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A measure theoretic basis theorem for Π_2^1

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Abstract. Using the covering game, we prove that every (lightface) Π_2^1-set of positive Lebesgue measure contains a member which is arithmetical in $0^\sharp$. This result generalizes a result for Π_1^1 due to Sacks and Tanaka.

1. Introduction.

The Sacks-Tanaka Theorem ([**7**], [**9**]) says that if a Π_1^1-set of real numbers has positive Lebesgue measure then it contains a hyperarithmetical element. (Here we are dealing with lightface Π_1^1-sets.) This theorem is a result about *basis problems*: whether definable sets of real numbers have definable members. The Sacks-Tanaka theorem was generalized by A. S. Kechris [**2**] to all odd levels of analytical hierarchy, under the assumption of determinacy of all infinite games associated with projective sets. In the present paper, we extend the Sacks-Tanaka theorem to the lowest even level of analytical hierarchy, namely to (lightface) Π_2^1.

THEOREM. *Assume* $0^\sharp$ *exists. Then every* Π_2^1-*set of real numbers with positive Lebesgue measure has a member which is arithmetical in* $0^\sharp$.

We prove this theorem by applying covering games. This kind of games has been used in order to show that determinacy of infinite games implies Lebesgue measurability and other regularity properties of pointsets. For general information about games and their role in descriptive set theory, see Moschovakis' textbook [**4**] or Martin and Kechris' survey paper [**3**]. In Section 2, we introduce covering games. Then we prove a few lemmas which we need. The proof of main theorem is given in Section 3, where we also give some remarks about the result.

2. Covering games.

Let $\mathscr{C}$ be the Cantor space, i.e. the set $\{0,1\}^\omega$ topologized with the product topology, taking $\{0,1\}$ discrete. Let $\{J_n\}_{n\in\omega}$ enumerate, in a straightforward way, the

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basic clopen sets in $\mathscr{C}$. Let $\{G_k\}_{k\in\omega}$ recursively enumerate all the finite unions of J_n's.

We give the standard product measure on $\mathscr{C}$. In what follows, this product measure is called *the Lebesgue measure*. This abuse of language would cause little confusion, since the Cantor space $\mathscr{C}$ and the unit interval [0, 1] are measure theoretically quite similar. In fact, removing an appropriate countable set (i.e., the sequences of 0's and 1's having only finitely many places for 1) from $\mathscr{C}$, we obtain a measure space which is (arithmetically) isomorphic to [0, 1]. For this reason, we also call elements of $\mathscr{C}$ *reals*.

Let us denote the Lebesgue measure, its inner and outer extensions by m, m_* and m^*, respectively. We may assume, without loss of generality, that the enumeration $\{J_n\}$ and $\{G_k\}$ have been made so that the relations $m(J_n) < p/(q+1)$ and $m(G_k) < p/(q+1)$ are recursive with respect to n, k, p, g (ranging over ω), as well as the relations with ">" replacing "<".

Let $E \subset \mathscr{C}\times\mathscr{C}$. The projection of E onto the first coordinate space is denoted by πE:

$$\pi E = \{\alpha\in\mathscr{C} : (\exists\beta\in\mathscr{C})[\langle\alpha,\beta\rangle\in E]\}.$$

Covering games have been introduced by L. Harrington in order to give a simpler proof of a theorem of J. Mycielski and S. Swierczkowski ([**5**]) that the Axiom of Determinacy implies every set of real numbers is Lebesgue measurable. The game which we are going to describe is so-called "unfolded version" of a covering game. This version has been invented by R. M. Solovay and A. S. Kechris.

Let $E \subset \mathscr{C}\times\mathscr{C}$. Let $P = \pi E$. Given a rational number $\varepsilon > 0$, we consider the following two-person infinite game:

$$\begin{array}{lcccccccc}
\text{(I)} & a_0, b_0 & & & & a_1, b_1 & & & \cdots \\
 & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\text{(II)} & & & k_0 & & & & k_1 & \cdots
\end{array}$$

where $a_i, b_i \in \{0,1\}$ and $k_i \in \omega$. We impose the following restriction on Player II's choices: k_i must satisfy $m(G_{k_i}) < \varepsilon/8^i$ for all $i\in\omega$. A course of choices of Player I specifies a pair of reals

$$\alpha = (a_0, a_1, \ldots, a_i, \ldots)$$

and

$$\beta = (b_0, b_1, \ldots, b_i, \ldots)$$

while Player II specifies an open subset G of $\mathscr{C}$:

$$G = G_{k_0} \cup G_{k_1} \cup \cdots \cup G_{k_i} \cup \cdots.$$

Player I wins if $\langle\alpha,\beta\rangle \in E$ and $\alpha \notin G$. Otherwise Player II wins. We call this game the *unfolded covering game associated with E and* ε and denote it by $\mathscr{G}^+(E:\varepsilon)$. The measure of P and winning strategies in $\mathscr{G}^+(E:\varepsilon)$ are related to each other as the next lemma shows.

Lemma 1. *Let* $E \subset \mathscr{C} \times \mathscr{C}$. *Let* $\varepsilon > 0$ *be rational. Let* $P = \pi E$. *Consider the game* $\mathscr{G}^+(E:\varepsilon)$.

(1) *If Player* I *has a winning strategy, then* $m_*(P) \geq \varepsilon$.

(2) *If Player* II *has a winning strategy, then* $m^*(P) < 8\varepsilon$.

Proof. (1) Suppose that Player I has a winning strategy σ in $\mathscr{G}^+(E:\varepsilon)$. Let S be the set of courses of legal moves of Player II:

$$S = \{\gamma \in \omega^\omega : (\forall i)[m(G_{\gamma(i)}) < \varepsilon/8^i]\}.$$

Let H be the set of pairs of reals which Player I specifies by playing according to σ while Player II plays legally:

$$H = \{\langle\alpha,\beta\rangle \in \mathscr{C} \times \mathscr{C} : (\exists\gamma \in S)(\forall i)[\langle\alpha(i),\beta(i)\rangle = \sigma(\gamma(0),\ldots,\gamma(i-1))]\}.$$

Let $Q = \pi H$. It is easy to see that S is closed (in fact, lightface Π_1^0) and H and Q are Σ_1^1. Since σ is a winning strategy of Player I, we have $H \subset E$. Therefore $Q \subset P$. Being Σ_1^1, Q is Lebesgue measurable. Therefore, in order to prove $m_*(P) \geq \varepsilon$, it is sufficient to show $m(Q) \geq \varepsilon$.

Suppose contrary, that $m(Q) < \varepsilon$. Then there exists a sequence $\{n_p\}_{p\in\omega}$ of integers such that

$$Q \subset \bigcup_{p\in\omega} J_{n_p} \quad \text{and} \quad \sum_{p\in\omega} m(J_{n_p}) < \varepsilon.$$

For each $i \in \omega$ let u_i be the smallest integer u such that

$$\sum_{p\geq u} m(J_{n_p}) < \frac{\varepsilon}{8^i}.$$

Let $\gamma(i) = k_i$ be an index of the finite sum

$$G_{k_i} = \bigcup\{J_{n_p} : u_i \leq p < u_{i+1}\}.$$

Then γ is a course of legal choices of Player II in $\mathscr{G}^+(E:\varepsilon)$ which defeats σ. Contradiction.

(2) Suppose that τ is a winning strategy of Player II in $\mathscr{G}^+(E:\varepsilon)$. Let D be the union of all open sets G_k which τ tells Player II to choose against Player I's choices:

$$D = \bigcup\{G_{\tau(a_0,b_0,\ldots,a_i,b_i)} : a_0,\ldots,a_i,b_0,\ldots,b_i \in \{0,1\},\ i \in \omega\}.$$

Straightforward computation shows $m(D) < 8\varepsilon$. Since τ is winning of Player II, we have $P \subset D$. $\square$

From the proof of Lemma 1, we can extract the following effective version. Note that we are dealing with relativized lightface pointclasses.

LEMMA 2. *Let $E \subset \mathscr{C} \times \mathscr{C}$. Let $\varepsilon > 0$ be rational. Let $P = \pi E$. Consider the game $\mathscr{G}^+(E : \varepsilon)$.*

(1) *If Player* I *has a winning strategy σ, then P contains a $\Sigma^1_1(\sigma)$-set Q whose Lebesgue measure is not less than ε.*

(2) *If Player* II *has a winning strategy τ, then P is contained in a $\Sigma^0_1(\tau)$-set D whose Lebesgue measure is less than* 8ε.

We need the following lightface version of the result of Mycielski and Swierczkowski.

LEMMA 3. *Let Γ be an adequate pointclass. Suppose that the game $\mathscr{G}^+(E : \varepsilon)$ is determined for all $E \subset \mathscr{C} \times \mathscr{C}$ in Γ and for every rational $\varepsilon > 0$. Then every $\exists^{\mathscr{C}}\Gamma$-set in $\mathscr{C}$ is Lebesgue measurable.*

PROOF. Suppose that a Lebesgue non-measurable set $P \subset \mathscr{C}$ in $\exists^{\mathscr{C}}\Gamma$ exists. Let B_i and B_o be Borel sets such that $B_i \subset P \subset B_o$, $m(B_i) = m_*(P)$ and $m(B_o) = m^*(P)$. Then $m(B_o \backslash B_i) > 0$. By the Lebesgue density theorem, there exists a finite binary sequence s such that for the corresponding basic clopen set $N_s = \{\alpha \in \mathscr{C} : \alpha \supset s\}$ we have

$$m(N_s \cap (B_o \backslash B_i)) > \frac{8}{9} m(N_s).$$

From this it follows that

$$m_*(N_s \cap P) < \frac{1}{9} m(N_s)$$

and

$$m^*(N_s \cap P) > \frac{8}{9} m(N_s).$$

Let $P' = \{\alpha \in \mathscr{C} : s^\frown\alpha \in P\}$. This set belongs to $\exists^{\mathscr{C}}\Gamma$ since this pointclass is closed under recursive substitutions. By the inequalities above, we have $m_*(P') < 1/9$ and $m^*(P') > 8/9$. Let $E \subset \mathscr{C} \times \mathscr{C}$ be a Γ-set such that $P' = \pi E$. Then by Lemma 1, neither player has a winning strategy in $\mathscr{G}^+(E : 1/9)$. $\square$

Finally, we see how $0^\sharp$ is related to existence of definable winning strategy. It is well-known that Π^1_1-Determinacy is equivalent to the existence of $0^\sharp$. See [**1**] and

Chapter 7 of [**6**] for the detail. From D. A. Martin's proof of Π^1_1-Determinacy from $0^\sharp$, we obtain the following

LEMMA 4. *Assume* $0^\sharp$ *exists. For every* Π^1_1*-game, either Player* I *has a winning strategy or Player* II *has a winning strategy which is recursive in* $0^\sharp$.

PROOF. We freely use terminology from [**4**]. Let $A \subset \omega^\omega$ be a Π^1_1-set. Let T be a recursive tree on $\omega \times \omega$ such that $\alpha \in A$ if and only if $T(\alpha)$ is a wellfounded tree. For each finite sequence s from ω, let $T(s)$ be the set of $t \in \omega^{<\omega}$ such that $\ell h(t) \leq \ell h(s)$ and $\langle (s|\ell h(t)), t\rangle \in T$. Then for every $\alpha \in \omega^\omega$, we have $T(\alpha) = \bigcup_{n\in\omega} T(\alpha|n)$.

Now we consider two games. The first is the ordinary game on ω with pay-off set A, for which we are to prove the lemma. Call this game $G(A)$. The second game is defined as follows: Player I chooses $a_{2n} \in \omega$ and an order preserving function f_n on $T(a_0, a_1, \dots, a_{2n})$ (under the Kleene-Brower ordering) into $\aleph_{\omega_1}$, while Player II chooses $a_{2n+1} \in \omega$. Moreover, we impose the following restriction on Player I's choices: f_{n+1} must be an extension of f_n. Player I wins if he can make all moves legally. Player II wins otherwise, i.e., if he can make his opponent impossible to carry on. Call this game $G^*(T)$. It is a closed game on a certain uncountable set. Hence by the Gale-Stewart Theorem, it is determined.

In $G^*(T)$, two players together specify a sequence $\alpha = (a_0, a_1, \dots)$. At the same time, Player I tries to build up an order preserving mapping of $T(\alpha)$ into $\aleph_{\omega_1}$ which witnesses the wellfoundedness of $T(\alpha)$, hence witnesses $\alpha \in A$.

If Player I has a winning strategy in $G^*(T)$, then it can be used in $G(A)$: simply forget f_n's. This yields a winning strategy of Player I in $G(A)$.

On the other hand, suppose Player II has a winning strategy τ^* in $G^*(T)$. We may assume without loss of generality that τ^* is definable in the constructible universe L with only one parameter $\aleph_{\omega_1}$. In $G(A)$, let Player II play using τ^* with f_n being the unique order preserving mapping of $T(a_0, \dots, a_{2n})$ into $\{\aleph_1, \dots, \aleph_k\}$ (where k is the cardinality of $T(a_0, \dots, a_{2n})$). It is easy to check that this is a winning strategy of Player II in $G(A)$, using the fact that the uncountable cardinals form a class of indiscernibles in L under the assumption of the existence of $0^\sharp$.

Let us denote this strategy of Player II by τ. Since τ^* is definable in L, the set-theoretic sentence $\tau(a_0, \dots, a_{2n}) = b$ can be written by a formula $\phi_{a_0,\dots,a_{2n},b}$, which depends on $(a_0, \dots, a_{2n}, b)$ in a recursive way, and the cardinals $\aleph_1, \dots, \aleph_k$ and $\aleph_{\omega_1}$:

$$\begin{aligned}\tau(a_0, \dots, a_{2n}) = b \quad &\Longleftrightarrow \quad L \models \phi_{a_0,\dots,a_{2n},b}[\aleph_1, \dots, \aleph_k, \aleph_{\omega_1}] \\ &\Longleftrightarrow \quad \phi_{a_0,\dots,a_{2n},b} \in 0^\sharp.\end{aligned}$$

Hence τ is recursive in $0^\sharp$. $\square$

3. Proof of the theorem and some remarks.

We are ready to prove the main theorem. Let A be a Π^1_2-set in $\mathscr{C}$ of positive Lebesgue measure. We know that every A is Lebesgue measurable (by Lemma 3). By the density argument just like the proof of Lemma 3, we may assume without loss of generality that $m(A) > 8/9$. Let $P = \mathscr{C}\backslash A$ and let $E \subset \mathscr{C} \times \mathscr{C}$ be a Π^1_1-set such that $P = \pi E$. Consider the game $\mathscr{G}^+(E : 1/9)$. This is a Π^1_1-game in which (by Lemma 1) Player I does not have a winning strategy. Then by Lemma 4, Player II has a winning strategy τ which is recursive in $0^\sharp$. By Lemma 2, there is a $\Sigma^0_1(\tau)$-set D such that $P \subset D$ and $m(D) < 1/9$. Let $K = \mathscr{C}\backslash D$. Then K is a compact $\Pi^0_1(\tau)$-set such that $A \supset K$ and $m(K) > 8/9$. In particular, K is not empty.

We show how to find a member of K which is arithmetical in τ. Since K is $\Pi^0_1(\tau)$, there exist a set R_1 of finite sequences of 0's and 1's such that

(1) R_1 is recursive in τ;

(2) if s is an initial segment of some $t \in R_1$, then $s \in R_1$;

(3) R_1 has infinitely many members;

(4) $K = \{\alpha \in \mathscr{C} : (\forall n)[\langle \alpha(0), \ldots, \alpha(n-1)\rangle \in R_1]\}$.

Using this set R_1, we define a real α_1 inductively: let $\alpha_1(n) = 0$ if infinitely many sequences extending $\langle \alpha_1(0), \ldots, \alpha_1(n-1), 0\rangle$ are in R_1. Otherwise let $\alpha_1(n) = 1$. It is easy to verify that $\alpha_1 \in K$ and α_1 is arithmetical in τ. Since τ is recursive in $0^\sharp$, the real α_1 is arithmetical in $0^\sharp$. Thus we have found a member of A which is arithmetical in $0^\sharp$. $\square$

REMARK 1. The condition "with positive Lebesgue measure" cannot be dropped from the theorem. To see this, let β be a Π^1_2-singleton which is not arithmetical in $0^\sharp$ (for example, the double sharp $0^{\sharp\sharp}$). Then let A be the set of $\alpha \in \mathscr{C}$ in which β is arithmetical. Then A is a Π^1_2-set which has the cardinality of the continuum. Clearly, it does not contain any member which is arithmetical in $0^\sharp$.

REMARK 2. If $0^\sharp$ does not exist, then some Π^1_2-set with positive measure may fail to contain definable members. To see this, let c be a Cohen real over L. Then in $L[c]$, the set of all non-constructible reals is a Π^1_2-set of measure 1 (see Theorem 3.1 of [**8**]). But it contains no ordinal-definable reals because $HOD = L$ holds in $L[c]$.

REMARK 3. Using unfolded Banach-Mazur games (see 6G.11 of [**4**]), we can get the Baire category version of the theorem: *if* $0^\sharp$ *exists, then every non-meager* Π^1_2*-set of reals contains a real which is arithmetical in* $0^\sharp$.

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