

## The index of a critical point for nonlinear elliptic operators with strong coefficient growth

By Athanassios G. KARTSATOS and Igor V. SKRYPNIK

(Received Jun. 15, 1998)

**Abstract.** This paper is devoted to the computation of the index of a critical point for nonlinear operators with strong coefficient growth. These operators are associated with boundary value problems of the type

$$\sum_{|\alpha|=1} \mathcal{D}^\alpha \{ \rho^2(u) \mathcal{D}^\alpha u + a_\alpha(x, \mathcal{D}^1 u) \} = \lambda a_0(x, u, \mathcal{D}^1 u), \quad x \in \Omega,$$

$$u(x) = 0, \quad x \in \partial\Omega,$$

where  $\Omega \subset \mathbf{R}^n$  is open, bounded and such that  $\partial\Omega \in \mathbf{C}^2$ , while  $\rho : \mathbf{R} \rightarrow \mathbf{R}_+$  can have exponential growth. An index formula is given for such densely defined operators acting from the Sobolev space  $W_0^{1,m}(\Omega)$  into its dual space. We consider different sets of assumptions for  $m > 2$  (the case of a real Banach space) and  $m = 2$  (the case of a real Hilbert space). The computation of the index is important for various problems concerning nonlinear equations: solvability, estimates for the number of solutions, branching of solutions, etc. The results of this paper are based upon recent results of the authors involving the computation of the index of a critical point for densely defined abstract operators of type  $(S_+)$ . The latter are based in turn upon a new degree theory for densely defined  $(S_+)$ -mappings, which has also been developed by the authors in a recent paper. Applications of the index formula to the relevant bifurcation problems are also included.

### 1. Introduction and preliminaries.

This paper is devoted to the computation of the index of a critical point for nonlinear elliptic operators with strong coefficient growth. It is well known [3], [10], [11] that the formula for the index plays a key role in problems of solvability, estimates for the number of solutions and branching of solutions of nonlinear equations. An application of this formula to the relevant bifurcation problem is also given herein. The simplest example of the type of equations included in the applications of this work is the following:

$$(1.1) \quad \sum_{i=1}^n \frac{\partial}{\partial x_i} \left\{ [e^u + a(x)] \frac{\partial u}{\partial x_i} \right\} - \lambda q(x)u = 0,$$

where  $a(x)$  is a positive, bounded and measurable function and  $q \in L_{n/2}(\Omega)$ , for some  $n > 2$ .

---

1991 *Mathematics Subject Classification.* Primary 47H15; Secondary 47H11, 47H12, 35B32.

*Key words and phrases.* Reflexive separable Banach space, densely defined operators of type  $(S_+)$ , degree theory for densely defined  $(S_+)$ -operators, index of an isolated critical point, non-linear elliptic PDE's with strong coefficient growth.

In what follows,  $X$  is a real separable reflexive Banach space with dual space  $X^*$ . The norm of the space  $X$  ( $X^*$ ) will be denoted by  $\|\cdot\|$  ( $\|\cdot\|_*$ ). We let  $\mathbf{R}^n$  denote the Euclidean space of dimension  $n$  and set  $\mathbf{R} = \mathbf{R}^1$ . For  $x_0 \in X$  and  $r > 0$ , we let  $B_r(x_0)$  denote the open ball  $\{x \in X : \|x - x_0\| < r\}$  of  $X$ . We use the same symbol for the open ball, with center at  $x_0$  and radius  $r$ , of any other real Banach space. Unless otherwise stated,  $N$  is the set of natural numbers. An operator  $A : X \supset D(A) \rightarrow X^*$  is “bounded” if it maps bounded subsets of its domain onto bounded sets in  $X^*$ . It is “compact” if it is strongly continuous and maps bounded subsets of  $D(A)$  onto relatively compact sets in  $X^*$ . In what follows, the single term “continuous” means “strongly continuous”. We denote by “ $\rightarrow$ ” (“ $\dashrightarrow$ ”) strong (weak) convergence. For  $u \in X$ ,  $h \in X^*$ , we denote by  $\langle h, u \rangle$  the value of the functional  $h$  at the element  $u$ .

Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  be a multi-index with nonnegative integer components. We let  $|\alpha| = \alpha_1 + \dots + \alpha_n$  and set  $\eta^\alpha = \eta_1^{\alpha_1} \dots \eta_n^{\alpha_n}$  for a multi-index  $\eta = (\eta_1, \dots, \eta_n) \in \mathbf{R}^n$ . Analogously, we define

$$\mathcal{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad \mathcal{D}^k u = \{\mathcal{D}^\alpha u : |\alpha| = k\},$$

where  $k$  is a nonnegative integer.

In Sections 1–5,  $\Omega$  denotes a bounded open subset of  $\mathbf{R}^n$  with boundary  $\partial\Omega \in C^2$ . We consider the boundary value problem

$$(1.2) \quad \sum_{|\alpha|=1} \mathcal{D}^\alpha \{\rho^2(u) \mathcal{D}^\alpha u + a_\alpha(x, \mathcal{D}^1 u)\} = \lambda a_0(x, u, \mathcal{D}^1 u), \quad x \in \Omega,$$

$$(1.3) \quad u(x) = 0, \quad x \in \partial\Omega.$$

From [5] we see that this problem can be reduced to an operator equation with a densely defined operator of type  $(S_+)$  if the following conditions are satisfied.

i) the real-valued functions  $a_\alpha(x, \xi')$  ( $|\alpha| = 1$ ),  $a_0(x, \xi)$ ,  $\rho(u)$  are defined for  $x \in \bar{\Omega}$ ,  $\xi' = \{\xi_\gamma : |\gamma| = 1\} \in \mathbf{R}^n$ ,  $\xi = \{\xi_\gamma : |\gamma| \leq 1\} \in \mathbf{R}^{n+1}$ . They are continuous with respect to  $\xi, \xi'$  and  $u$ , respectively, while the functions  $a_\alpha(x, \xi')$ ,  $a_0(x, \xi)$  are measurable with respect to  $x$ ;

ii) there exist positive numbers  $v_1, v_2, \mu$  such that, for  $x \in \bar{\Omega}$ ,  $\xi', \eta' \in \mathbf{R}^n$ ,  $\xi \in \mathbf{R}^{n+1}$  and  $u \in \mathbf{R}$ , the following inequalities

$$(1.4) \quad \sum_{|\alpha|=1} [a_\alpha(x, \xi') - a_\alpha(x, \eta')] (\xi_\alpha - \eta_\alpha) \geq v_1 |\xi' - \eta'|^m,$$

$$(1.5) \quad \sum_{|\alpha|=1} |a_\alpha(x, \xi')| + |a_0(x, \xi)| \leq v_2 (f(x) + |\xi_0|^{m_1} + |\xi'|)^{m-1},$$

$$(1.6) \quad 0 \leq \rho(u) \leq \mu \left\{ \left| \int_0^u \rho(s) ds \right| + 1 \right\}$$

hold, where  $2 \leq m < n$ ,  $0 \leq m_1 < n/(n-m)$ ,  $f \in L_m(\Omega)$ .

DEFINITION 1.1. We say that the pair  $\{\lambda_0, u_0\} \in \mathbf{R} \times W_0^{1,m}$  is a “solution” of the problem ((1.2), (1.3)) if

$$(1.7) \quad \rho^2(u_0)\mathcal{D}^\alpha u_0 \in L_{m'}(\Omega), \quad \text{for } |\alpha| = 1, \quad m' = \frac{m}{m-1},$$

and, for every function  $\phi \in W_0^{1,m}(\Omega)$ , the integral identity

$$(1.8) \quad \sum_{|\alpha|=1} \int_{\Omega} \{\rho^2(u_0)\mathcal{D}^\alpha u_0 + a_\alpha(x, \mathcal{D}^1 u_0)\} \mathcal{D}^\alpha \phi \, dx + \lambda_0 \int_{\Omega} a_0(x, u_0, \mathcal{D}^1 u_0) \phi \, dx = 0$$

holds.

Under the further assumptions

$$(1.9) \quad a_\alpha(x, 0) = 0, \quad \text{for } |\alpha| = 1, \quad a_0(x, 0, 0) = 0, \quad \text{for } x \in \Omega,$$

we study the bifurcation points for the problem ((1.2), (1.3)).

**DEFINITION 1.2.** We say that a real number  $\lambda_0$  is a bifurcation point for the problem ((1.2), (1.3)) if there exists a sequence  $\{\lambda_j, u_j\}$  of solutions of the problem ((1.2), (1.3)) such that

$$\lambda_j \rightarrow \lambda_0, \quad u_j \rightarrow 0 \in W_0^{1,m}(\Omega) \quad \text{and} \quad u_j \neq 0.$$

We study the bifurcation problem by using our recent development of a new degree theory for densely defined operators of type  $(S_+)$  [5] and an index theory for such operators in [6].

We recall some definitions connected with densely defined  $(S_+)$ -operators. Consider an operator  $A : X \supset D(A) \rightarrow X^*$  with  $D(A)$  dense in some open set  $D_0 \subset X$ . We assume that there exists a subspace  $L$  of  $X$  such that

$$(1.10) \quad D_0 \cap L \subset D(A), \quad \bar{L} = X.$$

**DEFINITION 1.3.** We say that the operator  $A$  satisfies Condition  $(S_+)_{0,L}$  if for every sequence  $\{u_j\} \subset D(A)$  such that

$$(1.11) \quad u_j \rightharpoonup u_0, \quad \limsup_{j \rightarrow \infty} \langle Au_j, u_j \rangle \leq 0, \quad \lim_{j \rightarrow \infty} \langle Au_j, v \rangle = 0,$$

for some  $u_0 \in X$  and every  $v \in L$ , we have

$$(1.12) \quad u_j \rightarrow u_0, \quad u_0 \in D(A), \quad Au_0 = 0.$$

We say that the operator  $A$  satisfies Condition  $(S_+)_L$  if the operator  $A_h : D(A) \rightarrow X^*$ ,  $A_h u = Au - h$ , satisfies Condition  $(S_+)_{0,L}$  for every  $h \in X^*$ . We say that the operator  $A$  satisfies Condition  $(S_+)$  if it satisfies Condition  $(S_+)_L$  with  $L = X$ .

In [5] we introduced the degree  $\text{Deg}(A, D, 0)$  of the operator  $A$  with respect to an arbitrary open bounded subset  $D$  of  $X$  provided that

$$(1.13) \quad Au \neq 0, \quad \text{for } u \in D(A) \cap \partial D, \quad \bar{D} \subset D_0,$$

and the operator  $A$  satisfies the following conditions:

$A_1)$  there exists a subspace  $L$  of  $X$  satisfying (1.10) and such that the operator  $A$  satisfies Condition  $(S_+)_L$ ;

$A_2$ ) for every  $v \in L$  and every finite-dimensional subspace  $F$  of  $L$  the mapping  $a(F, v) : F \rightarrow \mathbf{R}$ , defined by  $a(F, v)(u) = \langle Au, v \rangle$ , is continuous.

We define a nonlinear operator  $A : W_0^{1,m}(\Omega) \supset D(A) \rightarrow [W_0^{1,m}(\Omega)]^*$ , associated with the problem ((1.2), (1.3)), by

$$(1.14) \quad \langle Au, \phi \rangle = \sum_{|\alpha|=1} \int_{\Omega} \{ \rho^2(u) \mathcal{D}^\alpha u + a_\alpha(x, \mathcal{D}^1 u) \} \mathcal{D}^\alpha \phi(x) dx,$$

for  $u \in \mathcal{D}(A)$ ,  $\phi \in W_0^{1,m}(\Omega)$ , where

$$(1.15) \quad D(A) = \left\{ u \in W_0^{1,m}(\Omega) : \rho^2(u) \mathcal{D}^\alpha u \in L_{m'}(\Omega), \text{ for } |\alpha| = 1, m' = \frac{m}{m-1} \right\}.$$

From Theorem 5.1 in [5] we obtain the following result.

**THEOREM 1.1.** *Assume that Conditions i), ii) are satisfied. Then the operator  $A$ , defined by (1.14), satisfies Condition  $(S_+)_L$  with respect to the space  $L = C_0^\infty(\Omega)$ .*

We introduce a nonlinear operator  $C : W_0^{1,m}(\Omega) \rightarrow [W_0^{1,m}(\Omega)]^*$  by the equality

$$(1.16) \quad \langle Cu, \phi \rangle = \int_{\Omega} a_0(x, u, \mathcal{D}^1 u) \phi(x) dx.$$

Using Conditions i), ii) and the compactness of the embedding  $W_0^{1,m}(\Omega) \subset L_m(\Omega)$ , we obtain that the operator  $C$  is compact. Then Theorem 1.1 implies that the operator  $A + \lambda C$  satisfies Condition  $(S_+)_L$  for every  $\lambda$ . Therefore, for every open and bounded set  $D \subset W_0^{1,m}(\Omega)$ , we can define  $\text{Deg}(A + \lambda C, D, 0)$  if

$$Au + \lambda Cu \neq 0, \quad \text{for } u \in D(A) \cap \partial D,$$

where  $\text{Deg}$  is the degree function introduced in [5].

Using the degree of the operator  $A + \lambda C$  we can define and evaluate the index of a critical point of it. This index computation allows us to study the bifurcation problem by using a topological approach.

In Section 2 we recall the conditions of the result in [6] concerning the computation of the index of a critical point for abstract operators. In that section we also formulate a result about the index of a critical point for the operator  $A$  which is associated with the nonlinear elliptic boundary value problem. In Sections 2, 4 and 5 we consider the case of the Banach space  $W_0^{1,m}(\Omega)$ ,  $m > 2$ .

The index theorem for nonlinear elliptic operators is proved in Section 4. This proof is essentially based upon the regularity of solutions of linear and nonlinear elliptic equations. Some auxiliary results connected with the regularity of solutions are given in Section 3.

In Section 5 we establish necessary conditions for the existence of a bifurcation point for the problem ((1.2), (1.3)) (Theorem 5.2). We also establish sufficient conditions in terms of the degree function (Theorem 5.1) and the characteristic values of some linear operator connected with a special linearization of the unbounded operator  $A$  (Theorem 5.3). We show that every characteristic value of odd multiplicity is a point of bifurcation.

In the case of the Hilbert space  $W_0^{1,2}(\Omega)$  we establish in Sections 6, 7 analogous results with weaker assumptions about the smoothness of the coefficients of the equation (1.2) than those of Section 5. In the particular case of the problem ((1.1), (1.3)) the associated linear equation is

$$(1.17) \quad \sum_{|\alpha|=1} \mathcal{D}^\alpha \{ [1 + a(x)] \mathcal{D}^\alpha u \} - \lambda q(x)u = 0, \quad x \in \Omega.$$

Denote by  $\lambda_i, i = 1, 2, \dots$ , the eigenvalues of the problem ((1.17), (1.3)) and let  $\nu(\lambda_i)$  be the multiplicity of the eigenvalue  $\lambda_i$ . If  $\lambda_0$  is not an eigenvalue of the problem ((1.17), (1.3)) then the index  $\text{Ind}(A + \lambda_0 C, 0)$  of the operator  $A + \lambda_0 C$  with respect to zero is defined by

$$(1.18) \quad \text{Ind}(A + \lambda_0 C, 0) = \prod_{\lambda_i < \lambda_0} (-1)^{\nu(\lambda_i)},$$

where the product on the right-hand side of (1.18) is taken over all  $\lambda_i$  satisfying the inequality  $\lambda_i < \lambda_0$ . The operators  $A, C$  in (1.18) are defined for the equation (1.1) as in the case of the equalities (1.14), (1.16).

Formula (1.18) implies that  $\lambda_i$  is a bifurcation point of the problem ((1.1), (1.3)) if  $\nu(\lambda_i)$  is an odd number.

Bifurcation problems have been studied with different assumptions than ours by many authors. The reader is referred, for example, to [2], [3], [4], [6], [10], [11], [12], where conditions for the existence of bifurcation points have been established via linearization or asymptotic behavior of nonlinear boundary value problems. All these results involve nonlinear operators which are defined everywhere on some neighborhood of the critical point. Such assumptions generate corresponding growth conditions for the coefficients of the relevant differential equations. We are not aware of studies of bifurcation problems with strong coefficient growth.

## 2. Index of a critical point for $m > 2$ .

We first recall some definitions involving the index of a critical point of densely defined operators. We also state the theorem for the computation of the index from [6].

Let  $A : X \supset D(A) \rightarrow X^*$  satisfy Conditions  $A_1), A_2)$  of Section 1.

DEFINITION 2.1. A point  $u_0 \in D(A) \cap D_0$  is called a ‘‘critical point’’ of the operator  $A$  if  $Au_0 = 0$ . A critical point  $u_0$  is called ‘‘isolated’’ if there exists a ball  $B_r(u_0) \subset X$  which does not contain any other critical point of  $A$ .

DEFINITION 2.2. The number  $\text{Ind}(A, u_0)$ , defined by

$$(2.1) \quad \text{Ind}(A, u_0) = \lim_{\rho \rightarrow \infty} \text{Deg}(A, B_\rho(u_0), 0)$$

is called the ‘‘index’’ of the isolated critical point  $u_0$  of the operator  $A$ .

We formulate below a particular case of the general theorem from [6] which corresponds to the case of a bounded operator of linearization. We may assume that  $u_0 = 0$ .

We first introduce some classes of operators. We consider a linear operator  $A' : X \rightarrow X$  satisfying the following condition:

$A')$  the equation  $A'u = 0$  has only the zero solution. There exists a compact linear operator  $\Gamma : X \rightarrow X^*$  such that

$$(2.2) \quad \langle (A' + \Gamma)u, u \rangle > 0, \quad \text{for } u \in X, \quad u \neq 0,$$

and the operator  $T = (A' + \Gamma)^{-1}\Gamma : X \rightarrow X$  is well defined and compact.

We consider an operator  $A_0 : D_0 \rightarrow X^*$  which satisfies the following condition:

$A_0)$   $A_0$  is nonlinear, bounded, satisfies Conditions  $A_1)$ ,  $A_2)$  and is such that

$$(2.3) \quad \frac{A_0u}{\|u\|} \rightarrow 0, \quad \text{as } u \rightarrow 0, \quad A_0(0) = 0.$$

We assume that there exist a linear operator  $A' : X \rightarrow X^*$  satisfying Condition  $A')$  and a nonlinear operator  $A_0 : D_0 \rightarrow X^*$  satisfying Condition  $A_0)$  and such that the following condition holds:

$\omega)$  there exists a positive number  $\varepsilon$  such that

$$(2.4) \quad \frac{Au - A'u}{\|u\|} \rightarrow 0 \quad \text{as } u \rightarrow 0, \quad u \in Z'_\varepsilon,$$

where

$$(2.5) \quad Z'_\varepsilon = \bigcup_{t \in [0, 1]} \{u \in D(A) : tAu + (1-t)(A_0u + A'u) = 0, 0 < \|u\| \leq \varepsilon\}.$$

We note that for the case of a bounded operator  $A$ , which is Fréchet differentiable at 0, Condition  $\omega)$  is satisfied with  $A' = A'(0)$ , where  $A'(0)$  is the Fréchet derivative of  $A$  at 0. In the general case of an unbounded, or non-differentiable, operator  $A$  Condition  $\omega)$  introduces a new and workable linear approximation to the operator  $A$ .

We denote by  $F$  the direct sum of all invariant subspaces of the operator  $T$ , from Condition  $A')$ , corresponding to the characteristic values of it lying in the interval  $(0, 1)$ . Let  $R$  be the closure of the direct sum of all those invariant subspaces of the operator  $T$  which are not included in  $F$ . Then  $F$  and  $R$  are invariant subspaces of the operator  $T$ ,  $F$  is finite-dimensional and the splitting

$$(2.6) \quad X = F + R$$

holds true. We introduce a projection  $\Pi : X \rightarrow F$  as follows:

$$(2.7) \quad \Pi(f + r) = f, \quad \text{for } f \in F, r \in R.$$

**THEOREM 2.1 [6].** *Let  $A : X \supset D(A) \rightarrow X^*$  satisfy Conditions  $A_1)$ ,  $A_2)$  and be such that  $0 \in D(A) \cap D_0$ ,  $A(0) = 0$  and*

$$(2.8) \quad \langle Au, u - v \rangle \geq -c(v),$$

for every  $u, v \in L$  with  $\|u\| \leq r_0$ , where  $r_0, c(v)$  are positive numbers with  $c(v)$  depending only on  $v$ . Assume that there exist operators  $A_0 : D_0 \rightarrow X^*$ ,  $A' : X \rightarrow X^*$  satisfying  $A_0)$ ,  $A')$ ,  $\omega)$ , and such that the following conditions hold:

- 1) the operator  $\Pi(A' + \Gamma)^{-1} : (A' + \Gamma)X \rightarrow X$  is bounded, where the operators  $\Pi, \Gamma$  are defined by (2.7) and Condition A'), respectively;
- 2) the weak closure of the set

$$(2.9) \quad \sigma_\varepsilon = \left\{ v = \frac{u}{\|u\|} : u \in Z'_\varepsilon \cup Z''_\varepsilon \right\}$$

does not contain zero for a sufficiently small positive  $\varepsilon$ , where  $Z'_\varepsilon$  is defined by (2.5) and

$$(2.10) \quad Z''_\varepsilon = \bigcup_{t \in [0, 1]} \{u \in X : tA_0u + A'u = 0, 0 < \|u\| \leq \varepsilon\}.$$

Then zero is an isolated critical point of the operator  $A$  and its index is equal to  $(-1)^v$ , where  $v$  is the sum of the multiplicities of the characteristic values of the operator  $T$  lying in the interval  $(0, 1)$ .

We are now going to formulate our result about the index of the critical point of the operator  $A : W_0^{1,m}(\Omega) \supset D(A) \rightarrow [W_0^{1,m}(\Omega)]^*$  defined by

$$(2.11) \quad \langle Au, \phi \rangle = \int_\Omega \left\{ \rho^2(u) \sum_{|\alpha|=1} \mathcal{D}^\alpha u \cdot \mathcal{D}^\alpha \phi + \sum_{|\alpha| \leq 1} a_\alpha(x, u, \mathcal{D}^1 u) \mathcal{D}^\alpha \phi \right\} dx,$$

where  $D(A)$  is defined by (1.15).

We assume that the following conditions are satisfied:

- $\rho$ )  $\rho : \mathbf{R} \rightarrow \mathbf{R}$  is continuously differentiable on  $\mathbf{R}$  and satisfies (1.6);
- $a_1$ ) the real valued functions  $a_\alpha(x, \xi)$ ,  $|\alpha| \leq 1$ , are defined for  $x \in \bar{\Omega}$ ,  $\xi \in \mathbf{R}^{n+1}$  and are continuously differentiable with respect to  $\xi$ ; moreover,  $a_\alpha(x, 0) = 0$  for  $x \in \bar{\Omega}$ ,  $|\alpha| \leq 1$ ;
- $a_2$ ) there exist positive constants  $v_1, v_2$  such that for all  $x \in \bar{\Omega}$ ,  $\xi \in \mathbf{R}^{n+1}$ ,  $\eta' \in \mathbf{R}^n$  the inequalities

$$(2.12) \quad \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x, \xi) \eta_\alpha \eta_\beta \geq v_1 (1 + |\xi|)^{m-2} |\eta'|^2,$$

and

$$(2.13) \quad \sum_{|\alpha|, |\beta| \leq 1} |a_{\alpha\beta}(x, \xi)| (1 + |\xi|)^{|\alpha|+|\beta|} + \sum_{|\alpha| \leq 1} \sum_{i=1}^n |a_{\alpha i}(x, \xi)| (1 + |\xi|)^{|\alpha|} \leq v_2 (1 + |\xi|)^m$$

hold with some integer  $m$  from the interval  $(2, n)$ . In (2.12), (2.13)

$$(2.14) \quad a_{\alpha\beta}(x, \xi) = \frac{\partial}{\partial \xi_\beta} a_\alpha(x, \xi), \quad a_{\alpha i} = \frac{\partial}{\partial x_i} a_\alpha(x, \xi), \quad |\alpha|, |\beta| \leq 1, \quad i = 1, \dots, n.$$

We introduce a linear operator  $A' : W_0^{1,m}(\Omega) \rightarrow [W_0^{1,m}(\Omega)]^*$  by

$$(2.15) \quad \langle A'u, \phi \rangle = \sum_{|\alpha|, |\beta| \leq 1} \int_\Omega [\rho^2(0) \delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^\beta u \mathcal{D}^\alpha \phi dx,$$

where  $a_{\alpha\beta}^{(0)}(x) = a_{\alpha\beta}(x, 0)$ ,  $\delta_{\alpha\beta} = 1$  for  $|\alpha| = |\beta| = 1$  and  $\alpha = \beta$ , and  $\delta_{\alpha\beta} = 0$  for the rest of the values of  $\alpha, \beta$ .

Using Condition  $a_2$ ) we have the following estimate for  $u \in W_0^{1,m}(\Omega)$ :

$$(2.16) \quad \langle A'u, u \rangle \geq \int_{\Omega} [c_1 |\mathcal{D}^1 u|^2 - c_2 |u|^2] dx,$$

where the positive constants  $c_1, c_2$  depend only on  $v_1, v_2$  and  $\mu$ .

We define an operator  $\Gamma : W_0^{1,m}(\Omega) \rightarrow [W_0^{1,m}(\Omega)]^*$  by

$$(2.17) \quad \langle \Gamma u, \phi \rangle = \gamma \int_{\Omega} u(x) \phi(x) dx, \quad \gamma = c_1 + c_2,$$

where  $c_1, c_2$  are the constants from (2.16). From (2.16) and (2.17) we have

$$(2.18) \quad \langle (A' + \Gamma)u, u \rangle \geq c_1 \int_{\Omega} [|\mathcal{D}^1 u|^2 + |u|^2] dx, \quad u \in W_0^{1,m}(\Omega).$$

Using regularity results for linear elliptic equations (see Theorem 3.2 in Section 3) we will establish that the operator

$$(2.19) \quad T = (A' + \Gamma)^{-1} \Gamma : W_0^{1,m}(\Omega) \rightarrow W_0^{1,m}(\Omega)$$

is well defined and compact.

We will formulate the index theorem in terms of the characteristic values of the equation

$$(2.20) \quad \sum_{|\alpha|, |\beta| \leq 1} (-1)^{|\alpha|} \mathcal{D}^{\alpha} \{ [\rho^2 \delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^{\beta} u \} + \lambda \gamma u = 0.$$

We say that the pair  $\{\lambda_0, u_0\} \in \mathbf{R} \times W_0^{1,m}(\Omega)$  is a “solution” of the problem ((2.20), (1.3)) if  $A'u_0 + \lambda_0 \Gamma u_0 = 0$ .

**DEFINITION 2.3.** A number  $\lambda_0 \in \mathbf{R}$  is a “characteristic value” of the problem ((2.20), (1.3)) if there exists a solution  $\{\lambda_0, u_0\}$  of this problem with  $u_0 \neq 0$ .

It is clear that  $\lambda_0$  is a characteristic value of the problem ((2.20), (1.3)) if and only if  $1 - \lambda_0$  is a characteristic value of the operator  $T$ .

**DEFINITION 2.4.** The “multiplicity” of the characteristic value  $\lambda_0$  of the problem ((2.20), (1.3)) is the multiplicity of the characteristic value  $1 - \lambda_0$  of the operator  $T$  defined by (2.19).

**REMARK 2.1.** If the equality

$$(2.21) \quad \langle A'u, \phi \rangle = \langle A'\phi, u \rangle$$

is satisfied for every  $u, \phi \in W_0^{1,m}(\Omega)$ , then the multiplicity of the characteristic value  $\lambda_0$  of the problem ((2.20), (1.3)) coincides with the dimension of the space of solutions of the problem ((2.20), (1.3)) for  $\lambda = \lambda_0$ .

**THEOREM 2.2.** Assume that Conditions  $\rho$ ),  $a_1$ ),  $a_2$ ) are satisfied,  $a_{\alpha\beta}^{(0)} \in C^1(\bar{\Omega})$ , for  $|\alpha| = |\beta| = 1$ , and the equation



$$(2.22) \quad \sum_{|\alpha|, |\beta| \leq 1} (-1)^{|\alpha|} \mathcal{D}^\alpha \{[\rho^2(0)\delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^\beta u\} = 0$$

has only the zero solution in  $W_0^{1,m}(\Omega)$ . Then the index of the operator  $A$  defined by (2.11) is computed by the formula

$$(2.23) \quad \text{Ind}(A, 0) = (-1)^v,$$

where  $v$  is the sum of the multiplicities of the characteristic values of the problem ((2.20), (1.3)) lying in the interval  $(0, 1)$ .

The proof of this theorem is given in Section 4, where we verify that the operator  $A$  defined by (2.11) satisfies all the conditions of Theorem 2.1.

### 3. Auxiliary regularity results.

In this section we state some auxiliary results about the regularity of solutions of linear and nonlinear elliptic equations.

LEMMA 3.1. *Let  $g \in W^{1,p}(\Omega) \cap L_\infty(\Omega)$ ,  $1 < p < n$ , be nonnegative and assume that for every number  $r \geq 0$  the inequality*

$$(3.1) \quad \int_{\Omega} [g(x)]^r |\mathcal{D}^1 g(x)|^p dx \leq c(r+1)^q \int_{\Omega} [g(x)]^{r+p} dx$$

holds for some positive number  $q$ , where the constant  $c$  is independent of  $r$ . Then the estimate

$$(3.2) \quad \text{ess sup}\{g(x) : x \in \Omega\} \leq M_1 \left\{ \int_{\Omega} [g(x)]^{np/(n-p)} dx \right\}$$

holds, where the constant  $M_1$  depends only on  $n, p, q, c, \Omega$ .

The proof of this lemma can be found in [11, Chapter 8, Section 1].

We now formulate a regularity result for the solutions of the differential equation

$$(3.3) \quad -\sum_{|\alpha|=1} \mathcal{D}^\alpha \{\rho^2(u) \mathcal{D}^\alpha u\} + \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} \mathcal{D}^\alpha a_\alpha(x, u, \mathcal{D}^1 u) = 0, \quad x \in \Omega.$$

We understand that a solution  $u \in D(A)$  of Equation (3.3) is in the sense of an integral identity analogous to that of (1.8), where  $D(A)$  is defined by (1.15).

THEOREM 3.1. *Assume that Conditions  $\rho$ ,  $a_1$ ,  $a_2$  are satisfied and let  $u_0 \in D(A)$  be a solution of the equation (3.3). Then*

$$(3.4) \quad u_0 \in W^{2,2}(\Omega) \cap C^{1,\delta}(\bar{\Omega}),$$

for some  $\delta \in (0, 1)$ , and the estimate

$$(3.5) \quad \|u_0\|_{W^{2,2}(\Omega)} + \|u_0\|_{C^{1,\delta}(\bar{\Omega})} \leq M_2$$

holds with a constant  $M_1$  depending only on  $v_1, v_2, \mu, m, n, \Omega$  and the norm of the function  $u_0 \in W_0^{1,m}(\Omega)$ .

The assertions of this theorem follow from the results of Chapter 4, Sections 3–7, of the monograph [8].

We are also going to use a result concerning the regularity of solutions of linear elliptic equations. This result is for the equation

$$(3.6) \quad \sum_{|\alpha|, |\beta| \leq 1} (-1)^{|\alpha|} \mathcal{D}^\alpha \{a_{\alpha\beta}(x) \mathcal{D}^\beta u\} = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} \mathcal{D}^\alpha f_\alpha(x),$$

with coefficients  $a_{\alpha\beta}$  satisfying the inequalities

$$(3.7) \quad \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x) \xi_\alpha \xi_\beta \geq \nu^{(1)} \sum_{|\alpha|=1} \xi_\alpha^2,$$

$$(3.8) \quad \max_{x \in \bar{\Omega}} |a_{\alpha\beta}(x)| \leq \nu^{(2)}, \quad |\alpha|, |\beta| \leq 1,$$

for  $x \in \bar{\Omega}$ ,  $\xi_\alpha \in \mathbf{R}$  and positive constants  $\nu^{(1)}, \nu^{(2)}$ .

**THEOREM 3.2.** *Assume that  $a_{\alpha\beta} \in C(\bar{\Omega})$ , for  $|\alpha| = |\beta| = 1$ , and the conditions (3.7) and (3.8) are satisfied. Assume that the equation*

$$(3.9) \quad \sum_{|\alpha|, |\beta| \leq 1} (-1)^{|\alpha|} \mathcal{D}^\alpha \{a_{\alpha\beta}(x) \mathcal{D}^\beta u\} = 0$$

*has only the zero solution in  $W_0^{1,2}(\Omega)$ . Then for every  $p > 1$  and  $f_\alpha \in L_p(\Omega)$ ,  $|\alpha| \leq 1$ , the equation (3.6) has a unique solution  $u \in W_0^{1,p}(\Omega)$  and the following estimate holds:*

$$(3.10) \quad \sum_{|\alpha| \leq 1} \int_{\Omega} |\mathcal{D}^\alpha u(x)|^p dx \leq M_3 \sum_{|\alpha| \leq 1} \int_{\Omega} |f_\alpha(x)|^p dx,$$

*where the constant  $M_3$  depends only on  $n, p, \nu^{(1)}, \nu^{(2)}, \Omega$  and the moduli of continuity of the functions  $a_{\alpha\beta}$ ,  $|\alpha| = |\beta| = 1$ .*

The assertions of this theorem follow from [1] (see also [9, Theorem 6.21]).

#### 4. Proof of the index theorem for $m > 2$ .

We need to show that all the assumptions of Theorem 2.1 are satisfied for the operator  $A$  which is defined in (2.11).

We denote the norms of the spaces  $W_0^{1,m}(\Omega)$ ,  $L_m(\Omega)$ ,  $[W_0^{1,m}(\Omega)]^*$  by  $\|\cdot\|_{1,m}$ ,  $\|\cdot\|_m$ ,  $\|\cdot\|_{1,m}^*$ , respectively. Choosing  $D_0 = W_0^{1,m}(\Omega)$ ,  $L = C_0^\infty(\Omega)$  we see that Condition (1.10) is clearly satisfied.

The proof of Condition  $A_1)$  for the operator  $A$  is contained in [5]. The condition  $A_2)$  follows immediately from the assumptions  $\rho)$ ,  $a_1)$ . Let us prove the inequality (2.8).

**LEMMA 4.1.** *Assume that the conditions of Theorem 2.2 are satisfied. Then the inequality (2.8) holds for  $u, v \in C_0^\infty(\Omega)$ ,  $\|u\|_{1,m} \leq 1$ , and the operator  $A$  defined in (2.11).*

PROOF. Taking into account the condition  $a_2)$  we need to prove only the inequality

$$(4.1) \quad \sum_{|\alpha|=1} \int_{\Omega} \rho^2(u) \mathcal{D}^\alpha u \mathcal{D}^\alpha (u-v) dx \geq -c'(v),$$

where  $c'(v)$  depends only on  $v$  and  $u(x), v(x)$  satisfy the conditions of the lemma. Using the condition  $\rho$ ), we estimate the left-hand side of (4.1) as follows:

$$(4.2) \quad \sum_{|\alpha|=1} \int_{\Omega} \rho^2(u) \mathcal{D}^\alpha u \mathcal{D}^\alpha (u-v) dx \geq \frac{1}{2} \int_{\Omega} \rho^2(u) |\mathcal{D}^1 u|^2 dx - c''(v) \int_{\Omega} \rho^2(u) dx,$$

where

$$c''(v) = \frac{1}{2} \max\{|\mathcal{D}^1 v(x)|^2 : x \in \bar{\Omega}\}.$$

Let

$$(4.3) \quad \tilde{\rho}(u) = \int_0^u \rho(s) ds.$$

By the embedding theorem, we have

$$(4.4) \quad \int_{\Omega} |\tilde{\rho}(u)|^p dx \leq c_0 \left\{ \int_{\Omega} \rho^2(u) |\mathcal{D}^1 u|^2 dx \right\}^{p/2}, \quad p = \frac{2n}{n-2},$$

where  $c_0$  depends only on  $n$ .

For a function  $u \in C_0^\infty(\Omega)$  and any positive number  $N$  we define

$$(4.5) \quad E_N(u) = \{x \in \Omega : |u(x)| > N\}.$$

From  $\|u\|_{1,m} \leq 1$  we have

$$(4.6) \quad \text{meas } E_N(u) \leq \frac{1}{N^m}.$$

We estimate the second integral in the right-hand side of (4.2) by using (1.6), (4.4), (4.6) and Hölder's inequality. We set  $\rho_N = \max\{\rho(s) : |s| \leq N\}$  to obtain

$$(4.7) \quad \begin{aligned} \int_{\Omega} \rho^2(u) dx &= \int_{E_N(u)} \rho^2(u) dx + \int_{\Omega \setminus E_N(u)} \rho^2(u) dx \\ &\leq 2(\mu^2 + \rho_N^2) \text{meas } \Omega + 2\mu^2 \left[ \int_{\Omega} |\tilde{\rho}(u)|^p dx \right]^{(n-2)/n} \{\text{meas } E_N(u)\}^{2/n} \\ &\leq 2(\mu^2 + \rho_N^2) \text{meas } \Omega + 2\mu^2 c_0^{(n-2)/n} \left( \frac{1}{N^m} \right)^{2/n} \int_{\Omega} \rho^2(u) |\mathcal{D}^1 u|^2 dx. \end{aligned}$$

Inequality (2.8) follows now from (4.2) and (4.7) with  $N$  defined by

$$4c''(v)\mu^2 c_0^{(n-2)/n} \left( \frac{1}{N} \right)^{2m/n} = 1.$$

This completes the proof of Lemma 4.1. □

We define the operator  $A_0 : W_0^{1,m}(\Omega) \rightarrow [W_0^{1,m}(\Omega)]^*$  by the equality

$$(4.8) \quad \langle A_0 u, \phi \rangle = \sum_{|\alpha|=1} \int_{\Omega} |\mathcal{D}^1 u|^{m-2} \mathcal{D}^\alpha u \mathcal{D}^\alpha \phi \, dx.$$

LEMMA 4.2. *The operator  $A_0$  defined by (4.8) satisfies Condition  $A_0$ ) of Section 2.*

PROOF. From (4.8) we obtain

$$\|A_0 u\|_{1,m}^* = \|u\|_{1,m}^{m-1}$$

and, consequently, Condition (2.3) is satisfied. It is well known that the operator  $A_0$  is continuous and satisfies Condition  $(S_+)$  (see, e.g., [11, Chapter 1, Theorem 2.1]). This completes the proof.  $\square$

LEMMA 4.3. *Assume that the conditions of Theorem 2.2 are satisfied. Then the operator  $A'$ , defined by (2.15), satisfies Condition  $A'$ ) of Section 2.*

PROOF. Taking into account the fact that the equation (2.22) has only the zero solution in  $W_0^{1,2}(\Omega)$  and Theorem 3.2, we obtain that the equation  $A'u = 0$  has also only the zero solution. The operator  $\Gamma$ , which satisfies the relevant assumptions in Condition  $A'$ ), is defined now by (2.17). The compactness of this operator follows from the compactness of the embedding  $W_0^{1,p}(\Omega) \subset L_p(\Omega)$ . The inequality (2.2) is a consequence of (2.18).

We need to prove only the fact that the operator  $T = (A' + \Gamma)^{-1} \Gamma : W_0^{1,m}(\Omega) \rightarrow W_0^{1,m}(\Omega)$  is well defined and compact. For  $v \in W_0^{1,m}(\Omega)$ , we define  $u \in W_0^{1,m}(\Omega)$  as a solution of the equation

$$(4.9) \quad \sum_{|\alpha|, |\beta| \leq 1} (-1)^{|\alpha|} \mathcal{D}^\alpha \{ [\rho^2(0) \delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^\beta u \} + \gamma u = \gamma v.$$

From Theorem 3.2 we have the uniqueness of  $u(x)$  and the a priori estimate

$$(4.10) \quad \|u\|_{1,m} \leq K_1 \|v\|_m,$$

where the constant  $K_1$  is independent of  $v$ . Taking into account the definition of the operators  $A'$ ,  $\Gamma$ , we obtain  $(A' + \Gamma)u = \Gamma v$ . This means that  $u = Tv$ . We have established that the operator  $T$  is defined on the space  $W_0^{1,m}(\Omega)$ . The compactness of  $T$  follows from the estimate (4.10) and the compactness of the embedding  $W_0^{1,m}(\Omega) \subset L_m(\Omega)$ . The proof is complete.  $\square$

LEMMA 4.4. *Assume that the conditions of Theorem 2.2 are satisfied. Then the operators  $A, A', A_0$ , defined by (2.11), (2.15) and (4.8), respectively, satisfy Condition  $\omega$ ) of Section 2.*

PROOF. We will prove that the set  $Z'_\varepsilon$ , defined in (2.5), is empty for sufficiently small  $\varepsilon$ . Assume that the contrary is true. Then  $Z'_\varepsilon \neq \emptyset$  for every  $\varepsilon > 0$ . Thus, there exist sequences  $\{t_j\}, \{u_j\}$  such that

$$\{t_j\} \subset [0, 1], \quad \{u_j\} \subset D(A), \quad 0 < \|u_j\|_{1,m} < \frac{1}{j}, \quad t \rightarrow t_0$$

and

$$(4.11) \quad t_j A u_j + (1 - t_j)(A_0 u_j + A' u_j) = 0.$$

This means that the function  $u_j \in W_0^{1,m}(\Omega)$  is the solution of the equation

$$(4.12) \quad - \sum_{|\alpha|=1} \mathcal{D}^\alpha [\rho_j^2(u) \mathcal{D}^\alpha u] + \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} \mathcal{D}^\alpha a_{j,\alpha}(x, u, \mathcal{D}^1 u) = 0,$$

where

$$(4.13) \quad \begin{aligned} \rho_j^2(u) &= t_j \rho^2(u) + (1 - t_j) \rho^2(0), \\ a_{j,\alpha}(x, \xi) &= t_j a_\alpha(x, \xi) + (1 - t_j) \left\{ |\xi'|^{m-2} \xi'_\alpha + \sum_{|\beta| \leq 1} a_{\alpha\beta}^{(0)}(x) \xi_\beta \right\} \end{aligned}$$

and  $\xi'_\alpha = \xi_\alpha$  for  $|\alpha| = 1$ ,  $\xi'_\alpha = 0$  for  $\alpha = 0$ . It is easy to check that the functions  $\rho_j(u)$ ,  $a_{j,\alpha}(x, \xi)$  satisfy Conditions  $(\rho)$ ,  $(a_1)$ ,  $(a_2)$ , with some positive constants  $\bar{v}_1, \bar{v}_2, \bar{\mu}$ , instead of  $v_1, v_2, \mu$ , respectively. These constants  $\bar{v}_1, \bar{v}_2, \bar{\mu}$  are independent of  $j$ . Thus, by Theorem 3.1, we have the a priori estimate

$$(4.14) \quad \|u_j\|_{C^{1,\delta}(\bar{\Omega})} \leq M,$$

where the positive numbers  $\delta, M$  are independent of  $j$ . By the compactness of the embedding  $C^{1,\delta}(\bar{\Omega}) \subset C^1(\bar{\Omega})$  and  $\|u_j\|_{1,m} \rightarrow \infty$  we get

$$(4.15) \quad \lim_{j \rightarrow \infty} \|u_j\|_{C^1(\bar{\Omega})} = 0.$$

We rewrite  $a_{j,\alpha}(x, \xi)$  in the form

$$a_{j,\alpha}(x, \xi) = \sum_{|\beta| \leq 1} a_{\alpha\beta}^{(j)}(x, \xi) \xi_\beta + (1 - t_j) |\xi'|^{m-2} \xi'_\alpha,$$

where

$$a_{\alpha\beta}^{(j)}(x, \xi) = t_j \int_0^1 a_{\alpha\beta}(x, s\xi) ds + (1 - t_j) a_{\alpha\beta}^{(0)}(x)$$

with  $a_{\alpha\beta}(x, \xi)$  defined in (2.14). By the definition of a solution of the equation (4.12) we have the following integral identity for  $u_j(x)$ :

$$(4.16) \quad \sum_{|\alpha|, |\beta| \leq 1} \int_{\Omega} [\rho^{(2)}(u_j) \delta_{\alpha\beta} + |\mathcal{D}^1 u_j|^{m-2} \delta_{\alpha\beta} + a_{\alpha\beta}^{(j)}(x, u_j, \mathcal{D}^1 u_j)] \mathcal{D}^\beta u_j \mathcal{D}^\alpha \phi dx = 0,$$

for an arbitrary  $\phi \in W_0^{1,m}(\Omega)$  and the same  $\delta_{\alpha\beta}$  as in (2.15). From (4.16) with  $\phi(x) = u_j(x)$ , the estimate (4.14) and Conditions (1.6), (2.12), (2.13) we obtain the estimate

$$(4.17) \quad \|u_j\|_{1,2} \leq c \|u_j\|_2$$

with a positive constant  $c$  independent of  $j$ .

The sequence  $v_j(x) = u_j(x)/\|u_j\|_2^{-1}$  is bounded in  $W^{1,2}(\Omega)$ . By passing to a subsequence, if necessary, we may assume that  $v_j(x)$  converges to some function  $v_0(x)$  weakly in  $W_0^{1,2}(\Omega)$  and strongly in  $L_2(\Omega)$ . Since  $\|v_j\|_2 = 1$ , we have  $v_0 \neq 0$ .

Dividing (4.16) by  $\|u_j\|_2$  and then taking the limit as  $j \rightarrow \infty$  we obtain by virtue of (4.15)

$$\sum_{|\alpha|, |\beta| \leq 1} \int_{\Omega} [\rho^2(0)\delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^{\beta} v_0(x) \mathcal{D}^{\alpha} \phi \, dx = 0, \quad \phi \in W_0^{1,m}(\Omega).$$

The last equality is obviously true for every  $\phi \in W_0^{1,2}(\Omega)$ , which establishes the fact that  $v_0(x)$  is a solution of the equation (2.22) in  $W_0^{1,2}(\Omega)$ . Since  $v_0 \neq 0$ , we obtain a contradiction with the assumption of Theorem 2.2. Consequently,  $Z'_\varepsilon = \emptyset$  for a sufficiently small  $\varepsilon$  and the assertion of Lemma 4.4 has been proved.  $\square$

We consider the splitting

$$(4.18) \quad W_0^{1,m}(\Omega) = F + R$$

as we did in (2.6) for the operator  $T$  defined by (2.19). Let  $\Pi$  be the projection of  $W_0^{1,m}(\Omega)$  onto  $F$  associated with the splitting (4.18).

**LEMMA 4.5.** *Assume that the conditions of Theorem 2.2 are satisfied. Then the operator  $\Pi(A' + \Gamma)^{-1} : (A' + \Gamma)W_0^{1,m}(\Omega) \rightarrow W_0^{1,m}(\Omega)$  is bounded.*

**PROOF.** Let  $\lambda_1, \dots, \lambda_I$  be the characteristic values of the operator  $T$  defined by (2.19), and let  $v_i$  be the multiplicity of the characteristic value  $\lambda_i$ . Define, for the natural number  $N$ , the space

$$(4.19) \quad F_i(N) = \{u \in W_0^{1,m}(\Omega) : (I - \lambda_i T)^N u = 0\}.$$

It is well known [13] that  $F_i(N)$  is a finite-dimensional subspace of  $W_0^{1,m}(\Omega)$  and there exists a number  $N_i$  such that

$$(4.20) \quad F_i(N) = F_i(N_i), \quad \text{for } N \geq N_i, \quad \dim F_i(N_i) = v_i.$$

Now, the space  $F$  corresponding to the operator  $T$  is the direct sum of the spaces  $F_i(N_i)$ ,  $i = 1, \dots, I$ , and  $F$  can be defined (see [13, p. 317]) by

$$(4.21) \quad F = \{u \in W_0^{1,m}(\Omega) : (I - \tilde{T})u = 0\},$$

where

$$(4.22) \quad \tilde{T} = I - \prod_{i=1}^I (I - \lambda_i T)^{N_i}.$$

Analogously, the space  $R$  from the splitting (4.18) can be given by

$$(4.23) \quad R = \{(1 - \tilde{T})u : u \in W_0^{1,m}(\Omega)\},$$

where the operator  $\tilde{T}$  is defined by (4.22).

Let  $f_j(x)$ ,  $j = 1, \dots, \nu$ , be a basis for the space  $F$ . The projection  $\Pi : W_0^{1,m}(\Omega) \rightarrow F$  can be defined by

$$(4.24) \quad \Pi u = \sum_{i=1}^{\nu} \langle \zeta_i, u \rangle f_i(x),$$

where  $\zeta_j \in [W_0^{1,m}(\Omega)]^*$  and satisfies

$$(4.25) \quad \langle \zeta_j, f_i \rangle = \delta_{ij}, \quad \langle \zeta_j, r \rangle = 0,$$

for  $i, j = 1, \dots, \nu$ , and any function  $r(x)$  from  $R$ . In (4.25)  $\delta_{ij}$  is the Kronecker delta symbol.

Using the representation of a functional from  $[W_0^{1,m}(\Omega)]^*$  we can find functions  $z_{\alpha,j} \in L_{m'}(\Omega)$ , such that, for  $|\alpha| = 1$  and  $m' = m/(m-1)$ , we have

$$(4.26) \quad \langle \zeta_j, \phi \rangle = \sum_{|\alpha|=1} \int_{\Omega} z_{\alpha,j}(x) \mathcal{D}^{\alpha} \phi(x) dx, \quad \phi \in W_0^{1,m}(\Omega).$$

Define a function  $v \in W_0^{1,m'}$  as the solution of the equation

$$(4.27) \quad \Delta v_j(x) = \sum_{|\alpha|=1} \mathcal{D}^{\alpha} z_{\alpha,j}(x),$$

where  $\Delta$  is the Laplace operator. The existence of such a function  $v_j(x)$  follows from Theorem 3.2. From (4.26), (4.27) we obtain

$$(4.28) \quad \langle \zeta_j, \phi \rangle = \sum_{|\alpha|=1} \int_{\Omega} \mathcal{D}^{\alpha} v_j(x) \mathcal{D}^{\alpha} \phi(x) dx, \quad \phi \in W_0^{1,m}(\Omega).$$

We will prove that for every  $p > 1$  we have

$$(4.29) \quad v_j \in W_0^{1,p}(\Omega).$$

Using (4.23) and the second equality in (4.25) we obtain

$$(4.30) \quad \sum_{|\alpha|=1} \int_{\Omega} \mathcal{D}^{\alpha} v_j(x) \mathcal{D}^{\alpha} \phi(x) dx = \sum_{|\alpha|=1} \int_{\Omega} \mathcal{D}^{\alpha} v_j(x) \mathcal{D}^{\alpha} \tilde{T} \phi(x) dx,$$

for every  $\phi \in W_0^{1,m}(\Omega)$ .

Inclusion (4.29) is certainly true for  $p = m'$ . Let us assume that it is true for some  $p = p_1 \geq m'$ . Consider the functional

$$(4.31) \quad I_j(\phi) = \sum_{|\alpha|=1} \int_{\Omega} \mathcal{D}^{\alpha} v_j(x) \mathcal{D}^{\alpha} \tilde{T} \phi(x) dx,$$

for  $\phi \in C_0^{\infty}(\Omega)$ . By Hölder's inequality and (4.29) we have the estimate

$$(4.32) \quad |I_j(\phi)| \leq c_1 \|\tilde{T} \phi\|_{1,p_1}, \quad p_1' = \frac{p_1}{p_1 - 1},$$

for some constant  $c_1$  independent of  $\phi$ .

As in (4.10), we have the estimate

$$(4.33) \quad \|T\phi\|_{1,p'_1} \leq c_2 \|\phi\|_{p'_1}$$

with a constant  $c_2$  independent of  $\phi$ . From (4.22), (4.33) we have the analogous inequality for the operator  $\tilde{T}$ :

$$(4.34) \quad \|\tilde{T}\phi\|_{1,p'_1} \leq c_3 \|\phi\|_{p'_1}.$$

Using the estimates (4.32), (4.33) we obtain

$$(4.35) \quad |I_j(\phi)| \leq c_4 \|\phi\|_{p'_1},$$

which says that the functional  $I(\phi)$  can be extended to a continuous functional  $I_j : L_{p'_1}(\Omega) \rightarrow \mathbf{R}$ . From the representation of such a functional follows the existence of a function  $h_j \in L_{p_1}(\Omega)$  such that

$$(4.36) \quad I_j(\phi) = \int_{\Omega} h_j(x)\phi(x) dx,$$

for  $\phi \in L_{p'_1}(\Omega)$ .

From (4.30), (4.31) and (4.36) we obtain that  $v_j(x)$  is a solution to the equation

$$(4.37) \quad -\Delta v_j(x) = h_j(x).$$

Thus, from (4.29), (4.37) and a priori estimates for solutions of linear elliptic equations [1] we have  $v_j \in W^{2,p_1}(\Omega)$ . By the embedding theorem,

$$(4.38) \quad v_j \in W_0^{1,\tilde{p}}(\Omega),$$

where  $\tilde{p}$  is an arbitrary number if  $p_1 \geq n$  and  $\tilde{p} = p_1 n / (n - p_1)$  if  $p_1 < n$ . Starting from (4.29) with  $p = m'$  and using (4.38) we can establish after finitely many steps that the final value of  $\tilde{p}$  satisfies  $\tilde{p} > n$ . Thus, (4.29) holds for every  $p > 1$ .

We are now able to estimate the operator  $\Pi(A' + \Gamma)^{-1}$ . Let  $g = (A' + \Gamma)w$  be a functional in  $[W_0^{1,m}(\Omega)]^*$ , for some  $w \in W_0^{1,m}(\Omega)$ . Using the representation of the functional  $g$  we can define functions  $g_\alpha \in L_{m'}(\Omega)$ ,  $|\alpha| = 1$ , such that

$$(4.39) \quad \langle g, \phi \rangle = \sum_{|\alpha|=1} \int_{\Omega} g_\alpha(x) \mathcal{D}^\alpha \phi(x) dx, \quad \phi \in W_0^{1,m}(\Omega),$$

and

$$(4.40) \quad \sum_{|\alpha|=1} \|g_\alpha\|_{m'} \leq c_5 \|g\|_{1,m}^*,$$

where  $c_5$  is some positive constant independent of  $g$ . Using (4.39), (4.40) and Theorem 3.2 we have the estimate

$$(4.41) \quad \|w\|_{1,m'} \leq C_6 \|g\|_{1,m}^*,$$

where the function  $w \in W_0^{1,m}(\Omega)$  satisfies the equation  $(A' + \Gamma)w = g$ .



Using (4.24), (4.28) we obtain

$$\Pi(A' + \Gamma)^{-1}g = \sum_{j=1}^v \sum_{|\alpha|=1} \int_{\Omega} \mathcal{D}^{\alpha}v_j(x)\mathcal{D}^{\alpha}w(x) dx \cdot f_j(x).$$

From (4.29) with  $p = m$ , (4.41) and Hölder's inequality we get the estimate

$$\|\Pi(A' + \Gamma)^{-1}g\|_{1,m} \leq c_7 \|g\|_{1,m}^*$$

where the positive constant  $c_7$  is independent of  $g$ . This ends the proof.  $\square$

LEMMA 4.6. *Assume that the conditions of Theorem 2.2 are satisfied. Then there exists a positive number  $\varepsilon$  such that for  $t \in [0, 1]$  the equation*

$$(4.42) \quad t \sum_{|\alpha|=1} \mathcal{D}^{\alpha} \{ |\mathcal{D}^1 u|^{m-2} \mathcal{D}^{\alpha} u \} - \sum_{|\alpha|, |\beta| \leq 1} (-1)^{|\alpha|} \mathcal{D}^{\alpha} \{ [\rho^2(0)\delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^{\beta} u \} = 0$$

has only the zero solution in the ball  $B_{\varepsilon}(0)$ .

PROOF. Denote by  $N_1(t)$  the set of solutions of (4.42) in the ball  $B_1(0)$  and let  $N_1 = \bigcup_{t \in [0,1]} N_1(t)$ . At first, we are going to establish a priori estimates for an arbitrary  $u \in N_1$ . There exists a constant  $M^{(0)}$  independent of  $u$  such that

$$(4.43) \quad \max_{x \in \bar{\Omega}} |u(x)| \leq M^{(0)}, \quad u \in N_1.$$

Taking into account the fact that the equation (4.42) with  $t = 0$  has only the zero solution in  $B_1(0)$  we consider heretofore only the case where  $u \in N_1(t)$  for  $t > 0$ . By Theorem 3.1  $u \in W^{2,2}(\Omega) \cap C^{1,\delta}(\bar{\Omega})$  with  $\delta > 0$ .

In the integral identity

$$(4.44) \quad \int_{\Omega} \left\{ t \sum_{|\alpha|=1} |\mathcal{D}^1 u|^{m-2} \mathcal{D}^{\alpha} u \mathcal{D}^{\alpha} \phi + \sum_{|\alpha|, |\beta| \leq 1} [\rho^2(0)\delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^{\beta} u \mathcal{D}^{\alpha} \phi \right\} dx = 0,$$

which is valid for  $\phi \in W_0^{1,m}(\Omega)$ , we let  $\phi$  be the test function

$$\phi(x) = [1 + |u(x)|]^r u(x),$$

where  $r$  is an arbitrary positive number. After some standard calculations we obtain the inequality

$$\int_{\Omega} \{ t |\mathcal{D}^1 u|^m + |\mathcal{D}^1 u|^2 \} [1 + |u(x)|] dx \leq c_8 (1+r) \int_{\Omega} [1 + |u(x)|]^{r+2} dx,$$

and the estimate (4.43) from it and Lemma 3.1 with  $q = 0$ . By  $c_j, j = 8, \dots$ , we denote constants independent of the function  $u$ .

Now, we will show the estimate

$$(4.45) \quad \max_{x \in \bar{\Omega}} |\mathcal{D}^1 u(x)| \leq M^{(1)}, \quad u \in N_1,$$

under the assumption that  $u \in N_1(t)$ ,  $t > 0$ . As in the proof of Lemma 2.1 [8, Chapter 6], we can establish the estimate

$$(4.46) \quad \max_{x \in \partial\Omega} |\mathcal{D}^1 u(x)| \leq M', \quad u \in N_1,$$

with a constant  $M'$  independent of  $u$ .

Letting in (4.44)  $\phi(x) = (\partial/\partial x_i)\psi(x)$ , with  $\psi \in C_0^\infty(\Omega)$ , we obtain the equality

$$(4.47) \quad \int_{\Omega} \left\{ t \sum_{|\alpha|=1} \frac{\partial}{\partial x_i} (|\mathcal{D}^1 u|^{m-2} \mathcal{D}^\alpha u) \mathcal{D}^\alpha \psi \right. \\ \left. + \sum_{|\alpha|, |\beta| \leq 1} \frac{\partial}{\partial x_i} ([\rho^2(0)\delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^\alpha u) \mathcal{D}^\beta \psi \right\} dx = 0.$$

It is easy to verify that this equality is also true for an arbitrary function  $\psi \in W_0^{1,2}(\Omega)$ .

We will now obtain an  $L_2$ -estimate for the second derivatives of  $u \in N_1$ . Namely, we are going to establish the following inequality

$$(4.48) \quad \int_{\Omega} [t|\mathcal{D}^1 u|^{m-2} + 1] |\mathcal{D}^\alpha u|^2 \phi^2(x) dx \\ \leq c_9 \int_{\Omega} [t|\mathcal{D}^1 u|^{m-2} + 1] |\mathcal{D}^1 u|^2 (\phi^2(x) + |\mathcal{D}^1 \phi|^2) dx,$$

for  $|\alpha| = 2$  and some function  $\phi \in C^\infty(\bar{\Omega})$ . Letting  $\psi(x) = (\partial/\partial x_i)u(x)\phi_0^2(x)$ , with  $\phi_0 \in C_0^\infty(\Omega)$ , in (4.47) we obtain after a simple calculation the inequality (4.48) with  $\phi(x) = \phi_0(x)$ . If in some neighborhood  $\mathcal{U}(\bar{x})$  of a point  $\bar{x} \in \partial\Omega$  the equation of  $\mathcal{U}(\bar{x}) \cap \partial\Omega$  is given by the equality  $x_n = 0$ , then we let in (4.47)  $\psi(x) = (\partial/\partial x_i)u(x)\bar{\phi}(x)$ , with  $i < n$ ,  $\bar{\phi} \in C_0^\infty(\mathcal{U}(\bar{x}))$ , to obtain the inequality (4.48) with  $|\alpha| = 2$ ,  $\alpha_n \neq 2$ ,  $\phi(x) = \bar{\phi}(x)$ . Using this estimate and the equation (4.42) we get the inequality (4.48) with  $|\alpha| = 2$ ,  $\alpha_n = 2$ ,  $\phi(x) = \bar{\phi}(x)$ . Passing to local coordinates near the boundary  $\partial\Omega$  and choosing a corresponding partition of unity, we derive the estimate (4.48) with  $\phi(x) \equiv 1$ . Thus, we have proved the inequality

$$(4.49) \quad \int_{\Omega} [t|\mathcal{D}^1 u|^{m-2} + 1] |\mathcal{D}^2 u|^2 dx \leq M^{(2)}, \quad u \in N_1,$$

where the constant  $M^{(2)} > 0$  is independent of  $u$ ,  $t$ .

We define the function  $w$  as follows:

$$(4.50) \quad w(x) = \max\{|\mathcal{D}^1 u(x)|^2 - (M')^2, 0\},$$

where  $M'$  is the number from the estimate (4.46), and let in (4.47)

$$\psi(x) = \omega_r(w(x)) \frac{\partial u(x)}{\partial x_i}, \quad \text{where } \omega_r(s) = \frac{s^{r+1}}{1+s}, \quad r \geq 0.$$

After some standard calculations we derive the inequality

$$(4.51) \quad \int_{\Omega} \{t|\mathcal{D}^1 u(x)|^{m-2} + 1\} \omega_r(w(x)) |\mathcal{D}^2 u(x)|^2 dx \\ \leq c_{10} \int_{\Omega} \{\omega_r(w(x))(|\mathcal{D}^1 u|^2 + 1) + \omega'_r(w(x))(|\mathcal{D}^1 u|^4 + 1)\} dx,$$

where  $\omega'_r(s) = (d/ds)\omega_r(s)$ . Define

$$\bar{w}(x) = \max\{\sqrt{w(x)}, 1\}, \quad E = \{x \in \Omega; \bar{w}(x) > 1\}$$

and note that the following estimates hold:

$$(4.52) \quad \omega_r(w(x)) \geq c_{11} |\mathcal{D}^1 u(x)|^{2r}, \quad w(x) \geq c_{11} |\mathcal{D}^1 u(x)|^2, \quad x \in E,$$

$$(4.53) \quad |\mathcal{D}^1 u(x)| \leq c_{11} \bar{w}(x), \quad \omega_r(w(x)) \leq c_{11} [\bar{w}(x)]^{2r}, \\ \omega'_r(w(x)) \leq c_{11}(r+1) [\bar{w}(x)]^{2r-2}, \quad x \in \Omega.$$

Using the inequalities (4.51)–(4.53) we deduce

$$(4.54) \quad \int_{\Omega} [\bar{w}(x)]^{2r} |\mathcal{D}^1 \bar{w}(x)|^2 dx = \int_E [\bar{w}(x)]^{2r} |\mathcal{D}^1 \bar{w}(x)|^2 dx \\ \leq c_{12} \int_E \omega_r(w(x)) |\mathcal{D}^2 u(x)|^2 dx \\ \leq c_{13}(r+1) \int_{\Omega} [\bar{w}(x)]^{2r+2} dx.$$

From the estimates (4.49), (4.53) we have the following inequality:

$$(4.55) \quad \int_{\Omega} |\mathcal{D}^1 \bar{w}(x)|^2 dx \leq c_{14} \int_{\Omega} |\mathcal{D}^2 u(x)|^2 dx \leq c_{14} M^{(2)}.$$

The estimate (4.45) follows immediately from (4.54), (4.55) and Lemma 3.1.

Let us now show that  $N_{\varepsilon} = \{0\}$  for sufficiently small  $\varepsilon$ . Assume that this is not true. Then there exist sequences  $\{t_j\}$ ,  $\{u_j(x)\}$  such that  $u_j \in N_1(t_j)$ ,  $u_j \neq 0$ ,  $\|u_j\|_{1,m} \rightarrow 0$ . From (4.44) with  $t = t_j$ ,  $u(x) = u_j(x)$ ,  $\phi(x) = u_j(x)$  we obtain the estimate

$$(4.56) \quad \|u_j\|_{1,2} \leq c_{15} \|u_j\|_2$$

with a constant  $c_{15}$  independent of  $j$ .

Taking into account (4.56) we may assume that the sequence  $v_j(x) = u_j(x)/\|u_j\|_2$  converges weakly in  $W_0^{1,2}(\Omega)$  to some function  $v_0(x)$  such that  $v_0 \neq 0$ . Dividing the equality (4.44), with  $t = t_j$ ,  $u(x) = u_j(x)$ , by  $\|u_j\|_2$  and passing to the limit at  $j \rightarrow \infty$  we obtain by virtue of (4.45)

$$\sum_{|\alpha|, |\beta| \leq 1} \int_{\Omega} [\rho^2(0)\delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^{\beta} v_0(x) \mathcal{D}^{\alpha} \phi(x) dx = 0, \quad \phi \in W_0^{1,m}(\Omega).$$

This contradicts with our assumptions of Theorem 2.2 and completes the proof of the lemma.

We have verified that the operator  $A$ , defined by (2.11) satisfies all the assumptions of Theorem 2.1. Consequently, (2.23) follows from Theorem 2.1. This is the end of the proof of Theorem 2.2.  $\square$

### 5. Bifurcation of solutions for $m > 2$ .

We consider in this section the bifurcation of solutions of the problem ((1.2), (1.3)) for the case of the Banach space  $W_0^{1,m}(\Omega)$ ,  $m > 2$ . We first assume that the functions  $\rho(u)$ ,  $a_\alpha(x, \xi')$  ( $|\alpha| = 1$ ),  $a_0(x, \xi)$  satisfy (1.9) and Conditions i), ii) of Section 1. Let the operators  $A$ ,  $C$  be defined by (1.14), (1.16) and let  $\lambda_0$  be some real number. We assume that, for some  $\delta_0 > 0$ , zero is an isolated critical point of the operator  $A + \lambda C$  for every  $\lambda$  from the interval  $(\lambda_0 - \delta_0, \lambda_0 + \delta_0)$ . If this is not the case, then  $\lambda_0$  would be a bifurcation point. Taking into account Theorem 1.1 and Definition 2.2 we can define the index  $\text{Ind}(A + \lambda C, 0)$  of the operator  $A + \lambda C$  at zero for  $|\lambda - \lambda_0| < \delta_0$ .

Let

$$(5.1) \quad \bar{i}_\pm(\lambda_0) = \limsup_{\lambda \rightarrow \lambda_0 \pm} \text{Ind}(A + \lambda C, 0), \quad \underline{i}_\pm(\lambda_0) = \liminf_{\lambda \rightarrow \lambda_0 \pm} \text{Ind}(A + \lambda C, 0).$$

**THEOREM 5.1.** *Assume that Conditions i), ii) of Section 1 and (1.9) are satisfied and assume that at least two of the numbers*

$$(5.2) \quad \bar{i}^-(\lambda_0), \quad \bar{i}^+(\lambda_0), \quad \bar{i}_-(\lambda_0), \quad \bar{i}_+(\lambda_0), \quad \text{Ind}(A + \lambda_0 C, 0)$$

*are distinct for the operators  $A$ ,  $C$  defined by (1.14), (1.16), respectively. Then  $\lambda_0$  is a bifurcation point of the problem ((1.2), (1.3)).*

The assertion of Theorem 5.1 follows immediately from Theorem 5.1 of the paper [6].

Now, we will establish a necessary condition for the bifurcation point in terms of the linearized boundary value problem. We assume that the coefficients of the equation (1.2) satisfy Conditions  $\rho$ ),  $a_1$ ),  $a_2$ ) of Section 2 and introduce the operators  $A' : W_0^{1,m}(\Omega) \rightarrow [W_0^{1,m}(\Omega)]^*$ ,  $C' : W_0^{1,m}(\Omega) \rightarrow [W_0^{1,m}(\Omega)]^*$  by

$$(5.3) \quad \langle A'u, \phi \rangle = \sum_{|\alpha|=|\beta|=1} \int_{\Omega} [\rho^2(0)\delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^\beta u \mathcal{D}^\alpha \phi \, dx$$

$$(5.4) \quad \langle C'u, \phi \rangle = \sum_{|\alpha| \leq 1} \int_{\Omega} a_{0\alpha}^{(0)}(x) \mathcal{D}^\alpha u \phi(x) \, dx,$$

where

$$a_{\alpha\beta}^{(0)}(x) = a_{\alpha\beta}(x, 0), \quad a_{\alpha\beta}(x, \xi) = \frac{\partial}{\partial \xi_\beta} a_\alpha(x, \xi),$$

$\delta_{\alpha\beta} = 1$  for  $\alpha = \beta$  and  $\delta_{\alpha\beta} = 0$  otherwise.

From (2.12) we have

$$(5.5) \quad \langle A'u, u \rangle \geq v_0 \|u\|_{1,2}^2, \quad u \in W_0^{1,m}(\Omega),$$

where  $v_0$  is a positive constant independent of  $u$ . As in the proof of Lemma 4.3, we see that the operator

$$(5.6) \quad T = -(A')^{-1}C' : W_0^{1,m}(\Omega) \rightarrow W_0^{1,m}(\Omega)$$

is well defined and compact.

We introduce the linearized equation

$$(5.7) \quad \sum_{|\alpha|=|\beta|=1} \mathcal{D}^\alpha \{[\rho^2(0)\delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)]\mathcal{D}^\beta u\} - \lambda \sum_{|\alpha|\leq 1} a_{0\alpha}^{(0)}(x)\mathcal{D}^\alpha u, \quad x \in \Omega,$$

and define a pair  $\{\lambda_0, u_0\} \in \mathbf{R} \times W_0^{1,m}(\Omega)$  to be a ‘‘solution’’ of the problem ((5.7), (1.3)) if  $A'u_0 + \lambda C'u_0 = 0$ .

**DEFINITION 5.1.** A number  $\lambda_0 \in \mathbf{R}$  is said to be a ‘‘characteristic value’’ of the problem ((5.7), (1.3)) if there exists a solution  $\{\lambda_0, u_0\}$  of this problem such that  $u_0 \neq 0$ .

**DEFINITION 5.2.** The ‘‘multiplicity’’ of the characteristic value  $\lambda_0$  of the problem ((5.7), (1.3)) is the multiplicity of the characteristic value  $\lambda_0$  of the operator  $T$  defined by (5.6).

We note that the characteristic values of the problem ((5.7), (1.3)) and the operator  $T$  coincide.

**LEMMA 5.1.** Assume that the coefficients  $\rho(u)$ ,  $a_\alpha(x, \xi')$  ( $|\alpha| = 1$ ),  $a_0(x, \xi)$  of the equation (1.2) satisfy the relevant conditions  $\rho$ ,  $a_1$ ,  $a_2$  of Section 2. Let  $[\lambda_1, \lambda_2]$  be a closed interval which does not contain any characteristic values of the problem ((5.7), (1.3)). Then there exists a positive number  $\varepsilon$ , depending only on  $\lambda_1, \lambda_2$ , such that for every  $\lambda \in [\lambda_1, \lambda_2]$  the boundary value problem ((1.2), (1.3)) has only the zero solution in the ball  $B_\varepsilon(0) \subset W_0^{1,m}(\Omega)$ .

The proof of this lemma is similar to that of Lemma 4.4. It is therefore omitted.

**THEOREM 5.2.** Assume that Conditions  $\rho$ ,  $a_1$ ,  $a_2$  of Section 2 are satisfied by the coefficients of the equation (1.2). A necessary condition that  $\lambda_0$  be a bifurcation point of the problem ((1.2), (1.3)) is that  $\lambda_0$  is a characteristic value of the problem ((5.7), (1.3)).

**PROOF.** Assume that  $\lambda_0$  is not a characteristic value of the problem ((5.7), (1.3)). We need to show that  $\lambda_0$  is not a bifurcation point of the problem ((1.2), (1.3)). We know that the characteristic values of the problem ((5.7), (1.3)) and the operator  $T$  coincide and that the set of characteristic values of the compact operator  $T$  is discrete. Consequently, there exists  $\delta_1 > 0$  such that some interval  $(\lambda_0 - \delta_1, \lambda_0 + \delta_1)$  contains no characteristic values of the problem ((1.2), (1.3)). Thus, by Lemma 5.1, there is some ball  $B_{\varepsilon_1}(0) \subset W_0^{1,m}(\Omega)$  which contains only the zero solution of the boundary value problem ((1.2), (1.3)) for  $|\lambda - \lambda_0| \leq \delta_1$ . This says that  $\lambda_0$  is not a bifurcation point of the problem ((1.2), (1.3)) and completes the proof of Theorem 5.2.  $\square$

**THEOREM 5.3.** *Assume that the conditions of Theorem 5.2 are satisfied and let  $\lambda_0$  be a characteristic value of the problem ((5.7), (1.3)) of odd multiplicity. Then  $\lambda_0$  is a bifurcation point of the problem ((1.2), (1.3)).*

**PROOF.** Choose a positive number  $\delta_1$  such that the interval  $(\lambda_0 - \delta_1, \lambda_0 + \delta_1)$  contains only one characteristic value of the operator  $T$  which is defined by (5.6). Let  $\lambda_-, \lambda_+$  be arbitrary numbers from the intervals  $(\lambda_0 - \delta_1, \lambda_0), (\lambda_0, \lambda_0 + \delta_1)$ , respectively. Then the operators  $A_- = A + \lambda_- C, A_+ = A + \lambda_+ C$  satisfy all the assumptions on  $A$  of Theorem 2.2. Thus, by that theorem,

$$(5.8) \quad \text{Ind}(A_-, 0) = (-1)^{v_-}, \quad \text{Ind}(A_+, 0) = (-1)^{v_+},$$

where  $v_{\pm}$  is the sum of the multiplicities of the characteristic values of the operator  $T$  on the interval  $(0, \lambda_{\pm})$ . If  $v_0$  is the multiplicity of the characteristic value  $\lambda_0$ , then  $v_+ = v_- + v_0$  and, by the condition of the theorem, we have from (5.8)

$$(5.9) \quad \text{Ind}(A + \lambda_- C, 0) = -\text{Ind}(A + \lambda_+ C, 0),$$

for any  $\lambda_- \in (\lambda_0 - \delta_1, \lambda_0), \lambda_+ \in (\lambda_0, \lambda_0 + \delta_1)$ .

From (5.9) we have

$$\lim_{\lambda \rightarrow \lambda_0^-} \text{Ind}(A + \lambda C, 0) = - \lim_{\lambda \rightarrow \lambda_0^+} \text{Ind}(A + \lambda C, 0)$$

and the assertion of Theorem 5.3 follows from Theorem 5.1.  $\square$

## 6. Bifurcation of solutions for $m = 2$ .

We formulate first a result about the computation of the index of a critical point for abstract operators in the case of a real separable Hilbert space  $H$  instead of the Banach space  $X$ . We denote by  $\langle \cdot, \cdot \rangle$  the scalar product in the space  $H$ . The conditions  $A_1), A_2), A'), A_0), \omega), C)$ , which we will use below, are given in Sections 1, 2.

**THEOREM 6.1.** *Let  $H$  be a real separable Hilbert space and let  $A : H \supset D(A) \rightarrow H$  satisfy (2.8), Conditions  $A_1), A_2)$  and be such that  $0 \in D(A) \cap D_0, A(0) = 0$ . Assume that there exist a bounded linear operator  $A' : H \rightarrow H$  and compact linear operators  $\Gamma_0 : H \rightarrow H, \Gamma : H \rightarrow H$  such that*

$$\langle (A + \Gamma_0)u, u \rangle \geq c\|u\|^2, \quad u \in D(A), \quad \|u\| \leq 1,$$

$$\langle (A' + \Gamma)u, u \rangle \geq c\|u\|^2, \quad u \in H,$$

where  $c$  is a positive constant. Assume that the equation  $A'u = 0$  has only the zero solution and the condition  $\omega)$  is satisfied with  $A_0u = \|u\|A'u$ . Then zero is an isolated critical point of the operator  $A$  and its index equals  $(-1)^v$ , where  $v$  is the sum of the multiplicities of the characteristic values of the operator  $T = (A' + \Gamma)^{-1}\Gamma$  lying in the interval  $(0, 1)$ .

**PROOF.** We note that from the second inequality in (6.1) it follows that the operator  $(A' + \Gamma)^{-1}\Gamma$  is well defined on all of  $H$  and compact. Similarly, the operator  $A_0 : H \rightarrow H$ , defined by  $A_0u = \|u\|A'u$ , satisfies the condition  $A_0)$ . We are going to

verify that the conclusion of Theorem 6.1 follows from that of Theorem 2.1. From (6.1) we have the boundedness of the operator  $(A' + \Gamma)^{-1} : H \rightarrow H$  and, consequently, Condition 1) of Theorem 2.1 is satisfied.

Let us verify that Condition 2) of Theorem 2.1 is satisfied. From the choice of  $A_0$  and the conditions for the operator  $A'$  we have that the set  $Z_1''$ , defined by (2.10), is empty. Let  $\{t_j\}$ ,  $\{u_j\}$  be two sequences such that  $t_j \in [0, 1]$ ,  $u_j \in D(A)$  and

$$(6.2) \quad t_j A u_j + (1 - t_j)(A_0 u_j + A' u_j) = 0, \quad 0 < \|u_j\| \leq 1.$$

We may assume that the sequence  $v_j = u_j / \|u_j\|$  converges weakly to some  $v_0$ . From (6.1), (6.2) we have

$$c[t_j + (1 - t_j)(\|u_j\| + 1)]\|u_j\|^2 \leq t_j \langle \Gamma_0 u_j, u_j \rangle + (1 - t_j)(\|u_j\| + 1) \langle \Gamma u_j, u_j \rangle,$$

which says that  $v_0 \neq 0$ . This establishes the validity of Condition 2) of Theorem 2.1 and finishes the proof of Theorem 6.1.  $\square$

Throughout this and the following sections we assume that  $\Omega$  is an open and bounded subset of  $\mathbf{R}^n$ . Before we state and prove the index result of this section, we go over some auxiliary facts and assumptions. The operator  $A : W_0^{1,2}(\Omega) \supset D(A) \rightarrow W_0^{1,2}(\Omega)$  is defined by

$$(6.3) \quad \langle Au, \phi \rangle = \int_{\Omega} \left\{ \rho^2(u) \sum_{|\alpha|=1} \mathcal{D}^\alpha u \mathcal{D}^\alpha \phi + \sum_{|\alpha| \leq 1} a_\alpha(x, u, \mathcal{D}^1 u) \mathcal{D}^\alpha \phi \right\} dx,$$

where

$$(6.4) \quad D(A) = \{u \in W_0^{1,2}(\Omega) : \rho^2(u) \mathcal{D}^\alpha u \in L_2(\Omega) \text{ for } |\alpha| = 1\}.$$

We assume the following conditions.

$\tilde{\rho}$ )  $\rho : \mathbf{R} \rightarrow \mathbf{R}$  is continuous and satisfies (1.6);

$\tilde{a}_1$ ) the real valued functions  $a_\alpha(x, \xi)$ ,  $|\alpha| \leq 1$ , are defined on  $\bar{\Omega} \times \mathbf{R}^{n+1}$  and are measurable with respect to  $x$  and continuously differentiable with respect to  $\xi$ ; moreover,  $a_\alpha(x, 0) = 0$ , for  $x \in \Omega$ ,  $|\alpha| = 1$ ;

$\tilde{a}_2$ ) there exist positive constants  $v_1, v_2$  such that for  $x \in \bar{\Omega}$ ,  $\xi \in \mathbf{R}^{n+1}$ ,  $\eta_\alpha \in \mathbf{R}$  the inequalities

$$(6.5) \quad \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x, \xi) \eta_\alpha \eta_\beta \geq v_1 \sum_{|\alpha|=1} \eta_\alpha^2,$$

and

$$\sum_{|\alpha|, |\beta| \leq 1} |a_{\alpha\beta}(x, \xi)| \leq v_2$$

hold, where  $a_{\alpha\beta}(x, \xi) = (\partial / \partial \xi_\beta) a_\alpha(x, \xi)$ .

We introduce a linear operator  $A' : W_0^{1,2}(\Omega) \rightarrow W_0^{1,2}(\Omega)$  by (2.15). As in Section 2, we define an operator  $\Gamma : W_0^{1,2}(\Omega) \rightarrow W_0^{1,2}(\Omega)$  by (2.17) and then establish (2.18).

Using Condition  $\tilde{a}_2$ ), it is easy to verify the estimate

$$(6.6) \quad \langle (A + \Gamma_0)u, u \rangle \geq \kappa_1 \|u\|_{1,2}^2, \quad u \in D(A),$$

where  $\kappa_1 > 0$  and

$$\langle \Gamma_0 u, \phi \rangle = \gamma_0 \int_{\Omega} u(x) \phi(x) dx,$$

with the constant  $\gamma_0$  sufficiently large. From the compactness of the embedding  $W_0^{1,2}(\Omega) \subset L_2(\Omega)$  we have that the operators  $\Gamma, \Gamma_0$  are compact. As in Lemma 5.3, we can verify that the operator

$$(6.7) \quad T = (A' + \Gamma)^{-1} \Gamma : W_0^{1,2}(\Omega) \rightarrow W_0^{1,2}(\Omega)$$

is well defined and compact.

We consider the linearized equation

$$(6.8) \quad \sum_{|\alpha|, |\beta| \leq 1} (-1)^{|\alpha|} \mathcal{D}^\alpha \{ [\rho^2(0) \delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^\beta u \} + \lambda \gamma u = 0$$

with the same  $\delta_{\alpha\beta}, a_{\alpha\beta}^{(0)}$  as in (2.15). In view of Definitions 2.3, 2.4, we define analogously the characteristic value and the multiplicity of the characteristic value of the problem ((6.8), (1.3)).

**THEOREM 6.2.** *Assume that Conditions  $(\tilde{\rho}), (\tilde{a}_1), (\tilde{a}_2)$  are satisfied and that the equation (6.8) has only the zero solution in  $W_0^{1,2}(\Omega)$  for  $\lambda = 0$ . Then the index of the operator  $A$ , defined by (6.3), is computed by the formula*

$$(6.9) \quad \text{Ind}(A, 0) = (-1)^v,$$

where  $v$  is the sum of the multiplicities of the characteristic values of the problem ((6.8), (1.3)) lying in the interval  $(0, 1)$ .

**PROOF.** We need to prove that the operator  $A$ , defined by (6.3), satisfies all the conditions of Theorem 6.1. The inequality (2.8) follows immediately from Lemma 4.1, while the inequalities (6.1) follow from (2.18) and (6.6).

We need to check only Condition  $\omega$ ). Let  $\{u_j\} \subset D(A), \{t_j\} \subset [0, 1]$  be such that

$$(6.10) \quad t_j A u_j + (1 - t_j) [A_0 u_j + A' u_j] = 0, \quad 0 < \|u_j\|_{1,2} < \frac{1}{j}, \quad t_j \rightarrow t_0,$$

for some  $t_0 \in [0, 1]$ . The function  $u_j(x)$  satisfies the following integral identity:

$$(6.11) \quad \int_{\Omega} \left\{ t_j \left[ \sum_{|\alpha| \leq 1} a_{\alpha}(x, u_j, \mathcal{D}^1 u_j) \mathcal{D}^\alpha \phi(x) + \sum_{|\alpha|=1} \rho^2(u_j) \mathcal{D}^\alpha u_j(x) \mathcal{D}^\alpha \phi(x) \right] dx \right. \\ \left. + (1 - t_j) (1 + \|u_j\|_{1,2}) \sum_{|\alpha|, |\beta| \leq 1} [\rho^2(0) \delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^\beta u_j \mathcal{D}^\alpha \phi(x) \right\} dx \\ = 0,$$



for every  $\phi \in W_0^{1,2}(\Omega)$ . It is well known that Condition  $\tilde{a}_2$ ) guarantees the boundedness of the function  $u_j(x)$  (see [8, Chapter 4, Section 7]). We estimate the maximum of  $|u_j(x)|$ . Letting  $\phi(x) = |u_j(x)|^r u_j(x)$  in (6.11) with  $r \geq 0$  and repeating standard calculations we obtain the inequality

$$(6.12) \quad \int_{\Omega} |u_j(x)|^2 |\mathcal{D}^1 u_j(x)|^2 dx \leq \kappa_2(1+r) \int_{\Omega} |u_j(x)|^{2+r} dx,$$

where the constant  $\kappa_2$  is independent of  $j, r$ . Thus, from (6.10), (6.12) and Lemma 3.1 we have

$$(6.13) \quad \max\{|u_j(x)| : x \in \Omega\} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

For  $\phi \in W_0^{1,2}(\Omega)$  we have

$$(6.14) \quad \begin{aligned} \langle \|u_j\|_{1,2}^{-1} [Au_j - A'u_j], \phi \rangle &= \int_{\Omega} \left\{ \sum_{|\alpha|=1} [\rho^2(u_j) - \rho^2(0)] \mathcal{D}^\alpha v_j(x) \mathcal{D}^\alpha \phi(x) \right. \\ &\quad \left. + \sum_{|\alpha|, |\beta| \leq 1} \int_0^1 [a_{\alpha\beta}(x, su_j, s\mathcal{D}^1 u_j) - a_{\alpha\beta}(x, 0)] \mathcal{D}^\beta v_j(x) \mathcal{D}^\alpha \phi(x) ds \right\} dx, \end{aligned}$$

where  $v_j(x) = u_j(x)/\|u_j\|_{1,2}^{-1}$ , and the convergence of the right-hand side of (6.14) to zero follows from (6.10), (6.13), Conditions  $\tilde{\rho}$ ),  $\tilde{a}_1$ ),  $\tilde{a}_2$ ) and the continuity of the Nemytskii operator. We have shown the convergence in (2.4) and the proof is complete.  $\square$

Now, we consider the bifurcation of the solutions of the boundary value problem ((1.2), (1.3)) when the coefficients of the equation (1.3) satisfy the conditions  $\tilde{\rho}$ ),  $\tilde{a}_1$ ),  $\tilde{a}_2$ ). We introduce the operators  $A' : W_0^{1,2}(\Omega) \rightarrow W_0^{1,2}(\Omega)$ ,  $C' : W_0^{1,2}(\Omega) \rightarrow W_0^{1,2}(\Omega)$  and  $T : W_0^{1,2}(\Omega) \rightarrow W_0^{1,2}(\Omega)$  by (5.3), (5.4) and (5.6), respectively, and define a linearized equation in accordance with (5.7). Note that the inequality (5.5) for the operator  $A'$  is satisfied. We maintain the validity of Definitions 5.1, 5.2 for the boundary value problem ((5.7), (1.3)) under consideration.

**THEOREM 6.3.** *Assume that the coefficients  $\rho(u)$ ,  $a_\alpha(x, \xi')$  ( $|\alpha| = 1$ ),  $a_0(x, \xi)$  of the equation (1.2) satisfy the relevant conditions  $\tilde{\rho}$ ),  $\tilde{a}_1$ ),  $\tilde{a}_2$ ) and let  $\lambda_0$  be a bifurcation point of the problem ((1.2), (1.3)). Then  $\lambda_0$  is a characteristic value of the problem (5.7), (1.3).*

**PROOF.** Let  $\{\lambda_j, u_j\}$  be a sequence of solutions of the problem ((1.2), (1.3)) with

$$(6.15) \quad \lambda_j \rightarrow \lambda_0, \quad \|u_j\|_{1,2} \rightarrow 0, \quad u_j \neq 0.$$

The function  $u_j(x)$  satisfies the following integral identity:

$$(6.16) \quad \begin{aligned} \sum_{|\alpha|=1} \int_{\Omega} \{ \rho^2(u_j) \mathcal{D}^\alpha u_j + a_\alpha(a, \mathcal{D}^1 u_j) \} \mathcal{D}^\alpha \phi(x) dx \\ + \lambda_j \int_{\Omega} a_0(u, u_j, \mathcal{D}^1 u_j) \phi dx = 0, \quad \phi \in W_0^{1,2}(\Omega). \end{aligned}$$

We let  $\phi = u_j$  above and obtain, after some standard calculations,

$$(6.17) \quad \|u_j\|_{1,2} \leq \kappa_3 \|u_j\|_2,$$

with the constant  $k_3$  independent of  $j$ . We may thus assume that the sequence  $v_j(x) = u_j(x)/\|u_j\|_2$  converges weakly in  $W_0^{1,2}(\Omega)$  to  $v_0(x)$  and  $v_0 \neq 0$  by virtue of (6.17). As in the proof of Theorem 6.2, we can establish that (6.13) holds for the sequence  $u_j(x)$ . From (6.16) we have

$$(6.18) \quad \sum_{|\alpha|=1} \int_{\Omega} \left\{ \rho^2(u_j) \mathcal{D}^{\alpha} v_j + \sum_{|\beta|=1} \int_0^1 a_{\alpha\beta}(x, s \mathcal{D}^1 u_j) \mathcal{D}^{\beta} v_j ds \right\} \mathcal{D}^{\alpha} \phi(x) dx \\ + \lambda_j \sum_{|\alpha| \leq 1} \int_{\Omega} \int_0^1 a_{0\alpha}(x, s u_j, s \mathcal{D}^1 u_j) \mathcal{D}^{\alpha} v_j(x) \phi(x) ds dx = 0.$$

Using (6.13) and (6.15) we can pass to the limit in (6.18) to obtain that  $\{\lambda_0, v_0\}$  is a solution of the problem ((5.7), (1.3)). Taking into account that  $v_0 \neq 0$ , we see that the proof of the theorem is complete.  $\square$

**THEOREM 6.4.** *Assume that the conditions of Theorem 6.3 are satisfied and let  $\lambda_0$  be a characteristic value of the problem ((5.7), (1.3)) of odd multiplicity. Then  $\lambda_0$  is a bifurcation point of the problem ((1.2), (1.3)).*

The proof follows from Theorem 6.2 by repeating the arguments of the proof of Theorem 5.3.

## 7. Bifurcation of unbounded solutions.

In this section we demonstrate the possibility to study the bifurcation problem for the equation (1.2) even if the solutions of the problem ((1.2), (1.3)) are unbounded. We consider the case of the Hilbert space  $W_0^{1,2}(\Omega)$  and let, for simplicity,  $a_0(a, \xi)$  be the linear function defined by

$$(7.1) \quad a_0(x, \xi) = \sum_{|\alpha| \leq 1} q_{\alpha}(x) \xi_{\alpha},$$

where

$$(7.2) \quad q_{\alpha} \in L_n(\Omega) \quad \text{for } |\alpha| = 1, \quad q_0(x) \in L_{n/2}(\Omega).$$

It is known that the solutions of the problem ((1.2), (1.3)) can be unbounded under the assumptions (7.1), (7.2) (see, e.g., [8, Chapter 1, Section 2]).

In order to study the bifurcation problem with the function  $a_0(x, \xi)$ , defined by (7.1), (7.2), we will use a weak variant of Condition  $\omega$ ) that is suggested from a careful examination of the proof of Theorem 2.1 in [6].

**REMARK 7.1.** Assume that all the conditions of Theorem 2.1 are satisfied, except Condition  $\omega$ ). Then the conclusion of this theorem is true if the following modified condition  $\tilde{\omega}$  is satisfied:

$\tilde{\omega}$ ) there exists a positive number  $\varepsilon$  such that for every sequence  $\{u_j\}$  such that  $u_j \in Z'_\varepsilon(t_j)$ ,  $u_j \rightarrow 0$  we have

$$(7.3) \quad \lim_{j \rightarrow \infty} \frac{t_j}{\|u_j\|} \langle Au_j - A'u_j, v \rangle = 0,$$

for each  $v \in L$ , where, for  $t \in [0, 1]$ ,

$$(7.4) \quad Z'_\varepsilon(t) = \{u \in D(A) : tAu + (1-t)(A_0u + A'u) = 0, 0 < \|u\| \leq \varepsilon\}.$$

Define a linear operator  $C : W_0^{1,2}(\Omega) \rightarrow W_0^{1,2}(\Omega)$  by

$$(7.5) \quad \langle Cu, \phi \rangle = \sum_{|\alpha| \leq 1} \int_{\Omega} q_\alpha(x) \mathcal{D}^\alpha u(x) \phi(x) dx,$$

where  $q_\alpha(x)$  satisfies (7.2).

LEMMA 7.1. *Assume that  $n > 2$  and (7.2) is satisfied. Then the operator  $C$ , defined by (7.5), is compact.*

PROOF. We only check that if  $\{u_j\}, \{\phi_j\} \subset W_0^{1,2}(\Omega)$  converge weakly to  $u_0, \phi_0$ , respectively, then we have

$$(7.6) \quad \lim_{j \rightarrow \infty} \langle Cu_j, \phi_j \rangle = \langle Cu_0, \phi_0 \rangle.$$

By Hölder's inequality, we estimate

$$(7.7) \quad |\langle Cu_j, \phi_j - \phi_0 \rangle| \leq \kappa_4 \|u_j\|_{1,2} \left\{ \sum_{|\alpha|=1} \left[ \int_{\Omega} |q_\alpha(x) [\phi_j(x) - \phi_0(x)]|^2 dx \right] + \left[ \int_{\Omega} |q_0(x) [\phi_j(x) - \phi_0(x)]|^{p_0} dx \right]^{1/p_0} \right\},$$

where  $p_0 = 2n/(n+2)$  and  $\kappa_4$  is a constant independent of  $j$ . From the embedding theorem we have that the sequences  $\{q_\alpha(x) [\phi_j(x) - \phi_0(x)]\}$ ,  $|\alpha| \leq 1$ , converge to zero in measure as  $j \rightarrow \infty$ . Using (7.2) and Hölder's inequality we obtain that the integrals in the right-hand side of (7.7) are uniformly absolutely continuous. Consequently, the right-hand side of (7.7) tends to zero as  $j \rightarrow \infty$ . Thus, (7.6) is true and the proof is complete.  $\square$

LEMMA 7.2. *Assume that  $n > 2$  and that the functions  $\rho(u), a_\alpha(x, \xi)$  ( $|\alpha| = 1$ ) satisfy the conditions  $(\tilde{\rho}), (\tilde{a}_1), (\tilde{a}_2)$  of Section 6. Assume that the function  $a_0(x, \xi)$  satisfies Conditions (7.1), (7.2). Let  $A, A', C$  be the operators defined by (1.14), (5.3) and (7.5), respectively. Then for every  $\lambda \in \mathbf{R}$  the operators  $\tilde{A} = A + \lambda C, \tilde{A}' = A' + \lambda C$  and  $\tilde{A}_0$  (defined by  $\tilde{A}_0 u = \|u\|_{1,2} A' u$ ) satisfy Condition  $\tilde{\omega}$ ) with  $\varepsilon = 1$ .*

PROOF. Let  $\{u_j\} \subset W_0^{1,2}(\Omega)$  be such that

$$(7.8) \quad \int_{\Omega} \rho^2(u_j) |\mathcal{D}^1 u_j|^2 dx < +\infty, \quad \|u_j\|_{1,2} \rightarrow 0, \quad 0 < \|u_j\|_{1,2} \leq 1,$$

and, for some  $t_j \in [0, 1]$  and every  $\phi \in W_0^{1,2}(\Omega)$ ,

$$(7.9) \quad \int_{\Omega} \left\{ t_j \sum_{|\alpha|=1} (\rho^2(u_j) \mathcal{D}^\alpha u_j + a_\alpha(x, \mathcal{D}^1 u_j)) \mathcal{D}^\alpha \phi \right. \\ \left. + (1 - t_j) \sum_{|\alpha|=|\beta|=1} (1 + \|u_j\|_{1,2}) (\rho^2(0) \delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)) \mathcal{D}^\beta u_j \mathcal{D}^\alpha \phi \right. \\ \left. + \lambda \sum_{|\alpha| \leq 1} q_\alpha(x) \mathcal{D}^\alpha u_j \phi \right\} dx = 0.$$

From (7.9) with  $\phi = u_j$  we have

$$(7.10) \quad t_j \int_{\Omega} \rho^2(u_j) |\mathcal{D}^1 u_j|^2 dx \leq \kappa_5 \|u_j\|_{1,2}^2.$$

Using the notation (4.3) and the inequalities (4.4) and (7.10) we have

$$(7.11) \quad t_j^{p/2} \int_{\Omega} |\tilde{\rho}(u_j)|^p dx \leq \kappa_6 \|u_j\|_{1,2}^p.$$

For  $\phi \in C_0^\infty(\Omega)$  we have

$$(7.12) \quad t_j \|u_j\|_{1,2}^{-1} \langle \tilde{A} u_j - \tilde{A}' u_j, \phi \rangle \\ = t_j \|u_j\|_{1,2}^{-1} \int_{\Omega} \sum_{|\alpha|=1} [\rho^2(u_j) - \rho^2(0)] \mathcal{D}^\alpha u_j(x) \mathcal{D}^\alpha \phi(x) + t_j \|u_j\|_{1,2}^{-1} \\ \times \sum_{|\alpha|=|\beta|=1} \int_0^1 \int_{\Omega} [a_{\alpha\beta}(x, s \mathcal{D}^1 u_j) - a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^\beta u_j(x) \mathcal{D}^\alpha \phi(x) dx.$$

The convergence to zero of the second summand above follows from (7.8) and the continuity of the Nemytskii operator.

We see that the first summand of (7.12) is bounded above by

$$(7.13) \quad \kappa_7 \|\phi\|_{C^1(\bar{\Omega})} \cdot \left\{ \int_{\Omega} \frac{t_j}{\|u_j\|_{1,2}^2} |\rho^2(u_j) - \rho^2(0)| |\mathcal{D}^1 u_j|^2 \right\}^{1/2} \\ \cdot \left\{ \int_{\Omega} t_j |\rho^2(u_j) - \rho^2(0)| dx \right\}$$

The first integral in (7.13) is bounded above by virtue of (7.10). The integrand in the second integral of (7.13) converges to zero in measure by virtue of (7.8) and the embedding theorem. The sequence of the second integrals in (7.13) satisfies the condition of uniform absolute continuity. This follows from the condition  $\rho$ , (7.8) and (7.11). Consequently, the right-hand side of (7.12) tends to zero and the proof of the lemma is complete.  $\square$

We consider the linearized equation

$$(7.14) \quad \sum_{|\alpha|=|\beta|=1} \mathcal{D}^\alpha \{ [\rho^2(0) \delta_{\alpha\beta} + a_{\alpha\beta}^{(0)}(x)] \mathcal{D}^\beta u \} - \lambda \sum_{|\alpha| \leq 1} q_\alpha(x) \mathcal{D}^\alpha u = 0, \quad x \in \Omega.$$

Using Lemmas 7.1, 7.2 and Remark 7.1 we establish the following results in a manner analogous to that of Theorems 6.3 and 6.4.

**THEOREM 7.1.** *Assume that  $n > 2$  and the functions  $\rho(u)$ ,  $a_\alpha(x, \xi)$  ( $|\alpha| = 1$ ),  $a_0(x, \xi)$  satisfy the conditions of Lemma 7.2. Let  $\lambda_0$  be a bifurcation point of the problem ((1.2), (1.3)). Then  $\lambda_0$  is a characteristic value of the problem ((7.14), (1.3)).*

**THEOREM 7.2.** *Assume that  $n > 2$  and the functions  $\rho(u)$ ,  $a_\alpha(x, \xi)$  ( $|\alpha| = 1$ ),  $a_0(x, \xi)$  satisfy the conditions of Lemma 7.2. Let  $\lambda_0$  be a characteristic value of the problem ((7.14), (1.3)) of odd multiplicity. Then  $\lambda_0$  is a bifurcation point of the problem ((1.2), (1.3)).*

## References

- [1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, *Comm. Pure Appl. Math.* **12** (1959), 623–727.
- [2] M. A. Del Pino and R. F. Manasevich, Global bifurcation from the eigenvalue of the  $p$ -Laplacian, *J. Differential Equations* **92** (1991), 226–251.
- [3] P. Drábek, Solvability and bifurcation of Nonlinear Equations, *Pitman Res. Notes Math. Ser.*, #264, Longman, Harlow, 1992.
- [4] N. Fukagai, M. Ito and K. Narukawa, A bifurcation problem of some nonlinear degenerate equations, *Adv. Differential Equations* **2** (1997), 895–926.
- [5] A. G. Kartsatos and I. V. Skrypnik, Topological degree theories for densely defined mappings involving operators of type  $(S_+)$ , *Adv. Differential Equations* **4** (1999), 413–456.
- [6] A. G. Kartsatos and I. V. Skrypnik, The index of a critical point for densely defined operators of type  $(S_+)$  in Banach spaces (to appear).
- [7] M. Krasnoselskij, Topological methods in the theory of nonlinear integral equations, Pergamon Press, Oxford, 1964.
- [8] O. A. Ladyzhenskaya and N. N. Uraltseva, Linear and quasilinear elliptic equations, Academic Press, New York-London, 1968.
- [9] E. Magenes, Interpolation spaces and partial differential equations, *Uspekhi Math. Nauk* **21** (1966), 169–218.
- [10] I. V. Skrypnik, Nonlinear higher order elliptic equations, Naukova Dumka, Kiev, 1973.
- [11] I. V. Skrypnik, Methods for analysis of nonlinear elliptic boundary value problems, Amer. Math. Soc. Transl. Ser. II, #139, Providence, Rhode Island, 1994.
- [12] K. Taira and K. Umez, Bifurcation for nonlinear elliptic boundary value problems III, *Adv. Differential Equations* **1** (1996), 709–727.
- [13] A. E. Taylor, Introduction to functional analysis, John Wiley, New York, 1967.

**Athanasios G. KARTSATOS**

Department of Mathematics  
University of South Florida  
Tampa, Florida 33620-5700  
USA

**Igor V. SKRYPNIK**

Institute for Applied Mathematics and Mechanics  
R. Luxemburg Str. 74  
Donetsk 340114  
Ukraine

E-mail: hermes@math.usf.edu, skrypnik@iamm.ac.donetsk.ua