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A combinatorial approach to the conjugacy classes of the Mathieu simple groups, M_{24}, M_{23}, M_{22}

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Abstract. We determine the conjugacy classes of the Mathieu simple groups by using the combinatorial properties of the Steiner system and the binary Golay code.

0. Introduction.

Around 1860, Mathieu discovered the first five sporadic simple groups as multiply transitive permutation groups. In the 1970's, Conway [**2**], Curtis [**3**] et al. defined the Mathieu group M_{24} of degree 24 using the Binary Golay Code and the Steiner system $S(5,8,24)$. The methods in Conway [**2**] and Curtis [**3**] are very useful to study the structure of M_{24}. In 1904, Frobenius [**4**] determined the conjugacy classes of M_{24} and its character table. However he did not mention the way to classify the conjugacy classes of M_{24}. No such work has been published elsewhere. The purpose of this paper is to classify all the conjugacy classes of M_{24}, M_{23}, M_{22} using the combinatorial methods by Kondo [**5**] who determined the conjugacy classes of elements of orders 2 and 3 using its combinatorial properties. Aschbacher [**1**] also determined the conjugacy classes of elements of orders 2 and 3.

In section 1 we describe a number of basic results of M_{24} and Kondo's results which will be applied in our computations. In section 2 we determine the types of the elements in M_{24}, and in section 3 we investigate elements of order 4. In the last two sections 4 and 5, we determine the conjugacy classes of M_{24}, M_{23} and M_{22}.

1. Preliminaries.

For a finite set X, let S_X be the group of all permutations on X, and A_X be the group of all even permutations on X. Throughout this paper, permutations are multiplied from the left to the right, and $\boldsymbol{\Omega}$ denotes the set of 24 points.

Let $\mathscr{P}(\boldsymbol{\Omega})$ be the power set of $\boldsymbol{\Omega}$. $\mathscr{P}(\boldsymbol{\Omega})$ is a 24-dimensional vector space over GF(2) with $X+Y := X \cup Y - X \cap Y$ for $X, Y \in \mathscr{P}(\boldsymbol{\Omega})$. The *Binary Golay Code* $\boldsymbol{\Gamma}$ is a subspace of $\mathscr{P}(\boldsymbol{\Omega})$ which satisfies the following conditions:

$$\boldsymbol{\Gamma} \ni X \neq \varnothing \implies |X| \geq 8$$

$$\dim \boldsymbol{\Gamma} = 12.$$

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The code $\boldsymbol{\Gamma}$ exists and is uniquely determined up to isomorphism. Let $\mathcal{O} := \{X \in \boldsymbol{\Gamma} \mid |X| = 8\}$. Then $(\boldsymbol{\Omega}, \mathcal{O})$ forms a Steiner system $S(5, 8, 24)$ on $\boldsymbol{\Omega}$, that is, each 5-point subset of $\boldsymbol{\Omega}$ is contained in a unique element of $\mathcal{O}$. Elements of $\mathcal{O}$ are called octads.

DEFINITION. Let $\mathrm{Aut}(\boldsymbol{\Omega}, \mathcal{O}) := \{\sigma \in S_{\boldsymbol{\Omega}} \mid \mathcal{O}^{\sigma} = \mathcal{O}\}$. $\mathrm{Aut}(\boldsymbol{\Omega}, \mathcal{O})$ is called *the Mathieu group of degree* 24, which will be denoted by M_{24}. Let M_{24-i} be the stabilizer of i points in M_{24} for $i = 1, 2$. Since M_{24} is 5-fold transitive on $\boldsymbol{\Omega}$, the structure of M_{24-i} does not depend on a choice of i points. M_{24-i} is called *the Mathieu group of degree* $24 - i$.

LEMMA 1.1 ([2]). *For* $C_1 \neq C_2 \in \mathcal{O}$, $|C_1 \cap C_2| = 0, 2$ *or* 4.

DEFINITION. A partition $\boldsymbol{\Omega} = T_1 \cup T_2 \cup \cdots \cup T_6$ of $\boldsymbol{\Omega}$ into a 6-tuple of 4-point subsets is a *sextet* if and only if $T_i \cup T_j \in \mathcal{O}$ for any i, j $(i \neq j)$. Each T_i is called a *sextet component*.

DEFINITION ([2]). Let X be a subset of $\boldsymbol{\Omega}$.
(1) X is *special* if and only if there is an octad containing X.
(2) X is *non-special* if and only if there is no octad containing X.

DEFINITION ([5]). An ordered sequence $(x_1 x_2 \cdots x_7)$ of 7 mutually distinct points of $\boldsymbol{\Omega}$ is an *M-sequence* if and only if $\{x_1, x_2, \ldots, x_6\}$ is non-special and $\{x_2, x_3, \ldots, x_7\}$ is special.

DEFINITION ([5]). A 4×6 matrix $\mathscr{X}$ whose entries are distinct points of $\boldsymbol{\Omega}$ is an *M-matrix* if and only if $\mathscr{X}$ satisfies the following conditions:
(1) The partition of $\boldsymbol{\Omega}$ into 6 columns of $\mathscr{X}$ is a sextet.
(2) In the following 6 pictures, each 8-point subset of $\boldsymbol{\Omega}$ forms an octad.

$$\begin{pmatrix} & * & * & * & * & * \\ * & & & & & \\ * & & & & & \\ * & & & & & \end{pmatrix}, \quad \begin{pmatrix} * & * & * & * & & \\ * & * & * & * & & \\ & & & & & \\ & & & & & \end{pmatrix}, \quad \begin{pmatrix} * & * & * & * & & \\ & & & & & \\ * & * & * & * & & \\ & & & & & \end{pmatrix},$$

$$\begin{pmatrix} & & * & * & * & * \\ & & * & * & * & * \\ & & & & & \\ & & & & & \end{pmatrix}, \quad \begin{pmatrix} & & * & * & * & * \\ & & & & & \\ & & & & & \\ & & * & * & * & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * & & & \\ & & & * & & \\ * & & & & * & \\ * & & & & & * \end{pmatrix}.$$

The following theorem is the essential part of the combinatorial method of M_{24}.

THEOREM 1.2 ([5]). (1) *For an M-sequence* $(x_1 x_2 \cdots x_7)$, *there exists a unique M-matrix* $\mathscr{X}$ *such that*

$$\mathscr{X} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_7 & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}.$$

(2) *Let* $\mathscr{X}(i,j)$ *and* $\mathscr{Y}(i,j)$ $(1 \le i \le 4, 1 \le j \le 6)$ *be the* (i,j)*-entries of* M*-matrices* $\mathscr{X}$ *and* $\mathscr{Y}$ *respectively. Then a map* $\mathscr{X}(i,j) \mapsto \mathscr{Y}(i,j)$ *is an isomorphism of a Steiner system* $S(5,8,24)$, *that is, a position of each octad is uniquely determined, and does not depend on an* M*-matrix.*

Corollary 1.3 ([**5**]). *The Steiner system* $S(5,8,24)$ *is unique up to relabelling the points.*

Corollary 1.4 ([**5**]). M_{24} *acts regularly on the set of all* M*-sequences. In particular,*

$$|M_{24}| = \sharp\{\textit{all } M\textit{-sequences}\} = \binom{24}{5} \cdot 5! \cdot 3 \cdot 16 = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23.$$

This theorem shows that the cardinality of the set $\{(ax_1 \cdots x_6) : M\text{-sequence} \mid x_i \in \boldsymbol{\Omega}\}$ is $|M_{23}|$ and the cardinality of the set $\{(abx_1 \cdots x_5) : M\text{-sequence} \mid x_i \in \boldsymbol{\Omega}\}$ is $|M_{22}|$ for $a, b \in \boldsymbol{\Omega}$. These do not depend on the positions of a and b.

Theorems 1.5–1.11 are well-known.

Theorem 1.5 ([**2**]). (1) M_{24} *acts transitively on the set* $\mathcal{O}$ *of* 759 *octads.*
(2) M_{24} *acts transitively on the set of all sextets.*

Theorem 1.6 ([**2**]). *Let* $C \in \mathcal{O}$ *and*

$$H = H(C) := \{\sigma \in M_{24} \mid C^\sigma = C\}$$

$$N = N(C) := \{\sigma \in M_{24} \mid x^\sigma = x \ \ (\forall x \in C)\}.$$

Then the following holds:
(1) $N \simeq Z_2 \times Z_2 \times Z_2 \times Z_2$ *and* N *acts regularly on* $\boldsymbol{\Omega} - C$.
(2) $H/N \simeq A_8 \simeq GL(4,2)$ *and* H *splits over* N.

Corollary 1.7. *For* $\tau \in A_C$, *there are sixteen elements in* M_{24} *which contain* τ *in cycle notation.*

Theorem 1.8 ([**2**]). *Let* Y *be a non-special* 6*-point subset of* $\boldsymbol{\Omega}$. *Set*

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(Y) := \{\sigma \in M_{24} \mid Y^\sigma = Y\}$$

$$\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}_0(Y) := \{\sigma \in M_{24} \mid y^\sigma = y \ \ (\forall y \in Y)\}.$$

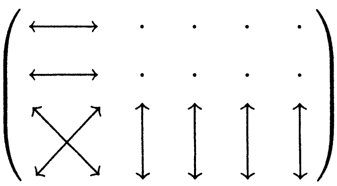
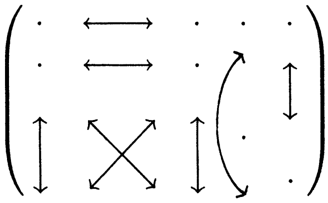
Then the following holds:
(1) $|\boldsymbol{\Sigma}_0| = 3$.
(2) $\boldsymbol{\Sigma}/\boldsymbol{\Sigma}_0 \simeq S_6$ *and* $\boldsymbol{\Sigma}$ *does not split over* $\boldsymbol{\Sigma}_0$.

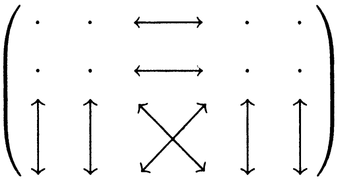
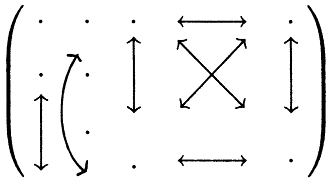
Corollary 1.9. *For* $\tau \in S_Y$, *there are three elements in* M_{24} *which contain* τ *in cycle notation.*

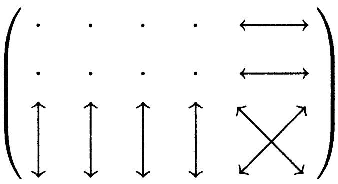
Let $Y = \{y_1, y_2, \ldots, y_6\}$ be a non-special 6-point subset of $\boldsymbol{\Omega}$. Then there exists an M-matrix $\mathscr{Y}$ with the first row $y_1, y_2, \ldots, y_6$. We give names $1, 2, \ldots, 24$ for the 24 points of $\boldsymbol{\Omega}$.

$$\mathscr{Y} = \begin{pmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \\ 19 & 20 & 21 & 22 & 23 & 24 \end{pmatrix}.$$

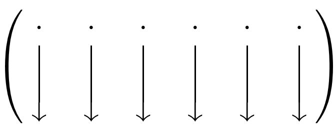
The following 5 permutations generate $\mathbf{\Sigma} = \mathbf{\Sigma}(Y)$.

$a_1 =$ , $a_2 =$ ,

$a_3 =$ , $a_4 =$ ,

$a_5 =$ .

Furthermore the following permutation generates $\mathbf{\Sigma}_0 = \mathbf{\Sigma}_0(Y)$.

$\alpha := (a_4a_5)^{-1}(a_1a_2)^{-1}(a_4a_5)(a_1a_2) =$ .

THEOREM 1.10 ([**2**]). *Let $\mathscr{T} = \{T_1, T_2, \dots, T_6\}$ be a sextet. Set*

$$K := \{\sigma \in M_{24} \mid \mathscr{T}^\sigma = \mathscr{T}\}$$

$$K_0 := \{\sigma \in M_{24} \mid (T_i)^\sigma = T_i \ \ (1 \le \forall i \le 6)\}.$$

Then the following holds:

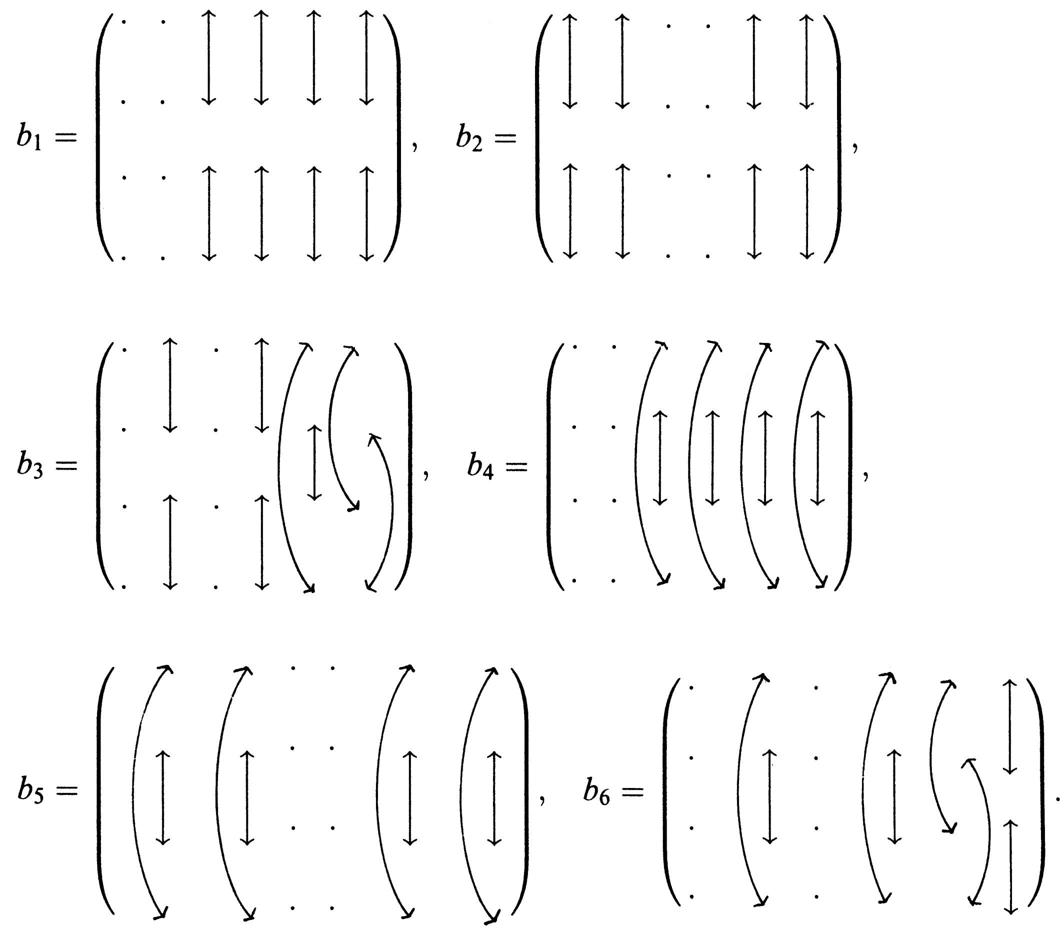
(1) *$K/K_0 \simeq S_6$ and K does not split over K_0.*

(2) *$|K_0| = 2^6 \cdot 3$, and K_0 has a unique elementary abelian Sylow 2-subgroup K_2.*

(3) *$K/K_2 \simeq \mathbf{\Sigma}(Y)$ (See Theorem 1.8) and K splits over K_2.*

By Theorem 1.5 (2), K does not depend on the sextet $\mathscr{T}$. Hence we may assume that $\mathscr{T}$ is the partition of $\boldsymbol{\Omega}$ into 6 columns of the M-matrix $\mathscr{Y}$.

The following 6 permutations generate K_2.



Theorem 1.11 ([**3**], [**5**]). *For an element $\sigma \in M_{24}$ of type (2^{12}), there exists a unique sextet all of whose components are fixed by σ.*

Theorem 1.12 ([**5**]). *For an element $1 \neq \sigma \in M_{24}$ fixing 6 points $x_1, x_2, \dots, x_6$ of $\boldsymbol{\Omega}$, one of the following holds*:

(1) *If $\{x_1, x_2, \dots, x_6\}$ is special, then σ is of type $(2^8 \cdot 1^8)$ such that the set of fixed points of σ forms an octad.*

(2) *If $\{x_1, x_2, \dots, x_6\}$ is non-special, then σ is of type $(3^6 \cdot 1^6)$.*

2. The types of the elements in M_{24}.

M_{24} is of order $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ by Corollary 1.4 and is a subgroup of S_{24}. In this section, we will study the types of the elements in M_{24}. Aschbacher [**1**] and Kondo [**5**] determined the conjugacy classes of elements of orders 2 and 3 using its combinatorial properties.

Theorem 2.1 ([**1**], [**4**], [**5**]). (1) *Each involution of M_{24} is a permutation on $\boldsymbol{\Omega}$ of type $(2^8 \cdot 1^8)$ or (2^{12}), and the orders of the centralizers are, respectively, $2^{10} \cdot 3 \cdot 7$ and $2^9 \cdot 3 \cdot 5$.*

(2) *Each element of M_{24} of order 3 is a permutation on $\boldsymbol{\Omega}$ of type $(3^6 \cdot 1^6)$ or (3^8), and the orders of the centralizers are, respectively, $2^3 \cdot 3^3 \cdot 5$ and $2^3 \cdot 3^2 \cdot 7$.*

Table 1. M_{24}

type	number of the conjugacy classes	order of the centralizer
$(2^8 \cdot 1^8)$	1	21504
(2^{12})	1	7680
$(3^6 \cdot 1^6)$	1	1080
(3^8)	1	504

Let σ be an element of order 5. By Theorem 1.12, σ fixes at most 5 points. It follows that σ is of type $(5^4 \cdot 1^4)$. Similarly elements of orders 7, 11, and 23 are, respectively, of types $(7^3 \cdot 1^3)$, $(11^2 \cdot 1^2)$ and $(23 \cdot 1)$. By Theorem 2.1 (1), there is an element σ of order 14. If σ contains a 14-cycle, then σ is of type $(14 \cdot 7 \cdot 2 \cdot 1)$. If σ is of type $(7^{e_1} \cdot 2^{e_2} \cdot 1^{e_3})$ $(e_1 \geq 1, e_2 \geq 1, e_3 \geq 0)$, then σ^7 is of type $(2^{e_2} \cdot 1^{7e_1+e_3})$. By Theorem 1.12, we have $e_2 = 8$ and $7e_1 + e_3 = 8$. It follows that σ is of type $(7 \cdot 2^8 \cdot 1)$ *i.e.* σ^2 is of type $(7 \cdot 1^{17})$. This contradicts the type of an element of order 7. This yields that an element of order 14 is of type $(14 \cdot 7 \cdot 2 \cdot 1)$. Similarly there are elements of orders 10, 15 and 21, and these elements are, respectively, of types $(10^2 \cdot 2^2)$, $(15 \cdot 5 \cdot 3 \cdot 1)$ and $(21 \cdot 3)$. Next we will consider the elements in the sextet stabilizer K (See Theorem 1.10).

$$a_3a_2a_1b_1 = (1,2,9,4)(3,10,7,8)(5,11,23,17)(6,12,18,24)(13,20,21,22)(14,15,16,19)$$

$$a_5a_3a_2a_1 = (1,2,3,4)(7,8,9,10)(13,14,15,16)(19,20,21,22)(5,6)(11,24)(12,17)(18,23)$$

$$a_3a_2a_1 = (1,2,3,4)(7,8,9,10)(13,20,15,22)(14,21,16,19)(11,17)(12,24)(5)(6)(18)(23)$$

$$a_5a_4a_3a_2a_1 = (1,2,3,4,5,6)(7,8,15,16,23,24)(9,22,17,12,19,14)(10,11,18,13,20,21)$$

$$a_2a_1a_4 = (7,20,21,13,14,9)(10,17,22,23,16,11)(1,2,3)(8,15,19)(4,5)(12,24)(6)(18)$$

$$a_3a_2a_1b_1\alpha = (1,2,15,22,19,20,9,4)(3,16,7,14,21,10,13,8)(6,18,12,24)(5,17)(11)(23)$$

$$a_5a_4a_3a_2a_1b_1b_2 = (1,8,21,4,5,6,7,2,9,16,23,24)(3,10,11,18,19,20,15,22,17,12,13,14)$$

$$a_5a_3a_2a_1\alpha = (7,14,21,10,13,20,9,16,19,8,15,22)(11,12,23,24,17,18)(1,2,3,4)(6,5)$$

Hence we have the following:

THEOREM 2.2. (1) *The elements of orders* 5, 7, 10, 11, 14, 15, 21 *and* 23 *are, respectively, of types* $(5^4 \cdot 1^4)$, $(7^3 \cdot 1^3)$, $(10^2 \cdot 2^2)$, $(11^2 \cdot 1^2)$, $(14 \cdot 7 \cdot 2 \cdot 1)$, $(15 \cdot 5 \cdot 3 \cdot 1)$, $(21 \cdot 3)$ *and* $(23 \cdot 1)$.

(2) *There are elements of types* (4^6), $(4^4 \cdot 2^4)$, $(4^4 \cdot 2^2 \cdot 1^4)$, (6^4), $(6^2 \cdot 3^2 \cdot 2^2 \cdot 1^2)$, $(8^2 \cdot 4 \cdot 2 \cdot 1^2)$, (12^2) *and* $(12 \cdot 6 \cdot 4 \cdot 2)$.

3. The relation between elements of order 4 and sextet components.

In this section, we will investigate the relation between elements of order 4 and sextet components. Let X be a subset of $\boldsymbol{\Omega}$ with $|X| = 4$. Then there exists a unique sextet containing X as a sextet component.

LEMMA 3.1. (1) *Let* $\sigma = (x_1, x_2, x_3, x_4) \cdots$ *be an element of type* (4^6) *in* M_{24}. *Then there exists a unique* 4*-cycle* (x_5, x_6, x_7, x_8) *of* σ *such that* $\{x_1, x_2, \dots, x_8\} \in \mathcal{O}$.

(2) *Let* $\mathcal{S}$ *be the sextet containing* $\{x_1, x_2, x_3, x_4\}$ *and* $\{x_5, x_6, x_7, x_8\}$. *Then* σ *induces a permutation of type* $(1^2 \cdot 4^1)$ *on the sextet components.*

PROOF. Let $\mathcal{S}$ be the sextet containing $X = \{x_1, x_2, x_3, x_4\}$, and let $\{X, S_1, S_2, S_3, S_4, S_5\}$ be the components of $\mathcal{S}$. Since σ acts on X and $\sigma^4 = 1$, we may assume that σ induces a permutation on the components as follows:

(a) $$(X)(S_1)(S_2)(S_3)(S_4)(S_5)$$

(b) $$(X)(S_1)(S_2)(S_3)(S_4, S_5)$$

(c) $$(X)(S_1)(S_2, S_3)(S_4, S_5)$$

(d) $$(X)(S_1)(S_2, S_3, S_4, S_5).$$

If σ induces (a), then $\sigma \in K_0$ (See Theorem 1.10). This contradicts that σ is of order 4. Next suppose that σ induces (b) or (c). Let U_1 be a σ-orbit on $S_4 \cup S_5$, and $\mathcal{U} = \{U_1, U_2, \dots, U_6\}$ be the sextet containing U_1. We may assume that $S_4 \cup S_5 = U_1 \cup U_2$. Then σ fixes two components U_1 and U_2. Assume that σ induces a permutation on $\mathcal{U}$ as follows:

$$\sigma = (U_1)(U_2)(U_3, U_4, U_5, U_6).$$

Since $|S_i \cap U_j| = 2$ $(i = 4, 5,\ j = 1, 2)$ and $|U_j \cap X| = 1$ $(j = 3, 4, 5, 6)$, we have $|(S_4 \cup X) \cap (U_1 \cup U_3)| = 3$. This is a contradiction by Lemma 1.1. It follows that σ induces a permutation on $\mathcal{U}$ of type $(1^4 \cdot 2^1)$ or $(1^2 \cdot 2^2)$. Then σ^2 is of type (2^{12}) and fixes every component of the two sextets $\mathcal{S}$ and $\mathcal{U}(\mathcal{S} \neq \mathcal{U})$. This is a contradiction by Theorem 1.11. Hence σ induces the permutation (d) on $\mathcal{S}$. This completes the proof.

LEMMA 3.2. *Let* $\sigma = (x_1, x_2, x_3, x_4) \cdots$ *be an element of type* $(4^4 \cdot 2^4)$ *in* M_{24}. *Then there exists a* 4*-cycle* (x_5, x_6, x_7, x_8) *of* σ *such that* $\{x_1, x_2, \dots, x_8\} \in \mathcal{O}$.

PROOF. By the same way as in the proof of Lemma 3.1, σ fixes at least two components $X = \{x_1, x_2, x_3, x_4\}$ and Y of the sextet containing X. Since σ acts on the octad $X \cup Y$ as A_8, σ acts on Y as 4-cycle. This completes the proof.

LEMMA 3.3. *Let $\sigma = (x_1, x_2, x_3, x_4) \cdots$ be an element of type $(4^4 \cdot 2^2 \cdot 1^4)$ in M_{24}. Then there exists a 4-cycle (x_5, x_6, x_7, x_8) of σ such that $\{x_1, x_2, \dots, x_8\} \in \mathcal{O}$.*

PROOF. The fixed points of σ^2 form an octad, and which is a union of two components of the sextet containing $\{x_1, x_2, x_3, x_4\}$. Since σ is of order 4, the proof is complete.

4. The conjugacy classes of M_{24}.

In this section, we will classify all the conjugacy classes of M_{24} which are given in TABLE 2.

TABLE 2. M_{24}

type	number of the conjugacy classes	order of the centralizer
$(4^4 \cdot 2^4)$	1	384
$(4^4 \cdot 2^2 \cdot 1^4)$	1	128
(4^6)	1	96
$(5^4 \cdot 1^4)$	1	60
$(6^2 \cdot 3^2 \cdot 2^2 \cdot 1^2)$	1	24
(6^4)	1	24
$(7^3 \cdot 1^3)$	2	42, 42
$(8^2 \cdot 4 \cdot 2 \cdot 1^2)$	1	16
$(10^2 \cdot 2^2)$	1	20
$(11^2 \cdot 1^2)$	1	11
$(12 \cdot 6 \cdot 4 \cdot 2)$	1	12
(12^2)	1	12
$(14 \cdot 7 \cdot 2 \cdot 1)$	2	14, 14
$(15 \cdot 5 \cdot 3 \cdot 1)$	2	15, 15
$(21 \cdot 3)$	2	21, 21
$(23 \cdot 1)$	2	23, 23

Types $(5^4 \cdot 1^4), (4^4 \cdot 2^2 \cdot 1^4)$ **and** $(6^2 \cdot 3^2 \cdot 2^2 \cdot 1^2)$.

THEOREM 4.1. *All elements of type $(5^4 \cdot 1^4)$ form one conjugacy class in M_{24}, and the order of the centralizer of the element in M_{24} is* 60.

PROOF. Let

$$\sigma = (x_1)(x_2)(x_3)(x_4)(y_1, y_2, y_3, y_4, y_5) \cdots$$

$$\tau = (s_1)(s_2)(s_3)(s_4)(t_1, t_2, t_3, t_4, t_5) \cdots$$

be elements of type $(5^4 \cdot 1^4)$. Let $C \in \mathcal{O}$ such that $C \supseteq Y := \{y_1, y_2, y_3, y_4, y_5\}$. Since $C^\sigma = C$, we may assume $C = Y \cup \{x_2, x_3, x_4\}$. It follows that $(x_1 x_2 y_1 y_2 y_3 y_4 y_5)$ is an M-sequence. Similarly we may assume that $(s_1 s_2 t_1 t_2 t_3 t_4 t_5)$ is an M-sequence. There exists a unique element $\rho \in M_{24}$ such that $\rho : s_i \mapsto x_i$ $(i = 1, 2), \rho : t_i \mapsto y_i$ $(1 \le i \le 5)$ by Corollary 1.4. Since $(\rho^{-1}\tau\rho)\sigma^{-1} = (x_1)(x_2)(y_1)(y_2)(y_3)(y_4)(y_5)\cdots$, $\rho^{-1}\tau\rho = \sigma$ by Corollary 1.4. It follows that all elements of type $(5^4 \cdot 1^4)$ are conjugate in M_{24}. Let

$$\mathfrak{M} = \left\{ (\sigma, (x_1 x_2 x_3 x_4 x_5 x_6 x_7)) \;\middle|\; \begin{array}{l} M_{24} \ni \sigma = (5^4 \cdot 1^4)\text{-type} \\ (x_1 x_2 x_3 x_4 x_5 x_6 x_7) : M\text{-sequence} \\ \sigma = (x_1)(x_2)(x_3, x_4, x_5, x_6, x_7)\cdots \end{array} \right\}.$$

For an element of type $(5^4 \cdot 1^4)$, there are $4 \cdot 5 \cdot 3 \cdot 1$ M-sequences which satisfy the conditions of $\mathfrak{M}$ (There are $4 \cdot 5$ choices for $x_3 x_4 x_5 x_6 x_7$, 3 choices for x_2, then x_1 is uniquely determined). Conversely, for an M-sequence $(x_1 x_2 x_3 x_4 x_5 x_6 x_7)$, if there are two elements $\sigma = (x_1)(x_2)(x_3, x_4, x_5, x_6, x_7)\ \cdots$ and $\tau = (x_1)(x_2)(x_3, x_4, x_5, x_6, x_7)\ \cdots$ of type $(5^4 \cdot 1^4)$, then $\sigma = \tau$ by Corollary 1.4. Hence for an M-sequence, there is a unique element which satisfies the conditions of $\mathfrak{M}$. Set $m := \sharp\{\sigma \in M_{24} \mid \sigma = (5^4 \cdot 1^4)\text{-type}\}$. Then

$$\begin{aligned} |\mathfrak{M}| &= m \cdot 4 \cdot 5 \cdot 3 \cdot 1 \\ &= \sharp\{\text{all } M\text{-sequences}\} \cdot 1 \\ &= |M_{24}| \cdot 1. \end{aligned}$$

Since $|M_{24}|/m = 60$, the order of the centralizer of an element of type $(5^4 \cdot 1^4)$ in M_{24} is 60.

We can determine the conjugacy classes of the elements of types $(4^4 \cdot 2^2 \cdot 1^4)$ and $(6^2 \cdot 3^2 \cdot 2^2 \cdot 1^2)$ by the same way as in the proof of Theorem 4.1.

Types $(4^4 \cdot 2^4), (6^4)$ **and** $(12 \cdot 6 \cdot 4 \cdot 2)$.

Theorem 4.2. *All elements of type* $(4^4 \cdot 2^4)$ *form one conjugacy class in* M_{24}, *and the order of the centralizer of the element in* M_{24} *is* 384.

Proof. Let

$$\begin{aligned} \sigma &= (y_1, y_2, y_3, y_4)(y_5, y_6)(y_7, y_8)\cdots \\ \tau &= (s_1, s_2, s_3, s_4)(s_5, s_6)\cdots \end{aligned}$$

be elements of type $(4^4 \cdot 2^4)$. Assume that there exists $C \in \mathcal{O}$ such that $C \supseteq Y := \{y_1, y_2, y_3, y_4, y_5, y_6\}$. Since $C^\sigma = C$, we may assume $C = Y \cup \{y_7, y_8\}$. This contradicts that σ acts on C as an even permutation. Hence Y is non-special. By Corollary 1.9, there are three elements which contain $(y_1, y_2, y_3, y_4)(y_5, y_6)$ in cycle notation. Using the generators of $\boldsymbol{\Sigma}(Y)$ (See Theorem 1.8), we find that these three elements are as follows:

$$\begin{aligned}\beta_1 &:= a_5a_3a_2a_1\\ &= (y_1, y_2, y_3, y_4)(y_5, y_6)(7,8,9,10)(13,14,15,16)(19,20,21,22)\\ &\quad \times (11,24)(12,17)(18,23)\end{aligned}$$

$$\begin{aligned}\beta_2 &:= \alpha\beta_1\\ &= (y_1, y_2, y_3, y_4)(y_5, y_6)(7,14,21,10,13,20,9,16,19,8,15,22)(11,12,23,24,17,18)\end{aligned}$$

$$\begin{aligned}\beta_3 &:= \alpha^2\beta_1\\ &= (y_1, y_2, y_3, y_4)(y_5, y_6)(7,20,15,10,19,14,9,22,13,8,21,16)(11,18,17,24,23,12).\end{aligned}$$

Therefore σ $(=\beta_1)$ is a unique element of type $(4^4 \cdot 2^4)$ containing $(y_1, y_2, y_3, y_4)\cdot(y_5, y_6)$ in cycle notation. Similarly we have that $\{s_1, s_2, \dots, s_6\}$ is non-special. By Corollary 1.4, there is an element $\rho \in M_{24}$ such that $\rho : s_i \mapsto y_i$ $(1 \le i \le 6)$. Since $\rho^{-1}\tau\rho = (y_1, y_2, y_3, y_4)(y_5, y_6)\cdots$, $\rho^{-1}\tau\rho = \sigma$. It follows that all elements of type $(4^4 \cdot 2^4)$ are conjugate in M_{24}. Let

$$\mathfrak{M} = \left\{ (\sigma, (x_1x_2x_3x_4x_5x_6)) \;\middle|\; \begin{array}{l} M_{24} \ni \sigma = (4^4 \cdot 2^4)\text{-type} \\ (x_1x_2x_3x_4x_5x_6) : \text{ordered sequence} \\ \{x_1, x_2, x_3, x_4, x_5, x_6\} : \text{non-special 6-point subset} \\ \sigma = (x_1, x_2, x_3, x_4)(x_5, x_6)\cdots \end{array} \right\}.$$

For an element of type $(4^4 \cdot 2^4)$, there are $4 \cdot 4 \cdot 4 \cdot 2$ ordered sequences which satisfy the conditions of $\mathfrak{M}$ (There are $4 \cdot 4$ choices for $x_1x_2x_3x_4$, $4 \cdot 2$ choices for x_5x_6). Conversely, for the ordered sequence, there is a unique element which satisfies the conditions of $\mathfrak{M}$. Set $m := \sharp\{\sigma \in M_{24} \mid \sigma = (4^4 \cdot 2^4)\text{-type}\}$. Then

$$\begin{aligned}|\mathfrak{M}| &= m \cdot 4 \cdot 4 \cdot 4 \cdot 2\\ &= \sharp\left\{ (x_1x_2x_3x_4x_5x_6) \;\middle|\; \begin{array}{l} (x_1x_2x_3x_4x_5x_6) : \text{ordered sequence} \\ \{x_1, x_2, x_3, x_4, x_5, x_6\} : \text{non-special 6-point subset} \end{array} \right\} \cdot 1\\ &= (|M_{24}|/3) \cdot 1.\end{aligned}$$

Since $|M_{24}|/m = 384$, the order of the centralizer of an element of type $(4^4 \cdot 2^4)$ in M_{24} is 384.

We can determine the conjugacy classes of the elements of types (6^4) and $(12 \cdot 6 \cdot 4 \cdot 2)$ by the same way as in the proof of Theorem 4.2.

Type $(15 \cdot 5 \cdot 3 \cdot 1)$.

Theorem 4.3. *All elements of type* $(15 \cdot 5 \cdot 3 \cdot 1)$ *form two conjugacy classes in* M_{24}, *and the order of the centralizer of the element in* M_{24} *is* 15.

Proof. Let

$$\sigma = (y_1, y_2, y_3, y_4, y_5)(y_6)\cdots$$
$$\tau = (s_1, s_2, s_3, s_4, s_5)(s_6)\cdots$$

be elements of type $(15 \cdot 5 \cdot 3 \cdot 1)$. By Theorem 1.12, $Y := \{y_1, y_2, \dots, y_6\}$ and $\{s_1, s_2, \dots, s_6\}$ are non-special. By Corollary 1.9, there are three elements which contain $(y_1, y_2, y_3, y_4, y_5)\ (y_6)$ in cycle notation. Using the generators of $\boldsymbol{\Sigma}(Y)$ (See Theorem 1.8), we find that these three elements are as follows:

$$
\begin{aligned}
\beta_1 :&= a_4 a_3 a_2 a_1 \\
&= (y_1, y_2, y_3, y_4, y_5)(y_6)(7, 8, 15, 10, 11, 19, 20, 9, 22, 23, 13, 14, 21, 16, 17)(12, 18, 24) \\
\beta_2 :&= \alpha\beta_1 \\
&= (y_1, y_2, y_3, y_4, y_5)(y_6)(7, 14, 9, 10, 17, 13, 20, 15, 16, 23, 19, 8, 21, 22, 11)(12, 24, 18) \\
\beta_3 :&= \alpha^2\beta_1 \\
&= (y_1, y_2, y_3, y_4, y_5)(y_6)(6)(12)(18)(24)(7, 20, 21, 10, 23)(8, 9, 16, 11, 13) \\
&\quad \times (14, 15, 22, 17, 19).
\end{aligned}
$$

The elements β_1 and β_2 are of type $(15 \cdot 5 \cdot 3 \cdot 1)$ which contain $(y_1, y_2, y_3, y_4, y_5)(y_6)$ in cycle notation. Assume that β_1 and β_2 are conjugate in M_{24}. Let ρ be an element such that $\rho^{-1}\beta_1\rho = \beta_2$. Then we may assume that $\rho = (y_1, y_2, y_3, y_4, y_5)^j (y_6) \cdots$ ($j = 1, 2, 3, 4$ or 5). If $j = 1, 2, 3$ or 4, then ρ is of type $(15 \cdot 5 \cdot 3 \cdot 1)$ or $(5^4 \cdot 1^4)$. If $j = 5$, then ρ is of type $(3^6 \cdot 1^6)$ by Theorem 1.12. On the other hand, the element ρ induces a permutation on $\{12, 18, 24\}$. We may assume that $\rho = (12)(18, 24) \cdots$, $(18)(12, 24) \cdots$ or $(24)(12, 18) \cdots$. This contradicts the type of ρ. Therefore β_1 and β_2 are not conjugate in M_{24}.

By Corollary 1.4, there is an element $\delta \in M_{24}$ such that $\delta : s_i \mapsto y_i$ $(1 \le i \le 6)$. Since $\delta^{-1}\tau\delta = (y_1, y_2, y_3, y_4, y_5)(y_6) \cdots$, $\delta^{-1}\tau\delta = \beta_1$ or β_2. It follows that all elements of type $(15 \cdot 5 \cdot 3 \cdot 1)$ form two conjugacy classes. Let $\mathscr{C}$ be one of them. Let

$$
\mathfrak{M} = \left\{ (\sigma, (x_1x_2x_3x_4x_5x_6)) \;\middle|\; \begin{array}{l} \mathscr{C} \ni \sigma \\ (x_1x_2x_3x_4x_5x_6) : \text{ordered sequence} \\ \{x_1, x_2, x_3, x_4, x_5, x_6\} : \text{non-special 6-point subset} \\ \sigma = (x_1, x_2, x_3, x_4, x_5)(x_6) \cdots \end{array} \right\}.
$$

For an element in $\mathscr{C}$, there are 5 ordered sequences which satisfy the conditions of $\mathfrak{M}$. Conversely, for the ordered sequence, there is a unique element in $\mathscr{C}$ which satisfies the conditions in $\mathfrak{M}$. Then

$$
\begin{aligned}
|\mathfrak{M}| &= |\mathscr{C}| \cdot 5 \\
&= \sharp \left\{ (x_1x_2x_3x_4x_5x_6) \;\middle|\; \begin{array}{l} (x_1x_2x_3x_4x_5x_6) : \text{ordered sequence} \\ \{x_1, x_2, x_3, x_4, x_5, x_6\} : \text{non-special 6-point subset} \end{array} \right\} \cdot 1 \\
&= (|M_{24}|/3) \cdot 1.
\end{aligned}
$$

Since $|M_{24}|/|\mathscr{C}| = 15$, the order of the centralizer of $\gamma \in \mathscr{C}$ in M_{24} is 15. This completes the proof.

Types (4^6) **and** $(8^2 \cdot 4 \cdot 2 \cdot 1^2)$.

LEMMA 4.4. *Let* $C = \{x_1, x_2, \dots, x_8\} \in \mathcal{O}$. *For an element* $\tau = (x_1, x_2, x_3, x_4)(x_5, x_6, x_7, x_8) \in A_C$, *there are eight elements in* M_{24} *of type* (4^6) *which contain* τ *in cycle notation and these elements are conjugate in* $H := \{\sigma \in M_{24} \mid C^\sigma = C\}$.

PROOF. By Corollary 1.7, there are sixteen elements in M_{24} which contain τ in cycle notation. Let

$$H := \{\sigma \in M_{24} \mid C^\sigma = C\}$$

$$N := \{\sigma \in M_{24} \mid x^\sigma = x \ \ (\forall x \in C)\} \quad \text{(See Theorem 1.6)}.$$

Each element of H can be written as (n, τ) $(n \in N, \tau \in GL(4,2))$, and the product in H is given by $(n_1, \tau_1)(n_2, \tau_2) := (n_1^{\tau_2} + n_2, \tau_1\tau_2)$. Let

$$\delta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in GL(4,2).$$

Then δ is of type (4^2) as an element of A_8, and (n, δ) $(n \in N)$ are the sixteen elements in M_{24} which contain δ in cycle natation.

Next we will investigate the types of the sixteen elements. Let

$$\sigma = (s_1, s_2, s_3, s_4)(s_5, s_6, s_7, s_8)\cdots$$

be an element of type $(4^6), (4^4 \cdot 2^4)$ or $(4^4 \cdot 2^2 \cdot 1^4)$. We may assume that $\{s_1, s_2, \dots, s_8\}$ is an octad by Lemmas 3.1, 3.2 and 3.3. Since M_{24} acts transitively on $\mathcal{O}$, and all elements of type (4^2) in A_8 form one conjugacy class in A_8, elements of types $(4^6), (4^4 \cdot 2^4)$ and $(4^4 \cdot 2^2 \cdot 1^4)$ are contained in the sixteen elements. Let

$$\begin{array}{llll} n_1 = (0,0,0,0) & n_2 = (1,0,0,0) & n_3 = (1,1,0,0) & n_4 = (1,0,1,0) \\ n_5 = (1,0,0,1) & n_6 = (1,1,1,0) & n_7 = (1,1,0,1) & n_8 = (1,0,1,1) \\ n_9 = (1,1,1,1) & n_{10} = (0,1,0,0) & n_{11} = (0,1,1,0) & n_{12} = (0,1,0,1) \\ n_{13} = (0,1,1,1) & n_{14} = (0,0,1,0) & n_{15} = (0,0,1,1) & n_{16} = (0,0,0,1) \end{array}$$

be all elements in N. By the following calculation, (n_i, δ) $(i = 1, 5, 8, 14)$ are conjugate in N.

$$(n_i, 1)^{-1}(n_1, \delta)(n_i, 1) = (n_i^\delta + n_1 + n_i, \delta).$$

Similarly the following holds:

$$(n_2, \delta) \underset{N}{\sim} (n_i, \delta) \quad (i = 4, 15, 16), \quad (n_3, \delta) \underset{N}{\sim} (n_i, \delta) \quad (i = 6, 12, 13),$$

$$(n_7, \delta) \underset{N}{\sim} (n_i, \delta) \quad (i = 9, 10, 11).$$

Furthermore, for

$$\rho = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \in C_{GL(4,2)}(\delta),$$

$(n_3,\rho)^{-1}(n_3,\delta)(n_3,\rho) = (n_2,\delta)$. It follows that the sixteen elements consist of three orbits, and the types of the representatives are (4^6), $(4^4 \cdot 2^4)$ and $(4^4 \cdot 2^2 \cdot 1^4)$. Moreover $(n_2,1)^{-1}(n_1,\delta)^2(n_2,1) = (n_7,\delta)^2$ shows that (n_i,δ) $(i = 2,3,4,6,12,13,15,16)$ are elements of type (4^6). Since there is an element $\gamma \in A_8$ such that $\gamma^{-1}\delta\gamma = \tau$,

$$(n_i^\gamma,\tau) = (n_1,\gamma)^{-1}(n_i,\delta)(n_1,\gamma) \quad (i = 2,3,4,6,12,13,15,16)$$

are elements of type (4^6) which contain τ in cycle notation, and the proof is complete.

Theorem 4.5. *All elements of type* (4^6) *form one conjugacy class in* M_{24}, *and the order of the centralizer of the element in* M_{24} *is* 96.

Proof. Let

$$\sigma = (x_1,x_2,x_3,x_4)(x_5,x_6,x_7,x_8)\cdots$$

$$\tau = (s_1,s_2,s_3,s_4)(s_5,s_6,s_7,s_8)\cdots$$

be elements of type (4^6). By Lemma 3.1, we may assume that $\{x_1,x_2,\dots,x_8\}$ and $\{s_1,s_2,\dots,s_8\}$ are octads. Then there exists an element ρ such that

$$\rho^{-1}\tau\rho = (x_1,x_2,x_3,x_4)(x_5,x_6,x_7,x_8)\cdots.$$

By Lemma 4.4, there is an element δ such that $\delta^{-1}(\rho^{-1}\tau\rho)\delta = \sigma$. It follows that all elements of type (4^6) are conjugate in M_{24}. Let

$$\mathfrak{M} = \left\{ (\sigma,(x_1x_2\cdots x_8)) \;\middle|\; \begin{array}{l} M_{24} \ni \sigma = (4^6)\text{-type} \\ (x_1x_2\cdots x_8) : \text{ordered sequence} \\ \{x_1,x_2,\dots,x_8\} \in \mathcal{O} \\ \sigma = (x_1,x_2,x_3,x_4)(x_5,x_6,x_7,x_8)\cdots \end{array} \right\}.$$

For an element of type (4^6), there are $6\cdot 4\cdot 4$ ordered sequences which satisfy the conditions of $\mathfrak{M}$ by Lemma 3.1. Conversely, for the ordered sequence, there are eight elements which satisfy the conditions of $\mathfrak{M}$ by Lemma 4.4. Set $m := \sharp\{\sigma \in M_{24} \mid \sigma = (4^6)\text{-type}\}$. Then

$$\begin{aligned} |\mathfrak{M}| &= m\cdot 6\cdot 4\cdot 4 \\ &= \sharp\left\{ (x_1x_2\cdots x_8) \;\middle|\; \begin{array}{l} (x_1x_2\cdots x_8) : \text{ordered sequence} \\ \{x_1,x_2,\dots,x_8\} \in \mathcal{O} \end{array} \right\}\cdot 8 \\ &= (|M_{24}|\cdot 8!/2^4\cdot |A_8|)\cdot 8. \end{aligned}$$

Since $|M_{24}|/m = 96$, the order of the centralizer of an element of type (4^6) in M_{24} is 96.

We can determine the conjugacy class of the element of type $(8^2 \cdot 4 \cdot 2 \cdot 1^2)$ by the same way as in the proof of Theorem 4.5.

Types $(7^3 \cdot 1^3), (10^2 \cdot 2^2), (11^2 \cdot 1^2), (14 \cdot 7 \cdot 2 \cdot 1), (21 \cdot 3)$ **and** $(23 \cdot 1)$.

Since M_{24} is simple, conjugacy classes of these types are determined easily using Sylow's theorem or Burnside's theorem.

Type (12^2)

THEOREM 4.6. *All elements of type* (12^2) *form one conjugacy class in* M_{24}, *and the order of the centralizer of the element in* M_{24} *is* 12.

PROOF. We will consider the elements in the sextet stabilizer K (See Theorem 1.10). For $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_4, b_5$ in K, put $\sigma := a_5 a_4 a_3 a_2 a_1 b_1 b_2$ and $\tau := \sigma^2$. Then

$$\sigma = (1, 8, 21, 4, 5, 6, 7, 2, 9, 16, 23, 24)(3, 10, 11, 18, 19, 20, 15, 22, 17, 12, 13, 14)$$

$$\tau = (1, 21, 5, 7, 9, 23)(8, 4, 6, 2, 16, 24)(3, 11, 19, 15, 17, 13)(10, 18, 20, 22, 12, 14).$$

Since $|C_{M_{24}}(\tau)| = 24$ (See Theorem 4.2), we have $|C_{M_{24}}(\sigma)| = 12$ or 24. Let $\delta := b_2 b_4 b_5 \tau$. Then

$$\delta = (1, 3, 5, 19, 9, 17)(2, 10, 24, 20, 4, 12)(6, 14, 16, 18, 8, 22)(7, 15, 23, 13, 21, 11).$$

Since δ is in $C_{M_{24}}(\tau) - C_{M_{24}}(\sigma)$, we have that $|C_{M_{24}}(\sigma)| = 12$ and

$$|\sigma^{M_{24}}| = |M_{24}|/|C_{M_{24}}(\sigma)| = 2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23.$$

The sum of the cardinalities of the conjugacy classes of M_{24} which we have determined equals $|M_{24}|$. It follows that all elements of type (12^2) are conjugate in M_{24}.

This yields that we have classified all the conjugacy classes of M_{24}.

5. The conjugacy classes of M_{23}, M_{22}.

In this section, we will classify all the conjugacy classes of $M_{23} := \{\sigma \in M_{24} \mid a^\sigma = a\}$ and $M_{22} := \{\sigma \in M_{24} \mid a^\sigma = a, b^\sigma = b\}$ $(a, b \in \boldsymbol{\Omega})$ which are given in TABLE 3 and TABLE 4.

TABLE 3. M_{23}

type	number of the conjugacy classes	order of the centralizer
$(2^8 \cdot 1^7)$	1	2688
$(3^6 \cdot 1^5)$	1	180
$(4^4 \cdot 2^2 \cdot 1^3)$	1	32
$(5^4 \cdot 1^3)$	1	15
$(6^2 \cdot 3^2 \cdot 2^2 \cdot 1)$	1	12
$(7^3 \cdot 1^2)$	2	14, 14
$(8^2 \cdot 4 \cdot 2 \cdot 1)$	1	8
$(11^2 \cdot 1)$	2	11, 11
$(14 \cdot 7 \cdot 2)$	2	14, 14
$(15 \cdot 5 \cdot 3)$	2	15, 15
(23)	2	23, 23

TABLE 4. M_{22}

type	number of the conjugacy classes	order of the centralizer
$(2^8 \cdot 1^6)$	1	384
$(3^6 \cdot 1^4)$	1	36
$(4^4 \cdot 2^2 \cdot 1^2)$	2	16, 32
$(5^4 \cdot 1^2)$	1	5
$(6^2 \cdot 3^2 \cdot 2^2)$	1	12
$(7^3 \cdot 1)$	2	7, 7
$(8^2 \cdot 4 \cdot 2)$	1	8
(11^2)	2	11, 11

Since M_{24} acts 5-fold transitively on the set $\boldsymbol{\Omega}$ of 24 letters, the types of the elements of M_{23} and M_{22} are as above.

THEOREM 5.1. *All involutions form one conjugacy class in M_{23}, and the order of the centralizer of the element in M_{23} is* 2688.

PROOF. Let

$$\sigma = (a)(x_1)(x_2)(x_3)(x_4)(y_1, y_2)(y_3, y_4)\cdots$$

$$\tau = (a)(s_1)(s_2)(s_3)(s_4)(t_1, t_2)(t_3, t_4)\cdots$$

be involutions in M_{23}. Let $C \in \mathcal{O}$ such that $C \supseteq X := \{a, x_1, x_2, y_1, y_2\}$. By Theorem 1.6, we may assume that $C = X \cup \{x_3, y_3, y_4\}$. It follows that $(x_4 a x_1 x_2 x_3 y_1 y_2)$ is an M-sequence. Similarly we may assume that $(s_4 a s_1 s_2 s_3 t_1 t_2)$ is an M-sequence. By Corollary 1.4, there exists an element $\rho \in M_{24}$ such that $\rho : s_i \mapsto x_i$ $(1 \le i \le 4)$, $\rho : a \mapsto a$, $\rho : t_i \mapsto y_i$ $(i = 1, 2)$. Since $(\rho^{-1}\tau\rho)\sigma^{-1} = (x_4)(a)(x_1)(x_2)(x_3)(y_1)(y_1)\cdots$, $\rho^{-1}\tau\rho = \sigma$ by Corollary 1.4. Moreover ρ is in M_{23}. It follows that all involutions are conjugate in M_{23}. Furthermore, counting the cardinality of the set

$$\left\{ (\sigma, (x_1 a x_2 x_3 x_4 x_5 x_6)) \;\middle|\; \begin{array}{l} M_{23} \ni \sigma = (2^8 \cdot 1^7)\text{-type} \\ (x_1 a x_2 x_3 x_4 x_5 x_6) : M\text{-sequence} \\ \sigma = (x_1)(a)(x_2)(x_3)(x_4)(x_5, x_6)\cdots \end{array} \right\},$$

we have $|C_{M_{23}}((2^8 \cdot 1^7)\text{-type})| = 2688$.

We can determine the conjugacy classes of the elements in M_{23}, M_{22} of orders 2, 5 and 6, and the conjugacy class of the element in M_{23} of order 4 by the same way as in the proof of Theorem 5.1.

THEOREM 5.2. *All elements of type $(3^6 \cdot 1^5)$ form one conjugacy class in M_{23}, and the order of the centralizer of the element in M_{23} is* 180.

PROOF. Let

$$\sigma = (a)(y_1)(y_2)(y_3)(y_4)(y_5)\cdots$$

$$\tau = (a)(t_1)(t_2)(t_3)(t_4)(t_5)\cdots$$

be elements of type $(3^6 \cdot 1^5)$. By Theorem 1.12, $Y := \{a, y_1, \dots, y_5\}$ and $\{a, t_1, \dots, t_5\}$ are non-special. There exists an element $\rho \in M_{24}$ such that $\rho : t_i \mapsto y_i$ $(1 \le i \le 5)$, $\rho : a \mapsto a$. Then ρ is in M_{23} and $\rho^{-1}\tau\rho = (a)(y_1)(y_2)(y_3)(y_4)(y_5)\cdots$. There is an M-matrix $\mathscr{Y}$ such that

$$\mathscr{Y} = \begin{pmatrix} a & y_1 & y_2 & y_3 & y_4 & y_5 \\ & & & & & \\ & & & & & \end{pmatrix}.$$

Using the elements in $\boldsymbol{\Sigma}(Y)$ (See Theorem 1.8), we find that the nonidentity elements in M_{24} fixing $a, y_1, \dots, y_5$ are α and α^2. Since $a_3^{-1}\alpha a_3 = \alpha^2$ $(a_3 \in M_{23})$ and $\sigma, \rho^{-1}\tau\rho \in \{\alpha, \alpha^2\}$, we have that σ and τ are conjugate in M_{23}. Furthermore, counting the cardinality of the set

$$\left\{ (\sigma, (ax_1x_2x_3x_4x_5)) \;\middle|\; \begin{array}{l} M_{23} \ni \sigma = (3^6 \cdot 1^5)\text{-type} \\ (ax_1x_2x_3x_4x_5) : \text{ordered sequence} \\ \{a, x_1, x_2, x_3, x_4, x_5\} : \text{non-special 6-point subset} \\ \sigma = (a)(x_1)(x_2)(x_3)(x_4)(x_5)\cdots \end{array} \right\},$$

we have $|C_{M_{23}}((3^6 \cdot 1^5)\text{-type})| = 180$.

We can determine the conjugacy class of the element in M_{22} of order 3 by the same way as in the proof of Theorem 5.2.

Using the same argument as in the proof of Lemma 4.4, we have the following result:

LEMMA 5.3. *Let $C = \{x_1, x_2, \dots, x_8\} \in \mathscr{O}$. For an element $\tau = (x_1)(x_2)(x_3, x_4)(x_5, x_6, x_7, x_8) \in A_C$, there are eight elements in M_{24} of type $(8^2 \cdot 4 \cdot 2 \cdot 1^2)$ which contain τ in cycle notation and these elements are conjugate in $N := \{\sigma \in M_{24} \mid x^\sigma = x\ (\forall x \in C)\}$.*

THEOREM 5.4. *All the elements of type $(8^2 \cdot 4 \cdot 2 \cdot 1)$ form one conjugacy class in M_{23}, and the order of the centralizer of the element in M_{23} is* 8.

PROOF. Let

$$\sigma = (a)(x_1)(x_2, x_3)(x_4, x_5, x_6, x_7)\cdots$$

$$\tau = (a)(s_1)(s_2, s_3)(s_4, s_5, s_6, s_7)\cdots$$

be elements of type $(8^2 \cdot 4 \cdot 2 \cdot 1)$. By Theorem 1.12, $\{a, x_1, \dots, x_7\}$ and $\{a, s_1, \dots, s_7\}$ are octads. Hence there exists an element ρ in M_{23} such that $\rho^{-1}\tau\rho = (a)(x_1)(x_2, x_3)(x_4, x_5, x_6, x_7)\cdots$. By Lemma 5.3, we have that σ and τ are conjugate in M_{23}. Moreover, counting the cardinality of the set

$$\left\{ (\sigma, (ax_1x_2\cdots x_7)) \;\middle|\; \begin{array}{l} M_{23} \ni \sigma = (8^2 \cdot 4 \cdot 2 \cdot 1)\text{-type} \\ (ax_1x_2\cdots x_7) : \text{ordered sequence} \\ \{a, x_1, x_2, \dots, x_7\} \in \mathscr{O} \\ \sigma = (a)(x_1)(x_2, x_3)(x_4, x_5, x_6, x_7)\cdots \end{array} \right\},$$

we have $|C_{M_{23}}((8^2 \cdot 4 \cdot 2 \cdot 1)\text{-type})| = 8$.

We can determine the conjugacy class of the element in M_{22} of order 8 by the same way as in the proof of Theorem 5.4.

Applying the results about the conjugacy classes of M_{24} (See TABLE 2), we can determine the conjugacy classes of M_{23} of types $(7^3 \cdot 1^2), (11^2 \cdot 1), (14 \cdot 7 \cdot 2), (15 \cdot 5 \cdot 3)$ and (23), and the conjugacy classes of M_{22} of types $(7^3 \cdot 1)$ and (11^2).

We have determined all the conjugacy classes of M_{23}, and the number of the elements in M_{22} of type $(4^4 \cdot 2^2 \cdot 1^2)$ is 41580.

THEOREM 5.5. *All elements of type* $(4^4 \cdot 2^2 \cdot 1^2)$ *form two conjugacy classes in* M_{22}, *and the orders of the centralizers in* M_{22} *are* 16 *and* 32.

PROOF. Let

$$\sigma = (a)(b)(x_1)(x_2)(y_1, y_2)(y_3, y_4)(z_1, z_2, z_3, z_4) \cdots (z_{13}, z_{14}, z_{15}, z_{16})$$

$$\tau = (a)(b)(s_1)(s_2)(t_1, t_2)(t_3, t_4)(w_1, w_2, w_3, w_4) \cdots (w_{13}, w_{14}, w_{15}, w_{16})$$

be elements of type $(4^4 \cdot 2^2 \cdot 1^2)$. Let $C_1 \in \mathcal{O}$ such that $C_1 \supseteq Z_1 := \{a, z_1, z_2, z_3, z_4\}$. By Theorem 1.6, we may assume that $C_1 = Z_1 \cup \{b, y_1, y_2\}$ or $C_1 = Z_1 \cup \{x_1, y_1, y_2\}$.

Case 1. $C_1 = Z_1 \cup \{b, y_1, y_2\}$.
Set $Z_2 := \{a, z_5, z_6, z_7, z_8\}, Z_3 := \{a, z_9, z_{10}, z_{11}, z_{12}\}$ and $Z_4 := \{a, z_{13}, z_{14}, z_{15}, z_{16}\}$. Let $C_i \in \mathcal{O}$ such that $C_i \supseteq Z_i$ $(2 \le i \le 4)$. Since $|C_i \cap C_j| = 2$ or 4 $(1 \le i \ne j \le 4)$ by Lemma 1.1, each C_i is as follows:

$$C_2 = Z_2 \cup \{b, y_1, y_2\} \text{ or } Z_2 \cup \{b, y_3, y_4\}$$

$$C_3 = Z_3 \cup \{b, y_1, y_2\} \text{ or } Z_3 \cup \{b, y_3, y_4\}$$

$$C_4 = Z_4 \cup \{b, y_1, y_2\} \text{ or } Z_4 \cup \{b, y_3, y_4\}.$$

This case implies that b is in the octad containing a and four points p_i $(1 \le i \le 4)$ with $(p_i)^\sigma = p_{i+1}$ $(1 \le i \le 3), (p_4)^\sigma = p_1$.

Case 2. $C_1 = Z_1 \cup \{x_1, y_1, y_2\}$.
Then

$$C_2 = Z_2 \cup \{x_1, y_1, y_2\} \text{ or } Z_2 \cup \{x_1, y_3, y_4\}$$

$$C_3 = Z_3 \cup \{x_1, y_1, y_2\} \text{ or } Z_3 \cup \{x_1, y_3, y_4\}$$

$$C_4 = Z_4 \cup \{x_1, y_1, y_2\} \text{ or } Z_4 \cup \{x_1, y_3, y_4\}.$$

This case implies that b is not in the octad containing a and four points p_i $(1 \le i \le 4)$ with $(p_i)^\sigma = p_{i+1}$ $(1 \le i \le 3), (p_4)^\sigma = p_1$.

Suppose that σ and τ satisfy Case 1. We may assume that $(x_1 a b z_1 z_2 z_3 z_4)$ and $(s_1 a b w_1 w_2 w_3 w_4)$ are M-sequences. By Corollary 1.4, there exists an element $\rho \in M_{24}$ such that $\rho : a \mapsto a, \rho : b \mapsto b, \rho : s_1 \mapsto x_1, \rho : w_i \mapsto z_i$ $(1 \le i \le 4)$. By Corollary 1.4, we have that $\rho^{-1} \tau \rho = \sigma$ $(\rho \in M_{22})$. Suppose that σ and τ satisfy the Case 2. Since we may assume that $(b a x_1 z_1 z_2 z_3 z_4)$ and $(b a s_1 w_1 w_2 w_3 w_4)$ are M-sequences, σ and τ are conjugate in M_{22} by the same argument as above. Suppose that σ satisfies Case 1, and τ satisfies Case 2. We assume that there exists an element $\rho \in M_{22}$ such that $\rho^{-1} \tau \rho = \sigma$. Then σ

satisfies Case 1, and $\rho^{-1}\tau\rho$ satisfies Case 2. This is a contradiction. Therefore σ and τ are not conjugate in M_{22}.

It follows that all elements of type $(4^4 \cdot 2^2 \cdot 1^2)$ form one or two conjugacy classes in M_{22}. We assume that σ satisfies Case 1. Let $\mathscr{C}$ be the conjugacy class of M_{22} containing σ. Counting the cardinality of the set

$$\left\{ (\tau, (x_1abx_2x_3x_4x_5)) \;\middle|\; \begin{array}{l} \mathscr{C} \ni \tau \\ (x_1abx_2x_3x_4x_5) : M\text{-sequence} \\ \tau = (x_1)(a)(b)(x_2, x_3, x_4, x_5)\cdots \end{array} \right\},$$

we have that $|C_{M_{22}}(\sigma)| = 32$ and $|\mathscr{C}| = 13860$. Then the order of $\mathscr{C}$ is less than 41580. It follows that all elements of type $(4^4 \cdot 2^2 \cdot 1^2)$ form two conjugacy classes in M_{22}. Moreover we have that the order of the other centralizer is 16.

This yields that we have classified all the conjugacy classes of M_{22}.

Our tables are, of course, the same as those in Frobenius [**4**] and Todd [**6**]!

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References

[1] M. Aschbacher, Sporadic Groups, Cambridge Tracts in Mathematics **104**, (1994), Cambridge University Press.
[2] J. Conway, Three lectures on exceptional groups, in "Finite Simple Groups" (Powell and Higman, eds.), Academic Press, (1971) 215–247.
[3] R. Curtis, A new combinatorial approach to M_{24}, Math. Proc. Cambridge Phil. Soc. **79**, (1976) 25–42.
[4] G. Frobenius, Über die Charaktere der mehrfach transitiven Gruppen, Berliner Berichte (1904) 558–571.
[5] T. Kondo, The Mathieu groups and the Conway groups (in Japanese), Lecture note at University of Tokyo.
[6] J. Todd, A representation of the Mathieu group M_{24} as a collineation group, Ann. Math. Pura Appl. series IV **71**, (1966) 199–238.

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