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Invariance of the Godbillon-Vey class by C^1-diffeomorphisms for higher codimensional foliations

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Abstract. G. Raby proved in [**4**] that the Godbillon-Vey invariant for codimension-one foliations is invariant by C^1-diffeomorphisms. In this paper we generalize the result for foliations of arbitrary codimension.

Introduction

Let $(M_1, \mathscr{F}_1)$ and $(M_2, \mathscr{F}_2)$ be closed foliated manifolds of codimension q. We assume the manifolds and the foliations are of class C^∞ and oriented. Let $\mathrm{GV}(\mathscr{F}_1)$ and $\mathrm{GV}(\mathscr{F}_2)$ denote the Godbillon-Vey class of $(M_1, \mathscr{F}_1)$ and $(M_2, \mathscr{F}_2)$, respectively. Let φ be a C^1 diffeomorphism from M_1 to M_2 which maps $\mathscr{F}_1$ to $\mathscr{F}_2$. Then we have the following theorem.

Theorem (Raby [**4**]). *Suppose that $q = 1$. It then follows $\varphi^*(\mathrm{GV}(\mathscr{F}_2)) = \mathrm{GV}(\mathscr{F}_1)$.*

In this paper we generalize his result to foliations of higher codimension.

1. Preliminaries.

In this section we review basic results about currents and forms after de Rham [**5**]. We put $n = \dim M_1 = \dim M_2$. The space of p-dimensional currents are the dual of the space of p-forms of class C^∞. If T is a current of dimension p, we say T is of degree $(n-p)$ or T is a p-current.

Let T be a p-current and η a p-form of class C^∞, then we denote by $\langle T, \eta \rangle$ the natural pairing.

Let ω be an integrable p-form, then ω defines a $(n-p)$-current T_ω by

$$\langle T_\omega, \eta \rangle = \int_{M_1} \omega \wedge \eta,$$

where η is an $(n-p)$-form. We denote the current T_ω by $[\omega]$.

Let T be a p-current and ω is a q-form of class C^∞, then we denote by $T \wedge \omega$ the $(p-q)$-current defined by

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$$\langle T \wedge \omega, \eta \rangle = \langle T, \omega \wedge \eta \rangle.$$

We define a $(p-q)$-current $\omega \wedge T$ by the formula $\omega \wedge T = (-1)^{pq} T \wedge \omega$.

Let T be a p-current on M_1 and f be a C^∞ map from M_1 to M_2, then we denote by $f_* T$ the p-current on M_2 defined by

$$\langle f_* T, \eta \rangle = \langle T, f^* \eta \rangle.$$

Finally for a p-current T we denote by dT the $(p-1)$-current defined by

$$\langle dT, \eta \rangle = \langle T, (-1)^{n-p+1} d\eta \rangle.$$

See [**5**] or [**6**] for the details of the nature of currents.

2. The invariance of the Godbillon-Vey class.

In this section we show the following.

Theorem. *Let $(M_1, \mathscr{F}_1)$ and $(M_2, \mathscr{F}_2)$ be closed foliated manifolds of codimension q. We assume that the manifolds and the foliations are of class C^∞ and oriented. Suppose that φ is a C^1-diffeomorphism from M_1 to M_2 which maps $\mathscr{F}_1$ to $\mathscr{F}_2$. Then $\varphi^*(\mathrm{GV}(\mathscr{F}_2)) = \mathrm{GV}(\mathscr{F}_1)$.*

Now we recall the definition of the Godbillon-Vey class GV. By the assumption $\mathscr{F}_1$ is transversely oriented. Then we have a C^∞ transverse volume form Ω_1 of $(M_1, \mathscr{F}_1)$. Due to the integrability of the subbundle $T\mathscr{F}_1$ of TM_1, there is a smooth 1-form α_1 satisfying

$$d\Omega_1 = \Omega_1 \wedge \alpha_1.$$

The Godbillon-Vey class is then defined to be the cohomology class of $H^*(M)$ given by the form $\alpha_1 \wedge (d\alpha_1)^q$.

Similarly we choose a C^∞ transverse volume form Ω_2 of $(M_2, \mathscr{F}_2)$ and define α_2. It suffices to show that two forms $\varphi^*(\alpha_2 \wedge (d\alpha_2)^q)$ and $\alpha_1 \wedge (d\alpha_1)^q$ defines the same $(n-(2q+1))$-current modulo boundaries. For simplicity we assume that φ preserves the orientations of the manifolds and foliations.

First we begin with the following lemmas.

Lemma 2.1 (Raby [**4**], Lemme i)). *Let ω be a p-form of class C^∞ on M_2, then we have*

$$d[\varphi^* \omega] = [\varphi^*(d\omega)].$$

Proof. We approximate $\psi = \varphi^{-1}$ by C^∞ maps in the C^1-topology and then see that

$$d(\psi_*[\omega]) = \psi_*[d\omega].$$

See Federer [**1**] and de Rham [**5**] for details.

On the other hand we have for any C^∞ form η

$$\varphi_* \psi_* [\eta] = [\eta] \quad \text{and} \quad \varphi_*[\varphi^* \eta] = [\eta].$$

Hence $[\varphi^* \eta] = \psi_*[\eta]$. It follows that

$$d[\varphi^* \omega] = d\psi_*[\omega] = \psi_*[d\omega] = [\varphi^*(d\omega)]. \qquad \square$$

Since φ maps $\mathscr{F}_1$ to $\mathscr{F}_2$ and preserves the orientation,

$$\varphi^*\Omega_2 = e^h \cdot \Omega_1,$$

where h is a certain continuous function.

Lemma 2.2 (see Raby [**4**], Lemme ii)).
i) *The function h is leafwise of class C^1.*
ii) *Let θ be a continuous $(q+1)$-form which satisfies*

$$d[e^h] \wedge \Omega_1 = [e^h\theta],$$

then we have

$$d[h] \wedge \Omega_1 = [\theta].$$

Proof. It suffices to show the equation locally. So we choose a foliated chart $U_i \cong \boldsymbol{R}^p \times \boldsymbol{R}^q$, where

$$\mathscr{F}_1|_{U_i} \cong \{\boldsymbol{R}^p \times \{y\}; y \in \boldsymbol{R}^q\}.$$

We denote points of U_i by $(x, y) = (x_1, \dots, x_p, y_1, \dots, y_q)$. Then we may assume

$$\Omega_1 = a(x, y)\, dy_1 \wedge \cdots \wedge dy_q,$$

where a is a C^∞ function.

The assumption shows that

$$\theta = \left(\sum_{i=1}^{p} b_i(x, y)\, dx_i\right) \wedge dy_1 \wedge \cdots \wedge dy_q,$$

where b_i are continuous functions. It follows that

$$\frac{\partial[e^h]}{\partial x_i} \cdot a = [e^h \cdot b_i].$$

Here the term $\partial[e^h]/\partial x_i$ is defined by taking the distributional derivative of the continuous function e^h regarded as a function of x_i. Therefore the function e^h is a C^0 function with distributional C^0 derivative with respect to x. Hence e^h is of class C^1 with respect to x.

Consequently h is also of class C^1 with respect to x, and the equation

$$\frac{\partial h}{\partial x_i} \cdot a = b_i.$$

holds. It follows that

$$d[h] \wedge \Omega_1 = \left[\left(\sum_{i=1}^{p} \frac{\partial h}{\partial x_i} dx_i\right) \wedge \Omega_1\right] = [\theta]. \qquad \square$$

We have

$$\begin{aligned} d[\varphi^*\Omega_2] &= [\varphi^*(d\Omega_2)] \\ &= [\varphi^*(\Omega_2 \wedge \alpha_2)] \\ &= [e^h\Omega_1 \wedge \varphi^*\alpha_2]. \end{aligned}$$

The left hand side is equal to

$$d[e^h\Omega_1] = d[e^h] \wedge \Omega_1 + [e^h\Omega_1 \wedge \alpha_1].$$

It follows that

$$d[e^h] \wedge \Omega_1 = [e^h(\Omega_1 \wedge \varphi^*\alpha_2 - \Omega_1 \wedge \alpha_1)].$$

Hence by ii) of Lemma 2.2,

$$d[h] \wedge \Omega_1 = [\Omega_1 \wedge \varphi^*\alpha_2 - \Omega_1 \wedge \alpha_1].$$

Consequently we have

$$(\varphi^*\alpha_2 - \alpha_1 + (-1)^{q+1}d[h]) \wedge \Omega_1 = 0.$$

Now we put $\lambda = \varphi^*\alpha_2 - \alpha_1$, then λ is a continuous 1-form and

$$(\lambda + (-1)^{q+1}d[h]) \wedge \Omega_1 = 0.$$

The derivative $d[\lambda]$ of λ is represented by a continuous 2-form. In fact, if we put $\partial\lambda = \varphi^*(d\alpha_2) - d\alpha_1$ then by virtue of Lemma 2.1

$$\begin{aligned} d[\lambda] &= d[\varphi^*\alpha_2 - \alpha_1] \\ &= [\varphi^*(d\alpha_2) - d\alpha_1] \\ &= [\partial\lambda]. \end{aligned}$$

Note that $d[\partial\lambda] = 0$ by definition.

Now we have an equality of currents as follows, namely,

$$\begin{aligned} &\varphi^*(\alpha_2 \wedge (d\alpha_2)^q) \\ &\quad = (\alpha_1 + \lambda) \wedge (d\alpha_1 + \partial\lambda)^q \\ &\quad = \alpha_1 \wedge (d\alpha_1)^q + \sum_{j=1}^{q} \binom{q}{j} \alpha_1 \wedge (d\alpha_1)^{q-j} \wedge (\partial\lambda)^j + \lambda \wedge (d\alpha_1 + \partial\lambda)^q \\ &\quad = \alpha_1 \wedge (d\alpha_1)^q - d\left(\sum_{j=1}^{q} \binom{q}{j} \alpha_1 \wedge (d\alpha_1)^{q-j} \wedge (\partial\lambda)^{j-1} \wedge \lambda \right) \\ &\qquad + \sum_{j=1}^{q} \binom{q}{j} (d\alpha_1)^{q-j+1} \wedge (\partial\lambda)^{j-1} \wedge \lambda + \lambda \wedge (d\alpha_1 + \partial\lambda)^q. \end{aligned}$$

Recall that

$$(\lambda + (-1)^{q+1} d[h]) \wedge \Omega_1 = 0$$

and h is leafwise of class C^1. Hence if we put

$$\lambda = (-1)^q \sum_{i=1}^{p} f_i dx_i + \sum_{j=1}^{q} g_j dy_j,$$

where f_i and g_j are continuous functions, then locally we have

$$f_i = \frac{\partial h}{\partial x_i}.$$

Now we put

$$d_x h = \sum_{i=1}^{p} \frac{\partial h}{\partial x_i} dx_i.$$

Note that the operator d_x can be defined globally as the exterior differentiation along the leaves.

Now we have the following lemma:

LEMMA 2.3 (see Raby [**4**], Lemme iii)). *Let ω be a continuous form such that $d[\omega]$ is defined by a continuous form of transverse degree q, i.e., of the form $\mu \wedge \Omega_1$, where μ is a continuous form. Then we have*

$$d[h \cdot d[\omega]] = (-1)^q \lambda \wedge d[\omega].$$

PROOF. We work on a foliated chart $U_i \cong \boldsymbol{R}^p \times \boldsymbol{R}^q$.

From the assumption and the above remark it follows that

$$\lambda \wedge d[\omega] = (-1)^q d_x h \wedge d[\omega].$$

Thus it suffices to show that

$$d[h \cdot d[\omega]] = d_x h \wedge d[\omega].$$

This equation is obvious when h is of class C^∞. In general we choose a sequence h_n of C^∞ functions such that h_n and $\partial h_n / \partial x_i$ converge to h and $\partial h / \partial x_i$ in the C^0-topology, respectively. Then it is easy to see $d[h_n \cdot d[\omega]]$ and $d_x h_n$ converge to $d[h \cdot d[\omega]]$ and $d_x h$ in the space of currents, respectively. □

Now we apply the above lemma to the continuous forms $\alpha_1 \wedge (d\alpha_1)^{q-j} \wedge (\partial\lambda)^{j-1}$ and $(\alpha_1 + \lambda) \wedge (d\alpha_1 + \partial\lambda)^{q-1}$. It suffices to see that the differential of these forms are of transverse degree not less than q. We have

$$d\Omega_1 = \Omega_1 \wedge \alpha_1.$$

It follows that $\Omega_1 \wedge d\alpha_1 = 0$. Similarly

$$\Omega_2 \wedge d\alpha_2 = 0$$

and hence

$$e^h\Omega_1 \wedge \varphi^*(d\alpha_2) = 0.$$

Noticing that $d[\lambda]$ is represented by the form $\varphi^*(d\alpha_2) - d\alpha_1$, we see that

$$\Omega_1 \wedge d[\lambda] = [\Omega_1 \wedge \partial\lambda] = 0.$$

These equalities show that the continuous forms $(td\alpha_1 + s\partial\lambda)^q$ are of transverse degree q for arbitrary real numbers t and s.

It follows that

$$(d\alpha_1)^{q-j+1} \wedge (\partial\lambda)^{j-1} \wedge \lambda = (-1)^q d[(d\alpha_1)^{q-j+1} \wedge (\partial\lambda)^{j-1} \cdot h],$$

and

$$\lambda \wedge (d\alpha_1 + \partial\lambda)^q = (-1)^q d[h \cdot (d\alpha_1 + \partial\lambda)^q].$$

Notice here that

$$\begin{aligned} d[(d\alpha_1 + \partial\lambda)^{q-1} \wedge (\alpha_1 + \lambda)] \\ &= d[\varphi^*(d\alpha_2)^{q-1} \wedge \varphi^*\alpha_2] \\ &= d[\varphi^*((d\alpha_2)^{q-1} \wedge \alpha_2)] \\ &= [\varphi^*(d\alpha_2)^q] \\ &= [(d\alpha_1 + \partial\lambda)^q]. \end{aligned}$$

Consequently the two continuous forms $\varphi^*(\alpha_2 \wedge (d\alpha_2)^q)$ and $\alpha_1 \wedge (d\alpha_1)^q$ define the same class as a current modulo boundaries. Namely we have $\varphi^*(\mathrm{GV}(\mathscr{F}_2)) = \mathrm{GV}(\mathscr{F}_1)$. □

REMARK 2.4. Quite recently, H. Moriyoshi has obtained the same result by a different approach [**3**].

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