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Conformal deformations of submanifolds in codimension two

By Marcos Dajczer and Ruy Tojeiro

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Abstract. Let $M^n \subset \boldsymbol{R}^{n+2}$, $n \geq 7$, be a conformally deformable submanifold of euclidean space in codimension two. In this paper we show that if the submanifold has sufficiently low conformal nullity, a generic conformal condition, then it can be realized as a hypersurface of a conformally deformable hypersurface. The latter have been classified by Cartan early this century. Furthermore, it turns out that all deformations of the former are induced by deformations of the latter.

Early in this century, E. Cartan ([**Ca**], see also [**Da**]) proved that a hypersurface $f : M^n \to \boldsymbol{R}^{n+1}$ of dimension $n \geq 5$ in euclidean space is conformally rigid if its *conformal nullity*, that is, the maximal dimension of an umbilical subspace, satisfies $\nu_f^c \leq n-3$ everywhere. Being *conformally rigid* means that any other conformal immersion of M^n into $\boldsymbol{R}^{n+1}$ must be conformally congruent to f, i.e., a composition of f with a conformal diffeomorphism of the ambient space. Moreover, Cartan (cf. [**DT**]) also gave a complete parametric description of all conformally deformable hypersurfaces. These will be referred to hereafter as *Cartan hypersurfaces*.

Cartan's rigidity theorem was extended by do Carmo and the first author ([**CD**]) to conformal immersions $f : M^n \to \boldsymbol{R}^{n+p}$ with dimension $n \geq 7$ and codimension $p \leq 4$. For codimension $p = 2$, their result states that f is conformally rigid if everywhere i) $\nu_f^c \leq n-5$ and ii) the second fundamental form A_ξ in any normal direction ξ has no principal curvatures with multiplicity greater than $n-3$. On the other hand, a large set of conformally deformable submanifolds in codimension 2 arises from Cartan hypersurfaces $N^{n+1} \subset \boldsymbol{R}^{n+2}$ simply by considering M^n to be an arbitrary hypersurface of N^{n+1}.

In this paper we show that, if only condition i) in do Carmo–Dajczer's rigidity theorem is assumed, then any conformally deformable submanifold M^n in $\boldsymbol{R}^{n+2}$, $n \geq 7$, can be realized as a hypersurface of a Cartan hypersurface. Furthermore, all deformations of the former are induced by deformations of the latter. More precisely, we prove the following conformal version of a result in [**DG**] on isometric immersions.

Theorem. *Let $f, g : M^n \to \boldsymbol{R}^{n+2}$, $n \geq 7$, be conformal immersions. Suppose that $\nu_f^c(x) \leq n-5$ everywhere. Then, there exists an open dense subset of M^n such that on any connected component U either f and g are conformally congruent or there exist conformal but nowhere conformally congruent Cartan hypersurfaces $\mathscr{F}, \mathscr{G} : N^{n+1} \to \boldsymbol{R}^{n+2}$ and an isometric embedding $i : U \to N^{n+1}$ such that $f|_U = \mathscr{F} \circ i$ and $g|_U = \mathscr{G} \circ i$.*

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Finally, we should point out that the classifications of all possible deformations when condition i) in do Carmo–Dajczer's result is not assumed remain an open problem even in the case of isometric deformations.

The *light cone* $\boldsymbol{V}^m$ is the degenerate totally umbilical hypersurface of nonzero null vectors in standard flat Lorentzian space $\boldsymbol{L}^{m+1}$, that is,

$$\boldsymbol{V}^m = \{X \in \boldsymbol{L}^{m+1} : \langle X, X\rangle = 0, X \neq 0\}.$$

Given an isometric immersion $f : M \to \boldsymbol{V}^m \subset \boldsymbol{L}^{m+1}$, the position vector f is a null normal vector field along the immersion f which is parallel in the normal connection. Moreover, the second fundamental form $A_f : TM \to TM$ in the normal direction f satisfies

$$A_f = -\mathrm{Id}. \tag{1}$$

Take the intersection $H_w \cap \boldsymbol{V}^m$ in $\boldsymbol{L}^{m+1}$ of the affine hyperplane orthogonal to $w \in \boldsymbol{V}^m$ given by

$$H_w = \{X \in \boldsymbol{L}^{m+1} : \langle X, w\rangle = 1\}$$

with the light cone. Observe that the normal bundle to $H_w \cap \boldsymbol{V}^m$ in $\boldsymbol{L}^{m+1}$ is the Lorentzian plane bundle spanned by w and the position vector. In particular, the metric induced on $H_w \cap \boldsymbol{V}^m$ is riemannian. Moreover, its second fundamental form with values in the normal bundle is given by $\alpha(X, Y) = -\langle X, Y\rangle w$. Hence, $H_w \cap \boldsymbol{V}^m$ is flat by the Gauss equation and is, in fact, the image of an isometric embedding $j_w : \boldsymbol{R}^{m-1} \to \boldsymbol{V}^m$.

The light cone turns out to be a useful tool in the study of conformal immersions. Namely, any conformal immersion $g : M \to \boldsymbol{R}^N$ can be made into an isometric immersion $G : M \to \boldsymbol{V}^{N+1} \subset \boldsymbol{L}^{N+2}$ by setting

$$G = \frac{1}{\varphi_g} j_w \circ g, \tag{2}$$

where $w \in \boldsymbol{V}^{N+1}$ is arbitrary and $\varphi_g > 0$ is the conformal factor given by $\langle g_* X, g_* Y\rangle = \varphi_g^2 \langle X, Y\rangle$. Conversely, any isometric immersion $G : M \to \boldsymbol{V}^{N+1}$ arises this way. For $w \in \boldsymbol{V}^{N+1}$ chosen so that $\langle G, w\rangle > 0$, the immersion $g : M \to \boldsymbol{R}^N$ defined by

$$j_w \circ g = \frac{1}{\langle G, w\rangle} G \tag{3}$$

is conformal with conformal factor $1/\langle G, w\rangle$.

The proof of our theorem will make use of the lemma below on flat bilinear forms. We denote by $S(\beta)$ the subspace spanned by the image of a symmetric bilinear form $\beta : V \times V \to W$. The *kernel* of β is defined as

$$N(\beta) = \{X \in V : \beta(X, Y) = 0 \text{ for all } Y \in V\}.$$

Also, we denote by $W^{r,s}$ an $(r+s)$-dimensional vector space with an inner product of type (r, s), being r the maximal dimension of a negative-definite subspace.

LEMMA 1 ([**CD**]). *Let $\beta : V \times V \to W^{3,3}$ be a nonzero symmetric bilinear form. Assume that* $\dim N(\beta) < \dim V - 6$ *and that β is flat, that is,*

$$\langle \beta(X,Z), \beta(Y,T)\rangle = \langle \beta(X,T), \beta(Y,Z)\rangle \quad \textit{for all } X,Y,Z,T \in V.$$

Then $W^{3,3} = W_1^{r,r} \oplus W_2^{3-r,3-r}$, $1 \le r \le 3$, *admits an orthogonal direct sum decomposition such that the W_j-component β_j of β, $j = 1, 2$, satisfies*

(i) β_1 *is null and* $\dim S(\beta_1) = r$,

(ii) β_2 *is flat and* $\dim N(\beta_2) \ge \dim V + 2r - 6$.

PROOF OF THE THEOREM. We may assume that M^n is endowed with the metric induced by f and that

$$\operatorname{rank} A_\delta^f \ge 3 \quad \text{for all non-zero element } \delta \text{ in } T_f^\perp M. \tag{4}$$

This follows easily from our hypothesis on ν_f^c and the fact that composing f with an inversion in $\boldsymbol{R}^{n+2}$ changes A_δ^f by a multiple of the identity map. Notice that this fact is what makes ν_f^c a conformal invariant.

Let $G : M^n \to \boldsymbol{V}^{n+3} \subset \boldsymbol{L}^{n+4}$ be given by (2). Clearly, the normal vector bundle $T_g^\perp M$ can be identified via j_{w*} to a vector subbundle of $T_G^\perp M$. Moreover, the $T_g^\perp M$-component α_G^* of α_G is given by

$$\alpha_G^* = \frac{1}{\varphi_g} j_{w*} \circ \alpha_g. \tag{5}$$

Since G is null and belongs to the Lorentzian plane-bundle $\boldsymbol{L}^2$ in $T_G^\perp M$ orthogonal to $T_g^\perp M$, one can easily see that there exists a unique orthonormal frame $\{\xi, \eta\}$ of $\boldsymbol{L}^2$ with $\|\xi\| = -1$ such that $G = \xi + \eta$. By (1), we then have

$$\langle \alpha_G, \eta\rangle + \langle \alpha_G, \xi\rangle = -\langle\, , \rangle. \tag{6}$$

At $x \in M^n$, let

$$W = T_f^\perp M \oplus \operatorname{span}\{\xi\} \oplus \operatorname{span}\{\eta\} \oplus T_g^\perp M$$

be endowed with the metric of signature $(3,3)$ which is negative-definite on the first two summands. For simplicity of notation we omit the "x" on pointwise computations. Define a symmetric bilinear form $\beta : TM \times TM \to W$ by $\beta = \alpha_f \oplus \alpha_G$, i.e.,

$$\beta(X,Y) = \alpha_f(X,Y) - \langle \alpha_G(X,Y), \xi\rangle \xi + \langle \alpha_G(X,Y), \eta\rangle \eta + \alpha_G^*(X,Y).$$

The Gauss equations for f and G imply that β is flat. Moreover, we have that $n - 6 > 0$ by assumption and that $N(\beta) = \{0\}$, since $\beta(X,X) \ne 0$ for $X \ne 0$ by (6). It follows from Lemma 1 that $\beta = \beta_1 \oplus \beta_2$, where β_1 is null with $\dim S(\beta_1) = r$, $1 \le r \le 3$, and β_2 is flat with $\dim N(\beta_2) \ge n + 2r - 6$.

Our first main step is to determine the pointwise structure of the second fundamental form of G on each subset

$$\mathcal{U}_j = \{x \in M^n : r(x) = j\}, \quad 1 \le j \le 3.$$

Lemma 2. i) $\mathscr{U}_1$ *is empty.*

At any $x \in M^n$ *there is a plane* $L \subset T_G^{\perp}M$ *and a linear isometry* $\tau : T_f^{\perp}M \to L$ *such that*:

ii) *When* $x \in \mathscr{U}_2$, *then*

$$\pi_L \circ \alpha_G = \tau \circ \alpha_f$$

where π_L $(\pi_{L^{\perp}})$ *denotes orthogonal projection onto* L $(L^{\perp})$. *Furthermore, there exists* $0 \neq \gamma_1 \in T_f^{\perp}M$ *such that*

$$A^f_{\gamma_1}|_{V_0} = \lambda\, Id, \quad \lambda \neq 0, \tag{7}$$

where $V_0 := N(\pi_{L^{\perp}} \circ \alpha_G)$ *satisfies* $\dim V_0 \geq n-2$.

iii) *If* $x \in \mathscr{U}_3$, *then*

$$\alpha_G = \tau \circ \alpha_f - \langle\, ,\rangle \vartheta,$$

where ϑ *is a null vector in the orthogonal complement of* L *in* $T_G^{\perp}M$.

Proof. Suppose $x \in \mathscr{U}_1$. Then, there are unit vectors $\gamma \in T_f^{\perp}M$, $\delta \in T_g^{\perp}M$ and a symmetric bilinear form $\phi : TM \times TM \to \boldsymbol{R}$ such that

$$\beta_1 = \phi(a_1\eta + a_2\delta + b_1\xi + b_2\gamma)$$

where $a_1^2 + a_2^2 = b_1^2 + b_2^2$. For any $Z \in N(\beta_2)$ and $X \in TM$, we have $\beta(X,Z) = \beta_1(X,Z)$. Using (6), we get

$$\alpha_f(X,Z) = b_2\phi(X,Z)\gamma, \quad \langle \alpha_G(X,Z), \eta\rangle = a_1\phi(X,Z)$$

and

$$\langle \alpha_G(X,Z), \eta\rangle + \langle X,Z\rangle = b_1\phi(X,Z).$$

We conclude that $b_1 - a_1 \neq 0$, and that

$$\alpha_f(X,Z) = \frac{b_2}{b_1 - a_1}\langle X,Z\rangle\gamma \quad \text{for all } Z \in N(\beta_2).$$

Hence, $N(\beta_2)$ is an umbilical subspace for f. Since $\dim N(\beta_2) \geq n-4$, this contradicts our assumption on ν_f^c and proves that $\mathscr{U}_1$ is empty.

At $x \in \mathscr{U}_2$ there exist vectors of unit length $\eta_1, \eta_2 \in T_f^{\perp}M$, $\zeta_1, \zeta_2 \in T_g^{\perp}M$ and symmetric bilinear forms $\phi, \phi' : TM \times TM \to \boldsymbol{R}$ such that

$$\beta_1 = \phi(c_1\eta + a_1\eta_1 + b_1\zeta_1) + \phi'(c_2\xi + a_2\eta_2 + b_2\zeta_2)$$

where

$$c_1^2 - a_1^2 + b_1^2 = 0 = c_2^2 + a_2^2 - b_2^2 \quad \text{and} \quad b_1b_2\langle\zeta_1,\zeta_2\rangle = a_1a_2\langle\eta_1,\eta_2\rangle. \tag{8}$$

Using (6), we get

$$\alpha_f(X,Z) = a_1\phi(X,Z)\eta_1 + a_2\phi'(X,Z)\eta_2,$$

$$\langle\alpha_G(X,Z),\eta\rangle + \langle X,Z\rangle = c_2\phi'(X,Z),$$

$$\langle\alpha_G(X,Z),\eta\rangle = c_1\phi(X,Z)$$

$$\alpha_G^*(X,Z) = b_1\phi(X,Z)\zeta_1 + b_2\phi'(X,Z)\zeta_2$$

for $Z \in N(\beta_2)$ and $X \in TM$. In particular,

$$c_2\phi'(X,Z) = c_1\phi(X,Z) + \langle X,Z\rangle.$$

We obtain that

$$\alpha_f(X,Z) = \rho(X,Z)\mu_1 + \langle X,Z\rangle\mu_2, \quad \alpha_G(X,Z) = \rho(X,Z)\delta_1 + \langle X,Z\rangle\delta_2, \tag{9}$$

where

$$\mu_1 = a_1\eta_1 + a_2c_1\eta_2, \quad \mu_2 = a_2\eta_2, \quad \delta_1 = c_1(\xi + \eta + b_2\zeta_2) + b_1\zeta_1, \quad \delta_2 = \xi + b_2\zeta_2$$

and $\rho = \phi$ when $c_2 \neq 0$ (say $c_2 = 1$) and

$$\mu_1 = \eta_2, \quad \mu_2 = -a_1\eta_1, \quad \delta_1 = b_2\zeta_2, \quad \delta_2 = -\eta - b_1\zeta_1$$

and $\rho = \phi'$ when $c_2 = 0$. Here, we took $a_2 = c_1 = 1$.

Notice that in any of the two cases above the vector μ_1 is nonzero. Let $\gamma_1 \in T_f^\perp M$ be a unit vector orthogonal to μ_1 and set $V_0 := N(\beta_2)$. By the first of equations (9), we get that $A^f_{\gamma_1}|_{V_0} = \lambda\,\mathrm{Id}$ for $\lambda = \langle \gamma_1, \mu_2\rangle$. It follows from (4) that $\lambda \neq 0$ and that $L = \mathrm{span}\{\mu_1, \mu_2\}$ has dimension two.

By (8) we have that

$$\|\mu_j\| = \|\delta_j\| \quad \text{for } 1 \leq j \leq 2, \quad \text{and} \quad \langle \mu_1, \mu_2\rangle = \langle \delta_1, \delta_2\rangle.$$

Hence,

$$\langle \alpha_G(X,Z), \alpha_G(Y,Z)\rangle = \langle \alpha_f(X,Z), \alpha_f(Y,Z)\rangle \quad \text{for all } Z \in V_0.$$

We conclude from the Gauss equations for f and G that

$$\langle \alpha_G(X,Y), \alpha_G(Z,Z)\rangle = \langle \alpha_f(X,Y), \alpha_f(Z,Z)\rangle.$$

Using (9), we get for $Z \in V_0$ of unit length that

$$\rho(Z,Z)(\langle \alpha_G, \delta_1\rangle - \langle \alpha_f, \mu_1\rangle) = -\langle \alpha_G, \delta_2\rangle + \langle \alpha_f, \mu_2\rangle. \tag{10}$$

Define $\tau : T_f^\perp \to L$ by $\tau(\mu_j) = \delta_j$, $j = 1,2$. Then, the claim of the lemma follows unless $\rho(Z,Z) = k$ for all unit $Z \in V_0$. But if this is the case, since $\dim V_0 \geq n-2$, we conclude that there exists a subspace $H \subset V_0$ with $\dim H \geq n-4$ such that

$$\rho(X,Z) = k\langle X,Z\rangle \quad \text{for all } Z \in H.$$

This and the first equation in (9) yield a contradiction with our assumption on ν_f^c.

Now take $x \in \mathscr{U}_3$ and set

$$\tilde\beta = -\langle \alpha_G, \xi\rangle\xi \oplus \alpha_f \quad \text{and} \quad \bar\beta = \langle \alpha_G, \eta\rangle\eta \oplus \alpha_G^*.$$

Since $\beta = \tilde\beta \oplus \bar\beta$ is null, we have

$$\langle \bar\beta(X,Y), \bar\beta(Z,T)\rangle = \langle \tilde\beta(X,Y), \tilde\beta(Z,T)\rangle \quad \text{for all } X,Y,Z,T \in TM.$$

Thus, there is a linear isometry $\tilde T : \mathrm{span}\{\xi\} \oplus T_f^\perp M \to \mathrm{span}\{\eta\} \oplus T_g^\perp M$ such that $\tilde T \circ \tilde\beta = \bar\beta$. Consider orthonormal vectors $\{\gamma_1, \gamma_2\}$ in $T_f^\perp M$ and $\{\zeta_1, \zeta_2\}$ in $T_g^\perp M$ such that

$$\tilde{T}(\xi) = a_1\eta + a_2\zeta_1, \quad \tilde{T}(\gamma_1) = -a_2\eta + a_1\zeta_1 \quad \text{and} \quad \tilde{T}(\gamma_2) = \zeta_2$$

where $a_1^2 + a_2^2 = 1$. We get,

$$\langle \alpha_G, \eta \rangle = -a_1\langle \alpha_G, \xi \rangle - a_2\langle \alpha_f, \gamma_1 \rangle \tag{11}$$

and

$$\langle \alpha_G, \zeta_1 \rangle = -a_2\langle \alpha_G, \xi \rangle + a_1\langle \alpha_f, \gamma_1 \rangle, \quad \langle \alpha_G, \zeta_2 \rangle = \langle \alpha_f, \gamma_2 \rangle. \tag{12}$$

From (6) and (11) we obtain that

$$\langle \alpha_G, \eta \rangle (1 - a_1) = a_1\langle\,,\,\rangle - a_2\langle \alpha_f, \gamma_1 \rangle.$$

Hence $a_1 \neq 1$ and a straightforward computation using (12) yields

$$\alpha_G = \langle \alpha_f, \gamma_1 \rangle \delta_1 + \langle \alpha_f, \gamma_2 \rangle \delta_2 - \langle\,,\,\rangle \vartheta,$$

where

$$\delta_1 = \frac{a_2}{a_1 - 1}(\xi + \eta) - \zeta_1, \quad \delta_2 = \zeta_2 \quad \text{and} \quad \vartheta = \frac{1}{a_1 - 1}(a_1\eta + \xi + a_2\zeta_1).$$

Observe that $\|\delta_1\| = 1, \|\vartheta\| = 0$ and $\langle \delta_1, \vartheta \rangle = 0 = \langle \delta_2, \vartheta \rangle$. To conclude the proof, set $L = \operatorname{span}\{\delta_1, \delta_2\}$ and define $\tau : T_f^\perp M \to L$ by $\tau(\gamma_j) = \delta_j$, $j = 1, 2$. □

We show next that f and g are conformally congruent on any connected component U of the interior of $\mathscr{U}_3$. The second fundamental form of $F := j_w \circ f|_U : U \to \boldsymbol{V}^{n+3} \subset \boldsymbol{L}^{n+4}$ is

$$\alpha_F = j_{w*} \circ \alpha_f - \langle\,,\,\rangle w.$$

Let τ and ϑ be given pointwise by part iii) of Lemma 2. Then, since $\alpha_G(X, Y) = \tau(\alpha_f(X, Y))$ if $X \perp Y$, we easily obtain that τ and ϑ are smooth.

Choose a smooth orthonormal frame $\{\gamma_1, \gamma_2\}$ in $T_f^\perp M$ and set $\delta_j = \tau(\gamma_j)$, $j = 1, 2$. Then, the Codazzi equations of F and G for $A^F_{\gamma_j} = A^f_{\gamma_j} = A^G_{\delta_j}$ yield

$$\begin{aligned}(\langle \nabla_X^\perp \delta_i, \delta_j \rangle - \langle \nabla_X^\perp \gamma_i, \gamma_j \rangle) A^f_{\gamma_j} Y - (\langle \nabla_Y^\perp \delta_i, \delta_j \rangle - \langle \nabla_Y^\perp \gamma_i, \gamma_j \rangle) A^f_{\gamma_j} X \\ = \langle \nabla_X^\perp \delta_i, \vartheta \rangle Y - \langle \nabla_Y^\perp \delta_i, \vartheta \rangle X, \quad i \neq j.\end{aligned} \tag{13}$$

Suppose that the linear functional $X \mapsto \langle \nabla_X^\perp \delta_1, \delta_2 \rangle - \langle \nabla_X^\perp \gamma_1, \gamma_2 \rangle$ does not vanish. Let H_1 be its kernel and let V_j, $j = 1, 2$, be the kernels of the linear functionals $X \mapsto \langle \nabla_X^\perp \delta_j, \vartheta \rangle$. Applying (13) for $X = X_0 \in H_1^\perp$ and $Y \in H = H_1 \cap V_1 \cap V_2$, we obtain that H is an umbilical subspace for f. Since $\dim H \geq n - 3$, this contradicts our assumption on ν_f^c. It follows that

$$\langle \nabla_X^\perp \delta_2, \delta_1 \rangle = \langle \nabla_X^\perp \gamma_2, \gamma_1 \rangle, \quad \text{and} \quad \langle \nabla_X^\perp \delta_2, \vartheta \rangle = 0 = \langle \nabla_X^\perp \delta_1, \vartheta \rangle \quad \text{for all } X \in TM.$$

Therefore, the vector bundle isometry $\tilde{\tau} : T_F^\perp U \to T_G^\perp U$ defined by

$$\tilde{\tau}(\gamma_j) = \delta_j \quad \text{for } j = 1, 2, \quad \text{and} \quad \tilde{\tau}(w) = \vartheta$$

preserves the second fundamental forms and normal connections. By the fundamental theorem for submanifolds in Lorentzian space, there exists an isometry T of $\boldsymbol{L}^{n+4}$ preserving the light cone such that $G|_U = T \circ F$.

Define $\bar{T} : \boldsymbol{R}^{n+2} \to \boldsymbol{R}^{n+2}$ by

$$j_w \circ \bar{T} = \frac{1}{\langle T \circ j_w, w \rangle} T \circ j_w.$$

Then $\bar{T}$ is conformal with conformal factor $1/\langle T \circ j_w, w \rangle$. Moreover, using that

$$\langle G, w \rangle = \left\langle \frac{1}{\varphi_g} j_w \circ g, w \right\rangle = \frac{1}{\varphi_g},$$

we obtain that along U

$$j_w \circ \bar{T} \circ f = \frac{T \circ j_w \circ f}{\langle T \circ j_w \circ f, w \rangle} = \frac{T \circ F}{\langle T \circ F, w \rangle} = \frac{G}{\langle G, w \rangle} = j_w \circ g.$$

Being j_w an embedding, this implies that $g|_U = \bar{T} \circ f|_U$.

Let $\mathscr{U}_2^*$ be the open dense subset of $\mathscr{U}_2$ where the dimension of the tangent subspaces V_0 is locally constant, and take a connected component $U \subset \mathscr{U}_2^*$. Then, the linear isometry τ and the subspaces V_0 in part ii) of Lemma 2 are easily seen to be smooth on U. We claim that also γ_1 is smooth. In fact, since $v_f^c \leq n - 5$, we can choose smooth orthogonal vector fields $Z, T \in V_0$ such that the smooth vector field $\alpha_f(Z, T)$ does not vanish. By (7), γ_1 is orthogonal to $\alpha_f(Z, T)$ and the claim follows.

Extend γ_1 to an orthonormal basis γ_1, γ_2 of $T_f^\perp M$ and set $\delta_j = \tau(\gamma_j)$, $j = 1, 2$. Comparing the Codazzi equations of f and G for $A_{\gamma_2}^f = A_{\delta_2}^G$, we get

$$\begin{aligned} &\langle \nabla_X^\perp \delta_2, \delta_1 \rangle A_{\delta_1}^G Y + A_{(\nabla_X^\perp \delta_2)_{L^\perp}}^G Y - \langle \nabla_Y^\perp \delta_2, \delta_1 \rangle A_{\delta_1}^G X - A_{(\nabla_Y^\perp \delta_2)_{L^\perp}}^G X \\ &\quad = \langle \nabla_X^\perp \gamma_2, \gamma_1 \rangle A_{\gamma_1}^f Y - \langle \nabla_Y^\perp \gamma_2, \gamma_1 \rangle A_{\gamma_1}^f X. \end{aligned} \tag{14}$$

Applying (14) for $X, Y \in V_0$, we obtain

$$\lambda(\langle \nabla_X^\perp \delta_2, \delta_1 \rangle - \langle \nabla_X^\perp \gamma_2, \gamma_1 \rangle) Y = \lambda(\langle \nabla_Y^\perp \delta_2, \delta_1 \rangle - \langle \nabla_Y^\perp \gamma_2, \gamma_1 \rangle) X,$$

hence

$$\langle \nabla_Y^\perp \delta_2, \delta_1 \rangle = \langle \nabla_Y^\perp \gamma_2, \gamma_1 \rangle \quad \text{for all} \quad Y \in V_0. \tag{15}$$

Now take $Y \in V_0$, $X \in V_0^\perp$. Then (14) and (15) yield

$$\lambda(\langle \nabla_X^\perp \delta_2, \delta_1 \rangle - \langle \nabla_X^\perp \gamma_2, \gamma_1 \rangle) Y = A_{(\nabla_Y^\perp \delta_2)_{L^\perp}}^G X.$$

Since the left-hand side belongs to V_0 and the right hand side to $V_0^\perp$, we conclude that

$$\langle \nabla_X^\perp \delta_2, \delta_1 \rangle = \langle \nabla_X^\perp \gamma_2, \gamma_1 \rangle \quad \text{for all} \quad X \in TM, \tag{16}$$

and that $A_{(\nabla_Y^\perp \delta_2)_{L^\perp}}^G X = 0$. Thus,

$$(\nabla_Y^\perp \delta_2)_{L^\perp \cap N_1} = 0 \quad \text{for all} \quad Y \in V_0, \tag{17}$$

where N_1 stands for the *first normal space* of G, that is, the subspace of the normal space spanned by the image of α_G. Comparing the Codazzi equations of f and G for $A^f_{\gamma_1} = A^G_{\delta_1}$ and using (16), we get

$$A^G_{(\nabla^\perp_Y \delta_1)_{L^\perp}} X = 0 \quad \text{for all } X \in TM \quad \text{and} \quad Y \in V_0,$$

hence,

$$(\nabla^\perp_Y \delta_1)_{L^\perp \cap N_1} = 0 \quad \text{for all } Y \in V_0. \tag{18}$$

Now, take $\zeta \in L^\perp \cap N_1^\perp$. Then, the Codazzi equation for A^G_ζ $(= 0)$ yields

$$A^G_{\nabla^\perp_X \zeta} Y = A^G_{\nabla^\perp_Y \zeta} X \quad \text{for all } X, Y \in TM.$$

For $X, Y \in V_0$, this gives

$$\lambda(\langle \nabla^\perp_X \zeta, \delta_1 \rangle Y - \langle \nabla^\perp_Y \zeta, \delta_1 \rangle X) = \langle \nabla^\perp_Y \zeta, \delta_2 \rangle A^G_{\delta_2} X - \langle \nabla^\perp_X \zeta, \delta_2 \rangle A^G_{\delta_2} Y. \tag{19}$$

If $\langle \nabla^\perp_Y \zeta, \delta_2 \rangle = 0$ for all $Y \in V_0$, then also $\langle \nabla^\perp_Y \zeta, \delta_1 \rangle = 0$ by (19). Otherwise, let H_j be the kernel of the linear functional $Y \in V_0 \mapsto \langle \nabla^\perp_Y \zeta, \delta_j \rangle$, $j = 1, 2$. Applying (19) for $X = Y_0 \in H_2^\perp \cap V_0$ and $Y \in H_1 \cap H_2$, we get

$$A^G_{\delta_2} Y = \frac{-\lambda \langle \nabla^\perp_{Y_0} \zeta, \delta_1 \rangle}{\langle \nabla^\perp_{Y_0} \zeta, \delta_2 \rangle} Y \quad \text{for all } Y \in H_1 \cap H_2.$$

Since $A^G_{\delta_2} = A^f_{\gamma_2}$ and $A^f_{\gamma_1} Y = \lambda Y$ for all $Y \in V_0$, we obtain that $H_1 \cap H_2$ is an umbilical subspace for f. Being $\dim H_1 \cap H_2 \geq \dim V_0 - 2 \geq n - 4$, this contradicts our assumption on ν^c_f. Therefore,

$$\langle \nabla^\perp_Y \zeta, \delta_2 \rangle = 0 = \langle \nabla^\perp_Y \zeta, \delta_1 \rangle \quad \text{for all } Y \in V_0 \quad \text{and} \quad \zeta \in L^\perp \cap N_1^\perp. \tag{20}$$

We conclude from (17), (18) and (20) that

$$(\nabla^\perp_Y \xi)_L = 0 \quad \text{for all } Y \in V_0, \quad \text{and} \quad \xi \in L^\perp. \tag{21}$$

The Codazzi equation for $\xi \in L^\perp$, $Y \in V_0$ and $X \in TM$ yields

$$\nabla_Y A^G_\xi X + A^G_{\nabla^\perp_X \xi} Y + A^G_\xi [X, Y] - A^G_{\nabla^\perp_Y \xi} X = 0. \tag{22}$$

Taking the inner product of (22) with $Z \in V_0$ and using (21), we get in terms of the derivative in $\boldsymbol{L}^{n+4}$ that

$$\langle \tilde{\nabla}_Y Z, (\tilde{\nabla}_X \xi)_{TM \oplus L} \rangle = 0 \quad \text{for all } X \in TM, \quad Y, Z \in V_0 \quad \text{and} \quad \xi \in L^\perp. \tag{23}$$

For each point $x \in U$, define

$$W = \operatorname{span}\{(\tilde{\nabla}_X \xi)_{TM \oplus L} : X \in TM, \xi \in L^\perp\}.$$

Clearly, W has codimension at most 2 in $V_0^\perp \oplus L$. We claim that, in fact, the codimension equals 2 everywhere on U. If $W = V_0^\perp \oplus L$ at some point in U, then we have by (23) that

$$\lambda\langle Y,Z\rangle = \langle\tilde{\nabla}_Z Y,\delta_1\rangle = 0 \quad \text{for all } Y,Z\in V_0,$$

a contradiction. Suppose now that W has codimension 1 at some point in U. Then there exist $X\in TM$ and $a,b\in \boldsymbol{R}$ such that the orthogonal complement of W in $V_0^\perp\oplus L$ is spanned by $X+a\delta_1+b\delta_2$. By (23), we have for all $Y,Z\in V_0$ that $\langle\tilde{\nabla}_Y Z,-b\delta_1+a\delta_2\rangle=0$, hence $\langle(aA^G_{\delta_2}-b\lambda\,\mathrm{Id})Y,Z\rangle=0$, which implies that $\langle(aA^f_{\gamma_2}-b\lambda\,\mathrm{Id})Y,Z\rangle=0$. Since $A^f_{\gamma_1}=\lambda\,\mathrm{Id}$ on V_0, we conclude that there exists an umbilical subspace of V_0 for f with dimension at least $n-4$. This contradicts our assumption on ν_f^c and proves the claim.

Let Γ be the vector subbundle of $V_0^\perp\oplus L$ orthogonal to W and consider the vector bundle isometry

$$\mathscr{T}=\mathrm{Id}\oplus\tau: TU\oplus T_f^\perp U\to TU\oplus L.$$

Being $\Omega=\mathscr{T}^{-1}(\Gamma)$ transversal to TU, the maps $\tilde F:\Omega\to\boldsymbol{R}^{n+2}$, $\tilde G:\Omega\to\boldsymbol{L}^{n+4}$ defined by

$$\tilde F(\vartheta(x))=f(x)+\vartheta(x) \quad\text{and}\quad \tilde G(\vartheta(x))=G(x)+\mathscr{T}(\vartheta(x))$$

are immersions if restricted to a neighborhood $\tilde U$ of the 0-section of Ω. We claim that $\tilde F$ and $\tilde G$ are isometric on $\tilde U$. Given a local section $\vartheta\in\Omega$, write $\vartheta=X+\delta$, for $X\in TU$ and $\delta\in T_f^\perp U$. Since $0=\langle(\tilde\nabla_X\vartheta)_{TM\oplus L},X+\tau\delta\rangle = -\langle\xi,\alpha_G(X,Z)+\nabla_Z^\perp\tau\delta\rangle$ for $\vartheta\in L^\perp$ and $Z\in TM$, we have that $\alpha_G(X,Z)+\nabla_Z^\perp\tau\delta\in L$. Moreover, it follows from (16) that τ is parallel with respect to the connection on L induced by ${}^G\nabla^\perp$. Therefore, we have

$$\begin{aligned}\tilde G_*(\vartheta(x))Z &= G_*(x)Z+\tilde\nabla_Z\mathscr{T}(X+\delta)\\ &= G_*(x)(Z+\nabla_Z X-A^G_{\tau(\delta)}Z)+\alpha_G(X,Z)+{}^G\nabla_Z^\perp\tau(\delta)\\ &= G_*(x)(Z+\nabla_Z X-A^f_\delta Z)+\tau(\alpha_f(X,Z)+{}^f\nabla_Z^\perp\delta).\end{aligned}$$

Since

$$\tilde F_*(\vartheta(x))Z=f_*(x)(Z+\nabla_Z X-A^f_\delta Z)+\alpha_f(X,Z)+{}^f\nabla_Z^\perp\delta,$$

the claim follows. Now, let N^{n+1} be the hypersurface of $\tilde U$ defined by $\tilde G(N^{n+1})=\tilde G(\tilde U)\cap \boldsymbol{V}^{n+3}$ and let $i:U\to N^{n+1}$ denote the inclusion map. Set $\mathscr{F}=\tilde F|_{N^{n+1}}$ and let $\mathscr{G}:N^{n+1}\to\boldsymbol{R}^{n+2}$ be the conformal immersion correspondent to $\tilde G|_{N^{n+1}}$ as in (3). Then, $\mathscr{F}$ and $\mathscr{G}$ are conformal and $f|_U=\mathscr{F}\circ i$, $g|_U=\mathscr{G}\circ i$, as we wished. Clearly, if $\mathscr{F}$ and $\mathscr{G}$ are conformally congruent, then the same holds for f and g. $\square$

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Marcos Dajczer
IMPA—Estrada Dona Castorina, 110
22460-320—Rio de Janeiro—Brazil
E-mail: marcos@impa.br

Ruy Tojeiro
Universidade Federal de Uberlândia
38400-089—Uberlândia—Brazil
E-mail: rtfjunior@ufu.br