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Curvature pinching for totally real submanifolds of a complex projective space

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Abstract. Montiel, Ros and Urbano [**M3**] showed a complete characterization of compact totally real minimal submanifold M of $CP^n(c)$ with Ricci curvature S of M satisfying $S \geq 3(n-2)c/16$. The purpose of this paper is to answer Ogiue's conjecture which the above result remains true under the weaker condition of the scalar curvature ρ of M satisfying $\rho \geq 3n(n-2)c/16$.

1. Introduction.

Let $CP^n(c)$ be an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $c(>0)$ and let M be an n-dimensional compact totally real minimal submanifold isometrically immersed in $CP^n(c)$. Let h be the second fundamental form of M in $CP^n(c)$.

Recently, Montiel, Ros and Urbano [**M3**] proved the following: Let M be an n-dimensional compact totally real minimal submanifold isometrically immersed in $CP^n(c)$. Then the Ricci curvature S of M satisfies

$$S \geq \frac{3(n-2)}{16}c$$

if and only if one of the following conditions holds: a) $S = (n-1)c/4$ and M is totally geodesic, b) $S = 0$, $n = 2$ and M is a finite Riemannian covering of a flat torus minimally embedded in $CP^2(c)$ with parallel second fundamental form, c) $S = 3(n-2)c/16$, $n > 2$ and M is an embedded submanifold congruent to the standard embedding of: $SU(3)/SO(3)$, $n = 5$; $SU(6)/Sp(3)$, $n = 14$; $SU(3)$, $n = 8$; or E_6/F_4, $n = 26$.

Ogiue [**O1**] conjectured the following: Under the weaker assumption of $\rho \geq 3n(n-2)c/16$, the above result remains true, where ρ is the scalar curvature of M.

With respect to this conjecture the author [**M1**] and, independently, [**X1**] showed: Let M be an n-dimensional compact totally real minimal submanifold isometrically immersed in $CP^n(c)$. Then

$$|h(v,v)|^2 \leq \frac{1}{8}c$$

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for any $v \in UM$ if and only if one of the following conditions is satisfied: A) $|h(v,v)|^2 \equiv 0$ and M is totally geodesic, B) $|h(v,v)|^2 \equiv (1/8)c$, $n = 2$ and M is a finite Riemannian covering of a flat torus minimally embedded in $CP^2(c)$ with parallel second fundamental form, C) $|h(v,v)|^2 \equiv (1/8)c$, $n > 2$ and M is an embedded submanifold congruent to the standard embedding of: $SU(3)/SO(3)$, $n = 5$; $SU(6)/Sp(3)$, $n = 14$; $SU(3)$, $n = 8$ or E_6/F_4, $n = 26$.

Gauchman [**G**] showed a similar result under the assumption of $|h(v,v)|^2 \leq (n+1)c/12n$.

The purpose of this paper is to answer Ogiue's conjecture.

Theorem. *Let M be an n-dimensional compact totally real minimal submanifold isometrically immersed in $CP^n(c)$. Then $\rho =$ constant and*

$$\rho \geq \frac{3n(n-2)}{16}c$$

if and only if one of the following conditions holds:

A) $\rho = n(n-1)c/4$ *and M is totally geodesic,*

B) $\rho = 0, n = 2$ *and M is a finite Riemannian covering of the unique flat torus minimally embedded in $CP^2(c)$ with parallel second fundamental form,*

C) $\rho = 3n(n-2)c/16$, $n > 2$ *and M is an embedded submanifold congruent to the standard embedding of*: $SU(3)/SO(3)$, $n = 5$; $SU(6)/Sp(3)$, $n = 14$; $SU(3)$, $n = 8$ *or* E_6/F_4, $n = 26$.

Xia [**X2**] showed a similar result under the assumption of $|h|^2 < (n+1)c/6$.

2. Preliminaries.

Let M be a compact Riemannian manifold, UM its unit tangent bundle, and UM_x the fibre of UM over a point x of M.

Now, we suppose that M is isometrically immersed in an $(n+p)$-dimensional Riemannian manifold $\overline{M}$. We denote by $\langle\, , \rangle$ the metric of $\overline{M}$ as well as the one induced on M. Let h be the second fundamental form of the immersion and A_ξ the Weingarten endomorphism associated a normal vector ξ, we define

$$L : T_xM \to T_xM$$

by the expression

$$Lv = \sum_{i=1}^{n} A_{h(v,e_i)}e_i.$$

Then, L is a self-adjoint linear map.

The first covariant derivative ∇h is symmetric and the second covariant derivatives $\nabla^2 h$ satisfies

$$(2.1) \qquad (\nabla^2 h)(X,Y,Z,W) = (\nabla^2 h)(Y,X,Z,W) + R^\perp(X,Y)h(Z,W) - h(R(X,Y)Z,W) - h(Z,R(X,Y)W),$$

where $R^{\perp}$ and R are the curvature tensors of the normal and tangent bundles over M, respectively.

Now let $v \in UM_x$, $x \in M$. If $e_2, \dots, e_n$ are orthonormal vectors in UM_x orthogonal to v, then we can consider $\{e_2, \dots, e_n\}$ as an orthonormal basis of $T_v(UM_x)$. We remark that $\{v = e_1, e_2, \dots, e_n\}$ is an orthonormal basis of T_xM. We denote the Laplacian of $UM_x \cong S^{n-1}$ by Δ.

If S and ρ is the Ricci tensor of M and the scalar curvature of M, respectively, and M is minimally immersed in $\overline{M}$, then from the Gauss equation we have

$$S(v, w) = \sum_{i=1}^{n} \overline{R}(v, e_i, e_i, w) - \sum_{i=1}^{n} \langle A_{h(v, e_i)} e_i, w \rangle, \tag{2.2}$$

$$\rho = \sum_{i,j=1}^{n} \overline{R}(e_j, e_i, e_i, e_j) - |h|^2, \tag{2.3}$$

where $\overline{R}$ is the curvature operator of $\overline{M}$.

Define a function f_1 on UM_x, $x \in M$, by

$$f_1(v) = |A_{h(v,v)} v|^2 = \sum_{i=1}^{n} \langle h(v, v), h(v, e_i) \rangle^2.$$

Using the minimality of M we can prove that

$$\begin{aligned}(\Delta f_1)(v) = {} & -6(n+4) f_1(v) + 8 \sum_{i=1}^{n} \langle A_{h(v,v)} v, A_{h(v,e_i)} e_i \rangle \\ & + 8 \sum_{i=1}^{n} \langle A_{h(v,v)} e_i, A_{h(v,e_i)} v \rangle + 8 \sum_{i=1}^{n} \langle A_{h(v,e_i)} v, A_{h(v,e_i)} v \rangle \\ & + 2 \sum_{i=1}^{n} \langle A_{h(v,v)} e_i, A_{h(v,v)} e_i \rangle.\end{aligned} \tag{2.4}$$

Similarly, define $f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}$ and f_{13} by

$$f_2(v) = \sum_{i=1}^{n} \langle A_{h(v,e_i)} v, A_{h(v,e_i)} v \rangle,$$

$$f_3(v) = \sum_{i=1}^{n} \langle A_{h(v,e_i)} v, A_{h(v,v)} e_i \rangle,$$

$$f_4(v) = \sum_{i,j=1}^{n} \langle A_{h(e_j,e_i)} e_j, A_{h(v,v)} e_i \rangle,$$

$$f_5(v) = \sum_{i=1}^{n} \langle A_{h(v,v)} v, A_{h(v,e_i)} e_i \rangle,$$

$$f_6(v) = \sum_{i,j=1}^{n} \langle A_{h(e_j,e_i)} e_j, A_{h(v,e_i)} v \rangle,$$

$$f_7(v) = \sum_{i,j=1}^{n} \langle A_{h(e_i,v)}e_i, A_{h(v,e_j)}e_j \rangle,$$

$$f_8(v) = \sum_{i=1}^{n} \langle A_{h(v,v)}e_i, A_{h(v,v)}e_i \rangle,$$

$$f_9(v) = |h(v,v)|^4,$$

$$f_{10}(v) = |h(v,v)|^2,$$

$$f_{11}(v) = \sum_{i=1}^{n} \langle A_{h(v,e_i)}e_i, v \rangle |h(v,v)|^2,$$

$$f_{12}(v) = \left(\sum_{i=1}^{n} \langle A_{h(v,e_i)}e_i, v \rangle \right)^2,$$

$$f_{13}(v) = |h|^2 |h(v,v)|^2,$$

respectively. Then we obtain

$$(2.5)\qquad (\varDelta f_2)(v) = -4(n+2)f_2(v) + 4f_6(v) + 4\sum_{i,j=1}^{n} \langle A_{h(e_j,e_i)}v, A_{h(v,e_i)}e_j \rangle$$
$$+ 2\sum_{i,j=1}^{n} \langle A_{h(e_j,e_i)}v, A_{h(e_j,e_i)}v \rangle + 2\sum_{i,j=1}^{n} \langle A_{h(v,e_i)}e_j, A_{h(v,e_i)}e_j \rangle,$$

$$(2.6)\qquad (\varDelta f_3)(v) = -4(n+2)f_3(v) + 2f_4(v) + 4\sum_{i,j=1}^{n} \langle A_{h(e_j,e_i)}v, A_{h(e_j,v)}e_i \rangle$$
$$+ 4\sum_{i,j=1}^{n} \langle A_{h(v,e_i)}e_j, A_{h(e_j,v)}e_i \rangle,$$

$$(2.7)\qquad (\varDelta f_4)(v) = -2nf_4(v),$$

$$(2.8)\qquad (\varDelta f_5)(v) = -4(n+2)f_5(v) + 4f_6(v) + 4f_7(v) + 2f_4(v),$$

$$(2.9)\qquad (\varDelta f_6)(v) = -2nf_6(v) + 2\sum_{i,j,k=1}^{n} \langle A_{h(e_j,e_i)}e_j, A_{h(e_k,e_i)}e_k \rangle,$$

$$(2.10)\qquad (\varDelta f_7)(v) = -2nf_7(v) + 2\sum_{i,j,k=1}^{n} \langle A_{h(e_j,e_i)}e_j, A_{h(e_k,e_i)}e_k \rangle,$$

$$(2.11)\qquad (\varDelta f_8)(v) = -4(n+2)f_8(v) + 8\sum_{i,j=1}^{n} \langle A_{h(e_j,v)}e_i, A_{h(e_j,v)}e_i \rangle,$$

$$(2.12)\qquad (\varDelta f_9)(v) = -8(n+6)f_9(v) + 32f_1(v) + 16\sum_{i=1}^{n} \langle A_{h(v,e_i)}e_i, v \rangle |h(v,v)|^2,$$

$$(\Delta f_{10})(v) = -4(n+2)f_{10}(v) + 8\sum_{i=1}^{n}\langle A_{h(v,e_i)}e_i, v\rangle. \tag{2.13}$$

$$(\Delta f_{11})(v) = -6(n+4)f_{11}(v) + 16f_5(v) + 2|h|^2|h(v,v)|^2 + 8f_{12}(v), \tag{2.14}$$

$$(\Delta f_{12})(v) = -4(n+2)f_{12}(v) + 8f_7(v) + 4|h|^2\sum_{i=1}^{n}\langle A_{h(v,e_i)}e_i, v\rangle, \tag{2.15}$$

$$(\Delta f_{13})(v) = -4(n+2)f_{13}(v) + 8|h|^2\sum_{i=1}^{n}\langle A_{h(v,e_i)}e_i, v\rangle. \tag{2.16}$$

Since

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i,e_i,v) &= \sum_{i=1}^{n}\langle(\nabla^2 h)(e_i,e_i,v,v), h(v,v)\rangle \\ &\quad + \sum_{i=1}^{n}\langle(\nabla h)(e_i,v,v), (\nabla h)(e_i,v,v)\rangle,\end{aligned}$$

we have the following (See [**M1**], [**M2**] and [**M3**]):

Lemma 1. *Let M be an n-dimensional totally real minimal submanifold isometrically immersed in $CP^n(c)$. Then for $v \in UM_x$ we have*

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i,e_i,v) &= \sum_{i=1}^{n}|(\nabla h)(e_i,v,v)|^2 + \frac{n+1}{4}c|h(v,v)|^2 \\ &\quad + 2\sum_{i=1}^{n}\langle A_{h(v,v)}e_i, A_{h(e_i,v)}v\rangle \\ &\quad - 2\sum_{i=1}^{n}\langle A_{h(v,e_i)}e_i, A_{h(v,v)}v\rangle \\ &\quad - \sum_{i=1}^{n}\langle A_{h(v,v)}e_i, A_{h(v,v)}e_i\rangle.\end{aligned} \tag{2.17}$$

The following lemma will be useful:

Lemma 2 ([**C**]). *Let M be as in Lemma 1. If ρ is constant and $n \le 3$, then M satisfies*

$$\frac{1}{2}\rho - \inf K(x) = \frac{1}{8}(n+1)(n-2)c$$

if and only if either:

(1) *M is totally geodesic, or*

(2) *$n = 3, c = 1/4$ and M is locally congruent to S^3 isometrically immersed in CP^3, where $K(x)$ is the sectional curvature at x.*

3. Proof of Theorem.

PROPOSITION. *Let M be an n-dimensional compact totally real minimal submanifold isometrically immersed in $CP^n(c)$. Then*

$$\rho \geq \frac{3n(n-2)}{16}c$$

if and only if one of the following conditions holds:

A) $\rho = n(n-1)c/4$ *and M is totally geodesic,*

B) $\rho = 0, n = 2$ *and M is a finite Riemannian covering of the unique flat torus minimally embedded in $CP^2(c)$ with parallel second fundamental form,*

C) $\rho = 3n(n-2)c/16, n > 2$ *and M is an embedded submanifold congruent to the standard embedding of*: $SU(3)/SO(3)$, $n = 5$; $SU(6)/Sp(3)$, $n = 14$; $SU(3)$, $n = 8$ *or* E_6/F_4, $n = 26$.

D) $\rho = n(n-1)c/4 - 4\lambda^2 \geq 3n(n-2)c/16$, $n \neq 2$ *and the second fundamental form of M takes the following form*: $h(e_1, e_1) = \lambda Je_1$, $h(e_2, e_2) = -\lambda Je_1$, $h(e_1, e_2) = -\lambda Je_2$, *otherwise zero for some non-constant function λ.*

By Lemma 2 we know that we have only to prove the above Proposition. For if ρ is constant, then the case of D cannot occur.

Now, from (2.3) we have

$$\rho = \frac{n(n-1)}{4}c - |h|^2.$$

Thus we prove Proposition under the assumption

$$|h|^2 \leq \frac{n(n+2)}{16}c. \tag{3.1}$$

We see the following equations hold for $v \in UM_x$, $x \in M$ (See [**M1**] and [**M2**]):

$$\sum_{i,j=1}^{n} \langle A_{h(e_j,e_i)}v, A_{h(e_j,v)}e_i\rangle = \sum_{i,j=1}^{n} \langle A_{h(v,e_i)}e_j, A_{h(v,e_j)}e_i\rangle, \tag{3.2}$$

$$\sum_{i,j=1}^{n} \langle A_{h(e_j,e_i)}v, A_{h(e_j,e_i)}v\rangle = \sum_{i,j=1}^{n} \langle A_{h(v,e_i)}e_j, A_{h(v,e_i)}e_j\rangle. \tag{3.3}$$

In terms of (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11), (2.17), (3.2) and (3.3) we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i, e_i, v) - \frac{1}{6}(\Delta f_1)(v) - \frac{1}{3(n+2)}(\Delta f_2)(v) \\
&\quad + \frac{1}{6(n+2)}(\Delta f_3)(v) + \frac{1}{3n(n+2)}(\Delta f_4)(v) + \frac{1}{6(n+2)}(\Delta f_5)(v) \\
&\quad - \frac{1}{3n(n+2)}(\Delta f_6)(v) + \frac{1}{3n(n+2)}(\Delta f_7)(v) + \frac{1}{6(n+2)}(\Delta f_8)(v) \\
&= \sum_{i=1}^{n} |(\nabla h)(e_i, v, v)|^2 + \frac{n+1}{4}cf_{10}(v) + (n+4)f_1(v) - 4f_5(v) - 2f_8(v).
\end{aligned} \tag{3.4}$$

Since M is totally real, the following equations also hold for $v \in UM_x$, $x \in M$ (See [**M2**]):

$$\sum_{i,j=1}^{n} \langle A_{h(e_i,e_j)}e_j, A_{h(v,e_i)}v\rangle = \sum_{i,j=1}^{n} \langle A_{Jv}^2 A_{Je_j}^2 e_i, e_i\rangle, \tag{3.5}$$

$$\sum_{i,j=1}^{n} \langle A_{h(v,e_j)}e_i, A_{h(v,e_j)}e_i\rangle = \sum_{i,j=1}^{n} \langle A_{Jv} A_{Je_j}^2 A_{Jv} e_i, e_i\rangle, \tag{3.6}$$

where J is the complex structure. Combining (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), (2.14), (2.15), (2.16), (3.5), (3.6) with (3.4), we obtain

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{n}&(\nabla^2 f_{10})(e_i,e_i,v) - \frac{1}{6}(\Delta f_1)(v) - \frac{1}{3(n+2)}(\Delta f_2)(v) \\
&+ \frac{1}{6(n+2)}(\Delta f_3)(v) + \frac{1}{3n(n+2)}(\Delta f_4)(v) + \frac{1}{6(n+2)}(\Delta f_5)(v) \\
&- \frac{1}{3n(n+2)}(\Delta f_6)(v) + \frac{1}{3n(n+2)}(\Delta f_7)(v) + \frac{1}{6(n+2)}(\Delta f_8)(v) \\
&+ \frac{(3n+2)(n+4)}{6n(n+2)^2}\left(\frac{1}{n}(\Delta f_4)(v) + (\Delta f_5)(v)\right) \\
&- \frac{(6n+8)(n-2)}{6n^2(n+2)^2}((\Delta f_6)(v) - (\Delta f_7)(v)) \\
&- \frac{1}{2(n+2)}(\Delta f_8)(v) + \frac{n+4}{8(n+2)}(\Delta f_9)(v) \\
&- \frac{4(n+1)}{6n(n+2)}\left((\Delta f_{11})(v) + \frac{2}{n+2}\left((\Delta f_{12})(v) - \frac{1}{2}(\Delta f_{13})(v)\right)\right) \\
= &\sum_{i=1}^{n} |(\nabla h)(e_i,v,v)|^2 + \frac{n+1}{4}cf_{10}(v) - \frac{4(n+1)}{n(n+2)}|h|^2|h(v,v)|^2 \\
&+ \frac{(n+6)(n+4)}{n+2}(f_1(v) - f_9(v)) + \frac{(6n+4)(n+4)}{n(n+2)}(f_{11}(v) - f_5(v))
\end{aligned} \tag{3.7}$$

Suppose that f_{10} attains its maximum at $u \in UM_x$. We shall call u a *maximal direction* at x. We choose an orthonormal basis $\{u = e_1, e_2, \dots, e_n\}$ of M at x such that A_{Ju} is diagonalized. Since u is a maximal direction, we have

$$\left|h\left(u + t\sum_{\alpha=2}^{n} x^\alpha e_\alpha, u + t\sum_{\alpha=2}^{n} x^\alpha e_\alpha\right)\right|^2 \le \left[1 + t^2\sum_{\alpha=2}^{n}(x^\alpha)^2\right]^2 |h(u,u)|^2. \tag{3.8}$$

at x for all $t, x^2, \dots, x^n \in R$. Expanding (3.8) in terms of t, we obtain

$$4t\sum_{\alpha=2}^{n} x^\alpha \langle h(u,u), h(u,e_\alpha)\rangle + O(t^2) \le 0.$$

It follows that

$$\langle h(u,u), h(u,e_\alpha)\rangle = 0, \quad \alpha = 2, \dots, n. \tag{3.9}$$

Expanding (3.8) again in terms of t, we obtain

$$\begin{aligned} 2t^2 \Bigg[&\sum_{\alpha=2}^{n} (|h(u,u)|^2 - \langle h(u,u), h(e_\alpha,e_\alpha)\rangle - 2|h(u,e_\alpha)|^2)(x^\alpha)^2 \\ &- \sum_{\alpha,\beta=2,\alpha\neq\beta}^{n} (\langle h(u,u), h(e_\alpha,e_\beta)\rangle + 2\langle h(u,e_\alpha), h(u,e_\beta)\rangle) x^\alpha x^\beta \Bigg] + O(t^3) \geq 0. \end{aligned} \tag{3.10}$$

Since (3.10) must hold for all real x^α, we obtain

$$|h(u,u)|^2 - \langle h(u,u), h(e_\alpha,e_\alpha)\rangle - 2|h(u,e_\alpha)|^2 \geq 0 \tag{3.11}$$

for $\alpha = 2, \dots, n$. From (3.9) we get $A_{h(u,u)}u = |h(u,u)|^2 u$ so that the equalities

$$f_1(u) = f_9(u),$$

$$f_5(u) = \sum_{i=1}^{n} |h(u,u)|^2 \langle A_{h(u,e_i)} e_i, u\rangle$$

hold. If $f_{10}(u) \neq 0$ at x, then it satisfies that $Ju = h(u,u)/|h(u,u)|$ at x, since $\langle h(u,u), Je_\alpha\rangle = \langle A_{Ju}u, e_\alpha\rangle = 0$, $\alpha = 2, \dots, n$, where we use the property $A_{Jv}w = A_{Jw}v$ for $v, w \in T_xM$. Then it holds that

$$f_8(u) = \sum_{i=1}^{n} |h(u,u)|^2 \langle A_{h(u,e_i)} e_i, u\rangle.$$

From (2.13) we also have

$$0 \geq (\Delta f_{10})(u) = -4(n+2) f_{10}(u) + 8 \sum_{i=1}^{n} \langle A_{h(u,e_i)} e_i, u\rangle. \tag{3.12}$$

Hence

$$\sum_{i=1}^{n} \langle A_{h(u,e_i)} e_i, u\rangle \leq \frac{n+2}{2} f_{10}(u). \tag{3.13}$$

Define f_{14} by

$$\begin{aligned} f_{14}(v) = &-\frac{1}{6} f_1(v) - \frac{1}{3(n+2)} f_2(v) + \frac{1}{6(n+2)} f_3(v) + \frac{1}{3n(n+2)} f_4(v) \\ &+ \frac{1}{6(n+2)} f_5(v) - \frac{1}{3n(n+2)} f_6(v) + \frac{1}{3n(n+2)} f_7(v) + \frac{1}{6(n+2)} f_8(v) \\ &+ \frac{(3n+2)(n+4)}{6n(n+2)^2} \left(\frac{1}{n} f_4(v) + f_5(v) \right) \end{aligned} \tag{3.14}$$

$$- \frac{(6n+8)(n-2)}{6n^2(n+2)^2}(f_6(v) - f_7(v))$$

$$- \frac{1}{2(n+2)} f_8(v) + \frac{n+4}{8(n+2)} f_9(v)$$

$$- \frac{4(n+1)}{6n(n+2)} \left(f_{11}(v) + \frac{2}{n+2} \left(f_{12}(v) - \frac{1}{2} f_{13}(v) \right) \right)$$

Combining (3.1) with (3.7), we obtain

$$\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i, e_i, u) + (\Delta f_{14})(u) \geq \sum_{i=1}^{n} |(\nabla h)(e_i, u, u)|^2 \geq 0. \tag{3.15}$$

On the other hand, from (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), (2.14), (2.15), (2.16), (3.2), (3.3) and (3.14) we have

$$(\Delta f_{14})(u) = 4\sum_{i=1}^{n} \langle A_{h(u,e_i)} e_i, u\rangle |h(u,u)|^2 - 2\sum_{i=1}^{n} \langle A_{h(u,u)} e_i, A_{h(u,e_i)} u\rangle$$
$$- \sum_{i=1}^{n} \langle A_{h(u,u)} e_i, A_{h(u,u)} e_i\rangle - \frac{4(n+1)}{n(n+2)} |h|^2 |h(u,u)|^2.$$

Assume that $\langle Lu, u\rangle \leq |h|^2/n$. Then

$$(\Delta f_{14})(u) = -2\sum_{i=1}^{n} \langle A_{h(u,u)} e_i, A_{h(u,e_i)} u\rangle + \left(\frac{3}{n} - \frac{4(n+1)}{n(n+2)}\right) |h|^2 |h(u,u)|^2$$

Also,

$$f_3(u) = |h(u,u)| \operatorname{trace} A_{Ju}^3.$$

By changing Ju by $-Ju$ if necessary, we may assume without loss of generality that $f_3(u) \geq 0$. Hence

$$(\Delta f_{14})(u) \leq 0. \tag{3.16}$$

Since u is a maximum direction at x, we get

$$\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i, e_i, u) \leq 0. \tag{3.17}$$

Thus from (3.15), (3.16) and (3.17) we have

$$\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i, e_i, u) + (\Delta f_{14})(u) = 0.$$

Therefore from (3.15) we obtain $(\nabla h)(e_i, u, u) = 0$ so that $|h(u,u)|^2 = \text{const.} \neq 0$ and $|h|^2 = n(n+2)c/16$ or $|h(u,u)|^2 = 0$. Hence the equalities of (3.12) and (3.13) hold. Next, from the definition of u we have

$$-|h(u,u)|^2 \leq b_\alpha \leq |h(u,u)|^2, \ \ \alpha = 2, \dots, n, \tag{3.18}$$

where $b_\alpha = \langle h(u,u), h(e_\alpha, e_\alpha)\rangle$. Because of the minimality of the immersion, we have

$$\sum_{\alpha=2}^{n} \langle h(u,u), h(e_\alpha, e_\alpha)\rangle = -|h(u,u)|^2. \tag{3.19}$$

Then note that $\langle A_{Ju} e_\alpha, e_\beta\rangle = 0$, $\alpha \neq \beta, \alpha, \beta = 2, \dots, n$ and

$$\sum_{\alpha=2}^{n} b_\alpha^2 = f_8(u) - |h(u,u)|^4.$$

In terms of (3.13) and the equality of (3.12) since the convex function $f(b_2, \dots, b_n) = \sum_{\alpha=2}^{n} b_\alpha^2$ of $(n-1)$ variables $b_2, \dots, b_n$ subject to the constraints (3.18) and (3.19) exactly attains its maximal value $(n/2)|h(u,u)|^4$, it holds that

$$b_2 = \cdots = b_r = -b_{r+1} = \cdots = -b_{2r} = |h(u,u)|^2,$$

$$b_{2r+1} = 0.$$

Hence if n is even, then we obtain

$$\langle h(u,u), h(e_\alpha, e_\alpha)\rangle^2 = |h(u,u)|^4, \quad \alpha = 2, \dots, n.$$

By the inequality of Schwarz we see that M is isotropic at x. If n is odd, then we get

$$\langle h(u,u), h(e_\alpha, e_\alpha)\rangle^2 = |h(u,u)|^4, \quad \alpha = 2, \dots, n-1,$$

$$\langle h(u,u), h(e_n, e_n)\rangle = 0.$$

Thus each $e_\alpha, \alpha = 2, \dots, n-1$ is also maximal direction. Hence we get

$$\langle h(e_n, e_n), h(e_n, e_i)\rangle = 0, \quad i \neq n.$$

Thus

$$\langle h(e_i, e_i), h(e_i, e_j)\rangle = 0, \quad \text{for any } i, j \ (i \neq j).$$

Therefore M is isotropic at x. That is, M is a submanifold of $CP^n(c)$ with parallel second fundamental form ([**N**]). It remains the case of $\langle Lu, u\rangle \geq |h|^2/n$. Then assume that f_{10} attains its minimal value at $u_0 \in UM_x$. Let $\{u = e_1, e_2, \dots, e_{n-1}, u_0 = e_n\}$ be an orthonormal basis of M at x. By the similar way with (3.8) and (3.9) we have

$$\langle h(u_0, u_0), h(u_0, e_\alpha)\rangle = 0, \quad \alpha = 1, \dots, n-1. \tag{3.20}$$

If $\langle Lu_0, u_0\rangle \geq |h|^2/n$, then $\langle Lu, u\rangle = |h|^2/n$. Thus we may assume that $\langle Lu_0, u_0\rangle \leq |h|^2/n$. That is, from (3.11) and (3.13) we have

$$|h(u_0, u_0)|^2 - \langle h(u_0, u_0), h(e_\alpha, e_\alpha)\rangle - 2|h(u_0, e_\alpha)|^2 \leq 0, \quad \alpha = 1, \dots, n-1, \tag{3.21}$$

$$\frac{n+2}{2} f_{10}(u_0) \leq \langle Lu_0, u_0\rangle \leq \frac{|h|^2}{n}$$

$$\left(\text{resp.} |h(u,u)|^2 - \langle h(u,u), h(e_\alpha, e_\alpha)\rangle - 2|h(u, e_\alpha)|^2 \geq 0, \quad \alpha = 2, \dots, n, \right.$$

$$\left. \frac{n+2}{2} f_{10}(u) \geq \langle Lu, u\rangle \geq \frac{|h|^2}{n} \right)$$

If $n \geq 3$, we may assume that there exists α, fixed $\alpha, 2 \leq \alpha \leq n-1$ such that $e_\alpha \neq u$ and $e_\alpha \neq u_0$. Let $\alpha(s)$ be any curve in the sphere UM_x such that $\alpha(0) = e_\alpha$, $\alpha'(0) = e_\beta$, $1 \leq \beta \neq \alpha \leq n$. Then we have

$$\frac{d}{ds}(f_{10} \circ \alpha)(0) = 4\langle h(e_\alpha, e_\alpha), h(e_\alpha, e_\beta)\rangle.$$

As we can choose an orthonormal basis $\{u = e_1, e_2, \ldots, e_\alpha, \ldots, e_{n-1}, u_0 = e_n\}$ such that $\langle h(e_\alpha, e_\alpha), h(e_\alpha, e_\beta)\rangle = 0$, $1 \leq \beta \neq \alpha \leq n$ (for example, we may choose an orthonormal basis such that A_{Je_α} is diagonalized), f_{10} attains its critical value at e_α. If f_{10} attains its critical value with the signature $(\overbrace{+, \ldots, +}^{n-1 \text{ times}})$, i.e., $|h(e_\alpha, e_\alpha)|^2 - \langle h(e_\alpha, e_\alpha), h(e_\beta, e_\beta)\rangle - 2|h(e_\alpha, e_\beta)|^2 \leq 0$ for $\beta \neq \alpha$, $1 \leq \beta \leq n$, (resp. $(\overbrace{-, \ldots, -}^{n-1 \text{ times}})$, i.e., $|h(e_\alpha, e_\alpha)|^2 - \langle h(e_\alpha, e_\alpha), h(e_\beta, e_\beta)\rangle - 2|h(e_\alpha, e_\beta)|^2 \geq 0$ for $\beta \neq \alpha$, $1 \leq \beta \leq n$) at e_α, then $f_{10}(e_\alpha) = f_{10}(u_0)$ (resp. $f_{10}(e_\alpha) = f_{10}(u)$). Now, if necessary, by renumbering, we may consider the case of $\{u = e_1, e_2, \ldots, e_{n-1}, u_0 = e_n\}$ which satisfies

$$f_{10}(e_1) \geq \cdots \geq f_{10}(e_\alpha) \geq \cdots \geq f_{10}(e_n), \tag{3.22}$$

f_{10} attains its critical value with the signature $(\overbrace{-, \ldots, -}^{p \text{ times}}, \overbrace{+, \ldots, +}^{n-1-p \text{ times}})$, $0 \leq p \leq n-1$, at e_α, $1 < \alpha < n$. At first, f_{10} attains the signature $(-, \ldots, \overset{\beta\text{th}}{+} \ldots)$ at e_α. Since we may assume that A_{Je_α} is diagonalized, we can put $h(e_\alpha, e_\alpha) = \langle h(e_\alpha, e_\alpha), Je_\alpha\rangle Je_\alpha$. Then

$$\langle h(e_\alpha, e_\alpha), Je_\alpha\rangle^2 - \langle h(e_\alpha, e_\alpha), Je_\alpha\rangle\langle h(e_1, e_1), Je_\alpha\rangle - 2\langle h(e_1, e_1), Je_\alpha\rangle^2 \geq 0, \tag{3.23}$$

$$\langle h(e_\alpha, e_\alpha), Je_\alpha\rangle^2 - \langle h(e_\alpha, e_\alpha), Je_\alpha\rangle\langle h(e_\beta, e_\beta), Je_\alpha\rangle - 2\langle h(e_\beta, e_\beta), Je_\alpha\rangle^2 \leq 0. \tag{3.24}$$

Since $\langle h(e_1, e_1), Je_\alpha\rangle^2 \geq \langle h(e_\beta, e_\beta), Je_\alpha\rangle^2$, from $h(e_1, e_\alpha) = \langle h(e_1, e_1), Je_\alpha\rangle Je_1$, $h(e_\beta, e_\alpha) = \langle h(e_\beta, e_\beta), Je_\alpha\rangle Je_\beta$, (3.23) and (3.24) we obtain $|h(e_1, e_\alpha)|^2 = |h(e_\beta, e_\alpha)|^2$. Thus we may assume that f_{10} attains its critical value with the signature $(-, \ldots, -)$ or $(+, \pm, \ldots)$ at e_α. Next, assume that f_{10} attains its critical value with the signature $(-, \ldots, -)$ and $(+, \pm, \ldots)$ at u and e_α, $\alpha = 2$, respectively. Let $f(v) = \langle h(v, v), Je_1\rangle$. If there exists $e_\beta, 3 \leq \beta < n$, at which f attains critical value, then we consider the case with the signature $+$ at e_β. Changing an orthonormal basis, we may choose an orthonormal basis such that A_{Je_1} is diagonalized. Then we may assume that if there exists e_β, $2 \leq \beta < n$, at which f attains critical value, then we consider the case with the signature $+$ at e_β, and we can put $h(e_1, e_1) = \langle h(e_1, e_1), Je_1\rangle Je_1$. Hence

$$|h(e_2, e_2)|^2 - \langle h(e_2, e_2), h(e_1, e_1)\rangle - 2|h(e_1, e_2)|^2 \leq 0,$$

i.e.,

$$-\langle h(e_2, e_2), Je_1\rangle\langle h(e_1, e_1), Je_1\rangle - \langle h(e_2, e_2), Je_1\rangle^2 \leq 0.$$

Thus, $\langle h(e_2, e_2), Je_1\rangle \geq 0$. From the above mention

$$\langle h(e_3, e_3), Je_1\rangle \geq 0, \ldots, \langle h(e_n, e_n), Je_1\rangle \geq 0.$$

Thus the case cannot occur. We assume that f_{10} attains the signature $(-,\dots,-)$ and $(-,\dots,-)$ at u and e_α, $\alpha = 2$, respectively. If there exists $e_\beta, 3 \le \beta < n$, at which f attains critical value, then we consider the case with the signature $+$ at e_β. Set $h(e_2, e_2) = \langle h(e_2, e_2), Je_2 \rangle Je_2$. Then since

$$|h(e_2, e_2)|^2 - \langle h(e_2, e_2), h(e_1, e_1)\rangle - 2|h(e_1, e_2)|^2 \ge 0,$$

$$-\langle h(e_2, e_2), Je_2\rangle \le \langle h(e_1, e_1), Je_2\rangle \le \frac{1}{2}\langle h(e_2, e_2), Je_2\rangle.$$

Hence $|h(e_1, e_1)|^2 = \langle h(e_2, e_2), Je_2\rangle^2$. On the other hand, $\langle h(e_3, e_3), Je_2\rangle \ge 0, \dots, \langle h(e_n, e_n), Je_2\rangle \ge 0$. Thus $f_{10}(e_1) = f_{10}(e_2) = \lambda^2$ for some real number λ and $f_{10}(e_3) = \cdots = f_{10}(e_n) = 0$. Then $\langle Lu, u\rangle = 2\lambda^2 > |h|^2/n$ $(n > 2)$ and $|h|^2 \le n(n+2)c/16$ for some λ. We remark that if $\lambda^2 < (1/8)c$ (resp. $\lambda = \text{const.}$), then M is totally geodesic ([**M1**] and [**X1**] (resp. [**C**])) and that by the similar way with [**C**] we can show there exists a non-constant function λ which satisfies Gauss, Codazzi and Ricci equations. It remains the case that f_{10} attains its critical value with the signature $(-,\dots,-),\dots,(-,\dots,-)$ at $e_1,\dots,e_\alpha (\alpha \ge 3)$, respectively. If there exists $e_\beta, \alpha+1 \le \beta < n$, at which f attains critical value, we consider the case with the signature $+$ at e_β. Put $h(e_\alpha, e_\alpha) = \langle h(e_\alpha, e_\alpha), Je_\alpha\rangle Je_\alpha$. If $\alpha = (n+4)/3$ $(n \ge 5)$, then $\langle Le_\alpha, e_\alpha\rangle = ((n+2)/2)|h(e_\alpha, e_\alpha)|^2$ and $\langle Le_n, e_n\rangle = 2|h(e_n, e_n)|^2$. Thus the case cannot occur. Assume that $n = 4$ and $\alpha = 3$. Then $h(e_1, e_1) = -\langle h(e_3, e_3), Je_3\rangle Je_3, h(e_2, e_2) = -\langle h(e_3, e_3), Je_3\rangle Je_3$ and $0 \le \langle h(e_4, e_4), Je_3\rangle \le 1/2\langle h(e_3, e_3), Je_3\rangle$ show that M is totally geodesic. If $n = 3\alpha - 3$ or $3\alpha - 2$, then changing an orthonormal basis, put $h(e_{\alpha+1}, e_{\alpha+1}) = \langle h(e_{\alpha+1}, e_{\alpha+1}), Je_{\alpha+1}\rangle Je_{\alpha+1}$. Then

$$\begin{aligned}
h(e_1, e_1) &= (-1-\beta_1)\langle h(e_{\alpha+1}, e_{\alpha+1}), Je_{\alpha+1}\rangle Je_{\alpha+1} + \langle h(e_1, e_1), Je_{\alpha+2}\rangle Je_{\alpha+2} \\
&\quad + \cdots + \langle h(e_1, e_1), Je_n\rangle Je_n, \\
&\cdots, \\
h(e_{\alpha-1}, e_{\alpha-1}) &= (-1-\beta_1)\langle h(e_{\alpha+1}, e_{\alpha+1}), Je_{\alpha+1}\rangle Je_{\alpha+1} + \langle h(e_1, e_1), Je_{\alpha+2}\rangle Je_{\alpha+2} \\
&\quad + \cdots + \langle h(e_1, e_1), Je_n\rangle Je_n, \\
h(e_\alpha, e_\alpha) &= (1+\beta_1)\langle h(e_{\alpha+1}, e_{\alpha+1}), Je_{\alpha+1}\rangle Je_{\alpha+1} - \langle h(e_1, e_1), Je_{\alpha+2}\rangle Je_{\alpha+2} \\
&\quad - \cdots - \langle h(e_1, e_1), Je_n\rangle Je_n, \\
h(e_{\alpha+1}, e_{\alpha+1}) &= \langle h(e_{\alpha+1}, e_{\alpha+1}), Je_{\alpha+1}\rangle Je_{\alpha+1}, \\
h(e_{\alpha+2}, e_{\alpha+2}) &= \beta_2\langle h(e_{\alpha+1}, e_{\alpha+1}), Je_{\alpha+1}\rangle Je_{\alpha+1} + \langle h(e_{\alpha+2}, e_{\alpha+2}), Je_{\alpha+2}\rangle Je_{\alpha+2} \\
&\quad + \cdots + \langle h(e_{\alpha+2}, e_{\alpha+2}), Je_n\rangle Je_n, \\
&\cdots, \\
h(e_n, e_n) &= \beta_{n-\alpha}\langle h(e_{\alpha+1}, e_{\alpha+1}), Je_{\alpha+1}\rangle Je_{\alpha+1} + \langle h(e_n, e_n), Je_{\alpha+2}\rangle Je_{\alpha+2} \\
&\quad + \cdots + \langle h(e_n, e_n), Je_n\rangle Je_n.
\end{aligned}$$

Since

$$|h(e_1,e_1)|^2 - \langle h(e_\alpha,e_\alpha), h(e_1,e_1)\rangle - 2|h(e_1,e_\alpha)|^2 = 0,$$
$$\cdots,$$
$$|h(e_{\alpha-1},e_{\alpha-1})|^2 - \langle h(e_\alpha,e_\alpha), h(e_{\alpha-1},e_{\alpha-1})\rangle - 2|h(e_{\alpha-1},e_\alpha)|^2 = 0([\mathbf{O2}]),$$

we have

$$\begin{aligned}(1+\beta_1)^2\langle h(e_{\alpha+1},e_{\alpha+1}), Je_{\alpha+1}\rangle^2 + \langle h(e_1,e_1), Je_{\alpha+2}\rangle^2 + \cdots + \langle h(e_1,e_1), Je_n\rangle^2 \\ = \langle h(e_1,e_\alpha), Je_{\alpha+2}\rangle^2 + \cdots + \langle h(e_1,e_\alpha), Je_n\rangle^2,\end{aligned}$$
$$\cdots,$$
$$\begin{aligned}(1+\beta_1)^2\langle h(e_{\alpha+1},e_{\alpha+1}), Je_{\alpha+1}\rangle^2 + \langle h(e_1,e_1), Je_{\alpha+2}\rangle^2 + \cdots + \langle h(e_1,e_1), Je_n\rangle^2 \\ = \langle h(e_{\alpha-1},e_\alpha), Je_{\alpha+2}\rangle^2 + \cdots + \langle h(e_{\alpha-1},e_\alpha), Je_n\rangle^2.\end{aligned}$$

Also, it holds

$$|h(e_\alpha,e_\alpha)|^2 - \langle h(e_\alpha,e_\alpha), h(e_{\alpha+2},e_{\alpha+2})\rangle - 2|h(e_\alpha,e_{\alpha+2})|^2 \geq 0,$$
$$\cdots,$$
$$|h(e_\alpha,e_\alpha)|^2 - \langle h(e_\alpha,e_\alpha), h(e_n,e_n)\rangle - 2|h(e_\alpha,e_n)|^2 \geq 0.$$

Hence, we obtain

$$\langle h(e_1,e_1), Je_{\alpha+2}\rangle = \cdots = \langle h(e_1,e_1), Je_n\rangle = \beta_2 = \cdots = \beta_{n-\alpha} = 0.$$

Hence M is totally geodesic. This proves Proposition.

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