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Hardy's inequalities for Laguerre expansions

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Abstract. For the real Hardy spaces, we shall establish Hardy's inequalities with respect to Laguerre expansions. The inequalities for the Hardy spaces with exponents smaller than one will be discussed.

1. Introduction.

The well-known Hardy inequality for the Fourier transforms says that

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^p |\xi|^{p-2}\, d\xi \le C\|f\|_{H^p(\boldsymbol{R})}^p, \tag{1}$$

$0 < p \le 1$ (see Garcia-Cuerva and Rubio de Francia [**3**, ch. III, Corollary 7.23], Stein [**7**, p. 128]). Here $H^p(\boldsymbol{R})$, $0 < p \le 1$, is the real Hardy space of the boundary distributions $f(x) = \Re F(x)$, where $F(z)$ is an element of the Hardy space $H^p(\boldsymbol{R}_+^2)$, that is, $F(z)$ is analytic on the upper half plane $\boldsymbol{R}_+^2 = \{z = x + iy;\, y > 0\}$ with the norm

$$\|f\|_{H^p(\boldsymbol{R})} = \|F\|_{H^p(\boldsymbol{R}_+^2)} = \sup_{y>0}\left(\int_{-\infty}^{\infty} |F(x+iy)|^p\, dx\right)^{1/p}.$$

In this paper, we shall establish the Hardy inequalities with respect to the Laguerre expansions.

The Laguerre function $\mathscr{L}_n^\alpha(x)$, $\alpha > -1$ is defined by

$$\mathscr{L}_n^\alpha(x) = \tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2},$$

where $\tau_n^\alpha = (\Gamma(n+1)/\Gamma(n+\alpha+1))^{1/2}$ and $L_n^\alpha(x) = (n!)^{-1} x^{-\alpha} e^x (d/dx)^n \{x^{n+\alpha} e^{-x}\}$ is the Laguerre polynomial of degree n and of order α. Then $\{\mathscr{L}_n^\alpha\}_{n=0}^\infty$ is a complete orthonormal system on the interval $[0, \infty)$ with respect to dx (see Szegö [**8**, 5.7]). We have the formal expansion

$$f(x) \sim \sum_{n=0}^{\infty} c_n^\alpha(f) \mathscr{L}_n^\alpha(x)$$

of a function $f(x)$ on $[0, \infty)$, where

$$c_n^\alpha(f) = \int_0^\infty \mathscr{L}_n^\alpha(x) f(x)\, dx$$

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is the n-th Fourier-Laguerre coefficient. The Hardy inequality was originally one of inequalities with respect to the Fourier coefficients (see Zygmund [**9**, ch. VII, (8.7)]), and the inequality for the Fourier transforms followed. Recently, Kanjin [**4**] changed the role of the Fourier transforms for that of the Laguerre coefficients and got the Hardy type inequality for $H^1(\boldsymbol{R})$ by using Askey's transplantation theorem (see [**1**]) for the Laguerre coefficients. The aim of this paper is to extend the Hardy inequality of the Laguerre expansions to $H^p(\boldsymbol{R})$ with p smaller than one by estimating the derivatives of $\mathscr{L}_n^\alpha(x)$ precisely.

Our theorem is as follows:

THEOREM. *Let* $\alpha \geq 0$. *Suppose* $\alpha/2 \neq$ *integer and* $(\alpha/2+1)^{-1} < p \leq 1$. *Then for* $f \in H^p(\boldsymbol{R})$ *supported in* $[0,\infty)$, *the Fourier-Laguerre coefficients* $c_n^\alpha(f)$ *are well-defined and satisfy*

$$\sum_{n=0}^{\infty} \frac{|c_n^\alpha(f)|^p}{(n+1)^{2-p}} \leq C_\alpha \|f\|_{H^p(\boldsymbol{R})}^p \tag{2}$$

with some constant C_α *independent of* f. *If* $\alpha/2 = 0,1,2,\dots,$ *then the above statement holds for each* p *with* $0 < p \leq 1$.

Here and below constants ($C, c_1, C_\alpha, C_{\alpha,p}$, etc.) may vary from inequality to inequality. They are always independent of f, n, etc. but may depend on α, p or other explicitly indicated parameters.

We shall give two lemmas (Lemma 1 and Lemma 2) in §2, and a proof of Theorem in §3, first for (p,∞)-atoms $a(x)$ supported in $[0,\infty)$ (Lemma 3), and next for $f(x) \in H^p(\boldsymbol{R})$ supported in $[0,\infty)$. The atomic decomposition characterization of $H^p(\boldsymbol{R})$, $0 < p \leq 1$ will play an essential role. For convenience, we state the characterization. Let $0 < p \leq 1$ and $N = [1/p - 1]$, where $[u]$ denotes the greatest integer not exceeding u. A (p,∞)-atom is a real-valued function $a(x)$ on $\boldsymbol{R}$ such that (i) $a(x)$ is supported in an interval $[b, b+h]$, (ii) $|a(x)| \leq h^{-1/p}$ a.e. x, and (iii) $\int_{\boldsymbol{R}} x^k a(x)\,dx = 0$ for all $k = 0,1,2,\dots,N$. An element $f(x)$ of $H^p(\boldsymbol{R})$ is characterized by the decomposition

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),$$

where each a_j is a (p,∞)-atom and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$, and

$$c_1 \|f\|_{H^p(\boldsymbol{R})} \leq \inf\left\{ \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\} \leq c_2 \|f\|_{H^p(\boldsymbol{R})}$$

with two positive constants c_1 and c_2 independent of f (see Garcia-Cuerva and Rubio de Francia [**3**, III.3], Stein [**7**, p. 107]). Further, the series $\sum_{j=1}^{\infty} \lambda_j a_j$ converges in H^p norm, consequently, also in the sense of tempered distributions. We shall deal with the elements $f \in H^p(\boldsymbol{R})$ supported in the interval $[0,\infty)$. These elements are also characterized by $f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x)$ with (p,∞)-atoms supported in $[0,\infty)$ and

$$c_1 \|f\|_{H^p(\boldsymbol{R})} \leq \inf \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \leq c_2 \|f\|_{H^p(\boldsymbol{R})}$$

where the infimum is taken over all such decompositions of f (see Miyachi [**5**], [**6**]).

2. Two Lemmas.

To prove Theorem we need orders of the m-th derivatives $(\mathscr{L}_n^\alpha)^{(m)}(x)$ with respect to n. Lemma 1 will be assigned to estimate the derivatives $(\mathscr{L}_n^\alpha)^{(m)}(x)$ for $m \le \alpha/2$ if $\alpha/2 \ne$ integer and for all m if $\alpha/2 = 0, 1, 2, \ldots$. Further, if $\alpha/2 \ne$ integer, $M = [\alpha/2]$ and $\delta = \alpha/2 - M$, then the bounds of the Lipschitz δ-norm of $(\mathscr{L}_n^\alpha)^{(M)}(x)$ are necessary, which will be given in Lemma 2.

Lemma 1. *Let $\alpha \ge 0$. If we set $M = [\alpha/2]$, then for each non-negative integer $m \le M$, the m-th derivative $(\mathscr{L}_n^\alpha)^{(m)}(x)$ of $\mathscr{L}_n^\alpha(x)$ with respect to x has an estimate,*

$$|(\mathscr{L}_n^\alpha)^{(m)}(x)| \le C_{\alpha,m} n^m, \quad m \le M. \tag{3}$$

Furthermore, if $\alpha/2 = 0, 1, 2, \ldots,$ then

$$|(\mathscr{L}_n^\alpha)^{(m)}(x)| \le C_{\alpha,m} n^m, \quad m = 0, 1, 2, \ldots. \tag{4}$$

Here $C_{\alpha,m}$ are positive constants independent of n.

Proof. Let $m \le M$. If we differentiate $(\mathscr{L}_n^\alpha)^{(m)}(x)$ m-times with respect to x, then we have an expression

$$(\mathscr{L}_n^\alpha)^{(m)}(x) = \sum_{0 \le j+k \le m} c_{j,k}^m \varphi_{n,j,k}^m(x), \tag{5}$$

where $c_{j,k}^m$ are some constants and

$$\varphi_{n,j,k}^m(x) = \tau_n^\alpha L_{n-j}^{\alpha+j}(x) e^{-x/2} x^{\alpha/2-k}. \tag{6}$$

Then it is enough to show $|\varphi_{n,j,k}^m(x)| \le C_\alpha n^{j+k}$. We divide the matter into two cases $nx \ge 1$ and $nx < 1$. First we argue the case $nx \ge 1$. We have

$$|\varphi_{n,j,k}^m(x)| = \tau_n^\alpha (\tau_{n-j}^{\alpha+j})^{-1} \tau_{n-j}^{\alpha+j} |L_{n-j}^{\alpha+j}(x)| e^{-x/2} x^{(\alpha+j)/2} x^{-j/2-k}.$$

We use two estimates

$$c_1 l^{-\beta/2} \le \tau_l^\beta \le c_2 l^{-\beta/2} \tag{7}$$

and

$$|\mathscr{L}_l^\beta(x)| \le c_3, \quad x > 0, \quad \beta \ge 0 \tag{8}$$

(see the table on p. 699 of [**2**]) where c_1, c_2 and c_3 are constants independent of l. It follows that

$$|\varphi_{n,j,k}^m(x)| \le C_\alpha n^{-\alpha/2} (n-j)^{(\alpha+j)/2} x^{-j/2-k}.$$

We have $|\varphi_{n,j,k}^m(x)| \le C_\alpha n^{j+k}$ by $x^{-1} \le n$.

Let $0 < nx < 1$. Since

$$|L_l^{\beta}(x)| \le C_\beta l^\beta, \quad 0 < lx \le 1, \tag{9}$$

(see [**8**, (7.6.8)]), it follows that

$$\begin{aligned} |\varphi_{n,j,k}^m(x)| &= \tau_n^\alpha |L_{n-j}^{\alpha+j}(x)| e^{-x/2} x^{\alpha/2-k} \\ &\le C_\alpha n^{-\alpha/2}(n-j)^{\alpha+j} x^{\alpha/2-k} \\ &\le C_\alpha n^{\alpha/2+j} x^{\alpha/2-k}. \end{aligned}$$

We have $|\varphi_{n,j,k}^m(x)| \le C_\alpha n^{j+k}$, by $x \le 1/n$, which completes the proof of (3).

Next we deal with the case that $\alpha/2 = 0, 1, 2, \dots$. Since $(x^{\alpha/2})^{(l)} = 0$, $l > \alpha/2$, we see easily $(\mathscr{L}_n^\alpha)^{(m)}$ has the same order, which completes the proof of Lemma 1.

If $\alpha/2$ is not an integer, in order to prove Theorem, we need more precise estimates which are given by the following lemma.

LEMMA 2. *Let $\alpha \ge 0$ and let $\alpha/2$ be not an integer. We put $\alpha/2 = M + \delta$, $0 < \delta < 1$. Then for the M-th derivative $(\mathscr{L}_n^\alpha)^{(M)}(x)$ of $\mathscr{L}_n^\alpha(x)$ with respect to x, we have an estimate*

$$|(\mathscr{L}_n^\alpha)^{(M)}(x+h) - (\mathscr{L}_n^\alpha)^{(M)}(x)| \le C_\alpha n^{\alpha/2} h^\delta, \tag{10}$$

where C_α is a constant independent of n.

PROOF. By (5), we see that it is enough to show

$$|\varphi_{n,j,k}^M(x+h) - \varphi_{n,j,k}^M(x)| \le C_\alpha n^{j+k+\delta} h^\delta \tag{11}$$

with some constant C_α independent of n, x and h, where $0 \le j+k \le M$. If $nh \ge 1$ and $nx \ge 1$, then

$$\begin{aligned} |\varphi_{n,j,k}^M(x+h)| &= \tau_n^\alpha |L_{n-j}^{\alpha+j}(x+h)| e^{-(x+h)/2} (x+h)^{\alpha/2-k} \\ &= \tau_n^\alpha (\tau_{n-j}^{\alpha+j})^{-1} |\mathscr{L}_{n-j}^{\alpha+j}(x+h)| (x+h)^{-j/2-k}. \end{aligned}$$

Thus by (7) and (8), we have

$$|\varphi_{n,j,k}^M(x+h)| \le C_\alpha n^{-\alpha/2}(n-j)^{(\alpha+j)/2}(x+h)^{-j/2-k},$$

which is bounded by $C_\alpha n^{j+k}$, since $n(x+h) \ge 2$. By $nh \ge 1$, we have $|\varphi_{n,j,k}^M(x+h)| \le C_\alpha n^{j+k+\delta} h^\delta$. Similarly, we have $|\varphi_{n,j,k}^M(x)| \le C_\alpha\, n^{-\alpha/2}(n-j)^{(\alpha+j)/2} x^{-j/2-k}$. Since $nx \ge 1$ and $nh \ge 1$, it follows that $|\varphi_{n,j,k}^M(x)| \le C_\alpha n^{j+k+\delta} h^\delta$. Therefore we have (11) for this case.

If $nh \ge 1$ and $nx \le 1$, then $n(x+h) \ge 1$, and as in the preceding case ($nx \ge 1$), we have

$$|\varphi_{n,j,k}^M(x+h)| \le C_\alpha n^{j+k+\delta} h^\delta.$$

By (7) and (9), we have

$$\begin{aligned}|\varphi^M_{n,j,k}(x)| &= \tau^\alpha_n |L^{\alpha+j}_{n-j}(x)| e^{-x/2} x^{\alpha/2-k}\\ &\le C_\alpha n^{-\alpha/2}(n-j)^{\alpha+j} x^{\alpha/2-k}\\ &\le C_\alpha n^{j+\alpha/2} x^{\alpha/2-k-\delta} x^\delta.\end{aligned}$$

Since $\alpha/2-k-\delta>0$ and $nx\le 1$, it follows that

$$|\varphi^M_{n,j,k}(x)| \le C_\alpha n^{j+\alpha/2} n^{-\alpha/2+k+\delta} x^\delta = C_\alpha n^{j+k+\delta} x^\delta, \tag{12}$$

which is bounded by $C_\alpha n^{j+k+\delta} h^\delta$ since $x\le h$. Therefore we have (11) for the case $nh\ge 1$ and $nx\le 1$.

Let $nh\le 1$. We divide the case into two cases $x\le h$ and $x\ge h$. We first treat the case $x\le h$. Then we have $nx\le 1$ and thus we have also (12) for this case. Further, since $n(x+h)\le 2$, we have (12) with $x+h$ instead of x.

We shall treat the case $nh\le 1$ and $x\ge h$. To get the inequality (11), we shall estimate

$$I_n = \tau^\alpha_n |L^{\alpha+j}_{n-j}(x+h) - L^{\alpha+j}_{n-j}(x)| e^{-(x+h)/2}(x+h)^{\alpha/2-k} \tag{13}$$

and

$$J_n = \tau^\alpha_n |L^{\alpha+j}_{n-j}(x)|\, |e^{-(x+h)/2}(x+h)^{\alpha-k} - e^{-x/2}x^{\alpha/2-k}|. \tag{14}$$

Let us first deal with I_n. By the mean value theorem, we have

$$\begin{aligned}I_n &= \tau^\alpha_n |L^{\alpha+j+1}_{n-j-1}(x+\theta h)| h e^{-(x+h)/2}(x+h)^{-\alpha/2-k}\\ &= \tau^\alpha_n (\tau^{\alpha+j+1}_{n-j-1})^{-1} |\mathscr{L}^{\alpha+j+1}_{n-j-1}(x+\theta h)| e^{-(1-\theta)h/2}\\ &\quad\times h\left(\frac{x+h}{x+\theta h}\right)^{\alpha/2}(x+\theta h)^{-(j+1)/2}(x+h)^{-k}\end{aligned} \tag{15}$$

where θ is some constant with $0<\theta<1$. Since $x\ge h$, it follows $1\le (x+h)/(x+\theta h) \le 2$. If $nx\ge 1$, then we have $(x+\theta h)^{-(j+1)/2}\le n^{(j+1)/2}$ and $(x+h)^{-k}\le n^k$. Thus, by (7) and (8) we have

$$\begin{aligned}I_n &\le C_\alpha n^{(j+1)/2} n^{(j+1)/2} n^k = C_\alpha n^{j+k}(nh)\\ &\le C_\alpha n^{j+k+\delta} h^\delta.\end{aligned}$$

For the last inequality, we used $nh\le 1$. If $nx\le 1$, then applying (9) to (15) we get

$$I_n \le C_\alpha n^{-\alpha/2}(n-j-1)^{\alpha+j+1} h e^{-(x+h)/2}(x+h)^{\alpha/2-k}. \tag{16}$$

Since $x\ge h$ and $nx\le 1$, it follows that $(x+h)^{\alpha/2-k}\le (2/n)^{\alpha/2-k}$. We have

$$I_n \le C_\alpha n^{j+k}(nh) \le C_\alpha n^{j+k+\delta} h^\delta. \tag{17}$$

We shall estimate J_n. It follows

$$J_n \le \tau_n^\alpha |L_{n-j}^{\alpha+j}(x)||(e^{-(x+h)/2} - e^{-x/2})(x+h)^{-\alpha-k}|$$
$$+ \tau_n^\alpha |L_{n-j}^{\alpha+j}(x)| e^{-x/2} |(x+h)^{-\alpha/2-k} - x^{-\alpha/2-k}|$$
$$= J_n^{(1)} + J_n^{(2)}, \quad \text{say.}$$

For $J_n^{(1)}$, by (7) and the mean value theorem we have

$$J_n^{(1)} = \tau_n^\alpha |L_{n-j}^{\alpha+j}(x)| e^{-(x+\theta h)/2} h (x+h)^{-\alpha/2-k} \tag{18}$$
$$\le C_\alpha n^{-\alpha/2} |\mathscr{L}_{n-j}^{\alpha+j}(x)| e^{-\theta h/2} h \left(\frac{x+h}{x}\right)^{\alpha/2} x^{-j/2}(x+h)^{-k}$$

for some θ with $0 < \theta < 1$. If $nx \ge 1$, then $(x+h)^{-1} \le n$. Since we have already assumed $h \le x$, the inequality $1 \le (x+h)/x \le 2$ holds. Thus by (8) we have $J_n^{(1)} \le C_\alpha n^{j+k}(nh) \le C_\alpha n^{j+k}(nh)^\delta \le C_\alpha n^{j+k+\delta} h^\delta$. If $nx \le 1$, then we apply (9) to (18) and get

$$J_n^{(1)} \le C_\alpha n^{\alpha/2+j} h x^{\alpha/2-k} \le C_\alpha n^{\alpha/2+j} h n^{-\alpha/2-k}$$
$$\le C_\alpha n^{j+k} nh \le C_\alpha n^{n+j+\delta} h^\delta.$$

Last we estimate $J_n^{(2)}$. Let $nx \ge 1$. If $\alpha/2 - k \ge 1$, then by the mean value theorem

$$J_n^{(2)} = \tau_n^\alpha |L_{n-j}^{\alpha+j}(x)| e^{-x/2} h (x+\theta h)^{\alpha/2-k-1} \tag{19}$$
$$= |\mathscr{L}_{n-j}^{\alpha+j}(x)| \left(\frac{x+\theta h}{x}\right)^{(\alpha+j)/2} (x+\theta h)^{\alpha/2-k-1}$$
$$\le C_\alpha n^{j+k+\delta} h^\delta.$$

If $0 < \alpha/2 - k < 1$, then $j = 0$, $\delta = \alpha/2 - k$, and

$$J_n^{(2)} = |\mathscr{L}_n^\alpha(x)| x^{\alpha/2} |(x+h)^{\alpha/2-k} - x^{\alpha/2-k}| \le C_\alpha n^{\alpha/2} h^\delta \tag{20}$$

because x^δ is of Lip δ $(0 < \delta < 1)$. Let $nx \le 1$. If $\alpha/2 - k \ge 1$, then we apply (9) to (19) and obtain

$$J_n^{(2)} \le C_\alpha n^{-\alpha/2}(n-j)^{\alpha+j} h \left(\frac{1}{n}\right)^{\alpha/2-k-1}$$
$$\le C_\alpha n^{j+k+\delta} h^\delta.$$

If $0 < \alpha/2 - k < 1$, then $j = 0$, $\delta = \alpha/2 - k$ and by (9) and the fact x^δ is of Lip δ we have $J_n^{(2)} \le C_\alpha n^{k+\delta} h^\delta$ which completes the proof of Lemma 2.

3. Proof of Theorem.

Now we shall prove Theorem. Because finite linear combinations of (p, ∞)-atoms are dense in $H^p(\boldsymbol{R})$, the following lemma is essential, which will be proved by using the lemmas in the previous section.

LEMMA 3. *Suppose* $\alpha \geq 0$, $\alpha/2 \neq$ *integer and* $1/p - 1 < \alpha/2$ $(2/(\alpha+2) < p \leq 1)$. *Then*

$$\sum_{n=0}^{\infty} \frac{|c_n^\alpha(a)|^p}{(n+1)^{2-p}} \leq C_\alpha \tag{21}$$

for all (p, ∞)*-atoms* $a(x)$ *supported in* $[0, \infty)$.

If $\alpha/2 = 0, 1, 2, \ldots,$ *then the above inequality holds for all* p *with* $0 < p \leq 1$.

PROOF. Let $M = [\alpha/2]$ and $N = [1/p - 1]$. We put $\alpha/2 = M + \delta$. Let $\alpha/2 \neq$ integer. We shall first deal with the case $N = M$. Let $I = [b, b+h]$ be an interval defining a (p, ∞)-atom $a(x)$. The Taylor expansion of $\mathscr{L}_n^\alpha(x)$ in x at $x = b$ leads to

$$\begin{aligned} c_n^\alpha(a) &= \frac{1}{M!} \int_b^{b+h} a(x) (\mathscr{L}_n^\alpha)^{(M)}(b + \theta(x-b))(x-b)^M \, dx \\ &= \frac{1}{M!} \int_b^{b+h} a(x) \times \{(\mathscr{L}_n^\alpha)^{(M)}(b + \theta(x-b)) - (\mathscr{L}_n^\alpha)^{(M)}(b)\}(x-b)^M \, dx \end{aligned} \tag{22}$$

for some θ with $0 < \theta < 1$. The last equality follows from the cancellation property of a (p, ∞)-atom $a(x)$. We have by (10)

$$|c_n^\alpha(a)| \leq \frac{1}{M!} \left(\int_b^{b+h} |a(x)| (x-b)^{M+\delta} \, dx \right) n^{M+\delta}, \tag{23}$$

and thus we have

$$\begin{aligned} |c_n^\alpha(a)| &\leq C_\alpha \frac{1}{M!} \|a\|_2 \left(\int_b^{b+h} (x-b)^{2(M+\delta)} \, dx \right)^{1/2} n^{M+\delta} \\ &\leq C_\alpha n^{M+\delta} h^{M+\delta+1/2} \|a\|_2. \end{aligned}$$

Since (p, ∞)-atoms satisfy $h \leq \|a\|_2^{-2p/(2-p)}$, it follows that

$$|c_n^\alpha(a)| \leq C_\alpha n^{M+\delta} \|a\|_2^{1-(2/(2-p))(M+\delta+1/2)}. \tag{24}$$

Let $R = \|a\|_2^{2p/(2-p)}$. It follows from the above inequality that

$$\sum_{n \leq R} \frac{|c_n^\alpha(a)|^p}{(n+1)^{2-p}} \leq C_\alpha^p \|a\|_2^{p\{1-(2p/(2-p))(M+\delta+1/2)\}} \sum_{n \leq R} n^{p(M+\delta)-2+p}. \tag{25}$$

Since $1/p - 1 < \alpha/2 = M + \delta$, it follows that $p(M+\delta) - 2 + p > -1$. Thus, we have

$$\sum_{n \leq R} \frac{|c_n^\alpha(a)|^p}{(n+1)^{2-p}} \leq C_{\alpha,p}, \tag{26}$$

where $C_{\alpha,p}$ depends only on α and p. For the sum over $n > R$, we have

$$\begin{aligned} \sum_{n>R} \frac{|c_n^\alpha(a)|^p}{(n+1)^{2-p}} &\leq C \left(\sum_{n>R} |c_n^\alpha(a)|^2 \right)^{p/2} \left(\sum_{n>R} \frac{1}{n^2} \right)^{(2-p)/2} \\ &\leq C \|a\|_2^p R^{-(2-p)/2} \leq C. \end{aligned} \tag{27}$$

Therefore (26) and (27) give (21).

Next we shall prove the case $N < M$. In this case, applying (3) to (22) with $N+1$ instead of M, we have (23) with $\delta = 0$ and $N+1$ instead of M, and thus (24) with $\delta = 0$ and $N+1$ instead of M holds. Thus we have (21) in the same way.

For the case $\alpha/2 = 0, 1, 2, \ldots$, we apply (4) to (22) with $N+1$ instead of M and easily get the desired inequality (21). This completes the proof of Lemma 3.

Now we shall finish the proof of Theorem. Let f be a general element in $H^p(\boldsymbol{R})$ such that $\operatorname{supp} f \subset [0, \infty)$. Then there exist real numbers λ_j and (p, ∞)-atoms a_j with $\operatorname{supp} a_j \subset [0, \infty)$, $j = 1, 2, \ldots$ such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ and

$$\sum_{j=1}^{\infty} |\lambda_j|^p \le C_p \|f\|_{H^p}^p. \tag{28}$$

We put $f_J = \sum_{j=1}^{J} \lambda_j a_j$. Since $0 < p \le 1$, we have

$$\sum_{n=0}^{\infty} \frac{|c_n^\alpha(f_J)|^p}{(n+1)^{2-p}} = \sum_{n=0}^{\infty} \frac{|\sum_{j=1}^{J} \lambda_j c_n^\alpha(a_j)|^p}{(n+1)^{2-p}} \le \sum_{j=1}^{J} |\lambda_j|^p \sum_{n=0}^{\infty} \frac{|c_n^\alpha(a_j)|^p}{(n+1)^{2-p}}. \tag{29}$$

Thus, Lemma 3 leads to

$$\sum_{n=0}^{\infty} \frac{|c_n^\alpha(f_J)|^p}{(n+1)^{2-p}} \le C_{\alpha,p} \|f\|_{H^p}^p. \tag{30}$$

By the density arguement, we see that $c_n^\alpha(f)$ are well-defined and the inequality (2) holds.

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