J. Math. Soc. Japan
Vol. 52, No. 2, 2000

Real Seifert form determines the spectrum for semiquasihomogeneous hypersurface singularities in C^3

To the memory of Professor Etsuo Yoshinaga

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(Received Apr. 22, 1998)
(Revised Oct. 12, 1998)

Abstract. We show that the real Seifert form determines the weights for nondegenerate quasihomogeneous polynomials in $\boldsymbol{C}^3$. Consequently the real Seifert form determines the spectrum for semiquasihomogeneous hypersurface singularities in $\boldsymbol{C}^3$. As a corollary, we obtain the topological invariance of weights for nondegenerate quasihomogeneous polynomials in $\boldsymbol{C}^3$, which has already been proved by the author [**Sae1**] and independently by Xu and Yau [**Ya1**], [**Ya2**], [**XY1**], [**XY2**]. The method in this paper is totally different from their approaches and gives some new results, as corollaries, about holomorphic function germs in $\boldsymbol{C}^3$ which are connected by μ-constant deformations to nondegenerate quasihomogeneous polynomials. For example, we show that two semiquasihomogeneous functions of three complex variables have the same topological type if and only if they are connected by a μ-constant deformation.

1. Introduction.

Let $f(z_1, \dots, z_{n+1})$ be a complex polynomial with $f(0) = 0$. We say that f is *quasihomogeneous* if there exists a sequence $(w_1, \dots, w_{n+1})$ of positive rational numbers, called *weights*, such that for all monomials $cz_1^{a_1} \cdots z_{n+1}^{a_{n+1}}$ of f with $c \neq 0$, we have $a_1/w_1 + \cdots + a_{n+1}/w_{n+1} = 1$. We say that such a polynomial is *nondegenerate* if it has an isolated critical point at the origin.

Saito [**Sai**] has proved that for every nondegenerate quasihomogeneous polynomial, we may assume that all the weights satisfy $w_j \geq 2$ after a suitable coordinate transformation. Furthermore, under this assumption, the weights are well-defined; i.e. the weights are analytic invariants of the germ $(\boldsymbol{C}^{n+1}, f^{-1}(0))$ at the origin, provided that

1991 *Mathematics Subject Classification*. Primary 32S50; Secondary 14B05, 32S25, 32S55.
Key Words and Phrases. Real Seifert form, quasihomogeneous polynomial, weights, spectrum, μ-constant deformation.
The author has been partially supported by Grant-in-Aid for Encouragement of Young Scientists (No. 08740057), Ministry of Education, Science and Culture, Japan.

they are greater than or equal to two. Then it arises the question whether they are topological invariants or not. The answer to this question has been shown to be affirmative for $n = 1$ by Yoshinaga and Suzuki [**YS1**] (see also [**Ni**]) and for $n = 2$ by the author [**Sae1**] and independently by Xu and Yau [**Ya1**], [**Ya2**], [**XY1**], [**XY2**]. The result for $n = 2$ has been obtained by using the fundamental group of the link 3-manifold $K_f = f^{-1}(0) \cap S_\varepsilon^5$, where S_ε^{2n+1} is a sufficiently small sphere in $\boldsymbol{C}^{n+1}$ centered at the origin. This method cannot be applied to higher dimensions, since for $n \geq 3$, the link manifold $K_f = f^{-1}(0) \cap S_\varepsilon^{2n+1}$ is always simply connected (see [**M**]). Note that an affirmative answer to the above question for n would imply an affirmative answer to Zariski's multiplicity problem [**Z**] for nondegenerate quasihomogeneous polynomials of $n+1$ variables (see [**HR**, §6] and [**Sae1**, Lemma 6]).

Note that for a nondegenerate quasihomogeneous polynomial f, the weights of f determine and are determined by the spectrum of the hypersurface $f^{-1}(0)$ (see [**SSS**, Example 5.2]). Recall that the spectrum is defined by using the mixed Hodge structure of the middle dimensional cohomology of the Milnor fiber (for a precise definition, see §2 of the present paper). Thus a priori it may not be a topological invariant of the hypersurface. Recently Némethi [**Ne**] has shown that the real Seifert form determines and is determined by the modulo 2 spectral pairs for an isolated hypersurface singularity in general. Note that in the case of nondegenerate quasihomogeneous polynomials, giving the spectral pairs is equivalent to giving the spectrum (see §2).

Our main result of this paper is the following.

THEOREM 1.1. *Let f and g be nondegenerate quasihomogeneous polynomials in $\boldsymbol{C}^3$. If their Seifert forms are equivalent over the real numbers, then the spectra of f and g coincide.*

For a definition of the Seifert form of a holomorphic function germ with an isolated critical point at the origin, see §2 or [**D**], [**K**], [**AGVII**]. Note that the Seifert form over the integers determines and is determined completely by the topology of the pair $(\boldsymbol{C}^{n+1}, f^{-1}(0))$ for all $n \geq 3$ (see [**D**], [**K**]). Thus our method is, in principle, applicable to higher dimensions as well. In fact, in [**Sae3**], it is shown that the real Seifert form determines the spectrum for a certain class of nondegenerate quasihomogeneous polynomials in $\boldsymbol{C}^{n+1}$ $(n \geq 1)$, by using the methods of the present paper.

In the following, for two holomorphic function germs f and $g : (\boldsymbol{C}^{n+1}, 0) \to (\boldsymbol{C}, 0)$ with an isolated critical point at the origin, we say that they are *topologically equivalent* (or they have the same *topological type*) if $(\boldsymbol{C}^{n+1}, f^{-1}(0))$ is locally homeomorphic to $(\boldsymbol{C}^{n+1}, g^{-1}(0))$ at the origin (as pairs). It has been known that this relation is equivalent to the *topological right-left equivalence* and that for nondegenerate quasihomogeneous polynomials, it is also equivalent to the *topological right equivalence* (for details, see [**Sae2**]).

Corollary 1.2 (Saeki [**Sae1**], Xu-Yau [**Ya1**], [**Ya2**], [**XY1**], [**XY2**]). *Let f (z_1, z_2, z_3) (resp. $g(z_1, z_2, z_3)$) be a nondegenerate quasihomogeneous polynomial with weights (w_1, w_2, w_3) (resp. (w_1', w_2', w_3')), where $w_j, w_k' \geq 2$. If f and g have the same topological type, then we have $w_j = w_j'$ up to order.*

Corollary 1.3. *Let f and $g : (\boldsymbol{C}^3, 0) \to (\boldsymbol{C}, 0)$ be holomorphic function germs with an isolated critical point at the origin. Suppose that for each of them there exists a μ-constant deformation which connects it with a nondegenerate quasihomogeneous polynomial. Then the following five are equivalent.*

(1) *The Seifert forms of f and g are equivalent over the real numbers.*
(2) *The Seifert forms of f and g are equivalent over the integers.*
(3) *f and g have the same characteristic polynomial and the same equivariant signatures.*
(4) *f and g have the same spectrum.*
(5) *f and g are connected by a μ-constant deformation.*

For a definition of a μ-constant deformation, see [**LR**], [**DG**], [**O2**]. For definitions of the characteristic polynomial and the equivariant signatures, see §2. Compare Corollary 1.3 (1) and (4) with [**Ne**].

Corollary 1.4. *Suppose that f and g are semiquasihomogeneous holomorphic function germs in $\boldsymbol{C}^3$. Then the seven conditions* (1)–(7) (*the five conditions in Corollary* 1.3 *and the following two*) *are equivalent to each other.*

(6) *f and g have the same topological type.*
(7) *The link* 3*-manifolds K_f and K_g have isomorphic fundamental groups and f and g have the same characteristic polynomial.*

Furthermore, if one of the seven conditions is satisfied, f and g have the same multiplicity. In particular, for semiquasihomogeneous holomorphic function germs in $\boldsymbol{C}^3$, the spectrum and the multiplicity are topological invariants.

For a definition of a semiquasihomogeneous function germ, see [**AGVI**].

Compare Corollary 1.4 with [**Ya2**]. Note that a μ-constant deformation is topologically constant for function germs in $\boldsymbol{C}^{n+1}$ with $n \neq 2$ ([**LR**]). For $n = 2$, this has not been known to be true or not in general. Note also that it has been known that the multiplicity is invariant under μ-constant deformations for nondegenerate quasihomogeneous polynomials (see [**G**], [**O'S**]). Furthermore, for quasihomogeneous polynomials of three variables, it has been known that the multiplicity is a topological invariant (see [**Sae1**], [**Ya1**], [**Ya2**], [**XY1**]). The new result contained in Corollary 1.4 is that for nondegenerate quasihomogeneous polynomials of three variables, the multiplicity is determined only by the (real) Seifert form.

Corollary 1.5. *Suppose that f and g are semiquasihomogeneous holomorphic function germs in $\boldsymbol{C}^{n+1}$ $(n \geq 1)$. Then they are topologically equivalent if and only if their stabilizations $\tilde{f}(z_1, \dots, z_{n+2}) = f(z_1, \dots, z_{n+1}) + z_{n+2}^2$ and $\tilde{g}(z_1, \dots, z_{n+2}) = g(z_1, \dots, z_{n+1}) + z_{n+2}^2$ are topologically equivalent.*

For the biholomorphic equivalence, a result similar to the above corollary has been known for all holomorphic function germs with an isolated critical point at the origin (see [**Ya3**]). Note that Corollary 1.5 remains true even without the semiquasihomogeneity condition, provided that $n \geq 3$. This follows from the facts that the Seifert forms over the integers of f and g are equivalent if and only if those of their stabilizations are equivalent (see Lemma 2.1) and that the Seifert form over the integers completely determines the topological type of f, provided that $n \geq 3$ ([**D**], [**K**]). For $n = 1, 2$, see Remark 3.11.

The paper is organized as follows. In §2, we recall the result of Steenbrink [**St2**] about the equivariant signatures of a nondegenerate quasihomogeneous polynomial and use it to obtain a necessary and sufficient condition for two such polynomials to have equivalent real Seifert forms in terms of their weights. As a by-product, we give a new proof of a result of Yoshinaga [**Yo1**] which gives a necessary and sufficient condition for two such polynomials to have the same characteristic polynomial. In §3, we use the results obtained in §2 to prove Theorem 1.1 and its corollaries. In §4, as an application of our method, we give normal forms of real and complex Seifert matrices of nondegenerate quasihomogeneous polynomials in terms of their weights.

The author deeply grieves over the death of Professor Etsuo Yoshinaga, who encouraged him a lot during his study on the subject of the present paper. The author would like to thank Takashi Nishimura for his invaluable comments. He would also like to thank the referee for his useful suggestions.

2. Real Seifert form and weights.

Let $f : (\boldsymbol{C}^{n+1}, 0) \to (\boldsymbol{C}, 0)$ be a holomorphic function germ with an isolated critical point at the origin. Denote by F the *Milnor fiber* of f; i.e. $F = f^{-1}(\delta) \cap D_\varepsilon^{2n+2}$, where δ and ε are sufficiently small positive real numbers with $0 < \delta \ll \varepsilon$ and D_ε^{2n+2} is the closed

$(2n+2)$-dimensional disk in $\boldsymbol{C}^{n+1}$ centered at the origin with radius ε. We put $E = D_\varepsilon^{2n+2} \cap f^{-1}(D_\delta^2)$, which is homeomorphic to the $(2n+2)$-dimensional disk ([**M**]). Then we define the *Seifert form* $L_{\boldsymbol{Z}} : H_n(F; \boldsymbol{Z}) \times H_n(F; \boldsymbol{Z}) \to \boldsymbol{Z}$ of f over the integers by $L_{\boldsymbol{Z}}(x, y) = \mathrm{lk}(\iota_* x, y)$, where $\iota : F \to \partial E - F$ is a parallel translation in the positive normal direction to F in ∂E and lk denotes the linking number in ∂E (for details, see [**D**]). Furthermore, we denote by $L_{\boldsymbol{R}} = L_{\boldsymbol{Z}} \otimes \boldsymbol{R} : H_n(F; \boldsymbol{R}) \times H_n(F; \boldsymbol{R}) \to \boldsymbol{R}$ the *real Seifert form* of f and by $L : H_n(F; \boldsymbol{C}) \times H_n(F; \boldsymbol{C}) \to \boldsymbol{C}$ the *sesquilinearized complex Seifert form* of f. We define the *cohomological monodromy operator* $T : H^n(F; \boldsymbol{C}) \to H^n(F; \boldsymbol{C})$ by $T = (h^*)^{-1}$ and the *homological monodromy operator* $H : H_n(F; \boldsymbol{C}) \to H_n(F; \boldsymbol{C})$ by $H = h_*$, where $h : F \to F$ is the geometric monodromy of the Milnor fibration $f : D_\varepsilon^{2n+2} \cap f^{-1}(\partial D_\delta^2) \to \partial D_\delta^2$ (see [**AGVII**, p. 30]). Furthermore, we denote the *sesquilinearized intersection form* of F by $S : H^n(F; \boldsymbol{C}) \times H^n(F; \boldsymbol{C}) \to \boldsymbol{C}$.

In the following, we often abuse the notations L, T, H and S to denote the matrix representatives of the corresponding maps with respect to fixed bases of $H^n(F; \boldsymbol{C})$ and $H_n(F; \boldsymbol{C})$ which are dual to each other. The following lemma can easily be proved (see [**D**], [**Sak**], [**K**]).

Lemma 2.1. *We have*

$$S = L + (-1)^n \overline{L^t}, \quad H = (-1)^{n+1} \overline{L^{-1}} L^t, \quad T = (-1)^{n+1} L \overline{(L^{-1})^t}, \quad S = L(I - \overline{H}),$$

where I denotes the identity matrix and for a matrix X, X^t denotes the transpose of X and $\overline{X}$ the complex conjugate of X. Furthermore, if we denote the sesquilinearized complex Seifert form of the stabilization $\tilde{f}(z_1, \ldots, z_{n+2}) = f(z_1, \ldots, z_{n+1}) + z_{n+2}^2$ by $\tilde{L}$, then we have $\tilde{L} = (-1)^{n+1} L$.

We denote by $\Delta_f(t)$ the characteristic polynomial of T. We also call $\Delta_f(t)$ the *characteristic polynomial* of f. Note that

$$H^n(F; \boldsymbol{C}) = \bigoplus_\lambda H^n(F; \boldsymbol{C})_\lambda,$$

where λ runs over all the eigenvalues of T (i.e. all the roots of $\Delta_f(t)$) and $H^n(F; \boldsymbol{C})_\lambda$ is the eigenspace corresponding to the eigenvalue λ. It is easy to see that the intersection form S decomposes as the orthogonal direct sum of $S_\lambda = S|_{H^n(F; \boldsymbol{C})_\lambda}$, since $S(T(x), T(y)) = S(x, y)$ for all $x, y \in H^n(F; \boldsymbol{C})$. Furthermore, we have similar decompositions for $H_n(F; \boldsymbol{C})$ and L with respect to H, since we have $L(H(x), H(y)) = L(x, y)$ for all $x, y \in H_n(F; \boldsymbol{C})$.

Next we recall the definitions of the spectral pairs and the spectrum of f. The mixed Hodge structure on $H^n(F; \boldsymbol{C})$ consists of an increasing *weight filtration* $W_\bullet$ and a decreasing *Hodge filtration* $F^{\bullet}$ (for details, see [**AGVII**, Chapter 14]). We write $T = T_s T_u = T_u T_s$ with T_s semisimple and T_u unipotent. Then T_s preserves the filtrations $F^{\bullet}$ and $W_{\bullet}$. For each eigenvalue λ of T, we define

$$H_{\lambda}^{p,q} = \ker(T_s - \lambda I : \mathrm{Gr}_{p+q}^{W} \mathrm{Gr}_F^p H^n(F; \boldsymbol{C})) \quad \text{and} \quad h_{\lambda}^{p,q} = \dim_{\boldsymbol{C}} H_{\lambda}^{p,q},$$

where $\mathrm{Gr}_i^W = W_i / W_{i-1}$ and $\mathrm{Gr}_F^p = F^p / F^{p+1}$. For $\alpha \in \boldsymbol{Q}$ and $w \in \boldsymbol{Z}$, we define the integers $m_{\alpha,w}$ as follows. It is easy to see that there exist a unique $p \in \boldsymbol{Z}$ and a unique $\beta \in \boldsymbol{Q}$ with $0 \le \beta < 1$ such that $\alpha = n - p - \beta$. Set $\lambda = \exp(-2\pi i \alpha)$, where $i = \sqrt{-1}$. If $\lambda \neq 1$, then we set $m_{\alpha,w} = h_{\lambda}^{p,w-p}$ and if $\lambda = 1$, then we set $m_{\alpha,w} = h_1^{p,w+1-p}$. The *spectral pairs* are collected in the invariant

$$Spp(f) = \sum_{\alpha, w} m_{\alpha,w}(\alpha, w),$$

which is considered to be an element of the free abelian group on $\boldsymbol{Q} \times \boldsymbol{Z}$. Forgetting the second factor w, we get the *spectrum* $Sp(f)$ of f defined by

$$Sp(f) = \sum_{\alpha} m_{\alpha}(\alpha), \quad m_{\alpha} = \sum_{w} m_{\alpha,w},$$

which is considered to be an element of the free abelian group on $\boldsymbol{Q}$ (see [**St3**, (2.1)] or [**SSS**, §1]. See also [**St4**]).

When the monodromy operator is semisimple (i.e. when $T = T_s$), the weight filtration $W^{\bullet}$ degenerates as

$$\{0\} = W_{n-1} \subset W_n \subset W_{n+1} = H^n(F; \boldsymbol{C}),$$

where $W_n = \bigoplus_{\lambda \neq 1} H^n(F; \boldsymbol{C})_{\lambda}$ (for example, see [**AGVII**, §13.2.3, Example 3 (p. 371)]). Thus $Spp(f)$ is of the form

$$\sum_{\alpha} m_{\alpha,n}(\alpha, n).$$

Hence, giving the spectral pairs is equivalent to giving the spectrum in this case.

Now suppose that n is even. In this case, S is hermitian by Lemma 2.1. Let $\mu_0(f)$ be the dimension of the kernel of S, $\mu(f)_{\lambda}^{+}$ (resp. $\mu(f)_{\lambda}^{-}$) the number of positive (resp. negative) eigenvalues of $S|_{H^n(F;\boldsymbol{C})_{\lambda}}$. We call $\mu(f)_{\lambda}^{+} - \mu(f)_{\lambda}^{-}$ the *equivariant signature* of f with respect to λ. By Steenbrink [**St1**], $\mu(f)_0$ and $\mu(f)_{\lambda}^{\pm}$ can be calculated from the Hodge numbers of f. Note that the Hodge numbers $h_{\lambda}^{p,q}$ and the spectral pairs mutually determine each other, and that when the monodromy operator T is semisimple, the spectral pairs are equivalent to the spectrum $Sp(f)$ of f, as has been seen above.

Suppose that f is a nondegenerate quasihomogeneous polynomial of weights $(w_1, w_2, \dots, w_{n+1})$. In this case, the monodromy operator is always semisimple (for example, see [**M**, §9] and [**AGVII**, §13.2.3, Example 3 (p. 371)]). We set

$$P_f(t) = \prod_{j=1}^{n+1} \frac{t - t^{1/w_j}}{t^{1/w_j} - 1} \in \boldsymbol{Z}[t^{1/m}],$$

where m is the least common multiple of the numerators u_j of $w_j = u_j/v_j$ and u_j and v_j are relatively prime positive integers. We write

$$P_f(t) = \sum_{\alpha \in \boldsymbol{Q}} c_\alpha t^\alpha.$$

Note that c_α is a nonnegative integer for each $\alpha \in \boldsymbol{Q}$ and that the spectrum $Sp(f)$ of f coincides with $\sum_{\alpha \in \boldsymbol{Q}} c_\alpha (\alpha - 1)$ (see [**SSS**, Example 5.2] or [**St4**, Examples in §1]). Using the above mentioned result, Steenbrink has shown the following.

Theorem 2.2 (Steenbrink [**St2**]). *Suppose that f is a nondegenerate quasihomogeneous polynomial in $\boldsymbol{C}^{n+1}$. When n is even, we have the following.*

$$\mu(f)_0 = \sum_{\alpha \in \boldsymbol{Z}} c_\alpha,$$

$$\mu(f)_\lambda^+ = \sum_{\lambda = \exp(-2\pi i \alpha),\, [\alpha]:\,\mathrm{even}} c_\alpha \quad (\lambda \neq 1),$$

$$\mu(f)_\lambda^- = \sum_{\lambda = \exp(-2\pi i \alpha),\, [\alpha]:\,\mathrm{odd}} c_\alpha \quad (\lambda \neq 1),$$

where $[\alpha]$ denotes the integer part of α.

Note that, in the above situation, the kernel of S coincides with $H^n(F; \boldsymbol{C})_1$. Using the above theorem, we prove the following.

Lemma 2.3. *Let f and g be nondegenerate quasihomogeneous polynomials in $\boldsymbol{C}^{n+1}$. Then their Seifert forms are equivalent over the real numbers if and only if $P_f(t) \equiv P_g(t) \mod t^2 - 1$.*

Proof. By considering the stabilization as in Lemma 2.1 (see [**AGVII**, §2.8]) if necessary, we may assume that n is even. First suppose that f and g have equivalent real Seifert forms. Then their characteristic polynomials of the monodromy operator coincide. This is because they are equal to

$$\det(t \cdot I - T) = \det(t \cdot I - (-1)^{n+1} L \overline{(L^{-1})^t})$$
$$= c \cdot \det(t \cdot \overline{L^t} - (-1)^{n+1} L)$$

for some nonzero complex number c, since $T = (-1)^{n+1} L \overline{(L^{-1})^t}$ by Lemma 2.1 and the Seifert form $L_{\boldsymbol{Z}}$ over the integers is always unimodular (see [**D**]). Thus the eigenvalues of their monodromy operators are the same. Furthermore, $\mu(f)_0 = \mu(g)_0$ and $\mu(f)^{\pm}_{\lambda} = \mu(g)^{\pm}_{\lambda}$ for each eigenvalue $\lambda \neq 1$. This follows from the facts that the sesquilinearized complex Seifert form L splits into the orthogonal direct sum of L_λ, where L_λ is the Seifert form restricted to the eigenspace corresponding to the eigenvalue λ, and that $S_\lambda = L_\lambda + (-1)^n \overline{L^t_\lambda}$ by Lemma 2.1.

Thus, by virtue of Theorem 2.2, we have

$$\sum_{\alpha \in \boldsymbol{Z}} c_\alpha = \sum_{\beta \in \boldsymbol{Z}} c'_\beta,$$

$$\sum_{\lambda = \exp(-2\pi i \alpha),\, [\alpha]:\text{even}} c_\alpha = \sum_{\lambda = \exp(-2\pi i \beta),\, [\beta]:\text{even}} c'_\beta, \tag{2.1}$$

$$\sum_{\lambda = \exp(-2\pi i \alpha),\, [\alpha]:\text{odd}} c_\alpha = \sum_{\lambda = \exp(-2\pi i \beta),\, [\beta]:\text{odd}} c'_\beta \tag{2.2}$$

for each $\lambda \neq 1$, where $P_f(t) = \sum_{\alpha \in \boldsymbol{Q}} c_\alpha t^\alpha$ and $P_g(t) = \sum_{\beta \in \boldsymbol{Q}} c'_\beta t^\beta$. Furthermore, by the symmetric property of the spectrum, we have

$$c_\alpha = c_{n-1-\alpha} \quad \text{and} \quad c'_\beta = c'_{n-1-\beta}$$

for every α and β (see [**AGVII**, p. 384 (i)] and [**SSS**, Theorem 1.1 (ii)]). In particular, we have

$$\sum_{\alpha \in \boldsymbol{Z},\, \alpha:\text{even}} c_\alpha = \sum_{\beta \in \boldsymbol{Z},\, \beta:\text{even}} c'_\beta, \tag{2.3}$$

$$\sum_{\alpha \in \boldsymbol{Z},\, \alpha:\text{odd}} c_\alpha = \sum_{\beta \in \boldsymbol{Z},\, \beta:\text{odd}} c'_\beta, \tag{2.4}$$

since $n-1$ is odd. Then by (2.1)–(2.4), it is not difficult to see that for each λ,

$$\sum_{\lambda = \exp(-2\pi i \alpha)} c_\alpha t^\alpha - \sum_{\lambda = \exp(-2\pi i \beta)} c'_\beta t^\beta$$

is divisible by $t^2 - 1$ in $\boldsymbol{Z}[t^{1/m}]$ for some positive integer m. Thus we have $P_f(t) \equiv P_g(t) \mod t^2 - 1$.

Conversely, suppose that $P_f(t) \equiv P_g(t) \mod t^2 - 1$. Then by the above argument, we see that the eigenvalues of the monodromy operators are the same and that $\mu(f)_0 = \mu(g)_0$ and $\mu(f)_\lambda^\pm = \mu(g)_\lambda^\pm$ for each eigenvalue $\lambda \neq 1$. Thus, it suffices to show that the characteristic polynomial of the monodromy operator and the equivariant signatures determine the real Seifert form (when n is even).

Recall that $H_n(F;\boldsymbol{C})$ decomposes as $\bigoplus_\lambda H_n(F;\boldsymbol{C})_\lambda$ and that L decomposes as the orthogonal direct sum of $L_\lambda = L|_{H_n(F;\boldsymbol{C})_\lambda}$, where $H_n(F;\boldsymbol{C})_\lambda$ is the eigenspace of H corresponding to the eigenvalue λ. If we put $H_\lambda = H|_{H_n(F;\boldsymbol{C})_\lambda}$, then we have $H_\lambda = \lambda \cdot \mathrm{id}$, where id denotes the identity map. Thus if $\lambda \neq 1$, then $\det(I - \overline{H_\lambda}) \neq 0$ and hence we have $L_\lambda = S_\lambda(I - \overline{H_\lambda})^{-1}$ by Lemma 2.1. Since the equivariant signatures determine S_λ, L_λ is also determined up to isomorphism when $\lambda \neq 1$. If $\lambda = 1$, then by Lemma 2.1 we have $0 = I - \overline{H_1} = L_1^{-1} S_1$, and hence $S_1 = 0$. Thus we have $L_1 = (-1)^{n+1}\overline{L_1^t}$; i.e. $L_1 = -\overline{L_1^t}$ by Lemma 2.1. Since $1 \in \boldsymbol{R}$, we may suppose that a matrix representative of L_1 is real. Then $\mu_0 = \operatorname{rank} L_1$ must be even, since otherwise we have $\det L_1 = \det(-L_1^t) = -\det(L_1) = 0$, which is a contradiction. Then, it is easy to see that there exists a nonsingular matrix $P \in GL(\mu_0, \boldsymbol{R})$ such that

$$P^t L_1 P = \begin{pmatrix} J & & & 0 \\ & J & & \\ & & \ddots & \\ 0 & & & J \end{pmatrix},$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence L_1 is determined only by μ_0.

Thus we have shown that the characteristic polynomial of the monodromy operator and the equivariant signatures completely determine the sesquilinearized complex Seifert form $L = \bigoplus_\lambda L_\lambda$. Since the characteristic polynomial is of real coefficients, λ is an eigenvalue if and only if $\bar{\lambda}$ is. Then for a fixed λ_0, we see easily that $(\bigoplus_{\lambda=\lambda_0} L_\lambda) \oplus (\bigoplus_{\lambda=\bar{\lambda}_0} L_\lambda)$ comes from a real form and is determined over $\boldsymbol{R}$ only by the rank. Thus we have shown that the characteristic polynomial of the monodromy operator and the equivariant signatures completely determine the real Seifert form $L_{\boldsymbol{R}}$ when n is even. This completes the proof. $\square$

We note that one can prove Lemma 2.3 by directly using the main theorem of [**Ne**]. Here we have given a proof which is somewhat elementary.

Yoshinaga [**Yo1**] has given a necessary and sufficient condition on the weights for two nondegenerate quasihomogeneous polynomials to have the same characteristic polynomial. He has used a result of Milnor and Orlik [**MO**]. Here we give an alternative proof of Yoshinaga's result by using the argument in the proof of Lemma 2.3 as follows.

Proposition 2.4 (Yoshinaga [**Yo1**]). *Let f and g be nondegenerate quasihomogeneous polynomials. Let the weights of f and g be denoted by $(w_1, \dots, w_{n+1}) = (u_1/v_1, \dots, u_{n+1}/v_{n+1})$ and $(w'_1, \dots, w'_{n+1}) = (u'_1/v'_1, \dots, u'_{n+1}/v'_{n+1})$ respectively, where u_j and v_j (resp. u'_j and v'_j) are relatively prime positive integers and $u_j/v_j, u'_j/v'_j \geq 2$. Then $\Delta_f(t) = \Delta_g(t)$ if and only if for every integer u with $u \geq 2$, we have*

$$\prod_{u_j = u} \left(1 - \frac{u_j}{v_j}\right) = \prod_{u'_j = u} \left(1 - \frac{u'_j}{v'_j}\right),$$

where a product over an empty set is equal to 1.

Remark 2.5. Yoshinaga [**Yo**] considers the condition that $\{2, u_1, \dots, u_{n+1}\} = \{2, u'_1, \dots, u'_{n+1}\}$ as well. However, this condition is redundant, since, as is easily seen, this is a consequence of the condition in Proposition 2.4.

Proof of Proposition 2.4. We see that $\Delta_f(t) = \Delta_g(t)$ if and only if the eigenvalues of their monodromy operators coincide. If we write

$$P_f(t) = \sum_{\alpha \in \boldsymbol{Q}} c_\alpha t^\alpha \quad \text{and} \quad P_g(t) = \sum_{\beta \in \boldsymbol{Q}} c'_\beta t^\beta,$$

then we have $Sp(f) = \sum_{\alpha \in \boldsymbol{Q}} c_\alpha(\alpha - 1)$ and $Sp(g) = \sum_{\beta \in \boldsymbol{Q}} c'_\beta(\beta - 1)$. Thus we have, by the definition of the spectrum,

$$\sum_{\alpha \equiv \gamma \mod 1} c_\alpha = \dim_{\boldsymbol{C}} H^n(F_f; \boldsymbol{C})_\lambda,$$

$$\sum_{\beta \equiv \gamma \mod 1} c'_\beta = \dim_{\boldsymbol{C}} H^n(F_g; \boldsymbol{C})_\lambda$$

for each $\gamma \in \boldsymbol{Q}$, where $\lambda = \exp(-2\pi i \gamma)$ and F_f and F_g are the Milnor fibers of f and g respectively. Thus if $\Delta_f(t) = \Delta_g(t)$, then $\dim_{\boldsymbol{C}} H^n(F_f; \boldsymbol{C})_\lambda = \dim_{\boldsymbol{C}} H^n(F_g; \boldsymbol{C})_\lambda$ for each λ, and then

$$\sum_{\lambda = \exp(-2\pi i \alpha)} c_\alpha t^\alpha - \sum_{\lambda = \exp(-2\pi i \beta)} c'_\beta t^\beta$$

is divisible by $t-1$ in $\boldsymbol{Z}[t^{1/m}]$, where m is a common multiple of $u_1, \dots, u_{n+1}, u_1', \dots, u_{n+1}'$. Hence $P_f(t) \equiv P_g(t) \mod t-1$. We can prove the converse similarly. Thus we have proved that $\Delta_f(t) = \Delta_g(t)$ if and only if $P_f(t) \equiv P_g(t) \mod t-1$. We note that this fact is also observed in [**Ne**, §1].

Putting $s = t^{1/m}$, we set

$$Q_f(s) = P_f(t) = \prod_{j=1}^{n+1} \frac{s^m - s^{mv_j/u_j}}{s^{mv_j/u_j} - 1},$$

$$Q_g(s) = P_g(t) = \prod_{j=1}^{n+1} \frac{s^m - s^{mv_j'/u_j'}}{s^{mv_j'/u_j'} - 1}.$$

Note that $Q_f(s)$ and $Q_g(s)$ are polynomials in s with nonnegative integer coefficients. Then $P_f(t) \equiv P_g(t) \mod t-1$ if and only if $Q_f(s) \equiv Q_g(s) \mod s^m - 1$, which in turn is equivalent to that $Q_f(\zeta^k) = Q_g(\zeta^k)$ for all $k \in \boldsymbol{Z}$, where $\zeta = \exp(2\pi i/m)$.

For $h(s) = (s^m - s^l)/(s^l - 1)$ with $0 < l < m$, we have

$$h(\xi) = \frac{m\xi^{m-1} - l\xi^{l-1}}{l\xi^{l-1}} = \frac{m}{l} - 1$$

for $\xi \in \boldsymbol{C}$ with $\xi^l = \xi^m = 1$ by the l'Hopital rule. Thus we have

$$\begin{aligned}
Q_f(\zeta^k) &= \prod_{u_j \nmid k} \frac{1 - \zeta^{kmv_j/u_j}}{\zeta^{kmv_j/u_j} - 1} \prod_{u_j \mid k} \left(\frac{m}{mv_j/u_j} - 1\right) \\
&= \prod_{u_j \nmid k} (-1) \prod_{u_j \mid k} \left(\frac{u_j}{v_j} - 1\right) \\
&= \prod_{u_j \nmid k} (-1) \prod_{u_j \mid k} (-1) \prod_{u_j \mid k} \left(1 - \frac{u_j}{v_j}\right) \\
&= (-1)^{n+1} \prod_{u_j \mid k} \left(1 - \frac{u_j}{v_j}\right).
\end{aligned}$$

Hence we see that $Q_f(\zeta^k) = Q_g(\zeta^k)$ if and only if

$$\prod_{u_j \mid k} \left(1 - \frac{u_j}{v_j}\right) = \prod_{u_j' \mid k} \left(1 - \frac{u_j'}{v_j'}\right).$$

Thus we see that $Q_f(\zeta^k) = Q_g(\zeta^k)$ for all $k \in \boldsymbol{Z}$ if and only if for every integer u with

$u \geq 2$, we have

$$\prod_{u_j=u}\left(1-\frac{u_j}{v_j}\right)=\prod_{u_j'=u}\left(1-\frac{u_j'}{v_j'}\right).$$

This completes the proof. $\square$

As to the condition for two nondegenerate quasihomogeneous polynomials to have equivalent real Seifert forms, we have the following.

Proposition 2.6. *Let f and g be nondegenerate quasihomogeneous polynomials as in Proposition* 2.4. *Then f and g have equivalent Seifert forms over the real numbers if and only if the following three are satisfied.*

(1) *For every integer u with $u \geq 2$, we have*

$$\prod_{u_j=u}\left(1-\frac{u_j}{v_j}\right)=\prod_{u_j'=u}\left(1-\frac{u_j'}{v_j'}\right),$$

where a product over an empty set is equal to 1.

(2) *For every odd integer k,*

$$\sharp\{j : u_j|k \text{ and } v_j \text{ is odd}\} = \sharp\{j : u_j|k \text{ and } v_j \text{ is even}\}$$

if and only if

$$\sharp\{j : u_j'|k \text{ and } v_j' \text{ is odd}\} = \sharp\{j : u_j'|k \text{ and } v_j' \text{ is even}\},$$

where $\sharp$ denotes the number of elements in the set.

(3) *For every odd integer k for which the equalities in* (2) *hold, we have*

$$\prod_{u_j \nmid k} i\cot\frac{kv_j\pi}{2u_j}\prod_{u_j|k,\, v_j:\text{odd}}\left(\frac{v_j}{u_j}-1\right)\prod_{u_j|k,\, v_j:\text{even}}\frac{u_j}{v_j}$$

$$=\prod_{u_j' \nmid k} i\cot\frac{kv_j'\pi}{2u_j'}\prod_{u_j'|k,\, v_j':\text{odd}}\left(\frac{v_j'}{u_j'}-1\right)\prod_{u_j'|k,\, v_j':\text{even}}\frac{u_j'}{v_j'}.$$

The above proposition easily follows from Lemma 2.3, Proposition 2.4 and the following lemma.

Lemma 2.7. *Set*

$$O_k = \sharp\{j : u_j|k \text{ and } v_j \text{ is odd}\},$$

$$E_k = \sharp\{j : u_j|k \text{ and } v_j \text{ is even}\}$$

for every odd integer k. Then we have $O_k \geq E_k$ for every odd integer k. Furthermore, under the same notation as in the proof of Proposition 2.4, *we have*

$$Q_f(\eta^k) = \begin{cases} 0 & (\text{if } O_k > E_k) \\ \displaystyle\prod_{u_j \nmid k} i \cot \frac{kv_j\pi}{2u_j} \prod_{u_j|k,\, v_j:\text{odd}} \left(\frac{v_j}{u_j} - 1\right) \prod_{u_j|k,\, v_j:\text{even}} \frac{u_j}{v_j} & (\text{if } O_k = E_k), \end{cases}$$

for every odd integer k, where $\eta = \exp(\pi i/m)$.

Proof. It is easy to see that for an odd integer k, $(\eta^k)^{mv_j/u_j} - 1 = 0$ if and only if $u_j|k$ and v_j is even. Furthermore, $(\eta^k)^m - (\eta^k)^{mv_j/u_j} = 0$ if and only if $u_j|k$ and v_j is odd. Then the first assertion of the lemma follows from the fact that $Q_f(s)$ is a polynomial in s and hence that $Q_f(\eta^k)$ is a complex number for each odd integer k. Furthermore, if $\sharp\{j : u_j|k \text{ and } v_j \text{ is odd}\} > \sharp\{j : u_j|k \text{ and } v_j \text{ is even}\}$, then $Q_f(\eta^k) = 0$. When $\sharp\{j : u_j|k \text{ and } v_j \text{ is odd}\} = \sharp\{j : u_j|k \text{ and } v_j \text{ is even}\}$, we have

$$\begin{aligned} Q_f(\eta^k) &= \prod_{u_j \nmid k} \frac{-(\eta^{kmv_j/u_j} + 1)}{\eta^{kmv_j/u_j} - 1} \prod_{u_j|k,\, v_j:\text{odd}} \frac{-(m - mv_j/u_j)}{\eta^{kmv_j/u_j} - 1} \prod_{u_j|k,\, v_j:\text{even}} \frac{-(\eta^{kmv_j/u_j} + 1)}{mv_j/u_j} \\ &= \prod_{u_j \nmid k} i \cot \frac{kv_j\pi}{2u_j} \prod_{u_j|k,\, v_j:\text{odd}} \left(\frac{v_j}{u_j} - 1\right) \prod_{u_j|k,\, v_j:\text{even}} \frac{u_j}{v_j}. \end{aligned}$$

This completes the proof. □

3. Proof of Theorem 1.1 and its corollaries.

Let f and g be nondegenerate quasihomogeneous polynomials in $\boldsymbol{C}^{n+1}$. We denote by $(w_1, \dots, w_{n+1}) = (u_1/v_1, \dots, u_{n+1}/v_{n+1})$ and $(w_1', \dots, w_{n+1}') = (u_1'/v_1', \dots, u_{n+1}'/v_{n+1}')$ the weights of f and g respectively, where u_j and v_j (resp. u_j' and v_j') are relatively prime positive integers and $w_j, w_k' \geq 2$.

Lemma 3.1. *Suppose that f and g have equivalent Seifert forms over the real numbers. If*

$$\sum_{j=1}^{n+1} \frac{1}{w_j} + \sum_{j=1}^{n+1} \frac{1}{w_j'} - 2 \min\left\{\frac{1}{w_1}, \dots, \frac{1}{w_{n+1}}, \frac{1}{w_1'}, \dots, \frac{1}{w_{n+1}'}\right\} < 1,$$

then we have $w_j = w_j'$ up to order.

Proof. Let m be a common multiple of $u_1, \dots, u_{n+1}, u_1', \dots, u_{n+1}'$. By Lemma 2.3 and the proof of Proposition 2.4, we have

$$\prod_{j=1}^{n+1}(s^m - s^{mv_j/u_j})\prod_{j=1}^{n+1}(s^{mv'_j/u'_j} - 1) - \prod_{j=1}^{n+1}(s^m - s^{mv'_j/u'_j})\prod_{j=1}^{n+1}(s^{mv_j/u_j} - 1)$$

$$= R(s)(s^{2m} - 1)\prod_{j=1}^{n+1}(s^{mv_j/u_j} - 1)\prod_{j=1}^{n+1}(s^{mv'_j/u'_j} - 1)$$

for some $R(s) \in \boldsymbol{Z}[s]$. Substituting η with $\eta^m = -1$ for s, we have

$$(-1)^{n+1}\prod_{j=1}^{n+1}(\eta^{mv_j/u_j} + 1)\prod_{j=1}^{n+1}(\eta^{mv'_j/u'_j} - 1)$$

$$-(-1)^{n+1}\prod_{j=1}^{n+1}(\eta^{mv'_j/u'_j} + 1)\prod_{j=1}^{n+1}(\eta^{mv_j/u_j} - 1) = 0.$$

Thus, putting

$$F(z) = \prod_{j=1}^{n+1}(z^{mv_j/u_j} + 1)\prod_{j=1}^{n+1}(z^{mv'_j/u'_j} - 1) - \prod_{j=1}^{n+1}(z^{mv'_j/u'_j} + 1)\prod_{j=1}^{n+1}(z^{mv_j/u_j} - 1),$$

we have $F(\eta) = 0$ for all η with $\eta^m = -1$. On the other hand, we have

$$F(z) = \left(\sum_{j=0}^{n+1} p_j\right)\left(\sum_{k=0}^{n+1}(-1)^{n+1-k}p'_k\right) - \left(\sum_{k=0}^{n+1} p'_k\right)\left(\sum_{j=0}^{n+1}(-1)^{n+1-j}p_j\right)$$

$$= \sum_{j-k\,:\,\mathrm{odd}}((-1)^{n+1-k} - (-1)^{n+1-j})p_j p'_k,$$

where p_k and p'_k are the k-th symmetric polynomials of $\{z^{mv_j/u_j}\}_{j=1}^{n+1}$ and $\{z^{mv'_j/u'_j}\}_{j=1}^{n+1}$ respectively. Thus we have $F(z) = z^{\beta}V(z)$ for some nonnegative integer β and for some $V(z) \in \boldsymbol{Z}[z]$ whose degree is smaller than or equal to

$$m\sum_{j=1}^{n+1}\left(\frac{v_j}{u_j} + \frac{v'_j}{u'_j}\right) - 2\alpha,$$

where $\alpha = \min\{mv_1/u_1, \dots, mv_{n+1}/u_{n+1}, mv'_1/u'_1, \dots, mv'_{n+1}/u'_{n+1}\}$. Note that $V(z)$ has at least m distinct zeros. By our hypothesis, we see that

$$m > m\sum_{j=1}^{n+1}\left(\frac{v_j}{u_j} + \frac{v'_j}{u'_j}\right) - 2\alpha \geq \text{degree } V(z).$$

Thus the polynomial $F(z)$ must be zero, and hence we have

$$\sum_{j-k:\text{odd}}((-1)^{n+1-k}-(-1)^{n+1-j})p_jp_k'=0.$$

By looking at the lowest degree term, we see that

$$\min\{1/w_1,\dots,1/w_{n+1}\}=\min\{1/w_1',\dots,1/w_{n+1}'\}.$$

Thus we may assume that $w_{n+1}=w_{n+1}'$. Then we have

$$\prod_{j=1}^{n}(z^{mv_j/u_j}+1)\prod_{j=1}^{n}(z^{mv_j'/u_j'}-1)-\prod_{j=1}^{n}(z^{mv_j'/u_j'}+1)\prod_{j=1}^{n}(z^{mv_j/u_j}-1)=0.$$

Looking at the lowest degree term, we obtain

$$\min\{1/w_1,\dots,1/w_n\}=\min\{1/w_1',\dots,1/w_n'\}.$$

Repeating this procedure, we have $w_j=w_j'$ up to order. This completes the proof. □

LEMMA 3.2. *Suppose that f and g have equivalent Seifert forms over the real numbers. If*

$$\min\left\{\sum_{j=1}^{n+1}\frac{1}{w_j},\sum_{j=1}^{n+1}\frac{1}{w_j'}\right\}\geq\frac{n-1}{2},$$

then we have $w_j=w_j'$ up to order.

PROOF. It is easy to see that the lowest degree term of $P_f(t)$ (resp. $P_g(t)$) is equal to t^{γ} (resp. $t^{\gamma'}$) and the highest one is equal to $t^{n+1-\gamma}$ (resp. $t^{n+1-\gamma'}$), where $\gamma=\sum_{j=1}^{n+1}1/w_j$ (resp. $\gamma'=\sum_{j=1}^{n+1}1/w_j'$). Thus, by our hypothesis, we see that $P_f(t)=t^{\gamma}+P_f'(t)+t^{n+1-\gamma}$ and $P_g(t)=t^{\gamma'}+P_g'(t)+t^{n+1-\gamma'}$, where each monomial of $P_f'(t)$ and $P_g'(t)$ is of degree in the interval $((n-1)/2,(n+3)/2)$ (when $\sum_{j=1}^{n+1}1/w_j=(n+1)/2$ (or $\sum_{j=1}^{n+1}1/w_j'=(n+1)/2$), we have $P_f(t)=t^{(n+1)/2}$ (resp. $P_g(t)=t^{(n+1)/2}$)). Thus $P_f(t)\equiv P_g(t)\bmod t^2-1$ if and only if $P_f(t)=P_g(t)$. Hence $w_j=w_j'$ up to order (see [**SSS**, Example 5.2]). This completes the proof. □

LEMMA 3.3. *Suppose that f and g have equivalent Seifert forms over the real numbers. If $n=2$, then the hypothesis of Lemma* 3.1 *or* 3.2 *is always satisfied.*

PROOF. Suppose that

$$\sum_{j=1}^{3}\frac{1}{w_j}+\sum_{j=1}^{3}\frac{1}{w_j'}-2\min\left\{\frac{1}{w_1},\frac{1}{w_2},\frac{1}{w_3},\frac{1}{w_1'},\frac{1}{w_2'},\frac{1}{w_3'}\right\}\geq 1$$

and $\gamma' = \sum_{j=1}^{3} 1/w'_j < 1/2$. Since $\sum_{j=1}^{3} 1/w_j + \sum_{j=1}^{3} 1/w'_j > 1$, we have $\sum_{j=1}^{3} 1/w_j > \sum_{j=1}^{3} 1/w'_j$ and $\sum_{j=1}^{3} 1/w'_j + 2 > 3 - \sum_{j=1}^{3} 1/w_j$. Thus, putting $Sp(f) = \sum_{\alpha \in \boldsymbol{Q}} c_\alpha(\alpha - 1)$ and $Sp(g) = \sum_{\beta \in \boldsymbol{Q}} c'_\beta(\beta - 1)$, we have $c_{\gamma'} = 0$, $c'_{\gamma'} = 1$, $c_{\gamma'+2} = 0$. By Theorem 2.2, we have $c_{\gamma'} + c_{\gamma'+2} = c'_{\gamma'} + c'_{\gamma'+2}$, since f and g have equivalent real Seifert forms and hence the same equivariant signatures. This implies that $c'_{\gamma'+2} = -1$, which is a contradiction. This completes the proof. □

Proof of Theorem 1.1. Combining Lemmas 3.1, 3.2 and 3.3, we obtain Theorem 1.1, since the weights determine the spectrum (see [**SSS**, Example 5.2] and also §2 of the present paper). □

Proof of Corollary 1.2. Since the Seifert form is a topological invariant (see [**Sae2**], [**D**], [**K**]), we easily obtain Corollary 1.2. □

Example 3.4. In [**Yo1**, Example 3], Yoshinaga considers the nondegenerate quasihomogeneous polynomials $f(z_1, z_2, z_3, z_4) = f_1(z_2, z_3, z_4) + z_1^2 = z_1^2 + z_2^2 + z_3^2 + z_4^{13}$ of weights $(2, 2, 2, 13)$ and $g(z_1, z_2, z_3, z_4) = g_1(z_2, z_3, z_4) + z_1^2 = z_1^2 + z_2^3 z_3 + z_3^2 z_4 + z_2 z_4^2$ of weights $(2, 13/3, 13/4, 13/5)$. The two polynomials do not have equivalent real Seifert forms, since the nondegenerate quasihomogeneous polynomials f_1 and g_1 of weights $(2, 2, 13)$ and $(13/3, 13/4, 13/5)$ respectively do not have equivalent real Seifert forms by Corollaries 1.2 and 1.4 and the Seifert forms of f and g are equivalent if and only if those of f_1 and g_1 are equivalent (see Lemma 2.1). Note that the characteristic polynomials of f and g coincide as is seen in [**Yo1**].

Lemma 3.5. *Let f and $g : (\boldsymbol{C}^{n+1}, 0) \to (\boldsymbol{C}, 0)$ be holomorphic function germs with an isolated critical point at the origin. If f and g are connected by a μ-constant deformation, then their Seifert forms are equivalent over the integers.*

Proof. As is shown in [**LR**, Proof of Theorem 2.1], we may assume that there exist sufficiently small positive real numbers $\varepsilon, \varepsilon', \delta$ with $0 < \delta \ll \varepsilon' \ll \varepsilon$ satisfying the following properties.

(1) S_ε^{2n+1} (resp. $S_{\varepsilon'}^{2n+1}$) intersects $f^{-1}(0)$ (resp. $g^{-1}(0)$) transversely, where $S_\varepsilon^{2n+1} = \partial D_\varepsilon^{2n+2}$ (resp. $S_{\varepsilon'}^{2n+1} = \partial D_{\varepsilon'}^{2n+2}$) and D_ε^{2n+2} (resp. $D_{\varepsilon'}^{2n+2}$) is the closed $(2n+2)$-dimensional disk in $\boldsymbol{C}^{n+1}$ of radius ε (resp. ε') centered at the origin.

(2) The maps $f : f^{-1}(S_\delta^1) \cap D_\varepsilon^{2n+2} \to S_\delta^1$ and $g : g^{-1}(S_\delta^1) \cap D_{\varepsilon'}^{2n+2} \to S_\delta^1$ are the Milnor fibrations of f and g respectively (see [**M**]), where S_δ^1 is the circle in $\boldsymbol{C}$ of radius δ centered at the origin.

(3) There exists an embedding $\varphi : g^{-1}(S_\delta^1) \cap D_{\varepsilon'}^{2n+2} \to f^{-1}(S_\delta^1) \cap D_\varepsilon^{2n+2}$ such that $f \circ \varphi = g$.

(4) On each Milnor fiber $g^{-1}(a) \cap D_{\varepsilon'}^{2n+2}$ of g $(a \in S_{\delta}^{1})$, the embedding $\varphi : g^{-1}(a) \cap D_{\varepsilon'}^{2n+2} \to f^{-1}(a) \cap D_{\varepsilon}^{2n+2}$ is a homotopy equivalence.

We warn the reader that the condition (3) does not necessarily imply that f and g are topologically right equivalent. Recall that the Seifert form of g is defined by using the linking numbers of n-cycles in $g^{-1}(a) \cap D_{\varepsilon'}^{2n+2}$ with their push-offs in $\partial(g^{-1}(D_{\delta}^{2}) \cap D_{\varepsilon'}^{2n+2})$, where D_{δ}^{2} is the disk in $\boldsymbol{C}$ of radius δ centered at the origin (see §2); i.e. the Seifert form of g is defined by using the intersection numbers of certain chains in $g^{-1}(D_{\delta}^{2}) \cap D_{\varepsilon'}^{2n+2}$, which is homeomorphic to the $(2n+2)$-dimensional disk. Since the embedding φ extends to an embedding of $g^{-1}(D_{\delta}^{2}) \cap D_{\varepsilon'}^{2n+2}$ into $f^{-1}(D_{\delta}^{2}) \cap D_{\varepsilon}^{2n+2}$, we see that the Seifert forms of f and g are isomorphic via $(\varphi|_{g^{-1}(a) \cap D_{\varepsilon'}^{2n+2}})_{*} : H_n(g^{-1}(a) \cap D_{\varepsilon'}^{2n+2}; \boldsymbol{Z}) \to H_n(f^{-1}(a) \cap D_{\varepsilon}^{2n+2}; \boldsymbol{Z})$. This completes the proof. $\square$

Proof of Corollary 1.3. Suppose that f (resp. g) is connected by a μ-constant deformation to a nondegenerate quasihomogeneous polynomial f_0 (resp. g_0). If f and g have equivalent real Seifert forms, then so do f_0 and g_0 by Lemma 3.5. Thus the spectra of f_0 and g_0 coincide by Theorem 1.1. On the other hand, it has been shown that the spectrum is invariant under μ-constant deformations by Varchenko [**V1**], [**V2**] (see also [**St3**]). Thus the spectra of f and g also coincide. Thus (1) implies (4).

If f and g have the same spectrum, so do f_0 and g_0. Thus f_0 and g_0 have the same weights (see [**SSS**, Example 5.2]). Then by [**O1**, §4], [**DG**, §6], or [**XY1**, Theorem 3.5], f_0 and g_0 are connected by a topologically constant and hence μ-constant deformation (see also Remark 3.8 below). Thus (4) implies (5).

By Lemma 3.5, (5) implies (2). Furthermore, (2) trivially implies (1). Finally, as has been pointed out in the proof of Lemma 2.3, (1) and (3) are equivalent to each other. This completes the proof. $\square$

Remark 3.6. Let f and g be nondegenerate quasihomogeneous polynomials in $\boldsymbol{C}^3$. If they are connected by a μ-constant deformation, then their Seifert forms are equivalent by Lemma 3.5 and hence they have the same spectrum by Theorem 1.1. Thus we have shown that if f and g are connected by a μ-constant deformation, then they have the same spectrum. In fact, this is always true for arbitrary holomorphic function germs in $\boldsymbol{C}^{n+1}$ with an isolated critical point at the origin, which is a result of Varchenko [**V1**], [**V2**]. Our proof above gives an alternative proof of this result in our special case.

Proof of Corollary 1.4. By [**DG**, Corollary 5] (see also [**O2**], [**O3**], [**Yo2**] and [**LR**, Remark 2.5]), we see that f and g can be connected to nondegenerate quasihomogeneous polynomials f_0 and g_0 respectively by topologically trivial (and hence μ-constant) deformations. Thus we can apply Corollary 1.3. Furthermore, if one of the conditions (1)–(5) is satisfied, f_0 and g_0 are connected by a topologically constant deformation. Thus f and g are also so connected and (6) holds. Conversely (6) implies (2). Furthermore, by [**Ya2**], [**Sae1**], (6) and (7) are equivalent to each other.

Note that f and f_0 (resp. g and g_0) have the same multiplicity. Since the multiplicity is determined by the weights for nondegenerate quasihomogeneous polynomials (see [**Sae1**]), we see that if one of the seven conditions is satisfied, f_0 and g_0 have the same multiplicity. Thus f and g have the same multiplicity. This completes the proof. $\square$

Remark 3.7. Let f and g be nondegenerate quasihomogeneous polynomials in $\boldsymbol{C}^{n+1}$ $(n \geq 1)$. Then the following four are equivalent.

(1) f and g are connected by a μ-constant deformation.

(2) f and g are connected by a topologically constant deformation.

(3) f and g have the same weights.

(4) f and g have the same spectrum.

When f and g are semiquasihomogeneous holomorphic function germs in $\boldsymbol{C}^{n+1}$, (1), (2) and (4) are equivalent to each other. Furthermore, if one of the conditions is satisfied, f and g have the same multiplicity.

The above facts can be proved by using a result of Varchenko [**V1**], [**V2**], an observation given in [**SSS**, Example 5.2], and Remark 3.8 below. Furthermore, the above facts imply that the "weights" of a semiquasihomogeneous holomorphic function germ are well-defined.

Remark 3.8. For a nondegenerate quasihomogeneous polynomial, the weights completely determine the topological type. This is a consequence of the fact that if two nondegenerate quasihomogeneous polynomials have the same weights, then they can be connected by a μ-constant deformation which has a uniform stable radius (see [**O1**], [**O2**], [**O3**]).

Remark 3.9. Note that in general the Seifert form (over the integers) does not determine the topological type of a holomorphic function germ in $\boldsymbol{C}^3$ with an isolated critical point at the origin (see [**AB1**]). Furthermore, in Corollary 1.4, (7) does not necessarily imply (6) for holomorphic function germs in $\boldsymbol{C}^3$ in general. See [**AB2**, Corollaire 5.6.6].

PROOF OF COROLLARY 1.5. The assertion holds for all holomorphic function germs in $\boldsymbol{C}^{n+1}$ with an isolated critical point at the origin for all $n \geq 3$, as has been pointed out in §1. When $n = 2$, if f and g are topologically equivalent, then so are $\tilde{f}$ and $\tilde{g}$, since they have equivalent Seifert forms. Conversely, if $\tilde{f}$ and $\tilde{g}$ are topologically equivalent, f and g have equivalent Seifert forms and hence they are topologically equivalent by Theorem 1.1 and Remark 3.8 (or by Corollary 1.4). The case where $n = 1$ follows from Remarks 3.8 and 3.10 below. This completes the proof. □

REMARK 3.10. When $n = 1$, the hypothesis of Lemma 3.2 is always satisfied. Thus for nondegenerate quasihomogeneous polynomials in $\boldsymbol{C}^2$, the weights are determined by the real Seifert form. Then it is not difficult to show that for nondegenerate quasihomogeneous polynomials f and g in $\boldsymbol{C}^2$, the following six are equivalent.

(1) f and g have the same topological type.

(2) f and g have the same characteristic polynomial and their stabilizations $\tilde{f}$ and $\tilde{g}$ have the same equivariant signatures.

(3) f and g have the same spectrum.

(4) f and g have the same weights (we assume that all the weights are greater than or equal to 2).

(5) f and g have equivalent Seifert forms over the real numbers.

(6) f and g have equivalent Seifert forms over the integers.

In fact, it has already been seen that (1) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Leftrightarrow$ (3) and (4) $\Rightarrow$ (1). Furthermore, by using our argument, we can show that (2) $\Leftrightarrow$ (5).

Note that the above statement is true also for $n = 2$ (see Corollary 1.4). We do not know if a corresponding statement of Theorem 1.1 for four variables is true or not. Probably one can use the list of nondegenerate quasihomogeneous polynomials of four variables obtained in [**YS2**] and can use the results obtained in §2 to attack this problem. Note that we do not even know if the multiplicity is an invariant of the real Seifert form for nondegenerate quasihomogeneous polynomials of four variables.

REMARK 3.11. As is noted in §1, the consequence of Corollary 1.5 holds for all holomorphic function germs in $\boldsymbol{C}^{n+1}$ with an isolated critical point at the origin, provided that $n \geq 3$. For $n = 2$, this is not true. For example, Artal-Bartolo [**AB1**] has given two function germs of three variables which have equivalent Seifert forms over the integers but which have different topological types. Then their stabilizations are topologically equivalent, since they have equivalent Seifert forms over the integers (see [**D**], [**K**]). For $n = 1$, the author does not know if the consequence of Corollary 1.5 holds for all holomorphic function germs in $\boldsymbol{C}^2$ with an isolated critical point at the origin. If the plane curve $f^{-1}(0)$ is irreducible, then this is true. This can be proved by using the fact that the topological type of such a function germ is determined by the characteristic polynomial of the monodromy (see [**B**], [**Le**]). For general function germs, Laufer [**La**] obtains some results, although they are not enough to answer to the problem[1].

4. Real and complex Seifert matrices.

In this section, using our method in the previous sections, we give normal forms of real and complex Seifert matrices of a nondegenerate quasihomogeneous polynomial.

PROPOSITION 4.1. *Let f be a nondegenerate quasihomogeneous polynomial in $\boldsymbol{C}^{n+1}$. We write $P_f(t) = \sum_{\alpha \in \boldsymbol{Q}} c_\alpha t^\alpha$. Then the sesquilinearized complex Seifert form L of f is equivalent to*

$$\bigoplus_{c_\alpha \neq 0} L_{\boldsymbol{C}}^n(c_\alpha, \alpha)$$

over the complex numbers, where $L_{\boldsymbol{C}}^n(c, \alpha)$ is the $c \times c$ diagonal matrix each of whose diagonal entries is equal to $i\exp(-\pi i \alpha)$ if n is even and is equal to $\exp(-\pi i \alpha)$ if n is odd $(i = \sqrt{-1})$.

PROPOSITION 4.2. *Let f be a nondegenerate quasihomogeneous polynomial in $\boldsymbol{C}^{n+1}$. We write $P_f(t) = \sum_{\alpha \in \boldsymbol{Q}} c_\alpha t^\alpha$. Then the real Seifert form $L_{\boldsymbol{R}}$ of f is equivalent to*

$$\bigoplus_{c_\alpha \neq 0,\, \alpha \leq (n+1)/2} L_{\boldsymbol{R}}^n(c_\alpha, \alpha)$$

over the real numbers, where $L_{\boldsymbol{R}}^n(c, \alpha)$ is defined as follows:

(1) *When n is even:*

$$L_{\boldsymbol{R}}^n(c, \alpha) = \begin{cases} \bigoplus_c \begin{pmatrix} \sin \pi\alpha & \cos \pi\alpha \\ -\cos \pi\alpha & \sin \pi\alpha \end{pmatrix} & \left(\text{if } \alpha < \dfrac{n+1}{2}\right) \\ \bigoplus_c (1) & \left(\text{if } \alpha = \dfrac{n+1}{2} \text{ and } n \equiv 0 \mod 4\right) \\ \bigoplus_c (-1) & \left(\text{if } \alpha = \dfrac{n+1}{2} \text{ and } n \equiv 2 \mod 4\right). \end{cases}$$

[1]The author is indebted to Lê Dũng Tráng for calling his attention to Laufer's work.

(2) *When n is odd*:

$$L_{\boldsymbol{R}}^{n}(c,\alpha)=\begin{cases}\bigoplus_{c}\begin{pmatrix}\cos\pi\alpha & -\sin\pi\alpha\\ \sin\pi\alpha & \cos\pi\alpha\end{pmatrix} & \left(\text{if } \alpha<\dfrac{n+1}{2}\right)\\ \bigoplus_{c}(1) & \left(\text{if } \alpha=\dfrac{n+1}{2} \text{ and } n\equiv 3 \mod 4\right)\\ \bigoplus_{c}(-1) & \left(\text{if } \alpha=\dfrac{n+1}{2} \text{ and } n\equiv 1 \mod 4\right).\end{cases}$$

Proof of Propositions 4.1 and 4.2. First suppose that n is even. We have shown that the real and the complex Seifert forms of f are completely determined by the characteristic polynomial and the equivariant signatures (see the proof of Lemma 2.3 in §2). On the other hand, when n is even, we see that the matrices appearing in the propositions have the same characteristic polynomial and the same equivariant signatures as those of f by using Theorem 2.2. Thus we have the result. When n is odd, consider the stabilization $\tilde{f}$ of f as in Lemma 2.1. Then setting

$$P_f(t)=\sum_{\alpha}c_\alpha t^\alpha \quad \text{and} \quad P_{\tilde{f}}(t)=\sum_{\beta}\tilde{c}_\beta t^\beta,$$

we have $\tilde{c}_\beta=c_{\beta-(1/2)}$, since

$$P_{\tilde{f}}(t)=P_f(t)\,\frac{t-t^{1/2}}{t^{1/2}-1}=P_f(t)t^{1/2}.$$

Thus by using Lemma 2.1, we have

$$\begin{aligned}L=\tilde{L}&=\bigoplus_{\tilde{c}_\beta\neq 0}L_{\boldsymbol{C}}^{n+1}(\tilde{c}_\beta,\beta)\\ &=\bigoplus_{c_\alpha\neq 0}L_{\boldsymbol{C}}^{n+1}\left(c_\alpha,\alpha+\frac{1}{2}\right)\\ &=\bigoplus_{c_\alpha\neq 0}L_{\boldsymbol{C}}^{n}(c_\alpha,\alpha).\end{aligned}$$

For the real Seifert forms, a similar argument can be applied. This completes the proof. □

We note that Proposition 4.1 is equivalent to [**Ne**, Theorem 6.3], although the formulations are different.

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