

Transfer maps of sphere bundles

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Abstract. Generalizing the transfer maps concerned with the projective spaces, we study some fundamental properties of transfer maps for sphere bundles. We show that their cofibers are represented by Thom spectra, which enables us to calculate the e -invariants of the transfer maps. We give some concrete formula for the e -invariants of them and its application.

1. Introduction.

In this paper, we treat the transfer maps of sphere bundles, and give some formulas for their e -invariants. Our results generalize the study of the S^1 -transfer map defined for the principal S^1 -bundle over the complex projective space (cf. [11], [12], [13]).

Let $(S(V), p, B)$ be the sphere bundle associated with a vector bundle (V, p, B) of fiber dimension $v > 0$ over a finite complex B . Then, for each vector bundle W over B , we have a map $\tau_W(p) : B^{V \oplus W} \rightarrow \Sigma S(V)^{p^*W}$ called an umkehr map as in [5] (see (2.1)). $\tau_W(p)$ is a stable map between the Thom spaces, and it also has a meaning as a map between the Thom spectra when W is a virtual vector bundle. Throughout the paper, X^α denotes the Thom space (resp. spectrum) of a (resp. virtual) vector bundle α over X .

We impose an assumption on W in the above. Let $j_V : B \rightarrow B^V$ be the map defined by the zero section of V , and assume that there exists a virtual vector bundle \tilde{W} over B^V with

$$(1.1) \quad j_V^* \tilde{W} \cong W \quad \text{over } B,$$

where \cong means an equivalence of virtual vector bundles. We also assume that the virtual fiber dimension of W is equal to that of \tilde{W} , and denote it by an integer b . Since $S(V) \xrightarrow{p} B \xrightarrow{j_V} B^V$ is a cofiber sequence, the equivalence of (1.1) gives a trivialization $\phi : p^*W \cong \underline{\mathbf{R}}^b$, where $\underline{\mathbf{R}}^b$ denotes the trivial vector bundle of dimension b .

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Let $q : \Sigma S(V)^{p^*W} \xrightarrow{T(\phi)} \Sigma S(V)^{\mathbb{R}^b} \xrightarrow{c} S^{b+1}$ be the composition of the stable homotopy equivalence $T(\phi)$ defined by ϕ and the map c defined by the collapsing map $S(V) \rightarrow \{*\}$. Then, we define a transfer map of the sphere bundle $S(V)$, which depends on \tilde{W} of (1.1) and a trivialization ϕ , to be

$$(1.2) \quad \tau_W = q \circ \tau_W(p) : B^{V \oplus W} \rightarrow S^{b+1}.$$

Then, it turns out that the transfer map is closely related with a map appearing in the James cofiber sequence [10], and we show that the cofiber $C(\tau_W)$ of τ_W is represented as follows:

THEOREM 1. $C(\tau_W)$ is stably homotopy equivalent to $\Sigma(B^V)^{\tilde{W}}$.

This theorem is based on a result in [11]. Let ξ be the canonical complex line bundle over the complex projective space CP^n . Then, the cofiber of p is CP^{n+1} , and the total space of the sphere bundle $S(\xi)$ is S^{2n+1} . For any virtual complex vector bundle W over CP^n there exists a virtual vector bundle \tilde{W} over CP^{n+1} satisfying the condition (1.1). The cofiber of the transfer map τ_W of $S(\xi)$ for $W = -\xi$ has played an important role in [11], [12], [6], [4], [8]. Thus, the following corollary is the source of Theorem 1.

COROLLARY 2. $(CP^n)^{\xi \oplus W} \xrightarrow{\tau_W} S^{b+1} \xrightarrow{i} \Sigma(CP^{n+1})^{\tilde{W}}$ is a cofiber sequence, where i is the inclusion to the bottom sphere.

Theorem 1 establishes a stable cofiber sequence $S^b \xrightarrow{i} (B^V)^{\tilde{W}} \xrightarrow{j} B^{V \oplus W} \xrightarrow{\tau_W} S^{b+1}$, which can be used several ways (see Corollary 6 for an example). If \tilde{W} is orientable with respect to a generalized cohomology theory $E^*(-)$, then it is immediately shown, by making use of the cofiber sequence, that $(\tau_W)^* = 0 : E^*(S^{b+1}) \rightarrow E^*(B^{V \oplus W})$, that is, the Adams–Novikov filtration of τ_W with respect to $E^*(-)$ is more than or equal to 1. But, a crucial use of the cofiber sequence enables us to calculate the e -invariant of τ_W , which is a starting point to apply the transfer maps and to generalize various results of [11], [4] and so on. In order to state it, we first recall the definition of the e -invariant of the transfer map (see [11]).

Let E be a ring spectrum with unit ι_E , and put $E^k G = \Sigma^k E \wedge SG$, where SG is the Moore spectrum for a group G (cf. [3, Part III]). Then, associated to the exact sequence $\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$ for the ring \mathbb{Z} of the integers and the field \mathbb{Q} of the rational numbers, we have a cofiber sequence $S^k \xrightarrow{i_{\mathbb{Q}}} S^k \mathbb{Q} \xrightarrow{\rho_{\mathbb{Z}}} S^k \mathbb{Q}/\mathbb{Z} \xrightarrow{\beta} S^{k+1}$ of the Moore spectra. Assume that \tilde{W} is orientable. Then, τ_W is a torsion element of the stable cohomotopy group $\pi_s^{b+1}(B^{V \oplus W})$, that is, $(i_{\mathbb{Q}})_*(\tau_W) = 0 \in \pi_s^{b+1}(B^{V \oplus W}; \mathbb{Q})$. Therefore, there is an element $\bar{\tau}_W \in \pi_s^b(B^{V \oplus W}; \mathbb{Q}/\mathbb{Z})$ with $\beta_*(\bar{\tau}_W) = \tau_W$. $\bar{\tau}_W$ is uniquely defined by τ_W since

$\pi_s^b(B^{V \oplus W}; \mathbf{Q}) = 0$. Then, the E -theory e -invariant $e_E(\tau_W)$ of the transfer map τ_W is defined by

$$(1.3) \quad e_E(\tau_W) = h^E(\bar{\tau}_W) \in E^b(B^{V \oplus W}; \mathbf{Q}/\mathbf{Z}),$$

where h^E is the E -theory Hurewicz homomorphism induced from ι_E . When $E = K$, the complex K -spectrum, the slant product $e_K(\tau_W) \setminus h^K(x) \in \mathbf{Q}/\mathbf{Z}$ for an element $x \in \pi_{2n+b}^s(B^{V \oplus W})$ is equal to the classical e -invariant [2] $e_C((\tau_W)_*(x))$ of the transfer image $(\tau_W)_*(x) \in \pi_{2n-1}^s(S^0)$.

Now, assume further that \tilde{W} and V are E -orientable (cf. [3, Part III, 10]), and choose Thom classes $U_{\tilde{W}}^E \in E^b((B^V)^{\tilde{W}})$ and $U_V^E \in E^v(B^V)$ to satisfy $(i_1)^*(U_{\tilde{W}}^E) = \iota_E$ and $(i_2)^*(U_V^E) = \iota_E$ for the inclusions $i_1 : S^b \rightarrow (B^V)^{\tilde{W}}$ and $i_2 : S^v \rightarrow B^V$ respectively. Let $U_{\tilde{W}}^H \in H^b((B^V)^{\tilde{W}}; \mathbf{Z})$ be the Thom class of \tilde{W} in ordinary cohomology, and define $sh^E(-\tilde{W}) \in E^0(B_+^V; \mathbf{Q})$ by the equality

$$(1.4) \quad U_{\tilde{W}}^H = U_{\tilde{W}}^E sh^E(-\tilde{W}) \quad \text{in } E^b((B^V)^{\tilde{W}}; \mathbf{Q}).$$

Here, we identify $U_{\tilde{W}}^E$ and $U_{\tilde{W}}^H$ with $(i_Q)_*(U_{\tilde{W}}^E)$ and $(\iota)_*(U_{\tilde{W}}^H)$ for $\iota : H^0 \mathbf{Z} \xrightarrow{i_Q} H^0 \mathbf{Q} = S^0 \mathbf{Q} \xrightarrow{\iota_E} E^0 \mathbf{Q}$ respectively.

Let $\Phi_V : E^i(B_+) \rightarrow E^{i+v}(B^V)$ be the Thom isomorphism given by the Thom class U_V^E , and $\Phi_{V \oplus W} : E^i(B_+) \rightarrow E^{i+b+v}(B^{V \oplus W})$ the Thom isomorphism given by an appropriately chosen Thom class of $V \oplus W$. Then, under those assumptions on \tilde{W} and V , the E -theory e -invariant of τ_W is represented as follows:

THEOREM 3. *Assume that $\pi_1(E; \mathbf{Q}) = \pi_1(E; \mathbf{Q}/\mathbf{Z}) = 0$. Then, we have*

$$e_E(\tau_W) = \Phi_{V \oplus W} \Phi_V^{-1} (sh^E(-\tilde{W}) - 1).$$

Let $e^E(V) = j_V^*(U_V^E)$ be the E -theory Euler class of V , where j_V is the map as in (1.1). Then, applying $j_V^* \Phi_V \Phi_{V \oplus W}^{-1}$ on the equality of Theorem 3, we have

$$e^E(V) \Phi_{V \oplus W}^{-1} (e_E(\tau_W)) = sh^E(-W) - 1$$

in $E^0(B_+; \mathbf{Q}/\mathbf{Z})$. Thus, the formula for $e_E(\tau_W)$ becomes simpler than the one in Theorem 3 under the following conditions: There exists a map $f : B \rightarrow C$ for a finite complex C ; V is the induced bundle $f^* V'$ of a vector bundle V' over C ; $\tilde{W} = T(f)^* \tilde{W}'$ for a virtual vector bundle \tilde{W}' over $C^{V'}$ and the map $T(f) : B^V \rightarrow C^{V'}$ induced from f ; $f^*(\text{Ann}(e^E(V'))) = 0$ for the annihilator $\text{Ann}(e^E(V')) = \{x \in E^0(C_+; \mathbf{Q}/\mathbf{Z}) \mid x e^E(V') = 0\}$ of $e^E(V')$. Under these conditions we have the following:

COROLLARY 4.

$$e_E(\tau_W) = \Phi_{V \oplus W} \left(\frac{sh^E(-W) - 1}{e^E(V)} \right).$$

According to these formulas, we give some concrete calculation for the e -invariants of generalized S^1 -transfer maps in §4.

The paper is organized as follows: In §2, we give the definition of transfer maps to be discussed, and prove Theorem 1. Some related properties of the transfer maps are also commented. In §3, we study the e -invariants of the transfer maps, and prove Theorem 3. In §4, we extend the classical calculation for the e -invariant of the S^1 -transfer map by applying the results in §3.

2. Transfer maps of sphere bundles.

First, we recall the definition of an umkehr map. Let (V, p, B, F^n, G) be a vector bundle over a finite complex B , where F is the field of the real, complex or quaternionic numbers and $G = O(n)$, $U(n)$ or $Sp(n)$ accordingly, and (\tilde{E}, p, B, G, G) the principal G -bundle associated with V . Then, the sphere bundle $(S(V), p, B, S(F^n), G)$ associated to V is given by $S(V) = \tilde{E} \times_G S(F^n)$ for the unit sphere $S(F^n)$ in F^n . Let $t : (F^n)_+ \rightarrow \Sigma S(F^n)_+$ be the Thom construction for the inclusion $S(F^n) \subset F^n$, and put $\tilde{t} = 1 \times_G t : \tilde{E} \times_G (F^n)_+ \rightarrow \tilde{E} \times_G \Sigma S(F^n)_+$, where $\{-\}_+$ denotes the one point compactification of $\{-\}$. We denote by $-V$ an inverse vector bundle of V , that is $V \oplus (-V) = \underline{\mathbf{R}}^N$ for some integer N , which actually exists since B is a finite complex. $\bar{\alpha}$ denotes the fiberwise one point compactification of a vector bundle α . $\bar{\alpha}$ and $\tilde{E} \times_G Y$ for a space Y with G -invariant base point are sectioned bundles in the sense of [9], and $\bar{\alpha}/B = B^\alpha$. We also denote by $\beta \wedge_B \gamma$ the fiberwise wedge sum of sectioned bundles β and γ . Then we have

$$\frac{(\tilde{E} \times_G (F^n)_+) \wedge_B \bar{W}}{B} \simeq \Sigma^N B^{V \oplus W} \quad \text{and} \quad \frac{(\tilde{E} \times_G \Sigma S(F^n)_+) \wedge_B \bar{W}}{B} \simeq \Sigma^{N+1} S(V)^{p^* W}.$$

Then, through these identification, the umkehr map $\tau_W(p)$ is defined to be the following stable map:

$$(2.1) \quad \tau_W(p) = \frac{\tilde{t} \wedge_B 1}{B} : B^{V \oplus W} \rightarrow \Sigma S(V)^{p^* W}.$$

Then, the transfer map τ_W of $S(V)$ is defined as $\tau_W = q \circ \tau_W(p)$ for the map $q : \Sigma S(V)^{p^* W} \rightarrow S^{b+1}$ as defined in (1.2).

On the other hand, applying the construction $((\tilde{E} \times_G \{-\}) \wedge_B \overline{W})/B$ on each space in the cofiber sequence $S(F^n)_+ \rightarrow D(F^n)_+ \rightarrow (F^n)_+ \rightarrow \Sigma S(F^n)_+$, we get the James cofiber sequence

$$S(V)^{p^*W} \xrightarrow{\hat{p}} B^W \xrightarrow{\hat{i}} B^{V \oplus W} \xrightarrow{\partial} \Sigma S(V)^{p^*W}.$$

Then, by the constructions of $\tau_W(p)$ and ∂ , they are equivalent as follows:

LEMMA 5 ([11, Lemma 2.12]). $\tau_W(p)$ is stably homotopic to ∂ up to sign.

PROOF OF THEOREM 1. In the definition of τ_W , we have used a stable homotopy equivalence $T(\phi) : S(V)^{p^*W} \rightarrow S(V)^{\mathbf{R}^b} \simeq \Sigma^b S(V)_+$. By Lemma 5, $\tau_W = q \circ \tau_W(p) \simeq q \circ \partial$, and thus we have a map $f : B^W \rightarrow \Sigma^{-1}C(\tau_W)$ which makes the following diagram stably homotopy commutative up to sign:

$$\begin{array}{ccccccc} S(V)^{p^*W} & \xrightarrow{\hat{p}} & B^W & \xrightarrow{\hat{i}} & B^{V \oplus W} & \xrightarrow{\partial} & \Sigma S(V)^{p^*W} \\ \downarrow q & & \downarrow f & & \parallel & & \downarrow q \\ S^b & \xrightarrow{i} & \Sigma^{-1}C(\tau_W) & \xrightarrow{j} & B^{V \oplus W} & \xrightarrow{\tau_W} & S^{b+1}. \end{array}$$

Then, the cofiber spectrum $C(f)$ of f is stably homotopy equivalent to the cofiber spectrum $C(q)$ of q , and $C(q) \simeq \Sigma S(V)$ by the equivalence $T(\phi)$. Put $h = \hat{p} \circ T(\phi)^{-1} \circ i' : \Sigma^b S(V) \rightarrow B^W$ for the inclusion $i' : \Sigma^b S(V) \rightarrow \Sigma^b S(V)_+$. In Held-Sjerve [7, Theorem 3.4], it is shown that there is a cofiber sequence $\Sigma^b S(V) \xrightarrow{h} B^W \xrightarrow{j'} (B^V)^{\tilde{W}}$. Then, we have a diagram

$$\begin{array}{ccccc} \Sigma^{b+1} S(V) & \xrightarrow{h} & \Sigma B^W & \xrightarrow{j'} & \Sigma (B^V)^{\tilde{W}} \\ \downarrow \simeq & & \parallel & & \\ C(f) & \longrightarrow & \Sigma B^W & \xrightarrow{f} & C(\tau_W), \end{array}$$

where two horizontal sequences are the cofiber sequences and the square is stably homotopy commutative. Hence, there is a stable equivalence $\Sigma (B^V)^{\tilde{W}} \simeq C(\tau_W)$, which is the required result of Theorem 1.

By definition, the transfer maps are natural in the following sense. Let X be a finite complex, and $g : X \rightarrow B$ be a map. g induces vector bundles $V' = (g^*V, q, X)$ and $W' = g^*W$, and also defines maps $T(g) : X^{V'} \rightarrow B^V$ and $T(g) : X^{V' \oplus W'} \rightarrow B^{V \oplus W}$. Let $\tilde{W}' = T(g)^* \tilde{W}$. Then, a trivialization $\phi' : q^*(W') \cong \mathbf{R}^b$ is induced from the trivialization ϕ given by (1.1), and thus we have a transfer map $\tau_{W'} : X^{V' \oplus W'} \rightarrow S^{b+1}$ of $S(V')$.

Then, it is obvious that

$$(2.2) \quad \tau_{W'} \cong \tau_W T(g) : X^{V' \oplus W'} \rightarrow S^{b+1}.$$

When $V = V_1 \oplus \underline{\mathbf{R}}$ for some vector bundle V_1 over B , τ_W can be interpreted as a J -map in the following sense. In this case, $B^V = \Sigma B^{V_1}$, and the classifying map of \tilde{W} has the adjoint map $B^{V_1} \rightarrow SO$. Thus, composing with the J -map $J : SO \rightarrow \Omega^\infty S^\infty$ and taking its adjoint, we get a stable map $J(\tilde{W}) : B^V \rightarrow S^1$. Then, as is well known, the cofiber of $J(\tilde{W})$ is stably homotopy equivalent to $\Sigma^{1-b}(B^V)^{\tilde{W}}$. Thus, $\Sigma(B^V)^{\tilde{W}} \simeq \Sigma^b(S^1 \cup_{J(\tilde{W})} C(B^V))$. Hence, we have the following corollary of Theorem 1:

COROLLARY 6. *When $V = V_1 \oplus \underline{\mathbf{R}}$, there exists an equivalence $h : B^{V \oplus W} \rightarrow \Sigma^b B^V$ that satisfies $\tau_W \simeq J(\tilde{W})h$.*

3. e -invariants of transfer maps.

By Theorem 1, we get the stable cofiber sequence

$$(3.1) \quad S^b \xrightarrow{i} (B^V)^{\tilde{W}} \xrightarrow{j} B^{V \oplus W} \xrightarrow{\tau_W} S^{b+1}.$$

We shall apply this sequence to study the e -invariant of τ_W . As defined in (1.3), the E -theory e -invariant $e_E(\tau_W)$ of τ_W is defined for a ring spectrum E with unit ι_E if \tilde{W} is orientable. When it is necessary, the Thom class $U_{\tilde{W}}^H \in H^b((B^V)^{\tilde{W}}; \mathbf{Z})$ of \tilde{W} is regarded as an element of $\pi_s^b((B^V)^{\tilde{W}}; \mathbf{Q})$ through $H \xrightarrow{i_Q} H^0 \mathbf{Q} = \pi^0 \mathbf{Q}$, and also as $E^b((B^V)^{\tilde{W}}; \mathbf{Q})$ through $\iota_E : \pi^0 \mathbf{Q} \rightarrow E^0 \mathbf{Q}$.

Now, assume further that E satisfies $\pi_1(E; \mathbf{Q}) = \pi_1(E; \mathbf{Q}/\mathbf{Z}) = 0$ and that \tilde{W} is E -oriented by a Thom class $U_{\tilde{W}}^E \in E^b((B^V)^{\tilde{W}})$ satisfying $i^*(U_{\tilde{W}}^E) = \iota_E$ for the inclusion $i : S^b \rightarrow (B^V)^{\tilde{W}}$. We denote $(i_Q)_*(U_{\tilde{W}}^E)$ simply by $U_{\tilde{W}}^E$. Then, under these assumptions, the calculation of the $e_E(\tau_W)$ is reduced to determine an element g in the following lemma.

LEMMA 7. *There exists a unique element $g \in E^b(B^{V \oplus W}; \mathbf{Q})$ with $j^*(g) = U_{\tilde{W}}^H - U_{\tilde{W}}^E$ for j of (3.1), and $e_E(\tau_W) = \rho_Z(g)$ holds.*

PROOF. Let $S^b \xrightarrow{i_Q} S^b \mathbf{Q} \xrightarrow{\rho_Z} S^b \mathbf{Q}/\mathbf{Z} \xrightarrow{\beta} S^{b+1}$ be the cofiber sequence of the Moore spectra. Then, since $i^*(U_{\tilde{W}}^H) = i_Q \in \pi^b(S^b; \mathbf{Q})$ for i of (3.1), there is an element $\bar{\tau}_W \in \pi_s^b(B^{V \oplus W}; \mathbf{Q}/\mathbf{Z})$ with $\beta_*(\bar{\tau}_W) = \tau_W$ and $j^* h^E(\bar{\tau}_W) = (\rho_Z)_*(U_{\tilde{W}}^H)$. Thus, $\bar{\tau}_W$ is the element of (1.3), and the E -theory e -invariant $e_E(\tau_W)$ is given by

$$(3.2) \quad e_E(\tau_W) = h^E(\bar{\tau}_W).$$

Put $u = U_{\tilde{W}}^H - U_{\tilde{W}}^E \in E^b((B^V)^{\tilde{W}}; \mathbf{Q})$. Since $i^*(u) = 0$ in $E^b(S^b; \mathbf{Q})$, we have an element $g \in E^b(B^{V \oplus W}; \mathbf{Q})$ with $j^*(g) = u$. The indeterminacy of g lies in $E^b(S^{b+1}; \mathbf{Q}) = \pi_1(E; \mathbf{Q}) = 0$, and thus g is unique. Furthermore, we have $j^*((\rho_Z)_*(g)) = (\rho_Z)_*(U_{\tilde{W}}^H) = j^*(h^E(\bar{\tau}_W))$, and, since j^* is injective by the assumption that $\pi_1(E; \mathbf{Q}/\mathbf{Z}) = 0$, we get the required equality $(\rho_Z)_*(g) = e_E(\tau_W)$.

We have introduced in (1.4) a class $sh^E(-\tilde{W}) \in E^0(B_+^V; \mathbf{Q})$ that satisfies $U_{\tilde{W}}^H = U_{\tilde{W}}^E sh^E(-\tilde{W})$. Then, by Lemma 7, we have

$$(3.3) \quad j^*(g) = U_{\tilde{W}}^E (sh^E(-\tilde{W}) - 1).$$

PROOF OF THEOREM 3. By assumption, V is E -oriented by a Thom class $U_V^E \in E^v(B^V)$, where v is the fiber dimension of V . Let $\kappa : B^{V \oplus W} \rightarrow (B^V)^{\tilde{W}} \wedge B^V$ be the map defined by $\kappa([x, (u, w)]) = [j_V(x), w] \wedge [x, u]$. Here, we represent an element of a Thom space Y^α by $[y, u]$ for $y \in Y$ and a vector u in the fiber over y , and $j_V : B \rightarrow B^V$ is the map given by the zero section as in (1.1). Since κ is of degree 1 when it is restricted on the bottom spheres, we can define a E -theory Thom class of $V \oplus W$ by $U_{V \oplus W}^E = \kappa^*(U_{\tilde{W}}^E \otimes U_V^E)$. We denote by Φ_V and $\Phi_{V \oplus W}$ the Thom isomorphisms associated to U_V^E and $U_{V \oplus W}^E$ respectively. Then, using Lemma 7, the following lemma establishes Theorem 3.

$$\text{LEMMA 8. } g = \Phi_{V \oplus W} \Phi_V^{-1} (sh^E(-\tilde{W}) - 1).$$

PROOF. Let $d : X^\alpha \rightarrow X^\alpha \wedge X_+$ be the map defined by $d[x, a] = [x, a] \wedge x$, and $\Phi_{\tilde{W}} : E^*(B_+^V) \rightarrow E^{*+b}((B^V)^{\tilde{W}})$ the Thom isomorphism associated to the Thom class $U_{\tilde{W}}^E$. By the definition of κ , the composition $(\kappa \wedge 1) \circ d \circ j : (B^V)^{\tilde{W}} \rightarrow B^{V \oplus W} \rightarrow B^{V \oplus W} \wedge B_+ \rightarrow (B^V)^{\tilde{W}} \wedge B^V \wedge B_+$ is homotopic to the composition $(1 \wedge d) \circ (1 \wedge c') \circ d : (B^V)^{\tilde{W}} \rightarrow (B^V)^{\tilde{W}} \wedge B_+^V \rightarrow (B^V)^{\tilde{W}} \wedge B^V \rightarrow (B^V)^{\tilde{W}} \wedge B^V \wedge B_+$, where j is the map of (3.1) and $c' : B_+^V \rightarrow B^V$ is the projection. Taking E -cohomology, we obtain the following commutative diagram:

$$\begin{array}{ccc} E^*(B_+) & \xrightarrow{\Phi_V} & E^{*+v}(B^V) \\ \downarrow \Phi_{V \oplus W} & & \downarrow \Phi_{\tilde{W}} \circ (c')^* \\ E^{*+b+v}(B^{V \oplus W}) & \xrightarrow{j^*} & E^{*+b+v}((B^V)^{\tilde{W}}). \end{array}$$

Thus, $j^*(\Phi_{V \oplus W}(x)) = \Phi_{\tilde{W}}(\Phi_V(x))$. Hence, by Lemma 7, $j^*(g) = \Phi_{\tilde{W}}(sh^E(-\tilde{W}) - 1) = j^*(\Phi_{V \oplus W} \Phi_V^{-1}(sh^E(-\tilde{W}) - 1))$. Since $j^* : E^b(B^{V \oplus W}; \mathbf{Q}) \rightarrow E^b((B^V)^{\tilde{W}}; \mathbf{Q})$ is injective, we have completed the proof.

Corollary 4 follows from Theorem 3 by using the naturality (2.2) of the transfer maps.

In the case $E = K$, the complex K -theory, V and \tilde{W} are assumed to be the complex vector bundles. Let $ch : K \rightarrow \prod_i H^{2i} \mathbf{Q}$ be the Chern character. Then, $ch : K^0(X; \mathbf{Q}) \rightarrow \sum_i H^{2i}(X; \mathbf{Q})$ is an isomorphism if X is a finite complex. The class $sh^K(-\tilde{W})$ is equal to $ch^{-1}(bh(-\tilde{W}))$ for the multiplicative characteristic class $bh(\tilde{W}) \in H^*(B^V; \mathbf{Q})$ (cf. [1]) defined by the relation $ch(U_{\tilde{W}}^K) = U_{\tilde{W}}^H bh(\tilde{W})$. In fact, we have $U_{\tilde{W}}^H = U_{\tilde{W}}^K ch^{-1} bh(-\tilde{W})$ by applying ch . Thus, in this case, Theorem 3 becomes

COROLLARY 9. $e_K(\tau_W) = \Phi_{V \oplus W} \Phi_V^{-1} (ch^{-1} bh(-\tilde{W}) - 1)$.

4. Generalized S^1 -transfer maps.

In this section, we apply Corollary 4 to the transfer maps which generalize the S^1 -transfer map. Let ξ be the canonical complex line bundle over the complex projective space CP^N . Then, $S(\xi)$ is the principal S^1 -bundle, and $(CP^N)^\xi = CP^{N+1}$.

Now, we take a virtual vector bundle $\tilde{W} = f(\xi) = \bigoplus_{k=m}^n a_k \xi^k$ over CP^{N+1} for the finite Laurent series $f(u) = \sum_{k=m}^n a_k u^k \in \mathbf{Z}[u, u^{-1}]$ with integer coefficients. Here, ξ^k denotes the k -fold (resp. $(-k)$ -fold) tensor product of ξ (resp. the conjugate bundle $\bar{\xi}$ of ξ) if k is a positive (resp. negative) integer, and $(\xi)^0 = \underline{\mathbf{C}}$. Then, we have the transfer map

$$(4.1) \quad \tau_{f(\xi)} : (CP^N)^{\xi \oplus f(\xi)} \rightarrow S^{a+1}$$

of $S(\xi)$ for $a = 2 \sum_{k=m}^n a_k$. The S^1 -transfer map mentioned in §1 is the transfer map $\tau_{-\xi}$ in the case of $f(u) = -u$. We shall calculate the e -invariant of $\tau_{f(\xi)}$.

Let E be a ring spectrum as in the previous section, and assume further that ξ is E -oriented with the E -theory Euler class $x^E = e^E(\xi) \in E^2(CP^\infty)$. Then, we have a formal group law F over E associated to x^E (cf. [3, Part II], [14, Appendix 2]). We put $1^F(z) = z$, $k^F(z) = F(z, (k-1)^F(z))$ for a positive integer $k \geq 2$, $0^F(z) = 0$, and $k^F(z) = (-k)^F(c^*(z))$ for $k < 0$, where $c^*(z)$ is a power series of z in which the coefficient of z^i is the same with that of $(x^E)^i$ in $c^*(x^E)$ for the conjugation $c : CP^\infty \rightarrow CP^\infty$. Also, \log^F denotes the logarithm series associated to F .

Let $m_k : CP^\infty \xrightarrow{d} \prod^k CP^\infty \xrightarrow{\mu_k} CP^\infty$ be the composition of the diagonal map d and the multiplication μ_k defined by the Hopf space structure on CP^∞ . Then, $\xi^k = m_k^*(\xi)$, and thus ξ^k is E -oriented with the E -theory Euler class $e^E(\xi^k) = m_k^*(x^E) = k^F(x^E)$. For the Thom class $U_k^E \in E^2((CP^{N+1})^{\xi^k})$ of ξ^k satisfying $j_\xi^*(U_k^E) = e^E(\xi^k)$ and $i^*(U_k^E) = \iota_E$, the class $sh^E(\xi^k)$ with $U_k^E = U_k^H sh^E(\xi^k)$ is given as follows:

LEMMA 10. For an integer $k \neq 0$,

$$sh^E(\xi^k) = \frac{k^F(x^E)}{k \log^F(x^E)} \quad \text{in } E^0(CP_+^{N+1}; \mathbf{Q}).$$

PROOF. We may assume $k > 0$, because $sh^E(\xi^{-k}) = c^*(sh^E(\xi^k))$ for the negative case. Then, $S(\xi^k)$ is homotopy equivalent to the standard mod p lens space $L(k)$ of dimension $2N+3$, and we have a cofiber sequence $L(k) \xrightarrow{p} CP^{N+1} \xrightarrow{j} (CP^{N+1})^{\xi^k}$, where we put $j = j_{\xi^k}$. Since $E^1(L(k); \mathbf{Q}) = 0$, $j^* : E^2((CP^{N+1})^{\xi^k}; \mathbf{Q}) \rightarrow E^2(CP^{N+1}; \mathbf{Q})$ is injective (in fact, isomorphic). Recall that $j^*(U_k^E) = e^E(\xi^k) = k^F(x^E)$. On the other hand, for the Thom class $U_k^H \in H^2((CP^{N+1})^{\xi^k}; \mathbf{Z})$ of ξ^k and the Euler class $x \in H^2(CP^{N+1}; \mathbf{Z})$ of ξ , we have $j^*(U_k^H) = kx$ and $x = \log^F(x^E)$. Hence, $j^*(U_k^E) = j^*(U_k^H k^F(x^E)/kx) = j^*(U_k^H k^F(x^E)/k \log^F(x^E))$, and we have the required result since j^* is injective.

We notice that Corollary 4 is valid for $e_E(\tau_{f(\xi)})$ since $\tau_{f(\xi)}$ is defined and natural for every $N > 0$. Thus, by Corollary 4 and Lemma 10, we get the following formula, in which $a = 2 \sum_{k=m}^n a_k$.

PROPOSITION 11. Let $\Phi : E^*(CP_+^N; \mathbf{Q}/\mathbf{Z}) \rightarrow E^{*+a+2}((CP^N)^{\xi \oplus f(\xi)}; \mathbf{Q}/\mathbf{Z})$ be the Thom isomorphism. Then,

$$(\Phi)^{-1}(e_E(\tau_{f(\xi)})) = \frac{1}{x^E} \left(\prod_{k=m}^n \left(\frac{k \log^F(x^E)}{k^F(x^E)} \right)^{a_k} - 1 \right) \quad \text{in } E^{-2}(CP_+^N; \mathbf{Q}/\mathbf{Z}).$$

EXAMPLE 12. For the case $E = K$, the complex K -theory, and $f(u) = -u^k$, the formula of Proposition 11 becomes

$$(\Phi)^{-1}(e_E(\tau_{-\xi^k})) = \frac{1}{X} \left(\frac{(1-tX)^k - 1}{k \log(1-tX)} - 1 \right) \quad \text{in } K^{-2}(CP_+^N; \mathbf{Q}/\mathbf{Z}),$$

where $t \in \pi_2(K)$ is the Bott class, $X = e^K(\xi) = t^{-1}(1 - \xi) \in K^2(CP_+^N)$ and \log is the power series expansion of the usual logarithm function.

References

- [1] J. F. Adams, On the groups $J(X)$ -II, *Topology*, **3** (1965), 137–171.
- [2] J. F. Adams, On the groups $J(X)$ -IV, *Topology*, **5** (1966), 21–71.
- [3] J. F. Adams, Stable homotopy and generalised homology, *Chicago Lectures in Mathematics*, Univ. Chicago Press, Chicago and London, 1974.
- [4] A. Baker, D. Carlisle, B. Gray, S. Hilditch, N. Ray and R. Wood, On the iterated complex transfer, *Math. Z.*, **199** (1988), 191–207.
- [5] J. C. Becker and D. H. Gottlieb, The transfer map and fiber bundles, *Topology*, **14** (1975), 1–12.
- [6] M. C. Crabb and K. Knapp, James numbers, *Math. Ann.*, **282** (1988), 395–422.

- [7] R. P. Held and D. K. Sjerve, The homotopy properties of Thom complexes, *Math. Z.*, **135** (1974), 315–323.
- [8] M. Imaoka, The double complex transfer at the prime 2, *Top. its Appl.*, **72** (1996), 199–207.
- [9] I. M. James, Bundles with special structure: I, *Ann. Math.*, **89** (1969), 359–390.
- [10] I. M. James, The topology of Stiefel manifolds, *London Math. Soc. Lecture Note Series* 24, Cambridge University Press, 1976.
- [11] K. Knapp, Some application of K-theory to framed bordism; e -invariant and transfer, *Habilitationsschrift*, Bonn, 1979.
- [12] H. Miller, Universal Bernoulli numbers and the S^1 -transfer, *Canadian Math. Soc. Conference Proc.*, **2** (1982), 437–449.
- [13] J. Mukai, On stable homotopy of the complex projective space, *Japan J. Math.*, **19** (1993), 191–216.
- [14] D. C. Ravenel, Complex cobordism and stable homotopy groups of spheres, *Mathematics* 121, Academic Press, London., 1986.

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