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Transversals for association schemes

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Abstract. Generalizing a well-known group-theoretical notion we define *transversals* for (association) schemes. Two results on transversals of schemes are offered. Firstly, we show that a closed subset in a scheme possesses a factor scheme if it possesses a transversal. Secondly, we characterize the Coxeter schemes in terms of transversals. (Coxeter schemes are exactly those schemes which can be identified with the buildings in the sense of Tits). The second result may be viewed as a "thick version" of the characterization of Coxeter groups by the existence of "minimal coset representatives". On the other hand, the characterizing conditions given in this result are similar to the well-known "gate property" defined for chamber systems having a Coxeter matrix as type. Thus, our second main result may be viewed as a unified treatment of these two results.

1. Introduction.

Let (X, G) be an association scheme.[1]

Let H be a closed subset of G, and let T be a subset of G. We shall say that T is a *transversal* of H *in* X if, for any two elements y and z in X, $|yT \cap zH| = 1$. If, for each element g in G, $|T \cap gH| = 1$, we shall say that T is a *transversal* of H *in* G.

It is straightforward that, via [**6**; Theorem A], both types of transversals, transversals in X as well as transversals in G, generalize naturally the well-known group-theoretical concept of (left) transversals (cf., e.g., [**2**]).

It is easy to see that each transversal of a closed subset H, say, in X is a transversal of H in G (cf., e.g., Theorem 1). However, the converse of this does not hold. We shall say more about the relationship between the two types of transversals in Theorem 1 and in Theorem 3. Apart from these two (elementary) theorems, the present note contains two main results in which we study the connection between transversals of closed subsets of G and the structure of (X, G) itself.

The first of these main result is Theorem 2. This theorem says that, for each closed subset H, say, of G, the scheme (X, G) possesses a factor scheme over H if H possesses a

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[1] For the basic notation and terminology concerning association schemes, the reader is referred to [**6**].

transversal in X. (It seems that there are only a few general conditions which guarantee that a closed subset of G gives rise to a factor scheme of (X, G). One such condition was given in [**5**; (1.9)].)

The second main result (Theorem 4) is a characterization of the Coxeter schemes in terms of transversals. Coxeter schemes were defined in [**6**]. They are exactly those schemes which, via [**6**; Theorem E], correspond to the (regular) buildings in the sense of Tits (cf., e.g., [**4**]). Using [**6**; Theorem A], our second main result may be viewed as a "thick version" of the characterization of Coxeter groups by the existence of "minimal coset representatives". On the other hand, the characterizing conditions given in this result are similar to the "gate property" defined in [**3**] for chamber systems having a Coxeter matrix as type.[2] Thus, our second main result may be viewed as a unified treatment of these two results.

We conclude this note with a corollary of the two main results. This corollary (which is due to M. Rassy) sheds some light on the exceptional character of the Coxeter schemes within the theory of association schemes.

2. The results.

Let us briefly fix the basic notation and terminology which we shall use throughout this note. (It is borrowed from [**6**].)

Let X be a set.

We define

$$1_X := \{(x, x) \mid x \in X\}.$$

Let r denote a subset of $X \times X$. We set

$$r^* := \{(y, z) \mid (z, y) \in r\}.$$

For each element x of X, we set

$$xr := \{y \in X \mid (x, y) \in r\}.$$

Let G be a partition of $X \times X$ such that $\emptyset \notin G$ and $1_X \in G$. Let us assume that, for each element g in G, g^* is an element of G. Then the pair (X, G) will be called an *association scheme* if, for any three elements d, e, and f in G, there exists a cardinal

[2] In fact, when translating our proof (more precisely the proofs of our Proposition 2 and our Proposition 3) into the language of chamber systems, we obtain an alternate (and, as we believe, a technically less complicated) proof of the main result of [**3**]. Let us also mention here that the main result of [**3**] as stated there is not correct. The definition of "gated stars" as given there leads to counterexamples. On the other hand, the main result of [**3**] remains true if one changes the definition of "gated stars" according to our definition of $T_L(N)$ as given below.

number a_{def} such that, for any two elements y and z in X, $(y,z) \in f$ implies that $|yd \cap ze^*| = a_{def}$.[3]

In the following, we shall always say "scheme" instead of "association scheme".

Let (X, G) be a scheme.

We shall always write 1 instead of 1_X.

For any two elements e and f in G, we define

$$ef := \{g \in G \mid a_{efg} \neq 0\}.$$

Let F denote a subset of G such that $\emptyset \neq F$. We shall say that F is *closed* if, for any two elements d and e in F, d^*e is a subset of F.

Now we are ready to state our first result.

Theorem 1. *Let (X, G) be a scheme, and let H denote a closed subset of G. Then, for each subset T of G, the following conditions are equivalent.*

(a) *T is a transversal of H in X.*

(b) *T is a transversal of H in G which, for each element t in T, satisfies $\{1\} = t^*t \cap H$.*

Again, let (X, G) be a scheme.

Let F denote a subset of G. For each element B in $\{X, G\}$ and, for each element b in B, we define

$$bF := \bigcup_{f \in F} bf.$$

For each subset E of G, we define

$$EF := \bigcup_{e \in E} eF.^4$$

Let H denote a closed subset of G. We set

$$X/H := \{xH \mid x \in X\}.$$

For each element g in G, we set

$$g^H := \{(yH, zH) \mid z \in yHgH\}.$$

We set

$$G//H := \{g^H \mid g \in G\}$$

[3] Let us emphasize here that this definition of association schemes is more general than the usual one (cf., e.g., [**1**]).

[4] This defines a multiplication on the power set of G. We shall call this multiplication the *complex multiplication*.

and

$$(X, G)^H := (X/H, G//H).$$

In general, it seems to be unknown whether or not the pair $(X, G)^H$ is a scheme. However, if $|X| \in \boldsymbol{N}$, $(X, G)^H$ is a scheme; see [**5**; (1.9)]. The following theorem offers another sufficient condition for $(X, G)^H$ to be a scheme.

Theorem 2. *Let (X, G) be a scheme, and let H denote a closed subset of G. Assume that H possesses a transversal in X. Then $(X, G)^H$ is a scheme.*

Once again, let (X, G) be a scheme.

For each subset F of G, we shall denote by $\langle F \rangle$ the intersection of all closed subsets of G which contain F.

We set

$$\mathrm{Inv}(G) := \{g \in G \mid |\langle \{g\} \rangle| = 2\}.$$

Let L denote a subset of $\mathrm{Inv}(G)$.

Let g denote an element of $\langle L \rangle$. Then, by [**6**; Theorem 1.4.1(i)] and [**6**; Lemma 1.4.5(i)],

$$\varnothing \neq \{j \in \boldsymbol{N} \mid g \in L^j\}.$$

We set

$$\mu_L(g) := \min\{j \in \boldsymbol{N} \mid g \in L^j\}.$$

For each subset F of G, we abbreviate $\mu_L(F) := \{\mu_L(f) \mid f \in F\}$.

For each subset N of L, we define

$$T_L(N) := \bigcap_{h \in \langle N \rangle} \{g \in \langle L \rangle \mid \{\mu_L(g) + \mu_N(h)\} = \mu_L(gh)\}.$$

Theorem 3. *Let (X, G) be a scheme, and let L denote a subset of* $\mathrm{Inv}(G)$. *Then, for each subset N of L, $T_L(N)$ is a transversal of $\langle N \rangle$ in X if and only if $T_L(N)$ is a transversal of $\langle N \rangle$ in G.*

The following theorem is the second main result of the present note.[5]

Theorem 4. *Let (X, G) be a scheme, and let L be a subset of* $\mathrm{Inv}(G)$ *such that (X, G) is L-constrained. Then the following conditions are equivalent.*

(a) *For each subset N of L, $T_L(N)$ is a transversal of $\langle N \rangle$ in G.*

(b) *For each subset N of L with $|N| \leq 2$, $T_L(N)$ is a transversal of $\langle N \rangle$ in G.*

(c) *(X, G) is a Coxeter scheme with respect to L.*

[5] "L-constrained" schemes as well as "Coxeter schemes with respect to L" were defined in [**6**; Section 5.1]. We shall repeat the definition in the beginning of Section 7.

Recall that, by [**6**; Theorem E], there exists a natural way to identify Coxeter schemes and (regular) buildings. Thus, Theorem 4 can be viewed as an algebraic characterization of (regular) buildings.

Note that Theorem 4 remains true if in condition (a) (or (b)) we replace the letter "G" with the letter "X". (This follows from Theorem 3.)

Corollary. *Let* (X, G) *be a scheme, and let* L *be a subset of* $\mathrm{Inv}(G)$ *such that* (X, G) *is a Coxeter scheme with respect to* L. *Then, for each subset* N *of* L, *we have the following.*

(i) *For each element* g *in* G, $|g\langle N\rangle| = |\langle N\rangle|$.

(ii) $|\langle N\rangle|$ *divides* $|G|$.

(iii) $(X, G)^{\langle N\rangle}$ *is a scheme.*

Let us conclude this section with an elementary observation the proof of which will be left to the reader. Let (X, G) be a scheme, let L denote a subset of $\mathrm{Inv}(G)$, and let N denote a subset of L. We define

$$U_L(N) = \bigcap_{l \in N} \{g \in \langle L\rangle \mid \{\mu_L(g) + 1\} = \mu_L(gl)\}.$$

It follows immediately from the definitions of $T_L(N)$ and $U_L(N)$ that $T_L(N) \subseteq U_L(N)$. However, if (X, G) is a Coxeter scheme with respect to L, we have $T_L(N) = U_L(N)$.

3. The proof of Theorem 1.

Let T denote a subset of G.

We first shall prove that condition (a) of Theorem 1 implies condition (b) of that theorem. In order to do so, we assume that T is a transversal of H in X.

Let g denote an element in G. We first wish to show that $|T \cap gH| = 1$.

Let y and z be elements in X such that $(y, z) \in g$.

First of all, as T is assumed to be a transversal of H in X, we have $\emptyset \neq yT \cap zH$. Thus, as $(y, z) \in g$, [**6**; Lemma 1.2.4] implies that $g \in TH$. Therefore, there exists an element t, say, in T such that $g \in tH$. Now, by [**6**; Lemma 1.2.5(i)], $t \in gH$. In particular, as $t \in T$, $1 \leq |T \cap gH|$.

We still have to show that $|T \cap gH| \leq 1$. Let r and s denote two elements in $T \cap gH$. We shall be done if we succeed in showing that $r = s$.

Since $r, s \in gH$, we have $g \in rH \cap sH$; see [**6**; Lemma 1.2.5(i)]. Since $g \in rH$ and $(y, z) \in g$, [**6**; Lemma 1.2.4] implies that $\emptyset \neq yr \cap zH$. Similarly, we obtain from $g \in sH$ and $(y, z) \in g$ that $\emptyset \neq ys \cap zH$. Let v denote an element in $yr \cap zH$, and let w denote an element in $ys \cap zH$. Since $r \in T$, $v \in yT \cap zH$. Similarly, as $s \in T$, $w \in yT \cap zH$. Thus, as T is assumed to be a transversal of H in X, $v = w$. It follows that $r = s$.

Thus, we have proved that T is a transversal of H in G. Let us now show that, for each element t in T, $\{1\} = t^*t \cap H$.

It is clear that $1 \in t^*t \cap H$. In order to show that $t^*t \cap H \subseteq \{1\}$, let us denote by g an element in $t^*t \cap H$. We have to prove that $g = 1$.

Let y and z be elements in X such that $(y, z) \in g$. Then, as $g \in t^*t$, [**6**; Lemma 1.2.4] implies that $\emptyset \neq yt^* \cap zt^*$. Let x denote an element in $yt^* \cap zt^*$. Then y, $z \in xT \cap yH$. (Recall that $g \in H$.) Thus, as T is assumed to be a transversal of H in X, $y = z$. It follows that $g = 1$.

Let us now prove that condition (b) of Theorem 1 implies (a) of that theorem. In order to do so, we assume that T is a transversal of H in G.

Let y and z denote two elements in X. We wish to show that $|yT \cap zH| = 1$.

Let g denote the uniquely determined element in G which satisfies $(y, z) \in g$. Since T is assumed to be a transversal of H in G, we have $\emptyset \neq T \cap gH$. Let t denote an element in $T \cap gH$. Since $t \in gH$, $g \in tH$; see [**6**; Lemma 1.2.5(i)]. Thus, as $(y, z) \in g$, we obtain from [**6**; Lemma 1.2.4] that $\emptyset \neq yt \cap zH$. In particular, as $t \in T$, $1 \leq |yT \cap zH|$.

We still have to show that $|yT \cap zH| \leq 1$. Let v and w denote two elements in $yT \cap zH$. We shall be done if we succeed in showing that $v = w$.

Let d (respectively e, f) denote the uniquely determined element in G which satisfies $(v, w) \in d$ (respectively $(y, v) \in e$, $(y, w) \in f$). Then we have $d \in H$ and $e, f \in T$. Moreover, we have $f \in ed$. Thus, as $|T \cap eH| \leq 1$, $e = f$. It follows that $e \in ed$. Thus, by [**6**; Lemma 1.2.5(ii)], $d^* \in e^*e$. Now we have $d^* \in e^*e \cap H$. Thus, by hypothesis, $d^* = 1$. It follows that $v = w$.

4. The proof of Theorem 2.

By [**6**; Proposition 1.5.3], $G//H$ is a partition of $X/H \times X/H$. Moreover, from [**6**; Theorem 1.3.1], we derive easily that $1_{X/H} \in G//H$. Finally, it follows from [**6**; Lemma 1.2.5(i)] that, for each element g in G, $(g^H)^*$ is an element of $G//H$.

Let us now denote by d, e and f three elements of G, and let us denote by y and z elements in X such that $(y, z) \in f$. In order to prove Theorem 2, we have to show that $|(yH)(d^H) \cap (zH)(e^H)^*|$ does not depend on the choice of the pair $(yH, zH) \in f^H$.

By hypothesis, H possesses a transversal T, say, in X. We set

$$U := y(T \cap H\,dH) \cap zHe^*H.$$

Then, as $(y, z) \in f$,

$$|U| = \sum_{b \in T \cap HdH} \sum_{c \in HeH} a_{bcf}.$$

On the other hand, we claim that

$$|U| = |(yH)(d^H) \cap (zH)(e^H)^*|.$$

First of all, it is clear that, for each element x in U,

$$xH \in (yH)(d^H) \cap (zH)(e^H)^*.$$

Let v and w denote two elements in U which satisfy $vH = wH$. Then, as $U \subseteq yT$, we must have $v, w \in yT$. On the other hand, $vH = wH$ says that $v \in wH$. It follows that $v, w \in yT \cap wH$. Thus, as T is a transversal of H in X, $v = w$.

Now we have shown that

$$|(yH)(d^H) \cap (zH)(e^H)^*| = \sum_{b \in T \cap HdH} \sum_{c \in HeH} a_{bcf}.$$

In particular, $|(yH)(d^H) \cap (zH)(e^H)^*|$ does not depend on the choice of the pair $(yH, zH) \in f^H$.

5. The proof of Theorem 3.

Let N denote a subset of L, and let t be an element of $T_L(N)$. By Theorem 1, we just have to show that $t^*t \cap \langle N \rangle \subseteq \{1\}$.

Let g denote an element in $t^*t \cap \langle N \rangle$. Since $g \in t^*t$, $t \in tg^*$; see [**6**; Lemma 1.2.5(ii)]. Thus, as $g \in \langle N \rangle$, the definition of $T_L(N)$ yields $\mu_L(t) + \mu_N(g^*) = \mu_L(t)$. It follows that $\mu_N(g^*) = 0$, so that we have $g^* = 1$. It follows that $g = 1$.

6. Free monoids and Coxeter maps.

Let L denote a set.

We shall denote by $\mathbf{F}(L)$ the free monoid over L. The multiplication of $\mathbf{F}(L)$ will be denoted by $*$, the unit element by $\mathbf{1}$.

A map from $\{N \subseteq L \mid |N| = 2\}$ to $(\boldsymbol{N} \backslash \{0, 1\}) \cup \{\aleph_0\}$ is called a *Coxeter map of L*.

We shall denote by $\mathbf{v}$ the set of pairs

$$(\mathbf{d} * l * l * \mathbf{e}, \mathbf{d} * \mathbf{e})$$

such that $l \in L$ and $\mathbf{d}, \mathbf{e} \in \mathbf{F}(L)$.

Let h and k be elements of L such that $h \neq k$. Let n be an element in $\boldsymbol{N}$. If $0 = n$, we set $\mathbf{f}_n(h,k) := \mathbf{1}$. If $1 \leq n$ and if $l_1, \dots, l_n$ are elements of $\{h, k\}$ such that, for each element i in $\{1, \dots, n\}$, $l_i = h$ if and only if i is odd, we define

$$\mathbf{f}_n(h,k) := l_1 * \cdots * l_n.$$

Let m denote a Coxeter map of L.

If $m(\{h,k\}) \in \boldsymbol{N}$, we abbreviate

$$\mathbf{f}_m(h,k) := \mathbf{f}_{m(\{h,k\})}(h,k).$$

We shall denote by $\mathbf{w}_m$ the set of pairs

$$(\mathbf{d} * \mathbf{f}_m(h,k) * \mathbf{e}, \mathbf{d} * \mathbf{f}_m(k,h) * \mathbf{e})$$

with $h, k \in L$, $h \neq k$, $m(\{h,k\}) \in \boldsymbol{N}$, and $\mathbf{d}, \mathbf{e} \in \mathbf{F}(L)$.

By $\langle \mathbf{w}_m \rangle$ we shall denote the smallest equivalence relation on L containing $\mathbf{w}_m$. Let $\mathbf{d}$ and $\mathbf{e}$ be elements of $\mathbf{F}(L)$. We occasionally shall write $\mathbf{d} \sim_m \mathbf{e}$ instead of $(\mathbf{d}, \mathbf{e}) \in \langle \mathbf{w}_m \rangle$.

We set

$$\mathbf{x}_m := \mathbf{w}_m \cup \mathbf{v},$$

and we shall denote by $\langle \mathbf{x}_m \rangle$ the smallest equivalence relation on L containing $\mathbf{x}_m$.

We shall denote by λ the uniquely determined monoid homomorphism from $\mathbf{F}(L)$ to the additive monoid $\boldsymbol{N}$ satisfying $L\lambda \subseteq \{1\}$.

We set

$$\mathbf{F}_m(L) := \{\mathbf{f} \in \mathbf{F}(L) \mid \min(\mathbf{f}\langle \mathbf{x}_m \rangle)\lambda = \mathbf{f}\lambda\}.^6$$

For each subset N of L, we abbreviate $\mathbf{F}_m(N) := \mathbf{F}(N) \cap \mathbf{F}_m(L)$.

7. The proof of Theorem 4.

Let (X, G) denote a scheme.

For any two elements h and k of $\mathrm{Inv}(G)$ with $h \neq k$, we set

$$M_G(\{h,k\}) := \{n \in \boldsymbol{N} \backslash \{0\} \mid 1 \in (hk)^n\}$$

and

$$m_G(\{h,k\}) := \begin{cases} \min M_G(\{h,k\}) & \text{if } M_G(\{h,k\}) \neq \emptyset \\ \aleph_0 & \text{if } M_G(\{h,k\}) = \emptyset. \end{cases}$$

Let T denote a subset of $\mathrm{Inv}(G)$.

By [**6**; Lemma 1.2.1(ii)], the power set $\mathbf{R}(G)$ of G is a monoid with respect to the complex multiplication. We shall denote by ρ_L the uniquely determined monoid homomorphism ρ from $\mathbf{F}(L)$ to $\mathbf{R}(G)$ such that, for each element l of $L, l\rho = \{l\}$.

We shall denote by m the restriction of m_G to $\{N \subseteq L \mid |N| = 2\}$. It is obvious that m is a Coxeter map of L.

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6 In the literature, the elements of $\mathbf{F}_m(L)$ are sometimes called *m-reduced*.

Assume that $\langle L \rangle = G$. The pair (X, G) will be called *L-constrained* if, for each element $\mathbf{f}$ of $\mathbf{F}_m(L)$, $|\mathbf{f}\rho| = 1$.

Assume that (X, G) is L-constrained. The pair (X, G) will be called a *Coxeter scheme with respect to L* if, for any two elements $\mathbf{d}$ and $\mathbf{e}$ of $\mathbf{F}_m(L)$, $\mathbf{d}\rho = \mathbf{e}\rho$ implies that $\mathbf{d} \sim_m \mathbf{e}$.

Lemma 1. *Let (X, G) be a scheme, and let L be a subset of* $\mathrm{Inv}(G)$ *such that (X, G) is a Coxeter scheme with respect to L. Then, for each subset N of L, we have the following.*

(i) *For each element g in $\langle N \rangle$, $\mu_L(g) = \mu_N(g)$.*

(ii) *For each element $\mathbf{f}$ in $\mathbf{F}_m(N)$, there exists an element g in $\langle N \rangle$ such that $\mu_N(g) = \mathbf{f}\lambda$ and $\{g\} = \mathbf{f}\rho$.*

Proof. (i) Let N denote a subset of L, and let g denote an element in $\langle N \rangle$. Then, by [**6**; Proposition 5.1.3(ii)], there exists an element $\mathbf{d}$ in $\mathbf{F}_m(N)$ such that $\mu_N(g) = \mathbf{d}\lambda$ and $\{g\} = \mathbf{d}\rho$.

By [**6**; Proposition 5.1.3(ii)], there also exists an element $\mathbf{e}$ in $\mathbf{F}_m(L)$ such that $\mu_L(g) = \mathbf{e}\lambda$ and $\{g\} = \mathbf{e}\rho$.

From $\{g\} = \mathbf{d}\rho$ and $\{g\} = \mathbf{e}\rho$ we obtain that $\mathbf{d}\rho = \mathbf{e}\rho$. Thus, as (X, G) is assumed to be a Coxeter scheme with respect to L, $\mathbf{d} \sim_m \mathbf{e}$. In particular, $\mathbf{d}\lambda = \mathbf{e}\lambda$. Now the claim follows from $\mu_N(g) = \mathbf{d}\lambda$ and $\mu_L(g) = \mathbf{e}\lambda$.

(ii) Let N denote a subset of L, and let $\mathbf{f}$ denote an element of $\mathbf{F}_m(N)$. Since $\mathbf{f} \in \mathbf{F}_m(L)$, there exists an element g in $\mathbf{f}\rho$ such that $\{g\} = \mathbf{f}\rho$. Thus, by [**6**; Theorem 5.1.5], $\mu_L(g) = \mathbf{f}\lambda$.

From $\mathbf{f} \in \mathbf{F}(N)$ we obtain that $\mathbf{f}\rho \subseteq \langle N \rangle$; use [**6**; Lemma 5.1.1] and [**6**; Theorem 1.4.1(i)]. Thus, as $g \in \mathbf{f}\rho$, $g \in \langle N \rangle$.

From $g \in \langle N \rangle$ and $\mu_L(g) = \mathbf{f}\lambda$ we finally obtain that $\mu_N(g) = \mathbf{f}\lambda$; see (i). $\square$

Proposition 1. *Let (X, G) be a scheme, and let L be a subset of* $\mathrm{Inv}(G)$ *such that (X, G) is a Coxeter scheme with respect to L. Then, for each subset N of L, $T_L(N)$ is a transversal of $\langle N \rangle$ in G.*

Proof. Let N denote a subset of L.

Let us first assume that there are two elements r and s in $T_L(N)$ such that $s \in r\langle N \rangle$. Since $s \in r\langle N \rangle$, there exists an element h, say, in $\langle N \rangle$ such that $s \in rh$.

Since $s \in rh$ and $r \in T_L(N)$,

$$\mu_L(r) + \mu_N(h) = \mu_L(s).$$

On the other side, as $s \in rh$, $r \in sh^*$; see [**6**; Lemma 1.2.5(i)]. Thus, as $s \in T_L(N)$,

$$\mu_L(s) + \mu_N(h^*) = \mu_L(r).$$

It follows that $\mu_N(h) = 0$, so that $h = 1$. Since $s \in rh$, this yields $r = s$.

What we have shown so far is that, for each element g in G, $|T_L(N) \cap g\langle N \rangle| \leq 1$. Let us now pick an element g in G, and let us denote by t an element in $g\langle N \rangle$ such that

$$\mu_L(t) = \min \mu_L(g\langle N \rangle).$$

We define

$$W := \{h \in \langle N \rangle \mid \min \mu_L(th) \leq \mu_L(t) + \mu_N(h) - 1\}.$$

Clearly, we shall be done if we succeed in showing that $\varnothing = W$.

Let us assume, by way of contradiction, that $\varnothing \neq W$. Then $\varnothing \neq \mu_N(W)$. We abbreviate

$$j := \min \mu_N(W),$$

and we pick an element h, say, in W such that

$$\mu_N(h) = j.$$

Since $h \in \langle N \rangle$, we obtain from [**6**; Proposition 5.1.3(ii)] that there exists an element $\mathbf{d}$, say, in $\mathbf{F}_m(N)$ such that

$$\mu_N(h) = \mathbf{d}\lambda$$

and

$$\{h\} = \mathbf{d}\rho.$$

Note that $1 \notin W$. Thus, as $h \in W$, $1 \neq h$. Thus, as $h \in \mathbf{d}\rho$, $\mathbf{1} \neq \mathbf{d}$. Therefore, there exist elements $\mathbf{d}'$ in $\mathbf{F}_m(N)$ and l in N such that

$$\mathbf{d} = \mathbf{d}' * l.$$

Since $\mathbf{d}' \in \mathbf{F}_m(N)$, we obtain from Lemma 1(ii) that there exists an element h', say, in $\langle N \rangle$ such that

$$\mu_N(h') = \mathbf{d}'\lambda$$

and

$$\{h'\} = \mathbf{d}'\rho.$$

From $\mu_N(h') = \mathbf{d}'\lambda$ and $\mu_N(h) = \mathbf{d}\lambda$ we obtain that $\mu_N(h') = j - 1$. Therefore,

$$h' \notin W.$$

By [**6**; Proposition 5.1.3(ii)], there exists an element $\mathbf{c}$ in $\mathbf{F}_m(L)$ such that

$$\mu_L(t) = \mathbf{c}\lambda$$

and

$$\{t\} = \mathbf{c}\rho.$$

Set

$$\mathbf{f} := \mathbf{c} * \mathbf{d}',$$

and let e be an element in th'. Then, as $h' \notin W$,

$$\mu_L(e) = \mu_L(t) + \mu_N(h') = \mathbf{c}\lambda + \mathbf{d}'\lambda = \mathbf{f}\lambda.$$

From $e \in th'$ we also obtain that

$$e \in \mathbf{c}\rho\mathbf{d}'\rho = (\mathbf{c} * \mathbf{d}')\rho = \mathbf{f}\rho.$$

Therefore, we have $\mathbf{f}\lambda \in \mu_L(\mathbf{f}\rho)$, and this implies that

$$\mathbf{f} \in \mathbf{F}_m(L);$$

see [**6**; Proposition 5.1.3(i)].

Assume first that $\mathbf{f} * l \in \mathbf{F}_m(L)$. Then, by Lemma 1(ii), there exists an element f in G such that $\mu_L(f) = (\mathbf{f} * l)\lambda$ and $\{f\} = (\mathbf{f} * l)\rho$. Since $\mathbf{f} * l = \mathbf{c} * \mathbf{d}$, the first equation yields

$$\mu_L(f) = (\mathbf{c} * \mathbf{d})\lambda = \mathbf{c}\lambda + \mathbf{d}\lambda = \mu_L(t) + \mu_N(h),$$

and the second one yields

$$\{f\} = (\mathbf{c} * \mathbf{d})\rho = \mathbf{c}\rho\mathbf{d}\rho = th.$$

Clearly, this contradicts the choice of $h \in W$.

Assume now that $\mathbf{f} * l \notin \mathbf{F}_m(L)$. Then, by [**6**; Corollary 3.1.6], there exist elements $\mathbf{a}$, $\mathbf{b}$ in $\mathbf{F}(L)$ and an element k in L such that

$$\mathbf{f} = \mathbf{a} * k * \mathbf{b}$$

and $\mathbf{b} * l \sim_m k * \mathbf{b}$.

Assume first that $\mu_L(\mathbf{c}) \leq \mu_L(\mathbf{a})$. Then, as $\mathbf{f} = \mathbf{c} * \mathbf{d}'$, there exists an element $\mathbf{e}$ in $\mathbf{F}(L)$ such that $\mathbf{c} * \mathbf{e} = \mathbf{a}$. It follows that $\mathbf{d}' = \mathbf{e} * k * \mathbf{b}$. Thus, as $\mathbf{b} * l \sim_m k * \mathbf{b}$, $\mathbf{d}' \sim_m \mathbf{e} * \mathbf{b} * l$, contrary to $\mathbf{d} \in \mathbf{F}_m(L)$.

Assume now that $\mu_L(\mathbf{a}) + 1 \leq \mu_L(\mathbf{c})$. Then, as $\mathbf{f} = \mathbf{c} * \mathbf{d}'$, there exists an element $\mathbf{e}$ in $\mathbf{F}(L)$ such that $\mathbf{c} = \mathbf{a} * k * \mathbf{e}$. It follows that $\mathbf{e} * \mathbf{d}' = \mathbf{b}$. Thus, as $\mathbf{d} = \mathbf{d}' * l$, $\mathbf{e} * \mathbf{d} = \mathbf{b} * l$. Thus, as $\mathbf{b} * l \sim_m k * \mathbf{b}$, $\mathbf{e} * \mathbf{d} \sim_m k * \mathbf{b}$. Thus, as $\mathbf{f} = \mathbf{a} * k * \mathbf{b}$,

$$\mathbf{f} \sim_m \mathbf{a} * \mathbf{e} * \mathbf{d}.$$

Since $\mathbf{f} \in \mathbf{F}_m(L)$, we now have that $\mathbf{a} * \mathbf{e} \in \mathbf{F}_m(L)$. Thus, by Lemma 1(ii), there exists an element s in G such that

$$\mu_L(s) = (\mathbf{a} * \mathbf{e})\lambda$$

and

$$\{s\} = (\mathbf{a} * \mathbf{e})\rho.$$

We now shall prove that $s \in g\langle N\rangle$ and that $\mu_L(s) = \mu_L(t) - 1$. This contradiction will conclude the proof of the proposition.

Since $\mathbf{f} \sim_m \mathbf{a} * \mathbf{e} * \mathbf{d}$, $\mathbf{f}\rho = (\mathbf{a} * \mathbf{e})\rho\mathbf{d}\rho$; see [**6**; Lemma 5.1.2]. Thus, as $e \in \mathbf{f}\rho$, $\{s\} = (\mathbf{a} * \mathbf{e})\rho$, and $\{h\} = \mathbf{d}\rho$, we have $e \in sh \subseteq s\langle N\rangle$. But we also have $e \in th' \subseteq t\langle N\rangle$ and $t \in g\langle N\rangle$. Therefore, by [**6**; Theorem 1.3.1], $s \in g\langle N\rangle$.

Since $\mathbf{f} \sim_m \mathbf{a} * \mathbf{e} * \mathbf{d}$, we also have $\mathbf{f}\lambda = (\mathbf{a} * \mathbf{e})\lambda + \mathbf{d}\lambda$. Thus, as

$$\mathbf{f}\lambda = \mathbf{c}\lambda + \mathbf{d}'\lambda = \mu_L(t) + \mathbf{d}\lambda - 1,$$

we have $(\mathbf{a} * \mathbf{e})\lambda = \mu_L(t) - 1$. Thus, as $\mu_L(s) = (\mathbf{a} * \mathbf{e})\lambda$, $\mu_L(s) = \mu_L(t) - 1$. □

Let (X, G) be a scheme, and let L denote a subset of $\mathrm{Inv}(G)$. We define

$$\mathbf{E}_m(L) := \{\mathbf{f} \in \mathbf{F}_m(L) \mid \mathbf{f}\lambda \in \mu_L(\mathbf{f}\rho)\}.$$

Just in order to get familiar with the meaning of the set $\mathbf{E}_m(L)$, let us state the following. Assume that (X, G) is L-constrained. Then, (X, G) is a Coxeter scheme with respect to L if and only if $\mathbf{E}_m(L) = \mathbf{F}_m(L)$. (This follows from [**6**; Theorem 5.1.5].)

PROPOSITION 2. *Let (X, G) be a scheme, and let L be a subset of* $\mathrm{Inv}(G)$ *such that (X, G) is L-constrained. Assume that the following two conditions hold.*

(i) *For any two elements $\mathbf{c}$ and $\mathbf{d}$ in $\mathbf{E}_m(L)$, $\mathbf{c}\rho = \mathbf{d}\rho$ implies that $\mathbf{c} \sim_m \mathbf{d}$.*

(ii) *For each subset N of L with $|N| = 1$, $T_L(N)$ is a transversal of $\langle N\rangle$ in G.*

Then (X, G) is a Coxeter scheme with respect to L.

Proof. Assume, by way of contradiction, that (X, G) is not a Coxeter scheme with respect to L. Then, by hypothesis (i), $\mathbf{F}_m(L) \not\subseteq \mathbf{E}_m(L)$. This means that $\emptyset \neq \mathbf{F}_m(L) \backslash \mathbf{E}_m(L)$. We abbreviate

$$n := \min\{\mathbf{f}\lambda \mid \mathbf{f} \in \mathbf{F}_m(L) \backslash \mathbf{E}_m(L)\},$$

and we pick an element $\mathbf{f}$, say, in $\mathbf{F}_m(L) \backslash \mathbf{E}_m(L)$ such that

$$\mathbf{f}\lambda = n.$$

Since $\mathbf{f} \in \mathbf{F}_m(L)$, there exists an element g in G such that

$$\{g\} = \mathbf{f}\rho.$$

Since $\mathbf{f} \notin \mathbf{E}_m(L)$,

$$\mu_L(g) \leq n - 1.$$

Note that $\mathbf{1} \in \mathbf{E}_m(L)$. Thus, as $\mathbf{f} \notin \mathbf{E}_m(L)$, $\mathbf{1} \neq \mathbf{f}$. In particular, there exist elements $\mathbf{e}$ in $\mathbf{F}_m(L)$ and l in L such that

$$\mathbf{f} = \mathbf{e} * l.$$

Since $\mathbf{e} \in \mathbf{F}_m(L)$, the choice of $\mathbf{f}$ forces $\mathbf{e} \in \mathbf{E}_m(L)$. Thus, there exists an element f in G such that

$$\mu_L(f) = \mathbf{e}\lambda$$

and

$$\{f\} = \mathbf{e}\rho.$$

Let us set $N := \{l\}$.

From $\{g\} = \mathbf{f}\rho$ and $\{f\} = \mathbf{e}\rho$ we obtain that $\{g\} = (\mathbf{e} * l)\rho = \mathbf{e}\rho l = fl$. Thus,

$$g\langle N\rangle = \{f, g\}.$$

On the other hand, as $|N| = 1$, we obtain from hypothesis (ii) that $\emptyset \neq T_L(N) \cap g\langle N\rangle$. Let t denote an element in $T_L(N) \cap g\langle N\rangle$.

Assume first that $t = f$. Then $g \in tl$, whence

$$\mu_L(t) + \mu_N(l) = \mu_L(g) \leq n - 1 = \mu_L(f) = \mu_L(t),$$

contrary to $\mu_N(l) = 1$.

Assume now that $t \neq f$. Then, as $t \in g\langle N\rangle = \{f, g\}$, $t = g$. In particular, as $f \in gl$, $f \in tl$. Therefore,

$$\mu_L(t) + \mu_N(l) = \mu_L(f) = n - 1.$$

Thus, as $\mu_N(l) = 1$, $\mu_L(t) = n - 2$. Thus, by [**6**; Lemma 5.1.1], there exists an element $\mathbf{d}$ in $\mathbf{F}(L)$ such that $\mathbf{d}\lambda = n - 2$ and $t \in \mathbf{d}\rho$.

From $\mathbf{d}\lambda = n - 2$ (and $\mu_L(f) = n - 1$) we conclude that

$$\mu_L(f) = \mathbf{d}\lambda + 1 = (\mathbf{d} * l)\lambda.$$

Moreover, from $t \in \mathbf{d}\rho$ we obtain that

$$f \in tl \subseteq \mathbf{d}\rho l = (\mathbf{d} * l)\rho.$$

Therefore, we have $(\mathbf{d} * l)\lambda \in \mu_L((\mathbf{d} * l)\rho)$, and this implies that $\mathbf{d} * l \in \mathbf{F}_m(L)$; see [**6**; Proposition 5.1.3(i)]. It follows that $\mathbf{d} * l \in \mathbf{E}_m(L)$.

On the other hand, we have $(\mathbf{d} * l)\rho = \{f\} = \mathbf{e}\rho$ and $\mathbf{e} \in \mathbf{E}_m(L)$. Thus, by hypothesis (i), $\mathbf{d} * l \sim_m \mathbf{e}$. It follows that $\mathbf{f} = \mathbf{e} * l \sim_m \mathbf{d} * l * l$. In particular, $\mathbf{f} \notin \mathbf{F}_m(L)$, contrary to our choice of $\mathbf{f}$. □

Note that the following proposition (together with Proposition 1 and Proposition 2) concludes the proof of Theorem 4.

PROPOSITION 3. *Let (X, G) be a scheme, and let L be a subset of* $\mathrm{Inv}(G)$ *such that (X, G) is L-constrained.*

Assume that, for each subset N of L with $|N| = 2$, $T_L(N)$ is a transversal of $\langle N \rangle$ in G. Then, for any two elements $\mathbf{c}$ and $\mathbf{d}$ in $\mathbf{E}_m(L)$, $\mathbf{c}\rho = \mathbf{d}\rho$ implies that $\mathbf{c} \sim_m \mathbf{d}$.

PROOF. We set

$$\mathscr{P} := \{(\mathbf{c}, \mathbf{d}) \in \mathbf{E}_m(L) \times \mathbf{E}_m(L) \mid \mathbf{c}\rho = \mathbf{d}\rho, \mathbf{c} \not\sim_m \mathbf{d}\}.$$

Let us assume, by way of contradiction, that $\emptyset \neq \mathscr{P}$. We abbreviate

$$n := \min\{\mathbf{c}\lambda \mid (\mathbf{c}, \mathbf{d}) \in \mathscr{P}\},$$

and we pick a pair $(\mathbf{c}, \mathbf{d})$ in $\mathscr{P}$ which satisfies $\mathbf{c}\lambda = n$.

Since $\mathbf{c} \in \mathbf{E}_m(L)$, there exists an element g in $\mathbf{c}\rho$ such that $\mu_L(g) = \mathbf{c}\lambda$. Since $\mathbf{c} \in \mathbf{F}_m(L)$ and $g \in \mathbf{c}\rho$,

$$\{g\} = \mathbf{c}\rho.$$

From $\{g\} = \mathbf{c}\rho$ and $\mathbf{c}\rho = \mathbf{d}\rho$ we obtain that $\{g\} = \mathbf{d}\rho$. Thus, as $\mathbf{d} \in \mathbf{E}_m(L)$, $\mu_L(g) = \mathbf{d}\lambda$. Thus, as $\mu_L(g) = \mathbf{c}\lambda$,

$$\mathbf{c}\lambda = \mathbf{d}\lambda.$$

From $(\mathbf{c}, \mathbf{d}) \in \mathscr{P}$ and $\mathbf{c}\lambda = \mathbf{d}\lambda$ we conclude that $1 \leq n$. Thus, there exist elements $\mathbf{a}$ in $\mathbf{E}_m(L)$ and h in L such that

$$\mathbf{c} = \mathbf{a} * h.$$

Since $\mathbf{a} \in \mathbf{F}_m(L)$, there exists an element e in G such that

$$\{e\} = \mathbf{a}\rho.$$

(Recall that (X, G) is assumed to be L-constrained.) It follows that

$$\{g\} = \mathbf{c}\rho = (\mathbf{a} * h)\rho = \mathbf{a}\rho h = eh.$$

Since $1 \leq \mathbf{c}\lambda = \mathbf{d}\lambda$, there exist elements $\mathbf{b}$ in $\mathbf{E}_m(L)$ and k in L such that

$$\mathbf{d} = \mathbf{b} * k.$$

We set $N := \{h, k\}$.

Since $h \neq k$, our hypothesis implies that $\emptyset \neq T_L(N) \cap g\langle N\rangle$. Let t denote an element in $T_L(N) \cap g\langle N\rangle$.

From $t \in g\langle N\rangle$, $g \in eh$, and $h \in N$ we obtain that $t \in e\langle N\rangle$. Thus, by [**6**; Lemma 1.2.5(i)], $e \in t\langle N\rangle$. Thus, there exists an element c in $\langle N\rangle$ such that

$$e \in tc.$$

Since $t \in T_L(N)$, we now obtain that

$$\mu_L(t) + \mu_N(c) = \mu_L(e).$$

By [**6**; Proposition 5.1.3(ii)], there exists an element $\mathbf{e}$ in $\mathbf{F}_m(L)$ such that

$$\mu_L(t) = \mathbf{e}\lambda$$

and

$$\{t\} = \mathbf{e}\rho.$$

Since $c \in \langle N\rangle$, we obtain from the same reference that there exists an element $\mathbf{a}'$ in $\mathbf{F}_m(N)$ with

$$\mu_N(c) = \mathbf{a}'\lambda$$

and

$$\{c\} = \mathbf{a}'\rho.$$

It follows that

$$\mu_L(e) = \mathbf{e}\lambda + \mathbf{a}'\lambda = (\mathbf{e} * \mathbf{a}')\lambda$$

and

$$e \in tc = \mathbf{e}\rho\mathbf{a}'\rho = (\mathbf{e} * \mathbf{a}')\rho.$$

Therefore, we have $(\mathbf{e} * \mathbf{a}')\lambda \in \mu_L((\mathbf{e} * \mathbf{a}')\rho)$, and this implies that $\mathbf{e} * \mathbf{a}' \in \mathbf{F}_m(L)$; see [**6**; Proposition 5.1.3(i)]. It follows that $\mathbf{e} * \mathbf{a}' \in \mathbf{E}_m(L)$.

On the other hand, we have $\mathbf{a} \in \mathbf{E}_m(L)$ and $\mathbf{a}\rho = \{e\} = (\mathbf{e} * \mathbf{a}')\rho$. Note also that $\mathbf{a}\lambda = \mathbf{c}\lambda - 1$, so that, by the choice of $(\mathbf{c}, \mathbf{d}) \in \mathscr{P}$, $(\mathbf{a}, \mathbf{e} * \mathbf{a}') \notin \mathscr{P}$. Thus, we must have

$$\mathbf{a} \sim_m \mathbf{e} * \mathbf{a}'.$$

From this (and from $\mathbf{c} = \mathbf{a} * h$) we obtain that $\mathbf{c} \sim_m \mathbf{e} * \mathbf{a}' * h$. Thus, as $\mathbf{c} \in \mathbf{F}_m(L)$, we also have

$$\mathbf{a}' * h \in \mathbf{F}_m(L).$$

Thus, as (X, G) is assumed to be L-constrained, there exist $c' \in G$ with

$$\{c'\} = (\mathbf{a}' * h)\rho = \mathbf{a}'\rho h = ch.$$

It follows that

$$\{g\} = eh \subseteq tch = tc'.$$

Similarly, we find $\mathbf{b}' \in \mathbf{F}(N)$ such that

$$\mathbf{b} \sim_m \mathbf{e} * \mathbf{b}'$$

and

$$\mathbf{b}' * k \in \mathbf{F}_m(L).$$

Moreover, we find d', $d \in \langle N \rangle$ such that $\{d\} = \mathbf{b}'\rho$, $\{d'\} = dk$, and $g \in td'$. (Recall that $\mathbf{c}\lambda = \mathbf{d}\lambda$.)

From $g \in tc' \cap td'$ we obtain that $\varnothing \neq t^*t \cap c'd'^*$; see [**6**; Lemma 1.2.5(iv)]. On the other hand, as c', $d' \in \langle N \rangle$, $c'd'^* \subseteq \langle N \rangle$. Therefore, by Theorem 1 and Theorem 3, $t^*t \cap c'd'^* \subseteq \{1\}$. It follows that $1 \in c'd'^*$. Thus, by [**6**; Lemma 1.2.5(iii)], $c' = d'$. Now we have

$$(\mathbf{a}' * h)\rho = \{c'\} = \{d'\} = (\mathbf{b}' * k)\rho.$$

On the other hand, we have $\mathbf{a}' * h$, $\mathbf{b}' * k \in \mathbf{F}_m(N)$ and $h \neq k$. Thus, by definition,

$$\{\mathbf{a}' * h, \mathbf{b}' * k\} = \{\mathbf{f}_m(h,k), \mathbf{f}_m(k,h)\}.$$

In particular, $\mathbf{a}' * h \sim_m \mathbf{b}' * k$. It follows that

$$\mathbf{c} = \mathbf{a} * h \sim_m \mathbf{e} * \mathbf{a}' * h \sim_m \mathbf{e} * \mathbf{b}' * k \sim_m \mathbf{b} * k = \mathbf{d},$$

contrary to the choice of $(\mathbf{c}, \mathbf{d})$. □

8. The proof of the corollary.

The second part of the corollary is an immediate consequence of its first part. (Use [**6**; Theorem 1.3.1].) The third part of the corollary follows from Theorem 2, Theorem 3, and Theorem 4. The first part of the corollary follows nearly immediately from the following little lemma.

Lemma 2. *Let (X,G) be a scheme, and let L be a subset of* Inv(G) *such that (X,G) is L-constrained. Let N denote a subset of L, and let t denote an element in $T_L(N)$. Then we have the following.*

(i) *For each element h in $\langle N\rangle$, $|th| = 1$.*

(ii) *Let e and f be elements of $\langle N\rangle$ such that $e \neq f$. Then, if (X,G) is a Coxeter scheme with respect to L, $\emptyset = te \cap tf$.*

Proof. (i) Let h denote an element in $\langle N\rangle$.

By [**6**; Proposition 5.1.3(ii)], there exists an element $\mathbf{d}$, say, in $\mathbf{F}_m(L)$ such that

$$\mu_L(t) = \mathbf{d}\lambda$$

and

$$\{t\} = \mathbf{d}\rho.$$

By the same reference, there exists an element $\mathbf{e}$ in $\mathbf{F}_m(N)$ such that

$$\mu_N(h) = \mathbf{e}\lambda$$

and

$$\{h\} = \mathbf{e}\rho.$$

Let g denote an element in th, and set $\mathbf{f} := \mathbf{d} * \mathbf{e}$. Then, by definition,

$$\mu_L(g) = \mu_L(t) + \mu_N(h) = \mathbf{d}\lambda + \mathbf{e}\lambda = \mathbf{f}\lambda.$$

Moreover,

$$th = \mathbf{d}\rho\mathbf{e}\rho = (\mathbf{d} * \mathbf{e})\rho = \mathbf{f}\rho.$$

In particular, $\mathbf{f}\lambda \in \mu_L(\mathbf{f}\rho)$, and this implies that $\mathbf{f} \in \mathbf{F}_m(L)$; see [**6**; Proposition 5.1.3(i)].

Since (X, G) is assumed to be L-constrained, $\mathbf{f} \in \mathbf{F}_m(L)$ implies that $|\mathbf{f}\rho| = 1$. On the other hand, we have $th = \mathbf{f}\rho$. Thus, we also have $|th| = 1$.

(ii) Assume, by way of contradiction, that $\emptyset \neq te \cap tf$. Then, by [**6**; Lemma 1.2.5(iv)], $\emptyset \neq t^*t \cap ef^*$. On the other hand, as $e, f \in \langle N \rangle$, $ef^* \subseteq \langle N \rangle$. Therefore, by Theorem 1 and Theorem 3, $t^*t \cap ef^* \subseteq \{1\}$. It follows that $1 \in ef^*$. Thus, by [**6**; Lemma 1.2.5(iii)], $e = f$. $\square$

In order to prove the first part of the corollary, let N denote a subset of L, and let t denote an element in $T_L(N)$. Then, by Lemma 2, $|t\langle N \rangle| = |\langle N \rangle|$. Thus, the first claim of the corollary follows from Theorem 4 (more precisely, from Proposition 1).

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