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The expressions of the Harish-Chandra C-functions of semisimple Lie groups $Spin(n,1)$, $SU(n,1)$

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Abstract. We give the recursion formula of the Harish-Chandra C-function with respect to the highest weight of the representations of K. Using this formula, we get the explicit expressions of the Harish-Chandra C-functions for $Spin(n,1)$ and $SU(n,1)$. As an application, by using these expressions, we get the realizations of discrete series representations of $SU(n,1)$ as subquotients of nonunitary principal series representations. We shall also get the decompositions of holomorphic and antiholomorphic discrete series when restricted to $U(n-1,1)$. By using the structures of K-spectra of discrete series representations, we can concretely construct the invariant subspaces of the representation spaces of holomorphic and antiholomorphic discrete series.

1. Introduction.

The Harish-Chandra C-function plays a basic role in studying harmonic analysis on semisimple Lie groups, because it closely relates to the Plancherel measure and the reducibility of the principal series representations. Moreover, the location of the singularities of the Harish-Chandra C-function is crucial for the proof of Paley-Wiener type theorems or various Schwartz type theorems. After a time, many peoples studied the Harish-Chandra C-function. However, even now, the explicit expressions of the Harish-Chandra C-functions are not known except for a few semisimple Lie groups and special cases, such as class one case or one dimensional K-type case.

The purpose of this paper is to give the explicit formulae of the Harish-Chandra C-functions for $Spin(n,1)$ and $SU(n,1)$. Here in order to describe the contents of this paper, we shall use some notation explained in §2. By the product formula for the Harish-Chandra C-function (cf. [**4**]), the problem of computing the Harish-Chandra C-functions of semisimple Lie groups of general rank is reduced to the real rank one case. For this reason, it is crucial to compute the Harish-Chandra C-function for $Spin(n,1)$ and $SU(n,1)$. For $\tau \in \hat{K}$, the Harish-Chandra C-function is given by

$$(1.1) \qquad C_\tau(\nu) = \int_{\bar{N}} e^{-(\nu+\rho)(H(\bar{n}))}\tau(\kappa(\bar{n}))^{-1}d\bar{n}, \quad (\nu \in \mathfrak{a}_c^*).$$

The reason for restricting our attention to the cases $Spin(n,1)$ and $SU(n,1)$ is that no multiple irreducible unitary representations of M occur in any irreducible unitary representation of K. In these cases, there exists a meromorphic function $C_\tau(\sigma:\nu)$ such

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that

$$TC_\tau(\nu) = C_\tau(\sigma : \nu)T, \quad (T \in \mathrm{Hom}_M(V_\tau, H_\sigma)). \tag{1.2}$$

We obtain in this paper the explicit expressions of $C_\tau(\sigma : \nu)$ for $Spin(n,1)$ and $SU(n,1)$. These expressions give us the precise informations on the zeros and the poles of the Harish-Chandra C-functions $C_\tau(\nu)$. On the other hand, Cohn [1] showed that for any semisimple Lie group G, there exist $p_{i,j}, q_{i,j} \in \boldsymbol{C}$ $(1 \le i \le r, 1 \le j \le j_i)$ and $\mu_1, \dots, \mu_r \in \mathfrak{a}^*$ such that

$$\det C_\tau(\nu) = \prod_{i=1}^{r} \prod_{j=1}^{j_i} \frac{\Gamma\left(\dfrac{-\langle \nu, \alpha_i \rangle}{2\langle \mu_i, \alpha_i \rangle} + q_{i,j}\right)}{\Gamma\left(\dfrac{-\langle \nu, \alpha_i \rangle}{2\langle \mu_i, \alpha_i \rangle} + p_{i,j}\right)}. \tag{1.3}$$

In [1], he conjectured that the constants $p_{i,j}$ and $q_{i,j}$ appearing in the above expression are rational numbers and depend linearly on the highest weight of τ. By using the expression of $C_\tau(\sigma : \nu)$ together with $V_\tau = \sum_{\sigma \in \hat{M}} [\tau : \sigma] H_\sigma$, we can get the explicit formula for $\det C_\tau(\nu)$ and this shows that Cohn's conjecture is true for $Spin(n,1)$ and $SU(n,1)$. Because the remaining rank one simple Lie groups $Sp(n,1)$ and $F_{4(-20)}$ have multiple irreducible unitary representations of M, we shall need more complicated argument for these groups and thus we will postpone the discussion for these groups to another paper.

This paper consists of two parts. The first part is devoted to the construction of the recursion formula of the Harish-Chandra C-function. To accomplish this, we shall use the formula of the infinitesimal operator of the principal series representation for semisimple Lie groups of real rank one, which was proved by Thieleker [10]. In this paper we will reform Thieleker's formula in terms of the M-invariant differential operators. With the help of this formula, we can obtain the recursion formula of the standard intertwining operator relative to D_K. From the relationship between the standard intertwining operator and the Harish-Chandra C-function, this formula leads to the recursion formula of the Harish-Chandra C-function. By using this recursion formula, for getting the expression of the Harish-Chandra C-function, it suffices to consider the case when the dominant, analytically integral form on $\mathfrak{t}_c$ is minimal in the sense of the betweenness condition of the Gel'fand-Tsetlin basis.

The second part is devoted to the calculation of integral in the definition of the Harish-Chandra C-function for the case when the dominant, analytically integral form on $\mathfrak{t}_c$ is minimal in the sense mentioned above. In order to carry out this integral, we shall realize the fundamental representations of K in terms of the alternating tensor products of $\boldsymbol{C}^n$ and compute the matrix element relative to a highest weight vector. From this, with the help of the integration formulae of the hypergeometric function, we can get the expression of the Harish-Chandra C-function associated with the above irreducible unitary representation of K.

In §8 and §9, as an application, we shall show that the information on zeros of the Harish-Chandra C-function can be used to get the realizations of discrete series representations of $SU(n,1)$ as subquotients of nonunitary principal series representations. We shall also get the decompositions of holomorphic and antiholomorphic discrete series when restricted to $U(n-1,1)$, which was proved in [8]. By using the structures of K-spectra of discrete series representations, we can concretely construct the invariant subspaces of the representation spaces of holomorphic and antiholomorphic discrete series. In the case of $SU(2,1)$, these decompositions were obtained by J. Xie [**14**] and our proof was inspired by his paper [**14**].

2. Notation and preliminaries.

The standard symbols $\boldsymbol{Z}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ shall be used for the integers, the real numbers and the complex numbers. If $x \in \boldsymbol{C}$, $\Re v$, $\Im v$ and $\bar{x}$ denote its real part, its imaginary part and its complex conjugate, respectively. If x is a column vector, ${}^t x$ denotes its transpose and $x^* = {}^t\bar{x}$. For $\boldsymbol{E} \subseteq \boldsymbol{R}$ and $p \in \boldsymbol{Z}_{>0}$, $\boldsymbol{E}^p_>$ denotes the subset of $\boldsymbol{E}^p$ comprised of all $x = (x_1, \dots, x_p)$ such that $x_j - x_{j+1} \in \boldsymbol{Z}_{\geq 0}$ for $1 \leq j \leq p-1$. If $x = (x_1, \dots, x_p) \in \boldsymbol{E}^p$ and $1 \leq q \leq p$, we write $|x| = \sum_{j=1}^p x_j$, $x_{\leq q} = (x_1, \dots, x_q) \in \boldsymbol{E}^q$ and $x_{\geq q} = (x_q, \dots, x_p) \in \boldsymbol{E}^{p-q+1}$. For $x \in \boldsymbol{E}^{p+1}_>$ and $y \in \boldsymbol{E}^p_>$, $x > y$ means $x_j - y_j \in \boldsymbol{Z}_{\geq 0}$ and $y_j - x_{j+1} \in \boldsymbol{Z}_{\geq 0}$ for $1 \leq j \leq p$. For a finite set F, $\mathrm{Card}\, F$ denotes its cardinal number. If V is a vector space over $\boldsymbol{R}$, V_c, V^* and V_c^* denote its complexification, its real dual and its complex dual, respectively.

For a Lie group L, $\hat{L}$ denotes the set of equivalence classes of irreducible unitary representations of L. As usual, we shall use lower case German letters to denote the corresponding Lie algebras and upper case German letters to denote their universal enveloping algebras. As is well-known, the elements of $\mathfrak{L}$ act on $C^\infty(L)$, as differential operator, on both sides. Following Harish-Chandra, we shall write $f(D; x; E)$ for the action of $D, E \in \mathfrak{L}$ on $f \in C^\infty(L)$ at $x \in L$.

Let G be a semisimple Lie group with finite center and K a maximal compact subgroup of G and θ the corresponding Cartan involution. Throughout this paper we assume $\mathrm{rank}\, G = 1$. Let $\langle \cdot, \cdot \rangle$ denote the Killing form on $\mathfrak{g}$ and define the inner product $\langle \cdot, \cdot \rangle_\theta$ on $\mathfrak{g}$ by $\langle X, Y \rangle_\theta = -\langle X, \theta Y \rangle$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ corresponding to θ. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra containing $\mathfrak{a}$ and set $\mathfrak{h}_\mathfrak{k} = \mathfrak{h} \cap \mathfrak{k}$. Let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{k}$ containing $\mathfrak{h}_\mathfrak{k}$. Fix an ordering on $\sqrt{-1}\mathfrak{h}_\mathfrak{k} + \mathfrak{a}$ that is compatible with the one on $\mathfrak{a}$ and fix the ordering on $\sqrt{-1}\mathfrak{t}$ that is compatible with the one on $\sqrt{-1}\mathfrak{h}_\mathfrak{k}$.

Let Σ denote the set of all nonzero roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and Σ^+ the subset of Σ consisting of all positive roots. For $\alpha \in \Sigma$, $\mathfrak{g}_\alpha$ denotes the corresponding root subspace of $\mathfrak{g}$ and $m_\alpha = \dim \mathfrak{g}_\alpha$. Let $\mathfrak{n}$ be the sum of all positive root subspaces. A and N denote the analytic subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$, respectively and $\bar{N} = \theta N$. Then $G = KAN$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ are the Iwasawa decompositions of G and $\mathfrak{g}$, respectively. For $g \in G$, g decomposes under $G = KAN$ as $g = \kappa(g) \exp H(g) n(g)$, where $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$. Let M and M' denote the centralizer and the normalizer of $\mathfrak{a}$ in K, respectively. Then $W(\mathfrak{a}) = M'/M$ is the Weyl group of G. For $w \in W(\mathfrak{a})$, $\sigma \in \hat{M}$ and $\nu \in \mathfrak{a}_c^*$, define $w\nu \in \mathfrak{a}_c^*$ and $w\sigma \in \hat{M}$ by $w\nu(H) = \nu(\mathrm{Ad}(w)^{-1} H)$ and $w\sigma(m) = \sigma(w^{-1} m w)$. Let Δ_K be the set of roots of $\mathfrak{k}_c$ relative to $\mathfrak{t}_c$ and Δ_K^+ the subset of Δ_K consisting of all positive roots. Put $\rho = (1/2) \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ and $\delta_K = (1/2) \sum_{\beta \in \Delta_K^+} \beta$.

Let D_K and D_M be the sets of dominant, analytically integral forms on $\mathfrak{t}_c$ and $\mathfrak{h}_{\mathfrak{k}c}$, respectively. If $\lambda \in D_K$ (resp. $\mu \in D_M$), we write τ_λ (resp. σ_μ) for the element in $\hat{K}$ (resp. $\hat{M}$) whose highest weight is equal to λ (resp. μ). For $\tau \in \hat{K}$ and $\sigma \in \hat{M}$, we denote by $[\tau:\sigma]$ the multiplicity of σ occurring in $\tau|_M$ and put $\hat{K}(\sigma)=\{\tau\in\hat{K}:[\tau:\sigma]\neq 0\}$ and $\hat{M}(\tau)=\{\sigma\in\hat{M}:[\tau:\sigma]\neq 0\}$. Similarly for $\lambda\in D_K$ and $\mu\in D_M$, we set $D_K(\mu)=\{\lambda\in D_K:\tau_\lambda\in\hat{K}(\sigma_\mu)\}$ and $D_M(\lambda)=\{\mu\in D_M:\sigma_\mu\in\hat{M}(\tau_\lambda)\}$.

Finally let dk and $d\bar{n}$ be the Haar measures on K and $\bar{N}$, respectively, normalized as $\int_K dk=1$ and $\int_{\bar{N}}\exp\{-2\rho(H(\bar{n}))\}d\bar{n}=1$.

3. Infinitesimal operator of the principal series.

In this section, we shall introduce the formula of the infinitesimal operator of the principal series representation that was shown by Thieleker [**10**]. We shall reform Thieleker's formula for our convenience so that we can get the recursion formula of the Harish-Chandra C-function. In the following discussion, for all $\tau\in\hat{K}$ (resp. $\sigma\in\hat{M}$), we fix representatives of τ (resp. σ) and by abuse of notation, write τ (resp. σ) for it again.

We shall first review the compact picture of the principal series to explain the notation and the parametrization. Let $(\sigma,H_\sigma)\in\hat{M}$ and $\nu\in\mathfrak{a}_c^*$. We set

$$C_\sigma^\infty(K)=\{\varphi\in C^\infty(K;H_\sigma):\varphi(km)=\sigma(m)^{-1}\varphi(k)\}. \tag{3.1}$$

Let $\mathscr{H}^{\sigma,\nu}$ denote the Hilbert space completion of $C_\sigma^\infty(K)$ relative to the inner product $\langle f,g\rangle=\int_K\langle f(k),g(k)\rangle_{H_\sigma}dk$. Define the action $\pi_{\sigma,\nu}$ of G on $\mathscr{H}^{\sigma,\nu}$ by

$$(\pi_{\sigma,\nu}(g)\varphi)(k)=e^{-(\nu+\rho)(H(g^{-1}k))}\varphi(\kappa(g^{-1}k)),\quad(\varphi\in\mathscr{H}^{\sigma,\nu}). \tag{3.2}$$

Then $(\pi_{\sigma,\nu},\mathscr{H}^{\sigma,\nu})$ is a representation of G and is unitary for $\nu\in\sqrt{-1}\mathfrak{a}^*$. These are called (nonunitary) principal series representations of G.

For $\varphi\in\mathscr{H}^{\sigma,\nu}$, we set $\varphi_\nu(g)=e^{-(\nu+\rho)(H(g))}\varphi(\kappa(g))$. For $\tau\in\hat{K}(\sigma)$, $\mathscr{H}^{\sigma,\nu}(\tau)$ denotes the τ-isotopic component of $\mathscr{H}^{\sigma,\nu}$. Then Frobenius reciprocity implies the following lemma.

Lemma 3.1. *The correspondence* $T\otimes v\mapsto f_{T\otimes v}(k)=T(\tau(k)^{-1}v)$ *is a* K*-module isomorphism of* $\mathrm{Hom}_M(V_\tau,H_\sigma)\otimes V_\tau$ *onto* $\mathscr{H}^{\sigma,\nu}(\tau)$. *Here* V_τ *denotes the representation space of* τ.

Hereafter we denote by $\alpha\in\Sigma^+$ the unique simple root and choose $H\in\mathfrak{a}$ so that $\alpha(H)=1$. Take $\{X_{\alpha,j}:1\leq j\leq m_\alpha\}$ and $\{U_j:1\leq j\leq m\}$ ($m=\dim\mathfrak{m}$) to be orthonormal bases of $\mathfrak{g}_\alpha$ and $\mathfrak{m}$, respectively and put $Y_{\alpha,i}=2^{-1/2}(X_{\alpha,i}+\theta X_{\alpha,i})$ and $Z_{\alpha,i}=2^{-1/2}(X_{\alpha,i}-\theta X_{\alpha,i})$. We set $\omega_{j\alpha}=-\sum_{i=1}^{m_{j\alpha}}Y_{j\alpha,i}^2$ and $\omega_{\mathfrak{k}}=-\sum_{i=1}^m U_i^2-\sum_{j=1}^2\omega_{j\alpha}$. We shall first prove the following lemma.

Lemma 3.2 (cf. [**10**, Lemma 1]). *Let* $Z\in\mathfrak{p}_c$ *and* $\varphi\in C_\sigma^\infty(K)$. *Then we have*

$$\begin{aligned}(\pi_{\sigma,\nu}(Z)\varphi)(k)&=\frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}(\phi_Z\varphi)(k)+\frac{1}{2\langle\alpha,\alpha\rangle}[(\phi_Z\varphi)(k;\omega_{\mathfrak{k}})-\phi_Z(k)\varphi(k;\omega_{\mathfrak{k}})]\\&\quad-\frac{1}{4\langle\alpha,\alpha\rangle}[(\phi_Z\varphi)(k;\omega_{2\alpha})-\phi_Z(k)\varphi(k;\omega_{2\alpha})].\end{aligned}$$

Here $\phi_Z(k)=\langle\mathrm{Ad}(k)^{-1}Z,H\rangle/\langle H,H\rangle$.

Proof. We first note that

$$\mathrm{Ad}(k)^{-1}Z=\frac{\langle\mathrm{Ad}(k)^{-1}Z,H\rangle H}{\langle H,H\rangle}+\sum_{j=1}^2\sum_{i=1}^{m_{j\alpha}}\langle\mathrm{Ad}(k)^{-1}Z,Z_{j\alpha,i}\rangle Z_{j\alpha,i}. \tag{3.3}$$

It follows from the definition of φ_ν that

$$\varphi_\nu(k;H) = -(\nu+\rho)(H)\varphi_\nu(k) \quad \text{for } H \in \mathfrak{a},\ k \in K, \tag{3.4}$$

$$\varphi_\nu(k;X) = 0 \quad \text{for } X \in \mathfrak{n},\ k \in K. \tag{3.5}$$

Noting $Z_{j\alpha,i} = -Y_{j\alpha,i} + \sqrt{2}X_{j\alpha,i}$, we obtain

$$\varphi_\nu(k;Z_{j\alpha,i}) = -\varphi_\nu(k;Y_{j\alpha,i}) = -\varphi(k;Y_{j\alpha,i}). \tag{3.6}$$

Taking into account (3.3) and (3.6), we have

$$\begin{aligned}(\pi_{\sigma,\nu}(Z)\varphi)(k) &= \varphi_\nu(-Z;k) = -\varphi_\nu(k;\mathrm{Ad}(k)^{-1}Z)\\ &= \langle \nu+\rho,\alpha\rangle\langle \mathrm{Ad}(k)^{-1}Z,H\rangle\varphi(k)\\ &\quad + \sum_{j=1}^{2}\sum_{i=1}^{m_{j\alpha}}\langle \mathrm{Ad}(k)^{-1}Z,Z_{j\alpha,i}\rangle\varphi(k;Y_{j\alpha,i}).\end{aligned} \tag{3.7}$$

A simple calculation yields that

$$[H,Y_{j\alpha,i}] = jZ_{j\alpha,i}, \quad [Y_{j\alpha,i},Z_{j\alpha,i}] = j\langle\alpha,\alpha\rangle H. \tag{3.8}$$

From (3.8), we obtain

$$\phi_Z(k;Y_{j\alpha,i}) = \frac{\langle \mathrm{ad}(-Y_{j\alpha,i})\,\mathrm{Ad}(k)^{-1}Z,H\rangle}{\langle H,H\rangle} = \frac{-j\langle \mathrm{Ad}(k)^{-1}Z,Z_{j\alpha,i}\rangle}{\langle H,H\rangle}. \tag{3.9}$$

Therefore, substituting (3.9) into (3.7), we obtain

$$\begin{aligned}(\pi_{\sigma,\nu}(Z)\varphi)(k) &= \frac{\langle \nu+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle}(\phi_Z\varphi)(k) - \frac{1}{\langle\alpha,\alpha\rangle}\sum_{i=1}^{m_\alpha}\phi_Z(k;Y_{\alpha,i})\varphi(k;Y_{\alpha,i})\\ &\quad - \frac{1}{2\langle\alpha,\alpha\rangle}\sum_{i=1}^{m_{2\alpha}}\phi_Z(k;Y_{2\alpha,i})\varphi(k;Y_{2\alpha,i}).\end{aligned} \tag{3.10}$$

A simple calculation using (3.8) gives that

$$\phi_Z(k;U_i) = 0, \quad \phi_Z(k;Y_{j\alpha,i}^2) = -j^2\langle\alpha,\alpha\rangle\phi_Z(k), \tag{3.11}$$

and hence

$$\phi_Z(k;\omega_{j\alpha}) = j^2 m_{j\alpha}\langle\alpha,\alpha\rangle\phi_Z(k), \quad \phi_Z(k;\omega_{\mathfrak{k}}) = (m_\alpha + 4m_{2\alpha})\langle\alpha,\alpha\rangle\phi_Z(k). \tag{3.12}$$

By using Leibniz's formula, we have

$$\begin{aligned}(\phi_Z\varphi)(k;\omega_{j\alpha}) &= \phi_Z(k)\varphi(k;\omega_{j\alpha}) + \phi_Z(k;\omega_{j\alpha})\varphi(k) - 2\sum_{i=1}^{m_{j\alpha}}\phi_Z(k;Y_{j\alpha,i})\varphi(k;Y_{j\alpha,i})\\ &= \phi_Z(k)\varphi(k;\omega_{j\alpha}) + j^2 m_{j\alpha}\langle\alpha,\alpha\rangle\phi_Z(k)\varphi(k)\\ &\quad - 2\sum_{i=1}^{m_{j\alpha}}\phi_Z(k;Y_{j\alpha,i})\varphi(k;Y_{j\alpha,i}).\end{aligned} \tag{3.13}$$

Therefore

$$(3.14)\qquad -\sum_{i=1}^{m_{j\alpha}} \phi_Z(k;\, Y_{j\alpha,i})\varphi(k;\, Y_{j\alpha,i}) = \frac{1}{2}[(\phi_Z\varphi)(k;\omega_{j\alpha}) - \phi_Z(k)\varphi(k;\omega_{j\alpha}) - j^2 m_{j\alpha}\langle\alpha,\alpha\rangle(\phi_Z\varphi)(k)].$$

Substituting (3.14) into (3.10), we have

$$\begin{aligned}(3.15)\quad (\pi_{\sigma,\nu}(Z)\phi)(k) &= \frac{\langle\nu+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle}(\phi_Z\varphi)(k) \\ &\quad + \frac{1}{2\langle\alpha,\alpha\rangle}[(\phi_Z\varphi)(k;\omega_\alpha) - \phi_Z(k)\varphi(k;\omega_\alpha) - m_\alpha\langle\alpha,\alpha\rangle(\phi_Z\varphi)(k)] \\ &\quad + \frac{1}{4\langle\alpha,\alpha\rangle}[(\phi_Z\varphi)(k;\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_{2\alpha}) - 4m_{2\alpha}\langle\alpha,\alpha\rangle(\phi_Z\varphi)(k)] \\ &= \frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}(\phi_Z\varphi)(k) + \frac{1}{2\langle\alpha,\alpha\rangle}[(\phi_Z\varphi)(k;\omega_\alpha+\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_\alpha+\omega_{2\alpha})] \\ &\quad - \frac{1}{4\langle\alpha,\alpha\rangle}[(\phi_Z\varphi)(k;\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_{2\alpha})].\end{aligned}$$

Noting (3.11) and using Leibniz's formula, we obtain

$$(3.16)\qquad (\phi_Z\varphi)(k;\omega_\alpha+\omega_{2\alpha}) = (\phi_Z\varphi)(k;\omega_{\mathfrak{k}}) + \phi_Z(k)\varphi\left(k;\sum_{i=1}^{m} U_i^2\right),$$

and hence

$$(3.17)\qquad (\phi_Z\varphi)(k;\omega_\alpha+\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_\alpha+\omega_{2\alpha}) = (\phi_Z\varphi)(k;\omega_{\mathfrak{k}}) - \phi_Z(k)\varphi(k;\omega_{\mathfrak{k}}).$$

Substituting (3.17) into the last expression in (3.15), we get the assertion. □

4. Recursion formula for C-function.

We shall first summarize some known results on the relationship between the standard intertwining operator and the Harish-Chandra C-function. In [**6**], Knapp and Stein constructed the integral expression of the intertwining operator between the principal series representations, which is called the standard intertwining operator. Let $(\sigma, H_\sigma) \in \hat{M}$ and $(\tau, V_\tau) \in \hat{K}(\sigma)$. Let $\nu \in \mathfrak{a}_c^*$ be such that $\Re\langle\nu,\alpha\rangle > 0$. Then the standard intertwining operator is defined by

$$(4.1)\qquad (A(w,\sigma,\nu)\varphi)(k) = \int_{\bar{N}} e^{-(\nu+\rho)(H(\bar{n}))}\varphi(kw\kappa(\bar{n}))\,d\bar{n}, \quad (\varphi(k) \in C_\sigma^\infty(K)).$$

As indicated in [**6**], we know $A(w,\sigma,\nu)\varphi \in C_{w\sigma}^\infty(K)$ and

$$(4.2)\qquad A(w,\sigma,\nu)\pi_{\sigma,\nu}(g)\varphi = \pi_{w\sigma,w\nu}(g)A(w,\sigma,\nu)\varphi.$$

Let $T \otimes v \in \mathrm{Hom}_M(V_\tau, H_\sigma) \otimes V_\tau$. Then it follows from Wallach (cf. [13, p. 270]) that

$$(A(w,\sigma,v)f_{T\otimes v})(k) = T(C_\tau(v)\tau(w)^{-1}\tau(k)^{-1}v), \tag{4.3}$$

where

$$C_\tau(v) = \int_{\bar{N}} e^{-(v+\rho)(H(\bar{n}))}\tau(\kappa(\bar{n}))^{-1}d\bar{n}. \tag{4.4}$$

Looking upon $C_\tau(v)$ as a linear mapping of $\mathrm{Hom}_M(V_\tau, H_\sigma)$, we write $C_\tau(\sigma : v)$ for the determinant of $C_\tau(v)$. We call $C_\tau(\sigma : v)$ the Harish-Chandra C-function associated with τ and σ. Our main concern in this paper is the case that $\dim \mathrm{Hom}_M(V_\tau, H_\sigma) = 1$. It is known that if $G = Spin(n,1)$ or $G = SU(n,1)$ then this assumption holds for all $\tau \in \hat{K}$ and $\sigma \in \hat{M}(\tau)$. Under this assumption, because $TC_\tau(v) = C_\tau(\sigma : v)T$, (4.3) can be written as follows.

Lemma 4.1. *Retain the above assumption. We have*

$$A(w,\sigma,v)f_{T\otimes v} = C_\tau(\sigma : v)f_{R_\tau(w)(T\otimes v)}.$$

Here $R_\tau(w)(T\otimes v) = T\tau(w)^{-1}\otimes v$.

Remark. The function $\det C_\tau(v)$ was first introduced by Cohn [**1**]. Later, Vogan and Wallach [**12**] studied the function $C_\tau(\sigma : v)$ for reductive Lie groups of arbitrary rank. In their paper, they proved that $C_\tau(\sigma : v)$, as a function of v, has a meromorphic extension on $\mathfrak{a}_c^*$ and it can be written as quotients of products of classical Γ functions.

We suppose the unitary representation $(\mathrm{Ad}, \mathfrak{p}_c)$ of K has no multiple weights and give a recursion formula of the Harish-Chandra C-function. Let $\Delta_\mathfrak{p}$ denote the set of all weights of $(\mathrm{Ad}, \mathfrak{p}_c)$ relative to $\mathfrak{t}_c$. Under this assumption, the following lemma is valid.

Lemma 4.2 ([**5**, p. 111]). *Let* $\lambda \in D_K$. *Then*

$$\mathrm{Ad}\otimes\tau_\lambda = \sum_{\beta\in\Delta_\mathfrak{p}} m(\lambda+\beta)\tau_{\lambda+\beta},$$

where $m(\lambda+\beta) = 0$ *or* 1.

In the following, V_λ is an abbreviation of V_{τ_λ} and when there is no possibility of confusion, we shall use similar abbreviations. We write $E_{\lambda+\beta}$ for the canonical projection of $\mathfrak{p}_c \otimes V_\lambda$ into $V_{\lambda+\beta}$ given by the decomposition in Lemma 4.2. Let $\lambda \in D_K$ and $\mu \in D_M(\lambda)$. For $T \in \mathrm{Hom}_M(V_\lambda, H_\mu)$, define $\tilde{T} \in \mathrm{Hom}_M(\mathfrak{p}_c \otimes V_\lambda, H_\mu)$ by

$$\tilde{T}(Z\otimes v) = \frac{\langle Z, H\rangle}{\langle H, H\rangle}T(v). \tag{4.5}$$

Define the linear mapping

$$\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)\colon \mathrm{Hom}_M(V_\lambda,H_\mu)\otimes V_\lambda \to \mathrm{Hom}_M(V_{\lambda+\beta},H_\mu)\otimes V_{\lambda+\beta} \tag{4.6}$$

by

$$\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v) = \tilde{T}E_{\lambda+\beta}^*\otimes E_{\lambda+\beta}(Z\otimes v). \tag{4.7}$$

We first prove the following lemma.

Lemma 4.3. *Retain the above notation and assumption. We have*

$$\phi_Z f_{T\otimes v} = \sum_{\beta\in\Delta_{\mathfrak{p}}} m(\lambda+\beta) f_{\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}.$$

Proof. We compute

$$\begin{aligned}(\phi_Z f_{T\otimes v})(k) &= \frac{\langle \mathrm{Ad}(k)^{-1}Z, H\rangle}{\langle H,H\rangle} T(\tau_\lambda(k)^{-1}v) \\ &= \tilde{T}((\mathrm{Ad}\otimes\tau_\lambda)(k)^{-1}(Z\otimes v)) \\ &= \tilde{T}\left((\mathrm{Ad}\otimes\tau_\lambda)(k)^{-1}\sum_{\beta\in\Delta_{\mathfrak{p}}} E^*_{\lambda+\beta}E_{\lambda+\beta}(Z\otimes v)\right) \\ &= \sum_{\beta\in\Delta_{\mathfrak{p}}} m(\lambda+\beta)\tilde{T}(E^*_{\lambda+\beta}\tau_{\lambda+\beta}(k)^{-1}E_{\lambda+\beta}(Z\otimes v)) \\ &= \sum_{\beta\in\Delta_{\mathfrak{p}}} m(\lambda+\beta) f_{\tilde{T}E^*_{\lambda+\beta}\otimes E_{\lambda+\beta}(Z\otimes v)}(k).\end{aligned}$$

Therefore the assertion holds. □

The next lemma is an easy consequence of Lemma 4.3.

Lemma 4.4. *If* $m(\lambda+\beta)\neq 0$ *then*

$$R_{\lambda+\beta}(w)\mathscr{M}_\mu(Z;\lambda+\beta,\lambda) = -\mathscr{M}_{w\mu}(Z;\lambda+\beta,\lambda)R_\lambda(w).$$

Proof. We compute

$$\begin{aligned}(R(w)(\phi_Z f_{T\otimes v}))(k) &= \frac{\langle \mathrm{Ad}(kw)^{-1}Z, H\rangle}{\langle H,H\rangle} T(\tau_\lambda(kw)^{-1}v) \\ &= \frac{\langle \mathrm{Ad}(k)^{-1}Z, \mathrm{Ad}(w)H\rangle}{\langle H,H\rangle} T\tau_\lambda(w)^{-1}(\tau_\lambda(k)^{-1}v) \\ &= -(\phi_Z f_{R_\lambda(w)(T\otimes v)})(k).\end{aligned} \tag{4.8}$$

Noting $f_{\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}\in\mathscr{H}^{\sigma_\mu,\nu}(\tau_{\lambda+\beta})$ and $f_{R_\lambda(w)(T\otimes v)}\in\mathscr{H}^{w\sigma_\mu,w\nu}(\tau_\lambda)$, we see that

$$\begin{aligned}R(w)(\phi_Z f_{T\otimes v}) &= \sum_{\beta\in\Delta_{\mathfrak{p}}} m(\lambda+\beta) f_{R_{\lambda+\beta}(w)\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}, \\ \phi_Z f_{R_\lambda(w)(T\otimes v)} &= \sum_{\beta\in\Delta_{\mathfrak{p}}} m(\lambda+\beta) f_{\mathscr{M}_{w\mu}(Z;\lambda+\beta,\lambda)R_\lambda(w)(T\otimes v)}.\end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8) and comparing side by side, we obtain the assertion. □

Combining Lemma 3.2 with Lemma 4.3, we have the following proposition.

Proposition 4.5. *Let* $\mu\in D_M$ *and* $\lambda\in D_K(\mu)$. *Then there exists* $\eta^\mu_\lambda(\omega_{2\alpha})\in\boldsymbol{C}$ *such that*

$$\pi_{\sigma_\mu,\nu}(Z)f_{T\otimes v} = \sum_{\beta\in\Delta_\mathfrak{p}}\left\{\frac{\langle \nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}+\frac{\langle 2\lambda+2\delta_K+\beta,\beta\rangle}{2\langle\alpha,\alpha\rangle}-\frac{\eta^\mu_{\lambda+\beta}(\omega_{2\alpha})-\eta^\mu_\lambda(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle}\right\}$$

$$\times\, m(\lambda+\beta)f_{\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)},$$

$$\pi_{w\sigma_\mu,w\nu}(Z)f_{R_\lambda(w)(T\otimes v)} = \sum_{\beta\in\Delta_\mathfrak{p}}\left\{-\frac{\langle \nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}+\frac{\langle 2\lambda+2\delta_K+\beta,\beta\rangle}{2\langle\alpha,\alpha\rangle}-\frac{\eta^\mu_{\lambda+\beta}(\omega_{2\alpha})-\eta^\mu_\lambda(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle}\right\}$$

$$\times\, m(\lambda+\beta)f_{\mathscr{M}_{w\mu}(Z;\lambda+\beta,\lambda)R_\lambda(w)(T\otimes v)}.$$

Proof. We obtain from Lemma 4.3 that

$$\begin{aligned}(\phi_Z f_{T\otimes v})(k;\omega_\mathfrak{k}) &= \sum_{\beta\in\Delta_\mathfrak{p}} m(\lambda+\beta)f_{\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k;\omega_\mathfrak{k})\\ &= \sum_{\beta\in\Delta_\mathfrak{p}}(\langle\lambda+\beta+\delta_K,\lambda+\beta+\delta_K\rangle-\langle\delta_K,\delta_K\rangle)m(\lambda+\beta)f_{\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k),\end{aligned}$$

$$\begin{aligned}\phi_Z(k)f_{T\otimes v}(k;\omega_\mathfrak{k}) &= (\langle\lambda+\delta_K,\lambda+\delta_K\rangle-\langle\delta_K,\delta_K\rangle)(\phi_Z f_{T\otimes v})(k)\\ &= \sum_{\beta\in\Delta_\mathfrak{p}}(\langle\lambda+\delta_K,\lambda+\delta_K\rangle-\langle\delta_K,\delta_K\rangle)m(\lambda+\beta)f_{\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k).\end{aligned}$$

Hence

$$\begin{aligned}&(\phi_Z f_{T\otimes v})(k;\omega_\mathfrak{k})-\phi_Z(k)f_{T\otimes v}(k;\omega_\mathfrak{k})\\ &\quad= \sum_{\beta\in\Delta_\mathfrak{p}}\langle 2\lambda+2\delta_K+\beta,\beta\rangle m(\lambda+\beta)f_{\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k).\end{aligned}$$

On the other hand, under the assumption that $\dim \mathrm{Hom}_M(V_\lambda,H_\mu)=1$, there exists $\eta^\mu_\lambda(\omega_{2\alpha})\in \boldsymbol{C}$ such that

$$T\tau_\lambda(\omega_{2\alpha}) = \eta^\mu_\lambda(\omega_{2\alpha})T,$$

and thus

$$f_{T\otimes v}(k;\omega_{2\alpha}) = T\tau_\lambda(\omega_{2\alpha})(\tau_\lambda(k)^{-1}v) = \eta^\mu_\lambda(\omega_{2\alpha})f_{T\otimes v}(k).$$

Similarly we have

$$f_{\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k;\omega_{2\alpha}) = \eta^\mu_{\lambda+\beta}(\omega_{2\alpha})f_{\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k).$$

Consequently we obtain

$$\begin{aligned}&(\phi_Z f_{T\otimes v})(k;\omega_{2\alpha})-\phi_Z(k)f_{T\otimes v}(k;\omega_{2\alpha})\\ &\quad= \sum_{\beta\in\Delta_\mathfrak{p}}(\eta^\mu_{\lambda+\beta}(\omega_{2\alpha})-\eta^\mu_\lambda(\omega_{2\alpha}))m(\lambda+\beta)f_{\mathscr{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k).\end{aligned}$$

Noting

$$\begin{aligned} f_{T\tau_\lambda(w)^{-1}\otimes v}(k;\omega_{2\alpha}) &= T\tau_\lambda(w)^{-1}\tau_\lambda(\omega_{2\alpha})(\tau_\lambda(k)^{-1}v) \\ &= T\tau_\lambda(\omega_{2\alpha})\tau_\lambda(w)^{-1}(\tau_\lambda(k)^{-1}v) \\ &= \eta_\lambda^\mu(\omega_{2\alpha}) f_{T\tau_\lambda(w)^{-1}\otimes v}(k), \end{aligned}$$

we can get immediately the second equation in Proposition 4.5. □

Combining Lemma 4.1 with Proposition 4.5, we can get the recursion formula of Harish-Chandra C-function.

THEOREM 4.6. *Let* $\mu \in D_M$, $\nu \in \mathfrak{a}_c^*$ *and* $\lambda \in D_K(\mu)$. *If* $m(\lambda+\beta) \neq 0$ *then*

$$\left\{-\frac{\langle \nu,\alpha\rangle}{\langle \alpha,\alpha\rangle} + \frac{\langle 2\lambda+2\delta_K+\beta,\beta\rangle}{2\langle\alpha,\alpha\rangle} - \frac{\eta_{\lambda+\beta}^\mu(\omega_{2\alpha}) - \eta_\lambda^\mu(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle}\right\} C_{\tau_\lambda}(\sigma_\mu : \nu)$$

$$= -\left\{\frac{\langle \nu,\alpha\rangle}{\langle \alpha,\alpha\rangle} + \frac{\langle 2\lambda+2\delta_K+\beta,\beta\rangle}{2\langle\alpha,\alpha\rangle} - \frac{\eta_{\lambda+\beta}^\mu(\omega_{2\alpha}) - \eta_\lambda^\mu(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle}\right\} C_{\tau_{\lambda+\beta}}(\sigma_\mu : \nu).$$

PROOF. We first recall that

$$A(w,\sigma_\mu,\nu)\pi_{\sigma_\mu,\nu}(Z) f_{T\otimes v} = \pi_{w\sigma_\mu,w\nu}(Z) A(w,\sigma_\mu,\nu) f_{T\otimes v}. \tag{4.10}$$

Combining Proposition 4.5 with (4.10), we have

the right-hand side of (4.10)

$$\begin{aligned} &= C_{\tau_\lambda}(\sigma_\mu : \nu)(\pi_{w\sigma_\mu,w\nu}(Z) f_{R_\lambda(w)(T\otimes v)}) \\ &= C_{\tau_\lambda}(\sigma_\mu : \nu)\sum_{\beta\in\Delta_\mathfrak{p}}\left\{-\frac{\langle \nu,\alpha\rangle}{\langle \alpha,\alpha\rangle} + \frac{\langle 2\lambda+2\delta_K+\beta,\beta\rangle}{2\langle\alpha,\alpha\rangle} - \frac{\eta_{\lambda+\beta}^\mu(\omega_{2\alpha}) - \eta_\lambda^\mu(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle}\right\} \\ &\quad \times m(\lambda+\beta) f_{\mathcal{M}_{w\mu}(Z;\lambda+\beta,\lambda)R_\lambda(w)(T\otimes v)}. \end{aligned}$$

Similarly taking into account Lemma 4.4, we have

the left-hand side of (4.10)

$$\begin{aligned} &= A(w,\sigma_\mu,\nu)\Bigg[\sum_{\beta\in\Delta_\mathfrak{p}}\left\{\frac{\langle \nu,\alpha\rangle}{\langle \alpha,\alpha\rangle} + \frac{\langle 2\lambda+2\delta_K+\beta,\beta\rangle}{2\langle\alpha,\alpha\rangle} - \frac{\eta_{\lambda+\beta}^\mu(\omega_{2\alpha}) - \eta_\lambda^\mu(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle}\right\} \\ &\quad \times m(\lambda+\beta) f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}\Bigg] \\ &= \sum_{\beta\in\Delta_\mathfrak{p}}\left\{\frac{\langle \nu,\alpha\rangle}{\langle \alpha,\alpha\rangle} + \frac{\langle 2\lambda+2\delta_K+\beta,\beta\rangle}{2\langle\alpha,\alpha\rangle} - \frac{\eta_{\lambda+\beta}^\mu(\omega_{2\alpha}) - \eta_\lambda^\mu(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle}\right\} \\ &\quad \times C_{\tau_{\lambda+\beta}}(\sigma_\mu : \nu) m(\lambda+\beta) f_{R_{\lambda+\beta}(w)\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)} \end{aligned}$$

$$= \sum_{\beta \in \Delta_{\mathfrak{p}}} \left[-\left\{ \frac{\langle \nu, \alpha \rangle}{\langle \alpha, \alpha \rangle} + \frac{\langle 2\lambda + 2\delta_K + \beta, \beta \rangle}{2\langle \alpha, \alpha \rangle} - \frac{\eta^{\mu}_{\lambda+\beta}(\omega_{2\alpha}) - \eta^{\mu}_{\lambda}(\omega_{2\alpha})}{4\langle \alpha, \alpha \rangle} \right\} \right]$$

$$\times\, C_{\tau_{\lambda+\beta}}(\sigma_\mu : \nu) m(\lambda+\beta) f_{\mathscr{M}_{w\mu}(Z;\lambda+\beta,\lambda) R_\lambda(w)(T\otimes v)}.$$

Therefore we obtain the desired formula. □

5. Representations of K and M.

In the remainder of this paper, we shall confine our attention to the cases of $Spin(n,1)$ and $SU(n,1)$. As is well-known, in these cases, the irreducible unitary representations of K and M are realized in terms of the Gel'fand–Tsetlin basis of $\mathfrak{u}(n)$ and $\mathfrak{o}(n)$. Later these realizations are used for getting the matrix element of the Harish-Chandra C-function relative to the highest weight vector. We shall borrow the notation from Knapp's book [**5**, pp. 60–64] and Vilenkin–Klimyk's book [**11**, pp. 361–365].

Let $G = SO_0(n,1)$, $(n \geq 3)$ or $G = SU(n,1)$, $(n \geq 2)$. We set $\boldsymbol{F} = \boldsymbol{R}$ if $G = SO_0(n,1)$ or $\boldsymbol{F} = \boldsymbol{C}$ if $G = SU(n,1)$ and $\boldsymbol{F}_I = \{x \in \boldsymbol{F} : x + \bar{x} = 0\}$. Then the Iwasawa decomposition of G is given as follows:

$$(5.1)\qquad K = \begin{cases} \left\{ \begin{pmatrix} X & \\ & 1 \end{pmatrix} : X \in SO(n) \right\}, & \text{for } G = SO_0(n,1), \\ \left\{ \begin{pmatrix} X & \\ & u \end{pmatrix} : X \in U(n), u \in U(1), u \det X = 1 \right\}, & \text{for } G = SU(n,1). \end{cases}$$

Let $H = E_{n,n+1} + E_{n+1,n} \in \mathfrak{p}$ and $\mathfrak{a} = \boldsymbol{R}H$, where $E_{p,q}$ denotes the matrix unit whose (k,l)-component is equal to $\delta_{p,k}\delta_{q,l}$.

$$(5.2)\qquad A = \left\{ a_t = \begin{pmatrix} I_{n-1} & & \\ & \cosh t & \sinh t \\ & \sinh t & \cosh t \end{pmatrix} : t \in \boldsymbol{R} \right\},$$

$$(5.3)\qquad N = \left\{ n(z,u) = \begin{pmatrix} I_{n-1} & z & -z \\ -z^* & 1-\omega/2 & \omega/2 \\ -z^* & -\omega/2 & 1+\omega/2 \end{pmatrix} : z \in \boldsymbol{F}^{n-1}, u \in \boldsymbol{F}_I, \omega = |z|^2 - 2u \right\},$$

$$(5.4)\qquad \bar{N} = \left\{ \bar{n}(z,u) = \begin{pmatrix} I_{n-1} & -z & -z \\ z^* & 1-\omega/2 & -\omega/2 \\ -z^* & \omega/2 & 1+\omega/2 \end{pmatrix} : z \in \boldsymbol{F}^{n-1}, u \in \boldsymbol{F}_I, \omega = |z|^2 - 2u \right\}.$$

It is easy to prove the following lemma and hence we omit its proof.

Lemma 5.1. *Let $\bar{n}(z,u)$ be as above. Then*

$$H(\bar{n}(z,u)) = \log|1+\omega| H,$$

$$\kappa(\bar{n}(z,u)) = \begin{pmatrix} I_{n-1} - \dfrac{2zz^*}{1+\omega} & \dfrac{-2z}{|1+\omega|} & 0 \\ \dfrac{2z^*}{1+\omega} & \dfrac{1-\omega}{|1+\omega|} & 0 \\ 0 & 0 & \dfrac{1+\omega}{|1+\omega|} \end{pmatrix}.$$

If $\boldsymbol{E} \subseteq \boldsymbol{R}$ and $p \in \boldsymbol{Z}_{>0}$, we denote by $\boldsymbol{E}^p_{\gg}$ the subset of $\boldsymbol{E}^p_{>}$ comprised of all $x = (x_1, \dots, x_p)$ such that $x_{p-1} \geq |x_p|$. If $x \in \boldsymbol{E}^{p+1}_{\gg}$ and $y \in \boldsymbol{E}^p_{>}$, we write $x \gg y$ for $x > y$ and $y_p > |x_{p+1}|$. If $x \in \boldsymbol{E}^p_{>}$ and $y \in \boldsymbol{E}^p_{\gg}$, we write $x \ggg y$ for $x > y_{\geq 2}$ and $y \gg x_{\geq 2}$. We first compute the second term $\langle 2\lambda + 2\delta_K + \beta, \beta \rangle / 2\langle \alpha, \alpha \rangle$ appeared in Theorem 4.6.

(1) $Spin(2n+1,1)$**-case.** Let $H_j = E_{2j-1,2j} - E_{2j,2j-1}$ for $1 \leq j \leq n$ and let $\{\varepsilon_j\}$ be the dual basis of $\mathfrak{t}_c^*$ relative to $\{H_j\}$. Then we have

$$\Delta_K^+ = \{\varepsilon_i \pm \varepsilon_j, (1 \leq i < j \leq n), \varepsilon_i, (1 \leq i \leq n)\}, \tag{5.5}$$

$$\Delta_{\mathfrak{p}} = \{\pm \varepsilon_i, (1 \leq i \leq n), 0\}, \tag{5.6}$$

$$D_K = \left(\frac{1}{2}\boldsymbol{Z}_{\geq 0}\right)^n_{>}, \quad D_M = \left(\frac{1}{2}\boldsymbol{Z}\right)^n_{\gg}. \tag{5.7}$$

It follows from [**10**] that $[\tau_\lambda : \sigma_\mu] = 1$ if and only if $\lambda \ggg \mu$. From these, we obtain

$$\frac{\langle 2\lambda + 2\delta_K + \varepsilon_j, \varepsilon_j \rangle}{2\langle \alpha, \alpha \rangle} = \lambda_j + n - j + 1, \quad (1 \leq j \leq n). \tag{5.8}$$

(2) $Spin(2n+2,1)$**-case.** Let H_j and ε_j be the same as in (1) for $1 \leq j \leq n+1$. Then we have

$$\Delta_K^+ = \{\varepsilon_i \pm \varepsilon_j, (1 \leq i < j \leq n+1)\}, \tag{5.9}$$

$$\Delta_{\mathfrak{p}} = \{\pm \varepsilon_i, (1 \leq i \leq n+1)\}, \tag{5.10}$$

$$D_K = \left(\frac{1}{2}\boldsymbol{Z}\right)^{n+1}_{\gg}, \quad D_M = \left(\frac{1}{2}\boldsymbol{Z}_{\geq 0}\right)^n_{>}. \tag{5.11}$$

It follows from [**10**] that $[\tau_\lambda : \sigma_\mu] = 1$ if and only if $\lambda \gg \mu$. From these, we obtain

$$\frac{\langle 2\lambda + 2\delta_K + \varepsilon_j, \varepsilon_j \rangle}{2\langle \alpha, \alpha \rangle} = \lambda_j + n - j + \frac{3}{2}, \quad (1 \leq j \leq n+1). \tag{5.12}$$

(3) $SU(n,1)$**-case.** Let $H_j = \sqrt{-1}E_{j,j}$ for $1 \leq j \leq n+1$ and let $\{\varepsilon_j\}$ be the dual basis of $\mathfrak{t}_c^*$ relative to $\{H_j\}$. Then we have

$$\Delta_K^+ = \{\varepsilon_i - \varepsilon_j, (1 \leq i < j \leq n)\}, \tag{5.13}$$

$$\Delta_{\mathfrak{p}} = \{\beta_j = \varepsilon_j - \varepsilon_{n+1}, -\beta_j, (1 \leq j \leq n)\}, \tag{5.14}$$

$$D_K = \left(\frac{1}{n+1}\boldsymbol{Z}\right)^n_{>}, \quad D_M = \left(\frac{1}{n+1}\boldsymbol{Z}\right)^{n-1}_{>}. \tag{5.15}$$

It follows from [7] that $[\tau_\lambda : \sigma_\mu] = 1$ if and only if $\lambda > \mu$. From these, we obtain

$$\frac{\langle 2\lambda + 2\delta_K + \beta_j, \beta_j \rangle}{2\langle \alpha, \alpha \rangle} = 2\lambda_j + 2|\tau_\lambda| + n - 2j + 3, \quad (1 \leq j \leq n). \tag{5.16}$$

Here $|\tau_\lambda| = |\lambda| = \sum_{p=1}^n \lambda_p$. In both $Spin(n,1)$ and $SU(n,1)$ cases, for $\lambda \in D_K$ and $\mu \in D_M$, we shall write $|\tau_\lambda|$ and $|\sigma_\mu|$ instead of $|\lambda|$ and $|\mu|$, respectively.

We remark that if $G = Spin(n,1)$ then $\eta_\lambda^\mu(\omega_{2\alpha}) = 0$. In the case of $SU(n,1)$, for computing $\eta_\lambda^\mu(\omega_{2\alpha})$, we need to construct the irreducible unitary representation $(\tau_\lambda, V_\lambda)$

and write down the action of $\tau_\lambda(\omega_{2\alpha})$. As is well-known, these are realized in terms of the Gel'fand–Tsetlin basis.

(1) $K = Spin(n)$**-case.** Let $\boldsymbol{M} = (m_n, m_{n-1}, \dots, m_3, m_2)$ be a sequence such that

$$m_{2p+1} = (m_{1,2p+1}, \dots, m_{p,2p+1}) \in \left(\frac{1}{2}\boldsymbol{Z}_{\geq 0}\right)^p_{>}, \tag{5.17}$$
$$m_{2p} = (m_{1,2p}, \dots, m_{p,2p}) \in \left(\frac{1}{2}\boldsymbol{Z}\right)^p_{\gg}.$$

The preceding sequence $\boldsymbol{M}$ is called a Gel'fand–Tsetlin data if $m_{2p+1} \ggg m_{2p}$ and $m_{2p+2} \gg m_{2p+1}$. For the Gel'fand-Tsetlin data $\boldsymbol{M}$, we write $v(\boldsymbol{M})$ for the corresponding Gel'fand–Tsetlin basis. For $\lambda \in D_K$, we denote by V_λ the Hilbert space generated by the orthonormal basis $v(\boldsymbol{M})$ with $m_n = \lambda$. We put $I_{p,q} = E_{p,q} - E_{q,p}, (p < q)$. Then there exists an irreducible unitary representation $(\tau_\lambda, V_\lambda)$ of K satisfying the following conditions:

$$\tau_\lambda(I_{2p,2p+1})v(\boldsymbol{M}) = \sum_{j=1}^{p} A^j_{2p}(\boldsymbol{M})v(\boldsymbol{M}^{+j}_{2p}) - \sum_{j=1}^{p} A^j_{2p}(\boldsymbol{M}^{-j}_{2p})v(\boldsymbol{M}^{-j}_{2p}), \tag{5.18}$$

$$\tau_\lambda(I_{2p+1,2p+2})v(\boldsymbol{M}) = \sum_{j=1}^{p} B^j_{2p+1}(\boldsymbol{M})v(\boldsymbol{M}^{+j}_{2p+1}) - \sum_{j=1}^{p} B^j_{2p+1}(\boldsymbol{M}^{-j}_{2p+1})v(\boldsymbol{M}^{-j}_{2p+1}) + \sqrt{-1}C_{2p}(\boldsymbol{M})v(\boldsymbol{M}), \tag{5.19}$$

where $\boldsymbol{M}^{\pm j}_{2p}$ is the Gel'fand-Tsetlin data obtained by replacing $m_{j,2p}$ with $m_{j,2p} \pm 1$ in m_{2p} of $\boldsymbol{M}$. For the explicit forms of $A^j_{2p}(\boldsymbol{M})$, $B^j_{2p+1}(\boldsymbol{M})$ and $C_{2p}(\boldsymbol{M})$, see [**11**, p. 364]. For $\lambda \in D_K$, define the Gel'fand-Tsetlin data $\boldsymbol{M}_\lambda = (m_n, \dots, m_1)$ by $m_{2p} = m_{2p+1} = \lambda_{\leq 2p}$. Then from (5.18), $v(\boldsymbol{M}_\lambda)$ is a highest weight vector of τ_λ.

(2) $K = S(U(n) \times U(1))$**-case.** Let $\boldsymbol{M} = (m_n, \dots, m_1)$ be a sequence such that

$$m_p = (m_{1,p}, \dots, m_{p,p}) \in \left(\frac{1}{n+1}\boldsymbol{Z}\right)^p_{>}. \tag{5.20}$$

Then preceding sequence $\boldsymbol{M}$ is called a Gel'fand-Tsetlin data if $m_{p+1} > m_p$. For the Gel'fand-Tsetlin data $\boldsymbol{M}$, we write $v(\boldsymbol{M})$ for the corresponding Gel'fand-Tsetlin basis. For $\lambda \in D_K$, we denote by V_λ the Hilbert space generated by the orthonormal basis $v(\boldsymbol{M})$ with $m_n = \lambda$. We put $X_p = E_{p,p+1}$, $Y_p = E_{p+1,p}$, $H_p = \sqrt{-1}(E_{p,p} - E_{p+1,p+1})$ and $H_0 = \sqrt{-1}\,\mathrm{diag}(-1, \dots, -1, n)$. Then there exists an irreducible unitary representation $(\tau_\lambda, V_\lambda)$ of K satisfying the following conditions:

$$\tau_\lambda(X_p)v(\boldsymbol{M}) = \sum_{j=1}^{p} A^j_p(\boldsymbol{M})v(\boldsymbol{M}^{+j}_p), \tag{5.21}$$

$$\tau_\lambda(Y_p)v(\boldsymbol{M}) = \sum_{j=1}^{p} B^j_p(\boldsymbol{M})v(\boldsymbol{M}^{-j}_p), \tag{5.22}$$

(5.23) $$\tau_\lambda(H_p)v(\boldsymbol{M}) = \left\{2\sum_{j=1}^{p} m_{j,p} - \sum_{j=1}^{p-1} m_{j,p-1} - \sum_{j=1}^{p+1} m_{j,p+1}\right\}\sqrt{-1}v(\boldsymbol{M}),$$

(5.24) $$\tau_\lambda(H_0)v(\boldsymbol{M}) = -(n+1)\sum_{j=1}^{n} m_{j,n}\sqrt{-1}v(\boldsymbol{M}),$$

where $\boldsymbol{M}_p^{\pm j}$ is the Gel'fand-Tsetlin data obtained by replacing $m_{j,p}$ with $m_{j,p} \pm 1$ in m_p of $\boldsymbol{M}$. For the explicit forms of $A_p^j(\boldsymbol{M})$, $B_p^j(\boldsymbol{M})$, see [**11**, p. 363]. For $\lambda \in D_K$, let $\boldsymbol{M}_\lambda = (\lambda, \lambda_{\leq n-1}, \dots, \lambda_{\leq 1})$. Then from (5.23) and (5.24), $v(\boldsymbol{M}_\lambda)$ is a highest weight vector of τ_λ.

We shall now compute $\eta_\lambda^\mu(\omega_{2\alpha})$. For $\lambda \in D_K$ and $\mu \in D_M(\lambda)$, define the Gel'fand-Tsetlin data $\boldsymbol{M}_{\lambda,\mu}$ by $\boldsymbol{M}_{\lambda,\mu} = (\lambda, \mu, \mu_{\leq n-1}, \dots, \mu_{\leq 1})$. Then it is obvious that $v(\boldsymbol{M}_{\lambda,\mu}) \in V_\lambda(\mu)$ and $v(\boldsymbol{M}_{\lambda,\mu})$ is a highest weight vector of the irreducible unitary representation $(\tau_\lambda|_M, V_\lambda(\mu))$ of M. Because

(5.25) $$\begin{aligned} Y_{2\alpha} &= \frac{1}{2\sqrt{n+1}}\,\mathrm{diag}(\overbrace{0,\dots,0}^{n-1}, i, -i) \\ &= \frac{-1}{2n\sqrt{n+1}}\{H_0 + H_1 + 2H_2 + \cdots + (n-1)H_{n-1}\}, \end{aligned}$$

we have from (5.23) and (5.24) that

(5.26) $$\tau_\lambda(Y_{2\alpha})v(\boldsymbol{M}_{\lambda,\mu}) = \frac{1}{2\sqrt{n+1}}(2|\tau_\lambda| - |\sigma_\mu|)\sqrt{-1}v(\boldsymbol{M}_{\lambda,\mu}).$$

Since $\omega_{2\alpha} = -Y_{2\alpha}^2$, it follows

(5.27) $$\tau_\lambda(\omega_{2\alpha})v(\boldsymbol{M}_{\lambda,\mu}) = \frac{1}{4(n+1)}(2|\tau_\lambda| - |\sigma_\mu|)^2 v(\boldsymbol{M}_{\lambda,\mu}).$$

Because $T\tau_\lambda(\omega_{2\alpha}) = \eta_\lambda^\mu(\omega_{2\alpha})T$ for $T \in \mathrm{Hom}_M(V_\lambda, H_\mu)$, we have

(5.28) $$T\tau_\lambda(\omega_{2\alpha})v(\boldsymbol{M}_{\lambda,\mu}) = \eta_\lambda^\mu(\omega_{2\alpha})Tv(\boldsymbol{M}_{\lambda,\mu}).$$

Therefore, it follows from $Tv(\boldsymbol{M}_{\lambda,\mu}) \neq 0$ that

(5.29) $$\eta_\lambda^\mu(\omega_{2\alpha}) = \frac{1}{4(n+1)}(2|\tau_\lambda| - |\sigma_\mu|)^2.$$

Taking into account $|\tau_{\lambda+\beta_j}| = |\tau_\lambda| + 1$, we obtain

(5.30) $$\frac{\eta_{\lambda+\beta_j}^\mu(\omega_{2\alpha}) - \eta_\lambda^\mu(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle} = 2|\tau_\lambda| - |\sigma_\mu| + 1.$$

6. Explicit expression of the recursion formula.

In this section we shall write down the recursion formula of the Harish-Chandra C-function in the cases of $Spin(n,1)$ and $SU(n,1)$. In these cases, because all noncompact roots have same length, we see that for $\lambda \in D_K$ and $\mu \in D_K(\lambda)$, $m(\lambda+\beta) = 1$ if and only if $[\tau_{\lambda+\beta} : \sigma_\mu] = 1$. In the remainder of this paper, we simply write v instead of $\langle v, \alpha\rangle / \langle \alpha, \alpha\rangle$.

(1) ***Spin*$(2n+1, 1)$-case.** Let $\lambda = (\lambda_1, \dots, \lambda_n) \in D_K$ and $\mu = (\mu_1, \dots, \mu_n) \in D_M(\lambda)$ and set $\lambda(\mu) = (\mu_1, \dots, \mu_{n-1}, |\mu_n|) \in D_K(\mu)$. We obtain from (5.8) and Theorem 4.6 that if $[\tau_{\lambda+\varepsilon_j} : \sigma_\mu] = 1$ then

$$(6.1) \qquad (-v + \lambda_j + n - j + 1) C_{\tau_\lambda}(\sigma_\mu : v) = -(v + \lambda_j + n - j + 1) C_{\tau_{\lambda+\varepsilon_j}}(\sigma_\mu : v).$$

Applying the preceding recursion formula and shifting the parameters as $\mu_j \mapsto \lambda_j$ $(1 \le j \le n-1)$ and $|\mu_n| \mapsto \lambda_n$, we can find

$$(6.2) \qquad C_{\tau_\lambda}(\sigma_\mu : v) = (-1)^{|\tau_\lambda| - |\tau_{\lambda(\mu)}|} \times \prod_{j=1}^{n-1} \frac{(-v + \mu_j + n - j + 1)_{\lambda_j - \mu_j}}{(v + \mu_j + n - j + 1)_{\lambda_j - \mu_j}} \frac{(-v + |\mu_n| + 1)_{\lambda_n - |\mu_n|}}{(v + |\mu_n| + 1)_{\lambda_n - |\mu_n|}} C_{\tau_{\lambda(\mu)}}(\sigma_\mu : v).$$

Here $(a)_n = \Gamma(a+n)/\Gamma(a)$.

By using the recursion formula (6.2), for getting the expression of $C_\tau(\sigma : v)$, it suffices to compute $C_{\tau_{\lambda(\mu)}}(\sigma_\mu : v)$. We remark that the Gel'fand-Tsetlin basis $v(\boldsymbol{M}_{\lambda(\mu)})$ is a highest weight vector of $(\tau_{\lambda(\mu)}|_M, V_{\lambda(\mu)}(\mu))$ and if $\mu_n \ge 0$ then $v(\boldsymbol{M}_{\lambda(\mu)})$ is a highest weight vector of $(\tau_{\lambda(\mu)}, V_{\lambda(\mu)})$. We first suppose $\mu_n \ge 0$. Take $T \in \mathrm{Hom}_M(V_{\lambda(\mu)}, V_{\lambda(\mu)}(\mu))$ to be a canonical projection. Noting $Tv(\boldsymbol{M}_{\lambda(\mu)}) = v(\boldsymbol{M}_{\lambda(\mu)}) \neq 0$ and $TC_{\tau_{\lambda(\mu)}}(v) = C_{\tau_{\lambda(\mu)}}(\sigma_\mu : v)T$, we have

$$\begin{aligned}(6.3) \qquad C_{\tau_{\lambda(\mu)}}(\sigma_\mu : v) &= \langle TC_{\tau_{\lambda(\mu)}}(v) v(\boldsymbol{M}_{\lambda(\mu)}), v(\boldsymbol{M}_{\lambda(\mu)})\rangle \\ &= \langle C_{\tau_{\lambda(\mu)}}(v) v(\boldsymbol{M}_{\lambda(\mu)}), v(\boldsymbol{M}_{\lambda(\mu)})\rangle \\ &= \int_{\bar{N}} e^{-(v+\rho)(H(\bar{n}))} \langle \tau_{\lambda(\mu)}(\kappa(\bar{n})^{-1}) v(\boldsymbol{M}_{\lambda(\mu)}), v(\boldsymbol{M}_{\lambda(\mu)})\rangle \, d\bar{n}.\end{aligned}$$

Setting $\phi_{\lambda(\mu)}(k) = \langle \tau_{\lambda(\mu)}(k) v(\boldsymbol{M}_{\lambda(\mu)}), v(\boldsymbol{M}_{\lambda(\mu)})\rangle$, we obtain

$$(6.4) \qquad C_{\tau_{\lambda(\mu)}}(\sigma_\mu : v) = \int_{\bar{N}} e^{-(v+\rho)(H(\bar{n}))} \phi_{\lambda(\mu)}(\kappa(\bar{n})^{-1}) d\bar{n}.$$

We shall next suppose $\mu_n < 0$. Since w is represented as $\exp(\pi I_{2n,2n+1})$, it follows from (5.19) that $w\mu = (\mu_1, \dots, \mu_{n-1}, -\mu_n)$. Thus $\lambda(\mu) = \lambda(w\mu)$ and $v(\boldsymbol{M}_{\lambda(w\mu)})$ is a highest weight vector of $\tau_{\lambda(\mu)}$. Take $T \in \mathrm{Hom}_M(V_{\lambda(\mu)}, V_{\lambda(\mu)}(w\mu))$ to be a canonical projection. Noting $T\tau_{\lambda(\mu)}(w)^{-1} \in \mathrm{Hom}_M(V_{\lambda(\mu)}, V_{\lambda(\mu)}(\mu))$, we have

$$(6.5) \qquad T(\tau_{\lambda(\mu)}(w)^{-1} C_{\tau_{\lambda(\mu)}}(v) \tau_{\lambda(\mu)}(w)) v(\boldsymbol{M}_{\lambda(w\mu)}) = C_{\tau_{\lambda(\mu)}}(\sigma_\mu : v) T v(\boldsymbol{M}_{\lambda(w\mu)}),$$

and hence

$$\begin{aligned}(6.6) \qquad C_{\tau_{\lambda(\mu)}}(\sigma_\mu : v) &= \int_{\bar{N}} e^{-(v+\rho)(H(\bar{n}))} \langle \tau_{\lambda(\mu)}(w^{-1}\kappa(\bar{n})^{-1} w) v(\boldsymbol{M}_{\lambda(w\mu)}), v(\boldsymbol{M}_{\lambda(w\mu)})\rangle \, d\bar{n} \\ &= \int_{\bar{N}} e^{-(v+\rho)(H(\bar{n}))} \phi_{\lambda(\mu)}(w^{-1}\kappa(\bar{n})^{-1} w) d\bar{n}.\end{aligned}$$

(2) *Spin*$(2n+2,1)$**-case.** Let $\lambda = (\lambda_1, \dots, \lambda_{n+1}) \in D_K$ and $\mu = (\mu_1, \dots, \mu_n) \in D_M(\lambda)$. We set $\lambda(\mu) = (\mu_1, \dots, \mu_n, \lambda_{n+1}) \in D_K(\mu)$. We obtain from (5.12) and Theorem 4.6 that if $[\tau_{\lambda+\varepsilon_j} : \sigma_\mu] = 1$ then

$$\left(-\nu + \lambda_j + n - j + \frac{3}{2}\right) C_{\tau_\lambda}(\sigma_\mu : \nu) = -\left(\nu + \lambda_j + n - j + \frac{3}{2}\right) C_{\tau_{\lambda+\varepsilon_j}}(\sigma_\mu : \nu). \tag{6.7}$$

Consequently, using the preceding recursion formula and shifting the parameters as $\mu_j \mapsto \lambda_j$ $(1 \le j \le n)$, we can find

$$C_{\tau_\lambda}(\sigma_\mu : \nu) = (-1)^{|\tau_\lambda| - |\tau_{\lambda(\mu)}|} \prod_{j=1}^{n} \frac{(-\nu + \mu_j + n - j + \frac{3}{2})_{\lambda_j - \mu_j}}{(\nu + \mu_j + n - j + \frac{3}{2})_{\lambda_j - \mu_j}} C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu). \tag{6.8}$$

In this case, because $v(\boldsymbol{M}_{\lambda(\mu)})$ is a highest weight vector of both $\tau_{\lambda(\mu)}|_M$ and $\tau_{\lambda(\mu)}$, we obtain

$$C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu) = \int_{\bar{N}} e^{-(\nu+\rho)(H(\bar{n}))} \phi_{\lambda(\mu)}(\kappa(\bar{n})^{-1}) d\bar{n}. \tag{6.9}$$

(3) $SU(n,1)$**-case.** Let $\lambda = (\lambda_1, \dots, \lambda_n) \in D_K$ and $\mu = (\mu_1, \dots, \mu_{n-1}) \in D_M(\lambda)$. We set $\lambda(\mu) = (\mu_1, \dots, \mu_{n-1}, \mu_{n-1}) \in D_K(\mu)$. We obtain from (5.16), (5.28) and Theorem 4.6 that if $[\tau_{\lambda+\beta_j} : \sigma_\mu] = 1$ then

$$(-\nu+2\lambda_j+|\sigma_\mu|+n-2j+2) C_{\tau_\lambda}(\sigma_\mu : \nu) = -(\nu+2\lambda_j+|\sigma_\mu|+n-2j+2) C_{\tau_{\lambda+\beta_j}}(\sigma_\mu : \nu). \tag{6.10}$$

Consequently, using the preceding recursion formula and shifting the parameters as $\mu_j \mapsto \lambda_j$ $(1 \le j \le n-1)$ and $\mu_{n-1} \mapsto \lambda_n$, we can find

$$\begin{aligned} C_{\tau_\lambda}(\sigma_\mu : \nu) &= (-1)^{|\tau_\lambda| - |\tau_{\lambda(\mu)}|} \\ &\times \prod_{j=1}^{n-1} \frac{\left(\frac{-\nu+n+|\sigma_\mu|}{2} + \mu_j - j + 1\right)_{\lambda_j - \mu_j}}{\left(\frac{\nu+n+|\sigma_\mu|}{2} + \mu_j - j + 1\right)_{\lambda_j - \mu_j}} \frac{\left(\frac{-\nu+n-|\sigma_\mu|}{2} - \mu_{n-1}\right)_{\mu_{n-1} - \lambda_n}}{\left(\frac{\nu+n-|\sigma_\mu|}{2} - \mu_{n-1}\right)_{\mu_{n-1} - \lambda_n}} \\ &\times C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu). \end{aligned} \tag{6.11}$$

In this case, because $v(\boldsymbol{M}_{\lambda(\mu)})$ is a highest weight vector of both $\tau_{\lambda(\mu)}|_M$ and $\tau_{\lambda(\mu)}$, we obtain

$$C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu) = \int_{\bar{N}} e^{-(\nu+\rho)(H(\bar{n}))} \phi_{\lambda(\mu)}(\kappa(\bar{n})^{-1}) d\bar{n}. \tag{6.12}$$

7. Fundamental representations of K.

In this section we shall give the explicit formulae of $\phi_{\lambda(\mu)}(\kappa(\bar{n})^{-1})$. We note that since K is connected compact, τ_λ can be extended to a holomorphic representation on K_c, which is a matrix group whose Lie algebra is $\mathfrak{k}_c$.

(1) *Spin*$(2n+1,1)$**-case.** The fundamental representations are listed as follows:

$$\omega_p = \varepsilon_1 + \cdots + \varepsilon_p, \quad (1 \le p \le n-1), \quad \omega_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n). \tag{7.1}$$

Let e_p be the standard basis for $\boldsymbol{C}^{2n+1}$ and set $e_p^+ = 2^{-1/2}(e_{2p-1} + \sqrt{-1}e_{2p})$ and $e_p^- = 2^{-1/2}(e_{2p-1} - \sqrt{-1}e_{2p})$ for $1 \le p \le n$. Let $(\Phi, \boldsymbol{C}^{2n+1})$ be the usual representation of $SO(2n+1)$ and $(\Phi_r, \bigwedge^r \boldsymbol{C}^{2n+1})$ the alternating tensor representation of $(\Phi, \boldsymbol{C}^{2n+1})$. Then Φ_r $(1 \le r \le n)$ is irreducible with the highest weight $\Lambda_r = \varepsilon_1 + \cdots + \varepsilon_r$ and $e_1^+ \wedge \cdots \wedge e_r^+$ is its highest weight vector. Let p: $Spin(2n+1) \to SO(2n+1)$ denote the covering mapping. Define the irreducible unitary representation of K by $\tilde{\Phi}_r(k) = \Phi_r(p(k))$. Then an easy computation yields that for $1 \le p,\ q \le n$,

$$\langle \Phi(\kappa(\bar{n}(x))^{-1})e_p^+, e_q^+ \rangle = \delta_{p,q} - \frac{(x_{2p-1} + \sqrt{-1}x_{2p})(x_{2q-1} - \sqrt{-1}x_{2q})}{1+\omega}. \tag{7.2}$$

Setting $z_p = x_{2p-1} + \sqrt{-1}x_{2p}$ $(1 \le p \le n)$ and $z = {}^t(z_1, \dots, z_n)$, we can write (7.2) as

$$\langle \Phi(\kappa(\bar{n}(x))^{-1})e_p^+, e_q^+ \rangle = \delta_{p,q} - \frac{z_p \bar{z}_q}{1+\omega} \quad \text{and} \quad \omega = |z|^2 = \sum_{p=1}^{n} |z_p|^2. \tag{7.3}$$

On the other hand, because w can be represented as $\mathrm{diag}(\overbrace{1, \dots, 1}^{2n-2}, -1, -1, 1)$, we obtain

$$\langle \Phi(w^{-1}\kappa(\bar{n}(x))^{-1}w)e_p^+, e_q^+ \rangle = \delta_{p,q} - \frac{z'_p \bar{z}'_q}{1+\omega}, \tag{7.4}$$

where $z' = {}^t(z_1, \dots, z_{n-1}, \bar{z}_n)$. Therefore

$$\begin{aligned} \phi_{\Lambda_r}(\kappa(\bar{n}(x))^{-1}) &= \langle \Phi_r(\kappa(\bar{n}(x))^{-1})e_1^+ \wedge \cdots \wedge e_r^+, e_1^+ \wedge \cdots \wedge e_r^+ \rangle \\ &= \det\left(\delta_{p,q} - \frac{z_p\bar{z}_q}{1+\omega}\right)_{1 \le p,q \le r} = 1 - \frac{\sum_{p=1}^r |z_p|^2}{1+\omega}. \end{aligned} \tag{7.5}$$

$$\phi_{\Lambda_r}(w^{-1}\kappa(\bar{n}(x))^{-1}w) = \phi_{\Lambda_r}(\kappa(\bar{n}(x))^{-1}) = 1 - \frac{\sum_{p=1}^r |z_p|^2}{1+\omega}.$$

We write $\mathfrak{k}_+$ (resp. $\mathfrak{k}_-$) for the sum of all positive root subspaces (resp. negative root subspaces) relative to $(\mathfrak{k}_c, \mathfrak{t}_c)$. Let K_+ and K_- denote the analytic subgroups of K_c corresponding to $\mathfrak{k}_+$ and $\mathfrak{k}_-$, respectively. For $\lambda \in D_K$, it follows from the definition of ϕ_λ that

$$\phi_\lambda(k_1 \exp H k_2) = \phi_\lambda(\exp H) = e^{\lambda(H)}, \quad (k_1 \in K_+,\ k_2 \in K_-,\ H \in \mathfrak{t}_c), \tag{7.6}$$

and thus we obtain $\phi_{\Lambda_n}(k_1 \exp H k_2) = \phi_{\omega_n}(k_1 \exp H k_2)^2$. Noting that $K_+ \exp \mathfrak{t}_c K_-$ is dense in K_c and ϕ_λ is holomorphic, we see that $\phi_{\Lambda_n}(k) = \phi_{\omega_n}(k)^2$ for any $k \in K_c$. Consequently we have

$$\phi_{\omega_n}(\kappa(\bar{n}(x))^{-1})^2 = \phi_{\Lambda_n}(\kappa(\bar{n}(x))^{-1}) = \frac{1}{1+\omega}. \tag{7.7}$$

Taking the branch of the square root so that $\phi_{\omega_n}(\kappa(\bar{n}(0))^{-1}) = 1$, we see that

$$\phi_{\omega_n}(\kappa(\bar{n}(x))^{-1}) = \frac{1}{\sqrt{1+\omega}}. \tag{7.8}$$

Let $\mu = (\mu_1, \dots, \mu_n) \in D_M$. Then noting $\lambda(\mu) = \sum_{p=1}^{n-2}(\mu_p - \mu_{p+1})\omega_p + (\mu_{n-1} - |\mu_n|)\omega_{n-1} +$

$2|\mu_n|\omega_n$, we obtain

$$\phi_{\lambda(\mu)}(k) = \prod_{p=1}^{n-1} \phi_{\omega_p}(k)^{\mu_p-\mu_{p+1}} \phi_{\omega_{n-1}}(k)^{\mu_{n-1}-|\mu_n|} \phi_{\omega_n}(k)^{2|\mu_n|}, \quad \text{for any } k \in K_c,$$

and thus

(7.9) $$\phi_{\lambda(\mu)}(\kappa(\bar{n}(x))^{-1})$$
$$= \prod_{p=1}^{n-1}\left(1 - \frac{\sum_{j=1}^{p} |z_j|^2}{1+\omega}\right)^{\mu_p-\mu_{p+1}} \left(1 - \frac{\sum_{j=1}^{n-1} |z_j|^2}{1+\omega}\right)^{\mu_{n-1}-|\mu_n|} \left(\frac{1}{\sqrt{1+\omega}}\right)^{2|\mu_n|}$$
$$= (1+\omega)^{-\mu_1} \prod_{p=1}^{n-2}\left(1 + \sum_{j=p+1}^{n} |z_j|^2\right)^{\mu_p-\mu_{p+1}} (1+|z_n|^2)^{\mu_{n-1}-|\mu_n|}.$$

(2) *Spin*$(2n+2,1)$**-case.** The fundamental representations are listed as follows:

(7.10) $$\omega_p = \varepsilon_1 + \cdots + \varepsilon_p, \quad (1 \le p \le n-1),$$
$$\omega_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n - \varepsilon_{n+1}),$$
$$\omega_{n+1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n+1}).$$

Define $\boldsymbol{C}_+^{2n+2}$ and $\boldsymbol{C}_-^{2n+2}$ to be the subspaces of $\boldsymbol{C}^{2n+2}$ generated by $\{e_1^+, \dots, e_{n+1}^+\}$ and $\{e_1^-, \dots, e_{n+1}^-\}$, respectively. Then Φ_{n+1} is reducible and has two irreducible components, which are $\bigwedge^n \boldsymbol{C}^{2n+2} \wedge \boldsymbol{C}_+^{2n+2}$ and $\bigwedge^n \boldsymbol{C}^{2n+2} \wedge \boldsymbol{C}_-^{2n+2}$. We denote by Φ_{n+1}^+ and Φ_{n+1}^- the irreducible unitary representations whose representation spaces are restricted to $\bigwedge^n \boldsymbol{C}^{2n+2} \wedge \boldsymbol{C}_+^{2n+2}$ and $\bigwedge^n \boldsymbol{C}^{2n+2} \wedge \boldsymbol{C}_-^{2n+2}$, respectively. Then the highest weight of Φ_{n+1}^+ (resp. Φ_{n+1}^-) is $\Lambda_{n+1}^+ = \varepsilon_1 + \cdots + \varepsilon_{n+1}$ (resp. $\Lambda_{n+1}^- = \varepsilon_1 + \cdots + \varepsilon_n - \varepsilon_{n+1}$) and $e_1^+ \wedge \cdots \wedge e_{n+1}^+$ (resp. $e_1^+ \wedge \cdots \wedge e_n^+ \wedge e_{n+1}^-$) is its highest weight vector. A simple calculation yields that for $1 \le p,\ q \le n$

(7.11) $$\langle \Phi(\kappa(\bar{n}(x))^{-1})e_p^+, e_{n+1}^+\rangle = -\frac{z_p\bar{z}_{n+1}}{1+\omega}, \quad \langle \Phi(\kappa(\bar{n}(x))^{-1})e_p^+, e_{n+1}^-\rangle = -\frac{z_p z_{n+1}}{1+\omega},$$
$$\langle \Phi(\kappa(\bar{n}(x))^{-1})e_{n+1}^+, e_q^+\rangle = -\frac{\bar{z}_{n+1}\bar{z}_q}{1+\omega}, \quad \langle \Phi(\kappa(\bar{n}(x))^{-1})e_{n+1}^-, e_q^+\rangle = -\frac{z_{n-1}\bar{z}_q}{1+\omega},$$
$$\langle \Phi(\kappa(\bar{n}(x))^{-1})e_{n+1}^+, e_{n+1}^+\rangle = -\frac{\bar{z}_{n+1}^2}{1+\omega}, \quad \langle \Phi(\kappa(\bar{n}(x))^{-1})e_{n+1}^-, e_{n+1}^-\rangle = -\frac{z_{n-1}^2}{1+\omega},$$

where $z_p = x_{2p-1} + \sqrt{-1}x_{2p}$ and $z_{n+1} = x_{2n+1} + \sqrt{-1}$. Accordingly, by a computation analogous to obtaining (7.4), we get

(7.12) $$\phi_{\Lambda_{n+1}^+}(\kappa(\bar{n}(x))^{-1}) = -\frac{\bar{z}_{n+1}^2}{1+\omega}, \quad \phi_{\Lambda_{n+1}^-}(\kappa(\bar{n}(x))^{-1}) = -\frac{z_{n+1}^2}{1+\omega}.$$

Taking the branch of the square root so that $\phi_{\omega_n}(\kappa(\bar{n}(0))^{-1}) = \phi_{\omega_{n+1}}(\kappa(\bar{n}(0))^{-1}) = 1$, we see that

$$\phi_{\omega_n}(\kappa(\bar{n}(x))^{-1}) = \frac{1+\sqrt{-1}x_{2n+1}}{\sqrt{1+\omega}}, \quad \phi_{\omega_{n+1}}(\kappa(\bar{n}(x))^{-1}) = \frac{1-\sqrt{-1}x_{2n+1}}{\sqrt{1+\omega}}. \tag{7.13}$$

Take $\lambda = (\lambda_1, \ldots, \lambda_{n+1}) \in D_K$ and $\mu = (\mu_1, \ldots, \mu_n) \in D_M(\lambda)$. Noting $\lambda(\mu) = \sum_{p=1}^{n-1}(\mu_p - \mu_{p+1})\omega_p + (\mu_n - \lambda_{n+1})\omega_n + (\mu_n + \lambda_{n+1})\omega_{n+1}$, we see that

$$\phi_{\lambda(\mu)}(\kappa(\bar{n}(x))^{-1}) = (1+\omega)^{-\mu_1} \prod_{p=1}^{n-1}\left(1 + \sum_{j=p+1}^{n} |z_j|^2 + x_{2n+1}^2\right)^{\mu_p - \mu_{p+1}} \tag{7.14}$$
$$\times (1 + x_{2n+1}^2)^{\mu_n}(1+\sqrt{-1}x_{2n+1})^{-\lambda_{n+1}}(1-\sqrt{-1}x_{2n+1})^{\lambda_{n+1}}.$$

(3) $SU(n,1)$-case. The fundamental representations are listed as follows:

$$\omega_p = \varepsilon_1 + \cdots + \varepsilon_p - p\varepsilon_{n+1}, \quad (1 \le p \le n-1), \quad \omega_0 = -\varepsilon_{n+1}. \tag{7.15}$$

Let $(\Phi, \boldsymbol{C}^n)$ be the usual representation of $K = S(U(n) \times U(1))$, that is, for $k = \begin{pmatrix} X & \\ & u \end{pmatrix} \in K$ and $z \in \boldsymbol{C}^n$, $\Phi(k)z = u^{-1}Xz$ and $(\Phi_r, \bigwedge^r \boldsymbol{C}^n)$ the alternating tensor representation of Φ. We denote by $(\Phi_0, \boldsymbol{C})$ the representation of K defined by $\Phi_0(k)z = u^{-1}z$. Then Φ_r $(1 \le r \le n-1)$ and Φ_0 are irreducible with highest weights ω_r and ω_0, respectively and $e_1 \wedge \cdots \wedge e_r$ and 1 are their highest weight vectors. An easy computation yields that for $1 \le p,\ q \le n-1$,

$$\langle \Phi(\kappa(\bar{n}(z,u))^{-1})e_p, e_q \rangle = \frac{1+\omega}{|1+\omega|}\left(\delta_{p,q} - \frac{2z_q\bar{z}_p}{1+\bar{\omega}}\right). \tag{7.16}$$

Therefore

$$\phi_{\Lambda_r}(\kappa(\bar{n}(z,u))^{-1}) = \left(\frac{1+\omega}{|1+\omega|}\right)^r \left(1 - \frac{2\sum_{p=1}^{r}|z_p|^2}{1+\bar{\omega}}\right), \tag{7.17}$$
$$\phi_0(\kappa(\bar{n}(z,u))^{-1}) = \left(\frac{1+\bar{\omega}}{|1+\omega|}\right)^{-1}.$$

Let $\mu = (\mu_1, \ldots, \mu_{n-1}) \in D_M$. Then $\lambda(\mu) = \sum_{p=1}^{n-2}(\mu_p - \mu_{p+1})\omega_p - (n+1)\mu_n\varepsilon_{n+1}$. Therefore by a calculation similar to that in (2) $Spin(2n+2,1)$ case, we have

$$\phi_{\lambda(\mu)}(\kappa(\bar{n}(z,u))^{-1}) \tag{7.18}$$
$$= (1+\omega)^{(|\sigma_\mu|+2\mu_{n-1})/2}(1+\bar{\omega})^{-(|\sigma_\mu|+2\mu_1)/2}\prod_{p=1}^{n-2}\left(1+\bar{\omega} - 2\sum_{j=1}^{p}|z_j|^2\right)^{\mu_p-\mu_{p+1}}.$$

8. Expressions of the Harish-Chandra C-functions.

Substituting the explicit expression of $\phi_{\lambda(\mu)}(\kappa(\bar{n})^{-1})$ into the integral formula of $C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu)$ and carrying out the integration, we can get the explicit expressions of $C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu)$. Combining the recursion formulae of the Harish-Chandra C-functions, we can get the expressions of the Harish-Chandra C-functions $C_\tau(\sigma : \nu)$.

(1) *Spin*$(2n+1,1)$**-case.** In this case, $\rho = n$, $\bar{N} \cong \boldsymbol{R}^{2n}$ and

(8.1) $$\int_{\boldsymbol{R}^{2n}} (1+\omega)^{-2n} dx = \frac{\pi^n (n-1)!}{(2n-1)!} (= c_n, \text{say}).$$

Thus we have under the identification $\boldsymbol{R}^{2n} \cong \boldsymbol{C}^n$ that

(8.2)
$$c_n C_{\tau_{\lambda(\mu)}}(\sigma_\mu : v) = \int_{\boldsymbol{C}^n} (1+\omega)^{-(v+n+\mu_1)} \prod_{p=1}^{n-2} \left(1 + \sum_{j=p+1}^{n} |z_j|^2\right)^{\mu_p - \mu_{p+1}} (1+|z_n|^2)^{\mu_{n-1}-|\mu_n|} \, dz d\bar{z},$$

by changing the integration variables with $z_j = r_j e^{\sqrt{-1}\theta_j}$ and $r_j^2 = s_j$, we have

$$= \pi^n \overbrace{\int_0^\infty \cdots \int_0^\infty}^{n} (1+s_1+\cdots+s_n)^{-(v+n+\mu_1)} \prod_{p=1}^{n-2} (1+s_{p+1}+\cdots+s_n)^{\mu_p-\mu_{p+1}}$$

$$\times (1+s_n)^{\mu_{n-1}-|\mu_n|} \, ds_1 \dots ds_n$$

$$= \pi^n \int_0^\infty (1+s_1+\cdots+s_n)^{-(v+n+\mu_1)} ds_1$$

$$\times \overbrace{\int_0^\infty \cdots \int_0^\infty}^{n-1} \prod_{p=1}^{n-2} (1+s_{p+1}+\cdots+s_n)^{\mu_p-\mu_{p+1}} (1+s_n)^{\mu_{n-1}-|\mu_n|} \, ds_2 \dots ds_n$$

$$= \frac{\pi^n}{v+n+\mu_1-1} \overbrace{\int_0^\infty \cdots \int_0^\infty}^{n-1} (1+s_2+\cdots+s_n)^{-(v+n+\mu_1-1)}$$

$$\times \prod_{p=1}^{n-2} (1+s_{p+1}+\cdots+s_n)^{\mu_p-\mu_{p+1}} (1+s_n)^{\mu_{n-1}-|\mu_n|} \, ds_2 \dots ds_n$$

$$= \frac{\pi^n}{v+n+\mu_1-1} \overbrace{\int_0^\infty \cdots \int_0^\infty}^{n-1} (1+s_2+\cdots+s_n)^{-(v+n+\mu_2-1)}$$

$$\times \prod_{p=2}^{n-2} (1+s_{p+1}+\cdots+s_n)^{\mu_p-\mu_{p+1}} (1+s_n)^{\mu_{n-1}-|\mu_n|} ds_2 \dots ds_n,$$

by continuing a similar calculation,

$$= \pi^n \prod_{j=1}^{n-1} \frac{1}{v+n+\mu_j-j} \cdot \frac{1}{v+|\mu_n|}.$$

Therefore taking into account (8.1), we have

(8.3) $$C_{\tau_{\lambda(\mu)}}(\sigma_\mu : v) = \frac{(2n-1)!}{(n-1)!} \prod_{j=1}^{n-1} \frac{1}{v+n+\mu_j-j} \cdot \frac{1}{v+|\mu_n|}.$$

(2) *Spin*$(2n+2,1)$**-case.** In this case, $\rho = n + 1/2$, $\bar{N} \cong \boldsymbol{R}^{2n+1}$ and

$$\int_{\boldsymbol{R}^{2n+1}} (1+\omega)^{-2n-1} dx = \frac{\pi^{n+1}}{2^{2n} n!} (= c_n, \text{say}). \tag{8.4}$$

Thus we have under the identification $\boldsymbol{R}^{2n+1} \cong \boldsymbol{C}^n \times \boldsymbol{R}$ that

$$\begin{aligned}
(8.5)\quad & c_n C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu) \\
&= \int_{\boldsymbol{C}^n \times \boldsymbol{R}} (1+\omega)^{-(\nu+n+(1/2)+\mu_1)} \prod_{p=1}^{n-1} \left(1 + \sum_{j=p+1}^{n} |z_j|^2 + x_{2n+1}^2\right)^{\mu_p - \mu_{p+1}} \\
&\quad \times (1 + x_{2n+1}^2)^{\mu_n} (1 + \sqrt{-1} x_{2n+1})^{-\lambda_{n+1}} (1 - \sqrt{-1} x_{2n+1})^{\lambda_{n+1}} \, dz d\bar{z} dx_{2n+1},
\end{aligned}$$

by changing the variables with $z_j = r_j e^{\sqrt{-1}\theta_j}$ and $r_j^2 = s_j$,

$$\begin{aligned}
&= \pi^n \overbrace{\int_0^\infty \cdots \int_0^\infty}^{n} \int_{-\infty}^{\infty} (1 + s_1 + \cdots + s_n + x_{2n+1}^2)^{-(\nu+n+1/2+\mu_1)} \\
&\quad \times \prod_{p=1}^{n-1} (1 + s_{p+1} + \cdots + s_n + x_{2n+1}^2)^{\mu_p - \mu_{p+1}} (1 + x_{2n+1}^2)^{\mu_n} \\
&\quad \times (1 + \sqrt{-1} x_{2n+1})^{-\lambda_{n+1}} (1 - \sqrt{-1} x_{2n+1})^{\lambda_{n+1}} \, ds_1 \ldots ds_n dx_{2n+1},
\end{aligned}$$

by carrying out a calculation similar to that in (8.2),

$$\begin{aligned}
&= \pi^n \prod_{j=1}^{n} \frac{1}{\nu + n + \mu_j - j + 1/2} \int_{-\infty}^{\infty} (1 + \sqrt{-1} x_{2n+1})^{-(\nu+\lambda_{n+1}+1/2)} \\
&\quad \times (1 - \sqrt{-1} x_{2n+1})^{-(\nu-\lambda_{n+1}+1/2)} dx_{2n+1}.
\end{aligned}$$

Here in order to compute the last expression in (8.5), we need the following lemma.

Lemma 8.1 (cf. [**3**, **9**]). *Let* $n \geq 1$, $\lambda \in \boldsymbol{C}$, $\ell \in \boldsymbol{Z}$, $q_j \in \boldsymbol{Z}_{\geq 0}$ $(1 \leq j \leq n-1)$ *and* $F = 1 + (1/2)(|z_1|^2 + \cdots + |z_{n-1}|^2) + \sqrt{-1}u$. *Then*

$$\begin{aligned}
&\int_{\boldsymbol{C}^{n-1} \times \boldsymbol{R}} \bar{F}^{(\lambda+\ell)/2} F^{(\lambda-\ell)/2} \prod_{p=1}^{n-1} \left(\bar{F} - \sum_{j=1}^{p} |z_j|^2\right)^{q_p} dz d\bar{z} du \\
&= \frac{(2\pi)^n 2^{\lambda+n+q_1+\cdots+q_{n-1}} \Gamma(-\lambda - n - q_1 - \cdots - q_{n-1})}{\prod_{j=1}^{n-1} \left(-\frac{\lambda+\ell}{2} - q_1 - \cdots - q_{j-1} - j\right) \Gamma\left(-\frac{\lambda+\ell}{2} - q_1 - \cdots - q_{n-1} - n + 1\right) \Gamma\left(-\frac{\lambda-\ell}{2}\right)}.
\end{aligned}$$

Taking into account (8.4), we obtain from Lemma 8.1 with $n = 1$ that

$$C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu) = \frac{n! 2^{-2\nu+2n+1} \Gamma(2\nu)}{\prod_{j=1}^{n} \left(\nu + n + \mu_j + \frac{1}{2} - j\right) \Gamma\left(\nu + \lambda_{n+1} + \frac{1}{2}\right) \Gamma\left(\nu - \lambda_{n+1} + \frac{1}{2}\right)}. \tag{8.6}$$

(3) $SU(n,1)$-case. In this case, $\rho = n$ and $\bar{N} \cong \boldsymbol{C}^{n-1} \times \boldsymbol{R}$ and

$$\int_{\boldsymbol{C}^{n-1}\times\boldsymbol{R}} |1+\omega|^{-2n} dz d\bar{z} du = \frac{\pi^n}{2^n(n-1)!} (= c_n, \text{ say}). \tag{8.7}$$

Thus we have

$$c_n C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu) = \int_{\boldsymbol{C}^{n-1}\times\boldsymbol{R}} (1+\omega)^{-(\nu+n-|\sigma_\mu|-2\mu_{n-1})/2} (1+\bar{\omega})^{-(\nu+n+|\sigma_\mu|+2\mu_1)/2} \times \prod_{p=1}^{n-2} \left(1+\bar{\omega} - 2\sum_{j=1}^{p} |z_j|^2\right)^{\mu_p - \mu_{p+1}} dz d\bar{z} du. \tag{8.8}$$

Taking into account (8.7), we obtain from Lemma 8.1 that

$$C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu) = \frac{(n-1)! 2^{-\nu+n} \Gamma(\nu)}{\prod_{j=1}^{n-1} \left(\frac{\nu+n+|\sigma_\mu|}{2} - j + \mu_j\right) \Gamma\left(\frac{\nu+n-|\sigma_\mu|}{2} - \mu_{n-1}\right) \Gamma\left(\frac{\nu-n+|\sigma_\mu|}{2} + 1 + \mu_{n-1}\right)}. \tag{8.9}$$

Combining the above expressions and the recursion formulae of the Harish-Chandra C-function, we can get the explicit expressions of the Harish-Chandra C-functions for $Spin(n,1)$ and $SU(n,1)$.

THEOREM 8.2. *The Harish-Chandra C-functions $C_{\tau_\lambda}(\sigma_\mu : \nu)$ for $Spin(n,1)$ and $SU(n,1)$ associated with $\tau_\lambda \in \hat{K}$ and $\sigma_\mu \in \hat{M}(\tau_\lambda)$ are given as follows:*

(1) *$Spin(2n+1,1)$-case.*

$$C_{\tau_\lambda}(\sigma_\mu : \nu) = \frac{(2n-1)!}{(n-1)!} \frac{\prod_{j=1}^{n} \Gamma(\nu - n + j - \mu_j) \prod_{j=1}^{n} \Gamma(\nu + n - j + \mu_j)}{\prod_{j=1}^{n} \Gamma(\nu - n + j - \lambda_j) \prod_{j=1}^{n} \Gamma(\nu + n - j + 1 + \lambda_j)}.$$

(2) *$Spin(2n+2,1)$-case.*

$$C_{\tau_\lambda}(\sigma_\mu : \nu) = \frac{n! 2^{-2\nu+2n+1} \Gamma(2\nu) \prod_{j=1}^{n} \Gamma(\nu - n + j - \frac{1}{2} - \mu_j) \prod_{j=1}^{n} \Gamma(\nu + n - j + \frac{1}{2} + \mu_j)}{\prod_{j=1}^{n+1} \Gamma(\nu - n + j - \frac{1}{2} - \lambda_j) \prod_{j=1}^{n+1} \Gamma(\nu + n - j + \frac{3}{2} + \lambda_j)}.$$

(3) *$SU(n,1)$-case.*

$$C_{\tau_\lambda}(\sigma_\mu : \nu) = \frac{(n-1)! 2^{-\nu+n} \Gamma(\nu) \prod_{j=1}^{n-1} \Gamma\left(\frac{\nu - n - |\sigma_\mu|}{2} + j - \mu_j\right) \prod_{j=1}^{n-1} \Gamma\left(\frac{\nu + n + |\sigma_\mu|}{2} - j + \mu_j\right)}{\prod_{j=1}^{n} \Gamma\left(\frac{\nu - n - |\sigma_\mu|}{2} + j - \lambda_j\right) \prod_{j=1}^{n} \Gamma\left(\frac{\nu + n + |\sigma_\mu|}{2} - j + 1 + \lambda_j\right)}.$$

We write $\det C_{\tau_\lambda}(v)$ for the determinant of the linear endomorphism $C_{\tau_\lambda}(v)$ of V_λ. Taking into account $V_\lambda = \sum_{\mu \in D_M(\lambda)} H_\mu$, we see that

$$\det C_{\tau_\lambda}(v) = \prod_{\mu \in D_M(\lambda)} C_{\tau_\lambda}(\sigma_\mu : v)^{\dim H_\mu}. \tag{8.10}$$

Thus, substituting the expression in Theorem 8.2 into (8.10), we obtain the explicit formula of $\det C_{\tau_\lambda}(v)$. On the other hand, in [**1**], Cohn obtained the expression of $\det C_\tau(v)$ for any semisimple Lie group. He showed that there exist $p_{i,j}, q_{i,j} \in \boldsymbol{C}$ $(1 \le i \le r, 1 \le j \le j_i)$ and $\mu_1, \dots, \mu_r \in \mathfrak{a}^*$ such that

$$\det C_\tau(v) = \prod_{i=1}^{r} \prod_{j=1}^{j_i} \frac{\Gamma\left(\dfrac{-\langle v, \alpha_i \rangle}{2\langle \mu_i, \alpha_i \rangle} + q_{i,j}\right)}{\Gamma\left(\dfrac{-\langle v, \alpha_i \rangle}{2\langle \mu_i, \alpha_i \rangle} + p_{i,j}\right)}. \tag{8.11}$$

He conjectured in his paper [**1**] that the constants $p_{i,j}$ and $q_{i,j}$ appearing in the above expression are rational numbers and depending linearly on the highest weight of τ. We can now concretely write the values of $p_{i,j}$ and $q_{i,j}$ and thus we obtain the following corollary.

COROLLARY 8.3. *Cohn's conjecture is true for Spin$(n,1)$ and $SU(n,1)$.*

Applying Theorem 6(i) and Theorem 7(ii) in [**7**], we can concretely construct the discrete series representations of $SU(n,1)$ as subquotients of the nonunitary principal series representations. Because these computations can be carried out without any difficulty, we shall only write the conclusions. For $\mu \in D_M$, $\lambda \in D_K(\mu)$ and $v \in \mathfrak{a}^*$, we set

$$a_\lambda(\mu, v) = \prod_{j=1}^{n-1} \frac{(-h_j + 1)_{\lambda_j - \mu_j}}{(k_j + 1)_{\lambda_j - \mu_j}} \frac{(-k_{n-1} + 1)_{\mu_{n-1} - \lambda_n}}{(h_{n-1} + 1)_{\mu_{n-1} - \lambda_n}}, \tag{8.12}$$

where $h_j = (v - n - |\sigma_\mu|)/2 + j - \mu_j$ and $k_j = (v + n + |\sigma_\mu|)/2 - j + \mu_j$. Let $\mathscr{H}^{\sigma_\mu, v}(K)$ denote the set of K-finite elements in $C_{\sigma_\mu}(K)$. For $f = \sum f_\lambda, g = \sum g_\lambda \in \mathscr{H}^{\sigma_\mu, v}(K)$, we set $\langle f, g \rangle_i = \sum \langle a_\lambda(\mu, v) f_\lambda, g_\lambda \rangle$.

COROLLARY 8.4. *Retain the above notation. For $\mu = (\mu_1, \dots, \mu_{n-1}) \in D_M$, we set $\mu'_p = \mu_p + n/2 - p$. Then the discrete series representations of $SU(n,1)$ are listed as follows*:

(1) *The holomorphic discrete series. We choose $\mu \in D_M$ and $v \in \mathfrak{a}^*$ so that $h_{n-1} \le -1$. Let*

$$\lambda_m = (\mu_1, \dots, \mu_{n-1}, \mu_{n-1} + h_{n-1} + 1),$$

$$\Lambda = \sum_{p=1}^{n-1} \mu'_p \varepsilon_p + \frac{v - |\sigma_\mu|}{2} \varepsilon_n - \frac{v + |\sigma_\mu|}{2} \varepsilon_{n+1},$$

$$S_\Lambda = \{\lambda \in D_K(\mu) : \mu_{n-1} \ge \lambda_n \ge \mu_{n-1} + h_{n-1} + 1\}.$$

Let V_Λ *be the Hilbert space completion of* $V_\Lambda(K) = \sum_{\lambda \in S_\Lambda} V_\lambda$ *relative to* $\langle \cdot, \cdot \rangle_i$ *and* $\pi_\Lambda(g) = \pi_{\sigma_\mu, \nu}(g)|_{V_\Lambda}$. *Then* (π_Λ, V_Λ) *is a holomorphic discrete series with the Harish-Chandra parameter* Λ *and the minimal K-type* λ_m.

(2) *The antiholomorphic discrete series. We choose* $\mu \in D_M$ *and* $\nu \in \mathfrak{a}^*$ *so that* $k_1 \leq -1$. *Let*

$$\lambda_m = (\mu_1 - k_1 - 1, \mu_1, \dots, \mu_{n-1}),$$

$$\Lambda = -\frac{\nu + |\sigma_\mu|}{2}\varepsilon_1 + \sum_{p=1}^{n-1} \mu_p' \varepsilon_{p+1} + \frac{\nu - |\sigma_\mu|}{2}\varepsilon_{n+1},$$

$$S_\Lambda = \{\lambda \in D_K(\mu) : \mu_1 \leq \lambda_1 \leq \mu_1 - k_1 - 1\}.$$

Let V_Λ *be the Hilbert space completion of* $V_\Lambda(K) = \sum_{\lambda \in S_\Lambda} V_\lambda$ *relative to* $\langle \cdot, \cdot \rangle_i$ *and* $\pi_\Lambda(g) = \pi_{\sigma_\mu, \nu}(g)|_{V_\Lambda}$. *Then* (π_Λ, V_Λ) *is an antiholomorphic discrete series with the Harish-Chandra parameter* Λ *and the minimal K-type* λ_m.

(3) *The nonholomorphic discrete series. We choose* $\mu \in D_M$ *and* $\nu \in \mathfrak{a}^*$ *so that* $h_a \leq -1$, $h_{a+1} > 0$, $k_{a+2} \leq -1$ *and* $k_{a+1} > 0$ *for some* $0 \leq a \leq n-2$. *Let*

$$\lambda_m = (\mu_1, \dots, \mu_a, \mu_a + h_a + 1, \mu_{a+2} - k_{a+2} - 1, \mu_{a+2}, \dots, \mu_{n-1}),$$

$$\Lambda = \sum_{p=1}^{a} \mu_p' \varepsilon_p + \frac{\nu - |\sigma_\mu|}{2}\varepsilon_{a+1} - \frac{\nu + |\sigma_\mu|}{2}\varepsilon_{a+2} + \sum_{p=a+2}^{n-1} \mu_p' \varepsilon_{p+1} + \mu_{a+1}' \varepsilon_{n+1},$$

$$S_\Lambda = \{\lambda \in D_K(\mu) : \mu_a \geq \lambda_{a+1} \geq \mu_a + h_a + 1, \mu_{a+2} \leq \lambda_{a+2} \leq \mu_{a+2} - k_{a+2} - 1\}.$$

If $a < n-2$, *let* V_Λ *be the Hilbert space completion of* $V_\Lambda(K) = \sum_{\lambda \in S_\Lambda} V_\lambda$ *relative to* $\Gamma(-h_{a+1} + 1)\langle \cdot, \cdot \rangle_i$. *If* $a = n-2$, *let* V_Λ *be the Hilbert space completion of* $V_\Lambda(K)$ *relative to* $\Gamma(-h_{n-1} + 1)\Gamma(-k_{n-1} + 1)\langle \cdot, \cdot \rangle_i$. *We put* $\pi_\Lambda(g) = \pi_{\sigma_\mu, \nu}(g)|_{V_\Lambda}$. *Then* (π_Λ, V_Λ) *is a nonholomorphic discrete series with the Harish-Chandra parameter* Λ *and the minimal K-type* λ_m.

9. Restriction of discrete series.

In this section we shall concretely construct the invariant subspaces of the representation spaces of holomorphic and antiholomorphic discrete series of $SU(n,1)$ when restricted to $U(n-1,1)$.

Let us embed $G_1 = U(n-1,1)$ into $G = SU(n,1)$ by $g = \begin{pmatrix} X & v \\ w^* & u \end{pmatrix} \mapsto \begin{pmatrix} X & 0 & v \\ 0 & a & 0 \\ w^* & 0 & u \end{pmatrix}$, where $a = (\det g)^{-1} \in U(1)$. Let

$$K_1 = \left\{ \begin{pmatrix} X & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & u \end{pmatrix} : X \in U(n-1),\ a, u \in U(1), au \det X = 1 \right\}, \tag{9.1}$$

$$Z = \{z(h) = \mathrm{diag}(\overbrace{h, \dots, h}^{n-1}, h^{-n}, h) : h \in U(1)\},$$

$$A_1 = \exp \boldsymbol{R} H_1,$$

where $H_1 = E_{n-1,n+1} + E_{n+1,n-1}$. Then $G_1 = K_1A_1K_1$ is a Cartan decomposition of G_1. For $\ell \in \boldsymbol{Z}$ and $\mu \in D_M$, define the unitary representation $(\chi_{(\ell,\mu)}, H_\mu)$ of $Z \times M$ by $\chi_{(\ell,\mu)}(z(h), m)v = h^\ell \sigma_\mu(m)v$, $(m \in M, v \in H_\mu)$. Since $K_1 = MZ$, $\chi_{(\ell,\mu)} \in \hat{K}_1$ if and only if $\ell + (n+1)\mu_{n-1} \in (n+1)\boldsymbol{Z}$. For $\alpha \in D_K$, it follows that

$$\tau_\alpha|_{K_1} = \sum_{\beta<\alpha} \chi_{(-(n+1)(|\alpha|-|\beta|),\beta)}, \quad V_\alpha = \sum_{\beta<\alpha} V_\alpha(\beta). \tag{9.2}$$

So when we look upon $V_\alpha(\beta)$ as a representation space of K_1, we write this representation space as $V_{(-(n+1)(|\alpha|-|\beta|),\beta)}$. We shall first rewrite the results in Proposition 4.5 in terms of the Clebsch-Gordan coefficients. Fix an orthonormal basis $\{E_i = E_{n+1,i}/\sqrt{2(n+1)}, F_i = E_{i,n+1}/\sqrt{2(n+1)} : 1 \le i \le n\}$ of $\mathfrak{p}_c$. Then E_i and F_i correspond to the Gel'fand-Tsetlin basis with data $(\bar{\mathbf{1}}_n, \dots, \bar{\mathbf{1}}_i, \mathbf{0}_{i-1}, \dots, \mathbf{0}_1)$ and $(\mathbf{1}_n, \dots, \mathbf{1}_i, \mathbf{0}_{i-1}, \dots, \mathbf{0}_1)$ respectively. Here $\mathbf{0}_i = (\overbrace{0, \dots, 0}^{i})$, $\mathbf{1}_i = (1, \mathbf{0}_{i-1})$ and $\bar{\mathbf{1}}_i = (\mathbf{0}_{i-1}, -1)$. Let $(\cdot,\cdot\,|\,\cdot)$ denote the Clebsch-Gordan coefficients relative to the decomposition $V_\lambda \otimes V_{\lambda'} = \sum_{\lambda'' \in D_K} V_{\lambda''}$, that is, for $v(\boldsymbol{M}) \in V_\lambda$, $v(\boldsymbol{M}') \in V_{\lambda'}$ and $v(\boldsymbol{M}'') \in V_{\lambda''}$

$$(v(\boldsymbol{M}), v(\boldsymbol{M}')\,|\,v(\boldsymbol{M}'')) = \langle E_{\lambda''}(v(\boldsymbol{M}) \otimes v(\boldsymbol{M}')), v(\boldsymbol{M}'')\rangle_{V_{\lambda''}}.$$

Here $E_{\lambda''}$ denotes the canonical projection of $V_\lambda \otimes V_{\lambda'}$ to $V_{\lambda''}$. We use the following fact concerning the Clebsch-Gordan coefficients of $U(n)$ (cf. [**11**, p. 385]).

Lemma 9.1. *For arbitrary Gel'fand-Tsetlin data* $\boldsymbol{M} = (m_n, \dots, m_1)$, $\boldsymbol{M}' = (m_n', \dots, m_1')$, $\boldsymbol{M}'' = (m_n'', \dots, m_1'')$, *there exist* $\left(\begin{matrix} m_j & m_j' \\ m_{j-1} & m_{j-1}' \end{matrix}\,\middle|\,\begin{matrix} m_j'' \\ m_{j-1}'' \end{matrix}\right) \in \boldsymbol{R}$ $(2 \le j \le n)$ *such that the Clebsch-Gordan coefficient* $(v(\boldsymbol{M}), v(\boldsymbol{M}')\,|\,(v(\boldsymbol{M}''))$ *can be expressed as follows*:

$$(v(\boldsymbol{M}), v(\boldsymbol{M}')\,|\,v(\boldsymbol{M}'')) = \prod_{j=2}^{n} \left(\begin{matrix} m_j & m_j' \\ m_{j-1} & m_{j-1}' \end{matrix}\,\middle|\,\begin{matrix} m_j'' \\ m_{j-1}'' \end{matrix}\right).$$

Moreover, $\left(\begin{matrix} m_j & m_j' \\ m_{j-1} & m_{j-1}' \end{matrix}\,\middle|\,\begin{matrix} m_j'' \\ m_{j-1}'' \end{matrix}\right)$ *has the following properties.*

(1) *If* $|m_j| + |m_j'| \ne |m_j''|$ *or* $|m_{j-1}| + |m_{j-1}'| \ne |m_{j-1}''|$, *then*

$$\left(\begin{matrix} m_j & m_j' \\ m_{j-1} & m_{j-1}' \end{matrix}\,\middle|\,\begin{matrix} m_j'' \\ m_{j-1}'' \end{matrix}\right) = 0.$$

(2) $\left(\begin{matrix} \mathbf{1}_j & m_j \\ \mathbf{1}_{j-1} & (m_j)_{\le j-1} \end{matrix}\,\middle|\,\begin{matrix} m_j' \\ (m_j')_{\le j-1} \end{matrix}\right) = \left(\begin{matrix} \bar{\mathbf{1}}_j & m_j \\ \bar{\mathbf{1}}_{j-1} & (m_j)_{\ge 2} \end{matrix}\,\middle|\,\begin{matrix} m_j' \\ (m_j')_{\ge 2} \end{matrix}\right) = 1.$

Remark. $\left(\begin{matrix} m_j & m_j' \\ m_{j-1} & m_{j-1}' \end{matrix}\,\middle|\,\begin{matrix} m_j'' \\ m_{j-1}'' \end{matrix}\right)$ are called scalar factors of the Clebsch-Gordan coefficients.

For each $\alpha \in D_K(\mu)$ and $\beta \in D_M(\alpha)$, let T_α^β be the canonical projection of V_α into $V_\alpha(\beta)$ and write $P_\alpha^\beta = \sqrt{\dim V_\alpha / \dim V_\alpha(\beta)}\, T_\alpha^\beta$. Throughout this section we shall identify $\mathscr{H}^{\sigma_\mu,\nu}(\tau_\alpha)$ with V_α and simply write v instead of $f_{P_\alpha^\beta \otimes v}$. In the following discussion, for

$x = (x_1, \dots, x_n) \in \boldsymbol{R}^n$ and $1 \le j \le n$, we shall use the notation $x^{\pm j}$ to denote $(x_1, \dots, x_{j-1}, x_j \pm 1, x_{j+1}, \dots, x_n)$. A simple calculation implies that

$$(9.3) \qquad \tilde{P}^{\beta}_{\alpha} E^*_{\alpha^{+j}} = \frac{1}{2\sqrt{2(n+1)}} \sqrt{\frac{\dim V_{\alpha^{+j}}}{\dim V_{\alpha}}} \left(\begin{matrix} \mathbf{1}_n & \alpha \\ \mathbf{0}_{n-1} & \mu \end{matrix} \middle| \begin{matrix} \alpha^{+j} \\ \mu \end{matrix} \right) P^{\beta}_{\alpha^{+j}},$$

$$\tilde{P}^{\beta}_{\alpha} E^*_{\alpha^{-j}} = \frac{1}{2\sqrt{2(n+1)}} \sqrt{\frac{\dim V_{\alpha^{-j}}}{\dim V_{\alpha}}} \left(\begin{matrix} \bar{\mathbf{1}}_n & \alpha \\ \mathbf{0}_{n-1} & \mu \end{matrix} \middle| \begin{matrix} \alpha^{-j} \\ \mu \end{matrix} \right) P^{\beta}_{\alpha^{-j}}.$$

For $\alpha \in D_K(\mu)$ and $\beta \in D_M(\alpha)$, we set $\boldsymbol{M}_{\alpha,\beta} = (\alpha, \beta, \beta_{\ge 2}, \dots, \beta_{\ge n-1})$ and $\tilde{\boldsymbol{M}}_{\alpha,\beta} = (\alpha, \beta, \beta_{\le n-2}, \dots, \beta_{\le 1})$. Then for $1 \le i \le n-1$, we have from Lemma 9.1 that

$$(9.4) \qquad F_i \otimes v(\tilde{\boldsymbol{M}}_{\alpha,\beta}) = \sum_{j=1}^{n} \sum_{k=1}^{i} \left(\begin{matrix} \mathbf{1}_n & \alpha \\ \mathbf{1}_{n-1} & \beta \end{matrix} \middle| \begin{matrix} \alpha^{+j} \\ \beta^{+k} \end{matrix} \right) \left(\begin{matrix} \mathbf{1}_i & \beta_{\le i} \\ \mathbf{0}_{i-1} & \beta_{\le i-1} \end{matrix} \middle| \begin{matrix} \beta^{+k}_{\le i} \\ \beta_{\le i-1} \end{matrix} \right) v(\tilde{\boldsymbol{M}}^{j,k}_{\alpha,\beta}),$$

$$E_i \otimes v(\boldsymbol{M}_{\alpha,\beta}) = \sum_{j=1}^{n} \sum_{k=n-i}^{n-1} \left(\begin{matrix} \bar{\mathbf{1}}_n & \alpha \\ \mathbf{0}_{n-1} & \beta \end{matrix} \middle| \begin{matrix} \alpha^{-j} \\ \beta^{-k} \end{matrix} \right) \left(\begin{matrix} \bar{\mathbf{1}}_i & \beta_{\ge n-i} \\ \mathbf{0}_{i-1} & \beta_{\ge n-i+1} \end{matrix} \middle| \begin{matrix} \beta^{-k+n-i-1}_{\ge n-i} \\ \beta_{\ge n-i+1} \end{matrix} \right) v(\boldsymbol{M}^{j,k}_{\alpha,\beta}),$$

where

$$\boldsymbol{M}^{j,k}_{\alpha,\beta} = (\alpha^{-j}, \beta^{-k}, \beta^{-k+1}_{\ge 2}, \dots, \beta^{-k+n-i-1}_{\ge n-i}, \beta_{\ge n-i+1}, \dots, \beta_{\ge n-1}),$$

$$\tilde{\boldsymbol{M}}^{j,k}_{\alpha,\beta} = (\alpha^{+j}, \beta^{+k}, \beta^{+k}_{\le n-2}, \dots, \beta^{+k}_{\le i}, \beta_{\le i-1}, \dots, \beta_{\le 1}).$$

REMARK. For the explicit forms of the scalar factors appeared in (9.3) and (9.4), see [**11**, p. 385].

Substituting (9.3) and (9.4) into the expressions in Proposition 4.5, we obtain

$$(9.5) \qquad \pi_{\sigma_\mu, v}(E_{i,n+1}) v(\tilde{\boldsymbol{M}}_{\alpha,\beta}) = \sum_{j=1}^{n} \sum_{k=1}^{i} \sqrt{\frac{\dim V_{\alpha^{+j}}}{\dim V_{\alpha}}} k_j(\alpha) \left(\begin{matrix} \mathbf{1}_n & \alpha \\ \mathbf{0}_{n-1} & \mu \end{matrix} \middle| \begin{matrix} \alpha^{+j} \\ \mu \end{matrix} \right) \left(\begin{matrix} \mathbf{1}_n & \alpha \\ \mathbf{1}_{n-1} & \beta \end{matrix} \middle| \begin{matrix} \alpha^{+j} \\ \beta^{+k} \end{matrix} \right)$$

$$\times \left(\begin{matrix} \mathbf{1}_i & \beta_{\le i} \\ \mathbf{0}_{i-1} & \beta_{\le i-1} \end{matrix} \middle| \begin{matrix} \beta^{+k}_{\le i} \\ \beta_{\le i-1} \end{matrix} \right) v(\tilde{\boldsymbol{M}}^{j,k}_{\alpha,\beta}),$$

$$\pi_{\sigma_\mu, v}(E_{n+1,i}) v(\boldsymbol{M}_{\alpha,\beta}) = \sum_{j=1}^{n} \sum_{k=n-i}^{n-1} \sqrt{\frac{\dim V_{\alpha^{-j}}}{\dim V_{\alpha}}} h_j(\alpha) \left(\begin{matrix} \bar{\mathbf{1}}_n & \alpha \\ \mathbf{0}_{n-1} & \mu \end{matrix} \middle| \begin{matrix} \alpha^{-j} \\ \mu \end{matrix} \right) \left(\begin{matrix} \bar{\mathbf{1}}_n & \alpha \\ \bar{\mathbf{1}}_{n-1} & \beta \end{matrix} \middle| \begin{matrix} \alpha^{-j} \\ \beta^{-k} \end{matrix} \right)$$

$$\times \left(\begin{matrix} \bar{\mathbf{1}}_i & \beta_{\ge n-i} \\ \mathbf{0}_{i-1} & \beta_{\ge n-i+1} \end{matrix} \middle| \begin{matrix} \beta^{-k+n-i-1}_{\ge n-i} \\ \beta_{\ge n-i+1} \end{matrix} \right) v(\boldsymbol{M}^{j,k}_{\alpha,\beta}),$$

where $h_j(\alpha) = (v - n - |\sigma_\mu|)/2 + j - \alpha_j$ and $k_j(\alpha) = (v + n + |\sigma_\mu|)/2 - j + \alpha_j + 1$. For $1 \le i < j \le n-1$, it follows from (5.21) and (5.22) that

$$(9.6) \qquad \pi_{\sigma_\mu, v}(E_{j,i}) v(\boldsymbol{M}_{\alpha,\beta}) = \pi_{\sigma_\mu, v}(E_{i,j}) v(\tilde{\boldsymbol{M}}_{\alpha,\beta}) = 0.$$

Let ω_1 be the Casimir operator of G_1, that is

$$\omega_1 = \frac{1}{2(n+1)} \sum_{\substack{1 \le i \le n+1 \\ i \ne n}} E_{i,i}^2 + \frac{1}{n+1} \sum_{1 \le i < j \le n-1} (E_{j,i}E_{i,j} + E_{i,j}E_{j,i}) + 2\sum_{j=1}^{n-1} (F_j E_j + E_j F_j). \tag{9.7}$$

We shall first consider the case of holomorphic discrete series. Fix $\mu \in D_M$ and $\nu \in \mathfrak{a}^*$ so that the condition in Corollary 8.4(1) is fulfilled. For simplicity we set $\mu_n = (\nu + n - |\sigma_\mu|)/2$. Then it follows from Corollary 8.4(1) that

$$V_\Lambda(K) = \sum_{\alpha < (\infty, \lambda_m)} V_\alpha = \sum_{\alpha < (\infty, \lambda_m)} \sum_{\beta < \alpha} V_\alpha(\beta) = \sum_{\substack{\ell - \mu_n \in \boldsymbol{Z}_{\ge 0} \\ \beta \in S}} \sum_{\substack{\beta < \alpha < (\infty, \lambda_m) \\ |\alpha| - |\beta| = \ell}} V_\alpha(\beta), \tag{9.8}$$

where $S = \{\beta \in D_M : \beta_1 \ge \mu_2, \mu_{j-1} \ge \beta_j \ge \mu_{j+1}, (2 \le j \le n-1)\}$. For our convenience, we introduce the following notation:

$$S_m = \{\beta \in S : \beta < \lambda_m\}, \quad S_c = \left\{\ell \in \frac{1}{n+1}\boldsymbol{Z} : \ell - \mu_n \in \boldsymbol{Z}_{\ge 0}\right\},$$

$$S_c^+ = \{\ell \in S_c : \ell - \mu_1 \in \boldsymbol{Z}_{\ge 0}\}, \quad S_c^- = S_c \backslash S_c^+.$$

For $\alpha \in D_K$ and $\beta \in D_M(\alpha)$, let

$$c(\alpha, j, k) = \sqrt{\frac{\dim V_{\alpha^{-j}}}{\dim V_\alpha}} h_j(\alpha) \left(\begin{array}{cc|c} \bar{\mathbf{1}}_n & \alpha & \alpha^{-j} \\ \mathbf{0}_{n-1} & \mu & \mu \end{array}\right) \left(\begin{array}{cc|c} \bar{\mathbf{1}}_n & \alpha & \alpha^{-j} \\ \bar{\mathbf{1}}_{n-1} & \beta & \beta^{-k} \end{array}\right),$$

$$d(k) = \left(\begin{array}{cc|c} \bar{\mathbf{1}}_i & \beta_{\ge n-i} & \beta_{\ge n-i}^{-k+n-i-1} \\ \mathbf{0}_{i-1} & \beta_{\ge n-i+1} & \beta_{\ge n-i+1} \end{array}\right).$$

For $\beta \in S_m$, let

$$\boldsymbol{Z}(\beta) = \{k \in \boldsymbol{Z}_{>0} : 1 \le k \le n-1, \beta^{-k} \in S_m\},$$

$$N_\ell(\beta) = \{\alpha \in D_K : \beta < \alpha < (\infty, \lambda_m), |\alpha| - |\beta| = \ell\}, \quad m(\ell, \beta) = \mathrm{Card}\, N_\ell(\beta),$$

$$V_\ell(\beta) = \sum_{\alpha \in N_\ell(\beta)} V_\alpha(\beta).$$

For $\beta \in S$, let

$$\beta_f = (\max(\beta_2, \mu_2), \ldots, \max(\beta_{n-1}, \mu_{n-1}), \mu_n),$$

$$\beta_\ell = (\min(\beta_1, \mu_1), \ldots, \min(\beta_{n-1}, \mu_{n-1})).$$

Taking into account $\beta_\ell, \beta_f \in S_m, \beta_f < (\beta_\ell, -\infty)$ and $\beta_\ell < (\beta, -\infty)$, we see that $N_\ell(\beta)$ can be written as

$$N_\ell(\beta) = \{\alpha \in D_K : \alpha_{\ge 2} \in S_m, \beta_f < (\alpha_{\ge 2}, -\infty), \alpha_{\ge 2} < (\beta_\ell, -\infty), |\alpha_{\ge 2}| \le |\beta| + \ell - \max(\beta_1, \mu_1)\}. \tag{9.9}$$

Taking into account $(\beta_\ell)_\ell = \beta_\ell$ and $(\beta_\ell)_f = (\lambda_m)_{\geq 2}$, we can easily see that $m(\ell,\beta) \leq m(\ell,\beta_\ell)$. For this reason, we write (9.8) as the following form:

$$V_A(K) = \sum_{\substack{\ell \in S_c^+ \\ \beta \in S_m}} \sum_{\alpha \in N_\ell(\beta)} V_\alpha(\beta) + \sum_{\substack{\ell \in S_c^+ \\ \beta \notin S_m}} \sum_{\alpha \in N_\ell(\beta)} V_\alpha(\beta) + \sum_{\substack{\ell \in S_c^- \\ \beta \in S_m}} \sum_{\alpha \in N_\ell(\beta)} V_\alpha(\beta) + \sum_{\substack{\ell \in S_c^- \\ \beta \notin S_m}} \sum_{\alpha \in N_\ell(\beta)} V_\alpha(\beta). \tag{9.10}$$

We shall here get expressions of eigenvectors of ω_1. Assume that an eigenvector v is represented as $v = \sum_{\alpha \in N_\ell(\beta)} c_\alpha v(\boldsymbol{M}_{\alpha,\beta})$, $(c_\alpha \in \boldsymbol{C})$. Then it follows from (9.6) that $\pi_A(E_{j,i})v = 0$ for $1 \leq i < j \leq n-1$. Thus for v being an eigenvector, it suffices to determine c_α such that $\pi_A(E_{n+1,i})v = 0$, $(1 \leq i \leq n-1)$. It follows from (9.5) that

$$\pi_A(E_{n+1,i})v = \sum_{\alpha \in N_\ell(\beta)} \left\{ c_\alpha \sum_{j=1}^{n} \sum_{k=n-i}^{n-1} c(\alpha,j,k) d(k) v(\boldsymbol{M}_{\alpha,\beta}^{j,k}) \right\}. \tag{9.11}$$

Noting $h_j(\alpha) < 0$ $(1 \leq j < n)$, $h_n(\alpha) \leq 0$ and $h_n(\alpha) = 0$ if and only if $\alpha_n = \mu_n$, we see that $c(\alpha,j,k) \neq 0$ if and only if $\beta^{-k} \in S_m$ and $\alpha^{-j} \in N_\ell(\beta^{-k})$. Letting $\boldsymbol{Z}(\beta,i) = \boldsymbol{Z}(\beta) \cap \{n-i,\dots,n-1\}$ and rewriting α^{-j} as α, we have from (9.11) that

$$\pi_A(E_{n+1,i})v = \sum_{k \in \boldsymbol{Z}(\beta,i)} \sum_{\alpha \in N_\ell(\beta^{-k})} \sum_{\alpha^{+j} \in N_\ell(\beta)} c_{\alpha^{+j}} c(\alpha^{+j},j,k) d(k) v(\boldsymbol{M}_{\alpha,\beta}^{0,k}). \tag{9.12}$$

Therefore $\pi_A(E_{n+1,i})v = 0$ implies that we have for $k \in \boldsymbol{Z}(\beta,i)$ and $\alpha \in N_\ell(\beta^{-k})$ that

$$\sum_{\alpha^{+j} \in N_\ell(\beta)} c_{\alpha^{+j}} c(\alpha^{+j},j,k) = 0. \tag{9.13}$$

From this, we see that it suffices to determine c_α such that $\pi_A(E_{n+1,n-1})v = 0$. To determine c_α, we use arguments similar to those in [**14**, Theorem 3.1].

Lemma 9.2. *Let* $\ell \in S_c$ and $\beta \in S_m$.

(1) *If* $\ell \in S_c^+$, *then there exists* $v = \sum_{\alpha \in N_\ell(\beta)} c_\alpha v(\boldsymbol{M}_{\alpha,\beta}) \in V_\ell(\beta)$ *such that* $\pi_A(E_{n+1,n-1})v = 0$. *Moreover, such a* v *is unique up to a scalar factor.*

(2) *If* $\ell \in S_c^-$ *and* $|\beta| \geq |\lambda_m| - \ell$, *then there exists* $v = \sum_{\alpha \in N_\ell(\beta)} c_\alpha v(\boldsymbol{M}_{\alpha,\beta}) \in V_\ell(\beta)$ *such that* $\pi_A(E_{n+1,n-1})v = 0$. *Moreover, such a* v *is unique up to a scalar factor.*

Proof. (1) We obtain from (9.7) that

$$N_\ell(\beta) = \{\alpha \in D_K : \alpha_{\geq 2} \in S_m, \ \beta < \alpha\}. \tag{9.14}$$

Then setting $N_\ell(\beta,p) = \{\alpha \in N_\ell(\beta) : \alpha_1 = p\}$, we have

$$N_\ell(\beta) = \bigcup_{\mu_1+\ell+|\beta|-|\lambda_m| \leq p \leq \ell} N_\ell(\beta,p).$$

We first remark the following fact. For $\lambda \in N_\ell(\beta,p)$, we put

$$\boldsymbol{Z}(\lambda,\beta) = \{k \in \boldsymbol{Z}_{>0} : k \in \boldsymbol{Z}(\beta), \lambda^{-1} \in N_\ell(\beta^{-k})\}.$$

Then setting $\alpha = \lambda^{-1}$, we have from (9.13) that

$$c_\lambda c(\lambda,1,k) + \sum_{j \in \boldsymbol{Z}(\lambda,\beta)} c_{\alpha^{+j+1}} c(\alpha^{+j+1}, j+1, k) = 0, \quad (k \in \boldsymbol{Z}(\lambda,\beta)). \tag{9.15}$$

By the orthogonality relations of the Clebsch-Gordan coefficients, it is easy to check that $c(\alpha^{+j+1}, j+1, k)$ are linearly independent and thus we can get $c_{\alpha^{+j+1}}$ $(j \in \boldsymbol{Z}(\lambda,\beta))$ from the above simultaneous equations.

We can find the constants c_α $(\alpha \in N_\ell(\beta))$ by induction on α_1. Let $\alpha_f = (\mu_1 + \ell + |\beta| - |\lambda_m|, \mu_2, \dots, \mu_n)$. We first choose c_{α_f} as an arbitrary nonzero real number. Suppose that c_α are determined for all $\alpha \in N_\ell(\beta,p)$. For $\alpha \in N_\ell(\beta,p-1)$, we pick $k \in \boldsymbol{Z}_{>0}$ so that $\alpha_{\geq 2}^{-k} \in S_m$. Then setting $\lambda = (\alpha^{+1})^{-k} \in N_\ell(\beta,p)$, we can get c_α from the simultaneous equations (9.15). By the orthogonality relations of the Clebsch-Gordan coefficients, it is easy to check that c_α is independent of the choice of k.

(2) Because $\beta \in S_m$ and $\ell \in S_c^-$, we have from (9.7) that

$$N_\ell(\beta) = \{\alpha \in D_K : \alpha_{\geq 2} \in S_m, \beta < \alpha, |\alpha_{\geq 2}| \leq |\beta| + \ell - \mu_1\}. \tag{9.16}$$

Thus $N_\ell(\beta) = \varnothing$ if $|\beta| < |\lambda_m| - \ell$. By a similar way to (1), we can also determine the constants c_α satisfying $\mu_1 + \ell + |\beta| - |\lambda_m| \leq \alpha_1 \leq \mu_1$. □

For $\ell \in S_c$ and $\beta \in S_m$, we choose v as in Lemma 9.2. We denote by $\mathscr{V}(\ell,\beta)(K)$ the $\pi_\Lambda(K_1)$-invariant subspace of $V_\Lambda(K)$ containing $\{\pi_\Lambda(F_{n-1})^j v : j \in \boldsymbol{Z}_{\geq 0}\}$. Then $\pi_\Lambda(E_{n-1})v = 0$ implies $\mathscr{V}(\ell,\beta)(K) = \sum_{\beta' < (\beta,-\infty)} V_{(-(n+1)\ell,\beta')}$. Taking into account $m(\ell,\beta) \leq m(\ell,\beta_\ell)$ for $\beta \notin S_m$, we obtain from (9.7) that

$$V_\Lambda(K) = \sum_{\substack{\ell \in S_c^+ \\ \beta \in S_m}} \mathscr{V}(\ell,\beta)(K) + \sum_{\substack{\ell \in S_c^- \\ \beta \in S_m \\ |\beta| \geq |\lambda_m| - \ell}} \mathscr{V}(\ell,\beta)(K). \tag{9.17}$$

We shall next consider the case of antiholomorphic discrete series. Fix $\mu \in D_M$ and $\nu \in \mathfrak{a}^*$ so that the condition indicated in Corollary 8.4(2) is fulfilled. For simplicity we set $\mu_0 = -(\nu + n + |\sigma_\mu|)/2$. In this case, if $\alpha \in S_\Lambda$, then $k_j(\alpha) < 0$ $(1 < j \leq n)$, $h_j(\alpha) > 0$ $(1 \leq j \leq n)$ and $k_1(\alpha) \leq 0$. Moreover, $k_1(\alpha) = 0$ if and only if $\alpha_1 = \mu_0$. We have from Corollary 8.4(2) that

$$V_\Lambda(K) = \sum_{\alpha < (\lambda_m, -\infty)} V_\alpha = \sum_{\alpha < (\lambda_m,-\infty)} \sum_{\beta < \alpha} V_\alpha(\beta) = \sum_{\substack{\ell \in \tilde{S}_c \\ \beta \in \tilde{S}}} \sum_{\substack{\beta < \alpha < (\lambda_m,-\infty) \\ |\alpha| - |\beta| = \ell}} V_\alpha(\beta), \tag{9.18}$$

where

$$\tilde{S} = \left\{\beta \in D_M : \mu_{j-1} \geq \beta_j \geq \mu_{j+1}, (1 \leq j \leq n-2), \mu_{n-2} \geq \beta_{n-1}\right\},$$

$$\tilde{S}_c = \left\{\ell \in \frac{1}{n+1}\boldsymbol{Z}_{\geq 0} : \mu_0 - \ell \in \boldsymbol{Z}_{\geq 0}\right\}.$$

For our convenience, we introduce the following notation:

$$\tilde{S}_m = \{\beta \in \tilde{S} : \beta < \lambda_m\},$$

$$\tilde{S}_c^+ = \{\ell \in \tilde{S}_c : \ell - \mu_1 \in \boldsymbol{Z}_{\geq 0}\}, \quad \tilde{S}_c^- = \tilde{S}_c \backslash \tilde{S}_c^+.$$

For $\beta \in \tilde{S}_m$, let

$$\tilde{\boldsymbol{Z}}(\beta) = \{k \in \boldsymbol{Z}_{>0} : 1 \le k \le n-1,\, \beta^{-k} \in \tilde{S}_m\},$$
$$\tilde{N}_\ell(\beta) = \{\alpha \in D_K : \beta < \alpha < (\lambda_m, -\infty), |\alpha| - |\beta| = \ell\}, \quad m(\ell,\beta) = \mathrm{Card}\, \tilde{N}_\ell(\beta),$$
$$W_\ell(\beta) = \sum_{\alpha \in \tilde{N}_\ell(\beta)} V_\alpha(\beta).$$

For $\beta \in \tilde{S}$, let

$$\beta_f = (\max(\beta_1, \mu_1), \dots, \max(\beta_{n-1}, \mu_{n-1})),$$
$$\beta_\ell = (\mu_0, \min(\beta_1, \mu_1), \dots, \min(\beta_{n-2}, \mu_{n-2})).$$

By the same reason as in the case of holomorphic discrete series, we have

$$\tilde{N}_\ell(\beta) = \{\alpha \in D_K : \alpha_{\le n-2} \in \tilde{S}_m,\, \beta_f < (\alpha_{\le n-2}, -\infty), \alpha_{\le n-2} < (\beta_\ell, -\infty), |\alpha_{\le n-2}| \le |\beta| + \ell - \min(\beta_{n-1}, \mu_{n-1})\}. \tag{9.19}$$

and thus $m(\ell,\beta) \le m(\ell,\beta_\ell)$. For this reason, we write (9.18) as follows:

$$V_A(K) = \sum_{\substack{\ell \in \tilde{S}_c^+ \\ \beta \in \tilde{S}_m}} W_\ell(\beta) + \sum_{\substack{\ell \in \tilde{S}_c^+ \\ \beta \notin \tilde{S}_m}} W_\ell(\beta) + \sum_{\substack{\ell \in \tilde{S}_c^- \\ \beta \in \tilde{S}_m}} W_\ell(\beta) + \sum_{\substack{\ell \in \tilde{S}_c^- \\ \beta \notin \tilde{S}_m}} W_\ell(\beta). \tag{9.20}$$

Assume that an eigenvector v is represented as $v = \sum_{\alpha \in M_\ell(\beta)} c_\alpha v(\tilde{\boldsymbol{M}}_{\alpha,\beta})$, $(c_\alpha \in \boldsymbol{C})$. Then by arguments similar to those in the case of holomorphic discrete series, for v being an eigenvector, it suffices to determine c_α such that $\pi_A(E_{n-1,n+1})v = 0$.

LEMMA 9.3. *Let $\ell \in \tilde{S}_c$ and $\beta \in \tilde{S}_m$.*

(1) *If $\ell \in \tilde{S}_c^+$, then there exists $v = \sum_{\alpha \in \tilde{N}_\ell(\beta)} c_\alpha v(\tilde{\boldsymbol{M}}_{\alpha,\beta}) \in W_\ell(\beta)$ such that $\pi_A(E_{n-1,n+1})v = 0$. Moreover, such a v is unique up to a scalar factor.*

(2) *If $\ell \in \tilde{S}_c^-$ and $|\beta| \ge |\lambda_m| - \ell$, then there exists $v = \sum_{\alpha \in \tilde{N}_\ell(\beta)} c_\alpha v(\tilde{\boldsymbol{M}}_{\alpha,\beta}) \in W_\ell(\beta)$ such that $\pi_A(E_{n-1,n+1})v = 0$. Moreover, such a v is unique up to a scalar factor.*

For $\ell \in \tilde{S}_c$ and $\beta \in \tilde{S}_m$, we choose v as in Lemma 9.3. We denote by $\mathscr{W}(\ell,\beta)(K)$ the $\pi_A(K_1)$-invariant subspace of $V_A(K)$ containing $\{\pi_A(E_{n-1})^j v : j \in \boldsymbol{Z}_{\ge 0}\}$. Then $\pi_A(F_{n-1})v = 0$ implies $\mathscr{W}(\ell,\beta)(K) = \sum_{\beta < (\beta', -\infty)} V_{(-(n+1)\ell, \beta')}$. Therefore we obtain from (9.19) that

$$V_A(K) = \sum_{\substack{\ell \in \tilde{S}_c^+ \\ \beta \in \tilde{S}_m}} \mathscr{W}(\ell,\beta)(K) + \sum_{\substack{\ell \in \tilde{S}_c^- \\ \beta \in \tilde{S}_m \\ |\beta| \ge |\lambda_m| - \ell}} \mathscr{W}(\ell,\beta)(K). \tag{9.21}$$

We summarize these into the following theorem.

THEOREM 9.4. *Let $\mathscr{V}(\ell,\beta)$ and $\mathscr{W}(\ell,\beta)$ denote the completions of $\mathscr{V}(\ell,\beta)(K)$ and $\mathscr{W}(\ell,\beta)(K)$ relative to $\langle \cdot, \cdot \rangle_i$, respectively.*

(1) *The holomorphic discrete series (π_A, V_A) is decomposed with no multiplicity as follows*:

$$V_A = \sum_{\substack{\ell \in S_c^+ \\ \beta \in S_m}} \mathscr{V}(\ell,\beta) + \sum_{\substack{\ell \in S_c^- \\ \beta \in S_m \\ |\beta| \geq |\lambda_m| - \ell}} \mathscr{V}(\ell,\beta).$$

The Blattner parameter of $\mathscr{V}(\ell,\beta)$ is $(-(n+1)\ell,\beta)$.

(2) *The antiholomorphic discrete series* (π_A, V_A) *is decomposed with no multiplicity as follows*:

$$V_A = \sum_{\substack{\ell \in \tilde{S}_c^+ \\ \beta \in \tilde{S}_m}} \mathscr{W}(\ell,\beta) + \sum_{\substack{\ell \in \tilde{S}_c^- \\ \beta \in \tilde{S}_m \\ |\beta| \geq |\lambda_m| - \ell}} \mathscr{W}(\ell,\beta).$$

The Blattner parameter of $\mathscr{W}(\ell,\beta)$ *is* $(-(n+1)\ell,\beta)$.

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