilde{\eta}_9 \wedge \tilde{\eta}_9) \circ \eta_{23}) = \tilde{\eta}_9[\iota_{11}, \iota_{11}]\eta_{21} = 0$. So Σ is a monomorphism. Hence, by (i), we have the relation of (ii). This completes the proof. $\square$

PROBLEM 4.4. (i) $\varDelta\delta_{4n} = i'\delta'_{4n-1}$?

(ii) *Do the elements* ε_3, μ_3, $\bar{\varepsilon}_3$, $\bar{\mu}_3$ *and* $\bar{\sigma}_7$ *have lifts*?

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Added in proof.

In the proof of Lemma 3.1 on p. 10, the statement of lines 21 and 22 is false. This part should be revised as follows. The statement of the last paragraph of the proof becomes unnecessary.

Since $p'_{2n}q'_{2n} = 0$ for the projection map $p'_{2n} : SO(2n+1) \to S^{2n}$, we have $p'_{2n} \circ \{q'_{2n}, 2\iota_{2n}, \eta_{2n}\} = -\{p'_{2n}, q'_{2n}, 2\iota_{2n}\} \circ \eta_{2n+1} = \eta_{2n}^2$ because $\{p'_{2n}, q'_{2n}, 2\iota_{2n}\} = \eta_{2n}$ for n odd, which is obtained by Lemma 4.1.(ii).a) of Mukai (Even maps from spheres to spheres, Proc. Japan Acad. **47**(1971), 1–5). So we have $H(\delta'_{2n+1}) = HJ\{q'_{2n}, 2\iota_{2n}, \eta_{2n}\} = \eta_{4n+1}^2$.

Let $i : SO(2n+1) \to SO(2n+2)$ be the inclusion. We know $\pi_{2n+1}(SO(2n+2)) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$ ([7]). We denote by $p'_{2n+1} : SO(2n+2) \to S^{2n+1}$ and $q'_{2n+1} : S^{2n+1} \to SO(2n+2)$ the corresponding elements to p'_{2n} and q'_{2n}, respectively. Since $p'_{2n+1}q'_{2n+1} = 2\iota_{2n+1}$, q'_{2n+1} is taken as a representative of a Toda bracket $\{i, q'_{2n}, 2\iota_{2n}\}$. Hence we have $\Sigma\delta'_{2n+1} = J(i\{q'_{2n}, 2\iota_{2n}, \eta_{2n}\}) = J(-\{i, q'_{2n}, 2\iota_{2n}\} \circ \eta_{2n}) = [\iota_{2n+2}, \iota_{2n+2}] \circ \eta_{4n+3} = [\iota_{2n+2}, \eta_{2n+2}]$ because $Jq'_{2n+1} = [\iota_{2n+2}, \iota_{2n+2}]$. This completes the proof.