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Wick product of white noise operators and quantum stochastic differential equations

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Abstract. The Wick product of operators on Fock space is introduced on the basis of the analytic characterization theorem for operator symbols established within the framework of white noise distribution theory. Existence and uniqueness of solutions are proved for a certain class of ordinary differential equations for Fock space operators. Quantum stochastic differential equations of Itô type and their generalizations involving higher powers of quantum white noises enter into our consideration.

Introduction.

This paper aims at fusing the ideas of infinite dimensional analysis (in particular, white noise distribution theory over Gaussian space) and quantum probability in order to solve differential equations for operators acting in a Boson Fock space. It is expected that our approach offers not only an interesting aspect to quantum stochastic differential equations of Itô type but also a prototype of general theory of non-commutative differential equations on an infinite dimensional space.

The white noise distribution theory was initiated by Hida [**15**] and has been discussed extensively in connection with stochastic analysis and harmonic analysis, see e.g., [**23**], [**28**] for recent progress. The fundamental framework is an infinite dimensional analogue of Schwartz type distribution theory and is based on the Gelfand triple:

$$(E)_\beta \subset L^2(E^*, \mu) \cong \Gamma(L^2(\boldsymbol{R})) \subset (E)_\beta^*, \tag{0.1}$$

where $E^* = \mathscr{S}'(\boldsymbol{R})$ and μ is the standard Gaussian measure on it. The triple (0.1) is referred to as the Hida-Kubo-Takenaka space [**22**] for $\beta = 0$ and as the Kondratiev-Streit space [**20**] for a general $0 \le \beta < 1$. Note that elements of these spaces are (generalized) functions on the infinite dimensional vector space E^*. Since $L^2(E^*, \mu)$ is canonically identified with the Boson Fock space $\Gamma(L^2(\boldsymbol{R}))$ through the Wiener-Itô-Segal isomorphism, the white noise distribution theory has been applied to some questions in quantum physics as well, see e.g., [**37**] and references therein.

Since the exponential vectors $\{\phi_\xi; \xi \in E_{\boldsymbol{C}}\}$ span a dense subspace of $(E)_\beta$, any operator $\Xi \in \mathscr{L}((E)_\beta, (E)_\beta^*)$ is determined uniquely by its action on exponential vectors, see §1. This leads us to the idea of the symbol of an operator, originally due to Berezin

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[**8**], [**9**], see also Krée-Rączka [**21**]. The symbol calculus is very successful at least at an algebraic level, see e.g. [**12**], where a certain non-commutative stochastic differential equation is solved by means of formal power series. It is therefore crucial to obtain analytic properties of operators from their symbols. From that aspect the most remarkable is that operators in $\mathscr{L}((E)_\beta, (E)_\beta^*)$ are completely characterized by simple analytic properties of their symbols, see Theorem 1.6. This result, known as the analytic characterization theorem for operator symbols, was established in [**27**], [**28**] with many applications, see e.g., [**10**], [**14**], [**30**], [**31**], [**32**]. In particular, it offers a natural method of defining a function of operators (generally speaking, there is no canonical method of defining a function of non-commuting objects). We also note that the idea of symbol shares a common spirit with those of pseudo-differential operators, see e.g., [**26**].

In this paper, keeping applications in mind, we discuss Hilbert space-valued white noise functions based on the triple:

$$(E)_\beta \otimes \mathscr{H} \subset L^2(E^*, \mu) \otimes \mathscr{H} \subset (E)_\beta^* \otimes \mathscr{H}, \tag{0.2}$$

where $\mathscr{H}$ is another Hilbert space. This scheme appears often in physical problems of an interacting system such as "System + Reservoir" model; whence $\mathscr{H}$ is called a system Hilbert space. In Section 1 we assemble a few preliminary results and develop a general theory of operators in $\mathscr{L} \equiv \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ along with [**29**]. In particular, we obtain a criterion for continuity of a map $t \mapsto \Xi_t \in \mathscr{L}$, t running over a locally compact space, in terms of symbols (Theorems 1.8 and 1.9).

In Section 2 we introduce the Wick product $\diamond$ of operators by means of the analytic characterization of symbols. This is an analytic extension of the well known notion of the Wick product (or normal-ordered product) in physics. Moreover, we prove (Theorems 2.5 and 2.8) that the Wick exponential function of Ξ and the time-ordered Wick exponential of $\{L_t\}$ converge in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ whenever Ξ and L_t are of finite degree $\leq 2/(1-\beta)$.

In Section 3 we discuss unique existence of a solution to a linear differential equation of the form:

$$\frac{d\Xi}{dt} = L_t \diamond \Xi + M_t, \tag{0.3}$$

where $t \mapsto L_t \in \mathscr{L}$, $t \mapsto M_t \in \mathscr{L}$ are continuous. In fact, if $\deg L_t \leq 2/(1-\beta)$ for all t there exists a unique solution in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ which is given by means of the time-ordered Wick exponential function (Theorem 3.1 and its corollaries). In a broad sense such an equation as in (0.3) might be called a quantum stochastic differential equation.

A quantum stochastic differential equation of Itô type is typically of the form

$$dU = (L_1 d\Lambda + L_2 dA + L_3 dA^* + L_4 dt)U, \tag{0.4}$$

where L_i are operators acting on $\mathscr{H}$, and $\{A_t\}, \{A_t^*\}, \{\Lambda_t\}$ are the annihilation process, the creation process and the number process, respectively. According to the standard theory originally due to Hudson and Parthasarathy [**18**], the equation (0.4) is solved by means of a quantum analogue of Itô theory where the role of infinitesimal increment of the Brownian motion dB_t in the classical Itô theory is played by dA_t, dA_t^* and $d\Lambda_t$. The quantum Itô theory has been developed extensively by Attal [**5**], [**6**], Belavkin [**7**], Lindsay [**24**], Meyer [**25**], Parthasarathy [**35**], among others. The angle in this paper is different from them. From our point of view (0.4) is brought into a normal form:

$$\begin{aligned}\frac{dU}{dt} &= L_1 a_t^* U a_t + L_2 U a_t + L_3 a_t^* U + L_4 U \\ &= (L_1 a_t^* a_t + L_2 a_t + L_3 a_t^* + L_4) \diamond U,\end{aligned} \tag{0.5}$$

which is, obviously, a particular case of (0.3). In the recent study of stochastic limit of quantum theory [**2**], see also [**3**], there appears a new type of a quantum stochastic differential equation such as

$$\frac{dU}{dt} = (M_1 a_t^{*2} + M_2 a_t^2) U, \tag{0.6}$$

which is highly singular from the usual aspect (though the above equation is understood in a formal sense at the moment). Obviously, the normal form of (0.6) is a simple example of our case (0.3).

In conclusion, having introduced a space of white noise distributions properly (that is, the Kondratiev-Streit space), we are able to grasp a unique solution of a differential equation of the form (0.3). Moreover, our approach covers typical quantum stochastic differential equations of Itô type and their generalizations. The next steps in this line of research are to study regularity properties of the solutions and to explore the possibility of non-linear extension. These are now in progress.

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General Notation. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ be locally convex spaces.

$\mathfrak{X}_C$: the complexification of $\mathfrak{X}$ when it is a real space.

$\mathscr{L}(\mathfrak{X}, \mathfrak{Y})$: the space of continuous linear operators from $\mathfrak{X}$ into $\mathfrak{Y}$; equipped with the topology of bounded convergence.

$\mathscr{B}(\mathfrak{X}, \mathfrak{Y}; \mathfrak{Z})$: the space of continuous bilinear maps from $\mathfrak{X} \times \mathfrak{Y}$ into $\mathfrak{Z}$; equipped with the topology of bi-bounded convergence.

$\mathfrak{X}^*$: the space of continuous linear functionals on $\mathfrak{X}$; equipped with the strong dual topology after our convention above.

$\mathfrak{X} \otimes \mathfrak{Y}$: the Hilbert space tensor product when both $\mathfrak{X}, \mathfrak{Y}$ are Hilbert spaces.

$\mathfrak{X} \otimes_\pi \mathfrak{Y}$: the completed π-tensor product. When there is no danger of confusion, $\otimes_\pi$ is denoted by $\otimes$ for simplicity.

1. Preliminary results on white noise operators.

1.1. White noise distributions.

We start with the real Gelfand triple

$$E = \mathscr{S}(\boldsymbol{R}) \subset H = L^2(\boldsymbol{R}, dt) \subset E^* = \mathscr{S}'(\boldsymbol{R}).$$

The norm of H is denoted by $|\cdot|_0$ and since compatible the real inner product of H and the canonical bilinear form on $E^* \times E$ are denoted by the same symbol $\langle \cdot, \cdot \rangle$. Let μ be the standard Gaussian measure on E^* and $L^2(E^*, \mu)$ the Hilbert space of $\boldsymbol{C}$-valued L^2-functions on E^*. The celebrated Wiener-Itô-Segal theorem says that $L^2(E^*, \mu)$ is unitarily isomorphic to the Boson Fock space $\Gamma(H_{\boldsymbol{C}})$. The isomorphism is a unique linear extension of the following correspondence between exponential functions and exponential vectors:

$$\phi_\xi(x) = e^{\langle x, \xi \rangle - \langle \xi, \xi \rangle/2} \quad \longleftrightarrow \quad \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right), \quad \xi \in E_{\boldsymbol{C}}.$$

If $\phi \in L^2(E^*, \mu)$ and $(f_n)_{n=0}^\infty \in \Gamma(H_{\boldsymbol{C}})$ are related through the Wiener-Itô-Segal isomorphism, we write

$$\phi \sim (f_n)$$

for simplicity. It is then noted that

$$\|\phi\|_0^2 = \sum_{n=0}^\infty n! |f_n|_0^2, \tag{1.1}$$

where $\|\phi\|_0$ is the L^2-norm of $\phi \in L^2(E^*, \mu)$.

In order to introduce white noise distributions we need a particular family of seminorms defining the topology of $E = \mathscr{S}(\boldsymbol{R})$. By means of the differential operator $A = 1 + t^2 - d^2/dt^2$ we introduce a sequence of norms in $H_{\boldsymbol{C}}$ in such a way that $|\xi|_p = |A^p \xi|_0$. The numbers:

$$0 < \rho = \|A^{-1}\|_{OP} = \frac{1}{2} < 1, \quad \delta = \|A^{-1}\|_{HS}$$

are frequently used. Let E_p be the Hilbert space obtained by completing E with respect to the norm $|\cdot|_p$. Then it is known that

$$E \cong \operatorname*{proj\,lim}_{p \to \infty} E_p, \quad E^* \cong \operatorname*{ind\,lim}_{p \to \infty} E_{-p}.$$

The norms $|\cdot|_p$ are naturally extended to the tensor products $E^{\otimes n}$ and their complexification $E_{\boldsymbol{C}}^{\otimes n}$. The canonical bilinear form $\langle \cdot, \cdot \rangle$ is also extended to a $\boldsymbol{C}$-bilinear form on $(E_{\boldsymbol{C}}^{\otimes n})^* \times E_{\boldsymbol{C}}^{\otimes n}$.

Let β be a fixed number with $0 \le \beta < 1$. For $\phi \in L^2(E^*, \mu)$ we introduce a new norm

$$\|\phi\|_{p,\beta}^2 = \sum_{n=0}^\infty (n!)^{1+\beta} |f_n|_p^2, \quad \phi \sim (f_n). \tag{1.2}$$

For any $p \ge 0$, $(E_p)_\beta = \{\phi ; \|\phi\|_{p,\beta} < \infty\}$ becomes a Hilbert space. We put

$$(E)_\beta = \operatorname*{proj\,lim}_{p \to \infty} (E_p)_\beta,$$

which becomes a countable Hilbert nuclear space. In fact, Lemma 1.1. *For any $p \ge 0$ the canonical map $\iota_p : (E_{p+1})_\beta \to (E_p)_\beta$ is of Hilbert-Schmidt type with $\|\iota_p\|_{HS} = \|\Gamma(A)^{-1}\|_{HS}$, where $\Gamma(A)$ is the second quantization of A acting in $\Gamma(H_{\boldsymbol{C}})$.*

The proof is straightforward modification of [**28**, Lemma 3.1.2]. We next consider the dual spaces. For $0 \le \beta < 1$ and $p \ge 0$ we put

$$\|\phi\|_{-p,-\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1-\beta} |f_n|_{-p}^2, \quad \phi \sim (f_n). \tag{1.3}$$

Then $\|\cdot\|_{-p,-\beta}$ is a Hilbertian norm on $L^2(E^*, \mu)$ and we denote by $(E_{-p})_{-\beta}$ the completion. The dual space of $(E)_\beta$ is obtained as

$$(E)_\beta^* \cong \operatorname*{ind\,lim}_{p\to\infty} (E_{-p})_{-\beta} = \bigcup_{p \ge 0} (E_{-p})_{-\beta},$$

and we come to a complex Gelfand triple:

$$(E)_\beta \subset L^2(E^*, \mu) \subset (E)_\beta^*. \tag{1.4}$$

This is called the *Kondratiev-Streit* space [**20**], see also [**23**]; while the case of $\beta = 0$ is referred to as the *Hida-Kubo-Takenaka space* [**22**] and is denoted simply by $(E) \subset L^2(E^*, \mu) \subset (E)^*$. Obviously, $(E)_\beta \subset (E)$ and $(E)^* \subset (E)_\beta^*$. The canonical bilinear form on $(E)_\beta^* \times (E)_\beta$ will be denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Then

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \quad \Phi \sim (F_n) \in (E)_\beta^*, \quad \phi \sim (f_n) \in (E)_\beta. \tag{1.5}$$

We note that (1.1), (1.2), (1.3) and (1.5) are all compatible.

Let $\mathscr{H}$ be another Hilbert space with norm $|\cdot|_{\mathscr{H}}$. We assume that $\mathscr{H} = \mathscr{H}_{\boldsymbol{R}} + i\mathscr{H}_{\boldsymbol{R}}$, where $\mathscr{H}_{\boldsymbol{R}}$ is a real Hilbert space with real inner product $\langle \cdot, \cdot \rangle_0$, and that $\mathscr{H}$ is equipped with the canonical $\boldsymbol{C}$-bilinear form $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ induced from $\langle \cdot, \cdot \rangle_0$. This rather curious assumption, which can be in fact removed, is posed in order to avoid notational trouble; thus both $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ are $\boldsymbol{C}$-bilinear forms. Then (1.4) is extended to a triple of $\mathscr{H}$-valued white noise functions:

$$(E)_\beta \otimes \mathscr{H} \subset L^2(E^*, \mu) \otimes \mathscr{H} \subset ((E)_\beta \otimes \mathscr{H})^* \cong (E)_\beta^* \otimes \mathscr{H}, \tag{1.6}$$

where $\mathscr{H}$ and $\mathscr{H}^*$ are identified. The canonical $\boldsymbol{C}$-bilinear form on $(E)_\beta^* \otimes \mathscr{H} \times (E)_\beta \otimes \mathscr{H}$ is denoted again by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Note that $(E)_\beta \otimes \mathscr{H}$ is nuclear if and only if $\dim \mathscr{H} < \infty$.

1.2. White noise operators.

On the basis of (1.6) we study operators in the class $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$. We first note the following

Proposition 1.2. *The canonical correspondence $\tilde{\Xi} \leftrightarrow \Xi$ given by*

$$\langle\!\langle \tilde{\Xi}(\phi \otimes u), \psi \otimes v \rangle\!\rangle = \langle \Xi(\phi \otimes \psi) u, v \rangle_{\mathscr{H}}, \quad \phi, \psi \in (E)_\beta, \quad u, v \in \mathscr{H}, \tag{1.7}$$

yields a topological isomorphism:

$$\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H}) \cong \mathscr{L}((E)_\beta \otimes (E)_\beta, \mathscr{L}(\mathscr{H})). \tag{1.8}$$

Proof. It is immediate that (1.8) holds in the algebraic sense. We shall prove that their topologies coincide. Let B_1, B_2 be two bounded subsets of $(E)_\beta$ and put

$$C_1 = \{\phi \otimes u; \phi \in B_1, |u|_{\mathscr{H}} \le 1\}, \quad C_2 = \{\psi \otimes v; \psi \in B_2, |v|_{\mathscr{H}} \le 1\}.$$

Obviously, C_1, C_2 are bounded subsets of $(E)_\beta \otimes \mathscr{H}$. In view of (1.7) we obtain

$$\|\Xi(\phi \otimes \psi)\|_{OP} = \sup\{|\langle\!\langle \tilde{\Xi}(\phi \otimes u), \psi \otimes v \rangle\!\rangle|; u, v \in \mathscr{H}, |u|_{\mathscr{H}} \le 1, |u|_{\mathscr{H}} \le 1\},$$

and hence

$$\sup\{\|\Xi(\phi \otimes \psi)\|_{OP}; \phi \in B_1, \psi \in B_2\} = \sup\{|\langle\!\langle \tilde{\Xi}(\omega), \omega' \rangle\!\rangle|; \omega \in C_1, \omega' \in C_2\}. \tag{1.9}$$

In general, any bounded subset of $\mathfrak{X} \otimes \mathfrak{Y}$, where $\mathfrak{X}$ is a Fréchet space and $\mathfrak{Y}$ is a nuclear Fréchet space, is contained in the closed convex balanced hull of a set of the form $B_1 \otimes B_2 \equiv \{\phi \otimes \psi; \phi \in B_1, \psi \in B_2\}$ where B_1 and B_2 are bounded subsets of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, see e.g., [**36**, Chapter IV, §9.8]. In other words, uniform convergence on any bounded subset of $\mathfrak{X} \otimes \mathfrak{Y}$ follows from uniform convergence on any set of the form $B_1 \otimes B_2$, where B_1 and B_2 are bounded subsets of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Then we see from (1.9) that the topologies of both sides of (1.8) coincide. □

We need mutual estimates of norms of $\tilde{\Xi}$ and Ξ.

Proposition 1.3. *We keep the notations as in Proposition* 1.2.

(1) *For each* $\Xi \in \mathscr{L}((E)_\beta \otimes (E)_\beta, \mathscr{L}(\mathscr{H}))$ *there exist* $C \ge 0$ *and* $p \ge 0$ *such that*

$$\|\Xi(\omega)\|_{OP} \le C\|\omega\|_{p,\beta}, \quad \omega \in (E)_\beta \otimes (E)_\beta,$$

where $\|\omega\|_{p,\beta}$ *is the Hilbertian norm of* $((E)_\beta \otimes (E)_\beta)_p$. *In that case*

$$\|\tilde{\Xi}(\tilde{\phi})\|_{-(p+1),-\beta} \le C\|\Gamma(A)^{-1}\|_{HS}^2 \|\tilde{\phi}\|_{p+1,\beta}, \quad \tilde{\phi} \in (E)_\beta \otimes \mathscr{H}.$$

(2) *For each* $\tilde{\Xi} \in \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ *there exist* $C \ge 0$ *and* $p \ge 0$ *such that*

$$\|\tilde{\Xi}(\tilde{\phi})\|_{-p,-\beta} \le C\|\tilde{\phi}\|_{p,\beta}, \quad \tilde{\phi} \in (E)_\beta \otimes \mathscr{H}.$$

In that case we have

$$\|\Xi(\omega)\|_{OP} \le C\|\Gamma(A)^{-1}\|_{HS}^2 \|\omega\|_{p+1,\beta}, \quad \omega \in (E)_\beta \otimes (E)_\beta.$$

Proof. This is a simple consequence of Lemma 1.1 and a general relation between Hilbert space tensor product and π-tensor product, see [**29**, Proposition A.9]. □

For simplicity we put

$$\mathscr{L} \equiv \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H}) \cong \mathscr{L}((E)_\beta \otimes (E)_\beta, \mathscr{L}(\mathscr{H})) \cong ((E)_\beta \otimes (E)_\beta)^* \otimes \mathscr{L}(\mathscr{H}),$$

where the second isomorphism is due to the kernel theorem, see (A.2) in the Appendix. From now on, we use the same symbol for corresponding elements under the isomorphism above. We shall introduce a stratification of $\mathscr{L} \cong \mathscr{L}((E)_\beta \otimes (E)_\beta, \mathscr{L}(\mathscr{H}))$. For $p \ge 0$ we put

$$\|\Xi\|_{\mathscr{L}_p} = \sup\{\|\Xi(\omega)\|_{OP}; \omega \in (E)_\beta \otimes (E)_\beta, \|\omega\|_{p,\beta} \le 1\}, \quad \Xi \in \mathscr{L}.$$

Then $\mathscr{L}_p = \{\Xi \in \mathscr{L}; \|\Xi\|_{\mathscr{L}_p} < \infty\}$ becomes a Banach space with norm $\|\cdot\|_{\mathscr{L}_p}$. It follows from Proposition 1.3 that

$$\mathscr{L} = \bigcup_{p \ge 0} \mathscr{L}_p.$$

By definition for any $\Xi \in \mathscr{L}_p$ we have

$$\|\Xi(\omega)\|_{OP} \le \|\Xi\|_{\mathscr{L}_p} \|\omega\|_{p,\beta}, \quad \omega \in (E)_\beta \otimes (E)_\beta,$$

$$\|\Xi(\phi \otimes u)\|_{-p,-\beta} \le \|\Xi\|_{\mathscr{L}_p} \|\phi\|_{p,\beta} |u|_{\mathscr{H}}, \quad \phi \in (E)_\beta, u \in \mathscr{H}.$$

1.3. Integral kernel operators and Fock expansion.

We have in [**28**], [**29**] established the theory of Fock space operators based on the Hida-Kubo-Takenaka space i.e., the case of $\beta = 0$. Most results obtained there admit straightforward generalization to the case of Kondratiev-Streit space, i.e., for general $0 \le \beta < 1$, in this connection see also [**23**].

The *annihilation operator at a point* $t \in \boldsymbol{R}$, denoted by a_t, is a unique operator in $\mathscr{L}((E)_\beta, (E)_\beta)$ having the property

$$a_t \phi_\xi = \xi(t) \phi_\xi, \quad \xi \in E_{\boldsymbol{C}}.$$

The adjoint operator $a_t^* \in \mathscr{L}((E)_\beta^*, (E)_\beta^*)$ is called the *creation operator at a point t*. It is known that both $t \mapsto a_t \in \mathscr{L}((E)_\beta, (E)_\beta)$ and $t \mapsto a_t^* \in \mathscr{L}((E)_\beta^*, (E)_\beta^*)$ are $\boldsymbol{C}^\infty$-maps. For $\kappa \in (E_{\boldsymbol{C}}^{\otimes (l+m)})^* \otimes \mathscr{L}(\mathscr{H}) \cong \mathscr{L}(E_{\boldsymbol{C}}^{\otimes (l+m)}, \mathscr{L}(\mathscr{H}))$ we put

$$\|\kappa\|_p = \sup\left\{ \sum_{\mathbf{i}} |\langle \kappa(e(\mathbf{i}))u, v \rangle_{\mathscr{H}}|^2 |e(\mathbf{i})|_p^2; |u|_{\mathscr{H}} \le 1, |v|_{\mathscr{H}} \le 1 \right\}^{1/2},$$

where $\{e(\mathbf{i})\}$ is the canonical orthonormal basis of $H^{\otimes (l+m)}$, see [**29**, §3]. For such a κ we associate an *integral kernel operator* whose formal integral expression is given by

$$\Xi_{l,m}(\kappa) = \int_{\boldsymbol{R}^{l+m}} \kappa(s_1, \dots, s_l, t_1, \dots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m,$$

where a_t stands for $a_t \otimes I$. It is known that $\Xi_{l,m}(\kappa) \in \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ for any $0 \le \beta < 1$. In fact, we have

Proposition 1.4. *Put*

$$\Delta_r = \frac{(1-\beta)^{1-\beta}\delta}{\rho^{r/2}(-e\log(\delta^2\rho^r))^{1-\beta}}, \quad r_0 = \inf\{r > 0; \log(\delta^2\rho^r) \le 0\}.$$

Then, for any $p > r_0/2$ *with* $\|\kappa\|_{-p} < \infty$ *it holds that*

$$\|\Xi_{l,m}(\kappa)\tilde{\phi}\|_{-(p+1),-\beta} \le \rho^{-p}\delta^{-1}(l^l m^m)^{(1-\beta)/2} \Delta_{2p}^{(l+m)/2} \|\kappa\|_{-p} \|\tilde{\phi}\|_{p+1,\beta}, \tag{1.10}$$

for $\tilde{\phi} \in (E)_\beta \otimes \mathscr{H}$. *Moreover,*

$$\|\Xi_{l,m}(\kappa)\|_{\mathscr{L}_{p+2}} \le \rho^{-p}\delta^{-1}\|\Gamma(A)^{-1}\|_{HS}^2 (l^l m^m)^{(1-\beta)/2} \Delta_{2p}^{(l+m)/2} \|\kappa\|_{-p}. \tag{1.11}$$

Proof. Inequality (1.10) is a simple generalization of [**29**, Theorem 3.9]. Then (1.11) follows from (1.10) with the help of Proposition 1.3. □

Moreover,

Theorem 1.5. *Any operator* $\Xi \in \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ *admits an infinite series expansion in terms of integral kernel operators*:

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \quad \kappa_{l,m} \in (E_{\boldsymbol{C}}^{\otimes (l+m)})^* \otimes \mathscr{L}(\mathscr{H}), \tag{1.12}$$

where the series converges in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$.

Expansion (1.12) is referred to as *Fock expansion* in [**28**] and has wide applications. In fact, such an expansion has appeared often in physical literatures since Haag [**13**]. In view of (1.12) we put

$$\deg \Xi = \sup\{l+m; \kappa_{l,m} \neq 0\} \le \infty.$$

Remark. By definition, for any $0 \le \beta < 1$ we have

$$\mathscr{L}((E) \otimes \mathscr{H}, (E)^* \otimes \mathscr{H}) \subset \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H}).$$

However, any operator in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ with finite degree belongs automatically to $\mathscr{L}((E) \otimes \mathscr{H}, (E)^* \otimes \mathscr{H})$. The parameter β is essential when we consider infinite series of integral kernel operators such as (1.12), see the sequel.

1.4. Operator symbols.

For $\Xi \in \mathscr{L} \equiv \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ an $\mathscr{L}(\mathscr{H})$-valued function $\hat{\Xi}$ on $E_{\boldsymbol{C}} \times E_{\boldsymbol{C}}$ defined by

$$\langle \hat{\Xi}(\xi,\eta)u, v\rangle_{\mathscr{H}} = \langle\!\langle \Xi(\phi_\xi \otimes u), \phi_\eta \otimes v \rangle\!\rangle, \quad \xi, \eta \in E_{\boldsymbol{C}}, \quad u, v \in \mathscr{H},$$

is called the *symbol* of Ξ. By means of the isomorphism (1.8) we may write $\hat{\Xi}(\xi,\eta) = \Xi(\phi_\xi \otimes \phi_\eta)$. In case of $\mathscr{H} = \boldsymbol{C}$ the symbol is reduced to a $\boldsymbol{C}$-valued function. Since $\{\phi_\xi \otimes u; \xi \in E_{\boldsymbol{C}}, u \in \mathscr{H}\}$ spans a dense subspace of $(E)_\beta \otimes \mathscr{H}$, an operator $\Xi \in \mathscr{L}$ is determined uniquely by the symbol. Note here simple relations:

$$\widehat{\Xi^*}(\xi,\eta) = \hat{\Xi}(\eta,\xi)^*, \quad \Xi \in \mathscr{L},$$

$$(\Xi_1 \otimes L)\hat{}\,(\xi,\eta) = \hat{\Xi}_1(\xi,\eta)L, \quad \Xi_1 \in \mathscr{L}((E)_\beta, (E)_\beta^*),\ L \in \mathscr{L}(\mathscr{H}),$$

where $\hat{\Xi}_1(\xi,\eta)$ is a complex number.

Remark. There is a similar notion called the *Wick symbol* also due to Berezin [**8**], [**9**], which is given by $e^{-\langle\xi,\eta\rangle}\hat{\Xi}(\xi,\eta)$ for $\Xi \in \mathscr{L}$. Although the Wick symbol has an advantage in some contexts, we do not use the Wick symbol in this paper just to avoid confusion.

The next assertion is known as the analytic characterization of symbols.

THEOREM 1.6. *An $\mathscr{L}(\mathscr{H})$-valued function Θ defined on $E_{\boldsymbol{C}} \times E_{\boldsymbol{C}}$ is the symbol of an operator $\Xi \in \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ if and only if*

(i) *for any $\xi, \xi_1, \eta, \eta_1 \in E_{\boldsymbol{C}}$ and $u, v \in \mathscr{H}$ the function*

$$(z, w) \mapsto \langle \Theta(z\xi + \xi_1, w\eta + \eta_1)u, v \rangle$$

is entire holomorphic on $\boldsymbol{C} \times \boldsymbol{C}$;

(ii) *there exist constant numbers $C \geq 0$, $K \geq 0$ and $p \geq 0$ such that*

$$\|\Theta(\xi, \eta)\|_{OP} \leq C \exp K(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)}), \quad \xi, \eta \in E_{\boldsymbol{C}}.$$

In that case, $\Xi \in \mathscr{L}_{p+q+3}$ for $q > q_1 = q_1(p, \beta) > 0$ and

$$\|\Xi\|_{\mathscr{L}_{p+q+3}} \leq CL(1 - M)^{-2}, \tag{1.13}$$

where $L = L(p, q) > 0$ and $0 < M = M(p, K, q) < 1$ are constant numbers defined for $q > q_1$.

PROOF. In case of $\beta = 0$ a complete proof is given in [**28**], [**29**]. The proof for general $0 \leq \beta < 1$ is a simple modification. In fact, given a function $\Theta : E_{\boldsymbol{C}} \times E_{\boldsymbol{C}} \to \mathscr{L}(\mathscr{H})$ satisfying (i) and (ii), we can construct $\Xi \in \mathscr{L}$ by an infinite series as in (1.12) where $\kappa_{l,m}$ satisfies

$$\|\kappa_{l,m}\|_{-(p+1)} \leq Ce(l^l m^m)^{-(1-\beta)/2} \{e\delta(2e(K+1))^{(1-\beta)/2}\}^{l+m}. \tag{1.14}$$

Then, in view of (1.11) we have

$$\|\Xi_{l,m}(\kappa_{l,m})\|_{\mathscr{L}_{p+q+3}} \leq \rho^{-(p+q+1)} \delta^{-1} \|\Gamma(A)^{-1}\|_{HS}^2 CM^{l+m},$$

where

$$M = M(q) = \rho^q \Delta_{2(p+q+1)}^{1/2} e\delta(2e(K+1))^{(1-\beta)/2}.$$

Keeping in mind that $\lim_{q \to \infty} M(q) = 0$, we put

$$q_1 = q_1(p, \beta) = \inf\{q > 0; M(q) \leq 1\}.$$

Then for $q > q_1$ the infinite series $\sum_{l,m=0}^{\infty} \|\Xi_{l,m}(\kappa_{l,m})\|_{\mathscr{L}_{p+q+3}}$ is convergent and we obtain (1.13) with $L = \rho^{-(p+q+1)} \delta^{-1} \|\Gamma(A)^{-1}\|_{HS}^2$ and M above. □

THEOREM 1.7. *Let $\Xi \in \mathscr{L}_p$ with Fock expansion* (1.12). *Then*

$$\|\kappa_{l,m}\|_{-(p+1)} \leq 2^\beta e(l^l m^m)^{(1-\beta)/2} K_\beta^{l+m} \|\Xi\|_{\mathscr{L}_p},$$

where

$$K_\beta = e\delta\{2e((1-\beta)2^{(2\beta-1)/(1-\beta)} + 1)\}^{(1-\beta)/2}.$$

PROOF. For an exponential vector ϕ_ξ we have

$$\|\phi_\xi\|_{p,\beta} \leq 2^{\beta/2} \exp\{(1-\beta)2^{(2\beta-1)/(1-\beta)} |\xi|_p^{2/(1-\beta)}\}, \quad \xi \in E_{\boldsymbol{C}}, \tag{1.15}$$

see [**23**, §5.2]. Then for any $\Xi \in \mathscr{L}_p$ it holds that

$$\|\hat{\Xi}(\xi,\eta)\|_{OP} \le 2^{\beta}\|\Xi\|_{\mathscr{L}_p}\exp\{(1-\beta)2^{(2\beta-1)/(1-\beta)}(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)})\}. \tag{1.16}$$

The assertion follows by combining (1.14) and (1.16). □

1.5. Convergence of operators.

The convergence of operators in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ is rephrased in terms of convergence of symbols.

THEOREM 1.8. *Let T be a locally compact space. Then for the map $t \mapsto \Xi_t \in \mathscr{L}$, $t \in T$, the following four conditions are equivalent*:

(i) *$t \mapsto \Xi_t \in \mathscr{L}$ is continuous;*

(ii) *for any $t_0 \in T$ there exist $p \ge 0$ and an open neighborhood U of t_0 such that*

$$\{\Xi_t; t \in U\} \subset \mathscr{L}_p \quad \textit{and} \quad \lim_{t\to t_0}\|\Xi_t - \Xi_{t_0}\|_{\mathscr{L}_p} = 0.$$

(iii) *for any $t_0 \in T$ there exist an open neighborhood U of t_0, a set of positive numbers $\{\varepsilon_t; t \in U\}$ converging to 0 as $t \to t_0$, constant numbers $K \ge 0$ and $p \ge 0$ such that*

$$\|\hat{\Xi}_t(\xi,\eta) - \hat{\Xi}_{t_0}(\xi,\eta)\|_{OP} \le \varepsilon_t \exp K(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)}), \quad \xi,\eta \in E_{\mathbf{C}}, \quad t \in U.$$

(iv) *for any $t_0 \in T$ there exist $C \ge 0$, $K \ge 0$, $p \ge 0$ and an open neighborhood U of t_0 such that*

$$\|\hat{\Xi}_t(\xi,\eta)\|_{OP} \le C\exp K(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)}), \quad \xi,\eta \in E_{\mathbf{C}}, \quad t \in U, \tag{1.17}$$

and

$$\lim_{t\to t_0}\|\hat{\Xi}_t(\xi,\eta) - \hat{\Xi}_{t_0}(\xi,\eta)\|_{OP} = 0, \quad \xi,\eta \in E_{\mathbf{C}}.$$

PROOF. (i) ⇔ (ii) The proof is deferred to the Appendix.

(ii) ⇒ (iii) In view of (1.16) we have

$$\|\hat{\Xi}_t(\xi,\eta) - \hat{\Xi}_{t_0}(\xi,\eta)\|_{OP} \le 2^{\beta}\|\Xi_t - \Xi_{t_0}\|_{\mathscr{L}_p}\exp\{(1-\beta)2^{(2\beta-1)/(1-\beta)}(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)})\},$$

from which the assertion is clear.

(iii) ⇒ (iv) is obvious.

(iv) ⇒ (i) Let t_0 be fixed and we shall prove the continuity of $t \mapsto \Xi_t$ at $t = t_0$. For that purpose it may be assumed without loss of generality that $\Xi_{t_0} = 0$. Applying Theorem 1.6, we see from (1.17) that there exist $q > 0$ and $M > 0$ such that

$$\|\Xi_t\|_{\mathscr{L}_{p+q}} \le M, \quad t \in U. \tag{1.18}$$

On the other hand, by assumption

$$\|\Xi_t(\phi_\xi \otimes \phi_\eta)\|_{OP} = \|\hat{\Xi}_t(\xi,\eta)\|_{OP} \to 0, \quad t \to t_0. \tag{1.19}$$

Since the exponential vectors span a dense subspace of $(E)_\beta$, for any $\omega \in (E)_\beta \otimes$

$(E)_\beta$ and $\varepsilon > 0$ there exists a finite linear combination of exponential vectors $\omega' = \sum_i \phi_{\xi_i} \otimes \phi_{\eta_i}$ such that $\|\omega - \omega'\|_{p+q,\beta} < \varepsilon$. Then by the triangle inequality

$$\begin{aligned}\|\Xi_t(\omega)\|_{OP} &\le \|\Xi_t(\omega - \omega')\|_{OP} + \|\Xi_t(\omega')\|_{OP} \\ &\le \|\Xi_t\|_{\mathscr{L}_{p+q}} \|\omega - \omega'\|_{p+q,\beta} + \left\| \sum_i \Xi_t(\phi_{\xi_i} \otimes \phi_{\eta_i}) \right\|_{OP} \\ &\le \varepsilon \|\Xi_t\|_{\mathscr{L}_{p+q}} + \sum_i \|\Xi_t(\phi_{\xi_i} \otimes \phi_{\eta_i})\|_{OP},\end{aligned}$$

and in view of (1.18) and (1.19) we come to

$$\limsup_{t \to t_0} \|\Xi_t(\omega)\|_{OP} \le \varepsilon M.$$

Consequently,

$$\lim_{t \to t_0} \|\Xi_t(\omega)\|_{OP} = 0, \quad \omega \in (E)_\beta \otimes (E)_\beta.$$

It then follows from the Banach-Steinhaus theorem that Ξ_t converges to 0 uniformly on any compact subset of $(E)_\beta \otimes (E)_\beta$, and hence on any bounded subset due to the nuclearity of $(E)_\beta \otimes (E)_\beta$. □

THEOREM 1.9. *For $n = 1, 2, \dots$ let $\Xi_n \in \mathscr{L}$ be given. Then the sequence Ξ_n converges to some Ξ in $\mathscr{L}$ if and only if*

(i) *there exist $C \ge 0$, $K \ge 0$ and $p \ge 0$ such that*

$$\|\hat{\Xi}_n(\xi, \eta)\|_{OP} \le C \exp K(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)}), \quad \xi, \eta \in E_C, \quad n = 1, 2, \dots.$$

(ii) *for any $\xi, \eta \in E_C$ the limit $\Theta(\xi, \eta) \equiv \lim_{n\to\infty} \hat{\Xi}_n(\xi, \eta)$ exists in $\mathscr{L}(\mathscr{H})$.*

PROOF. The "only if" part is straightforward by Theorem 1.8. We shall prove the "if" part. Given $\xi, \xi_1, \eta \in E_C$ and $u, v \in \mathscr{H}$, we consider

$$g_n(z) = \langle \hat{\Xi}_n(z\xi + \xi_1, \eta)u, v\rangle_{\mathscr{H}}, \quad g(z) = \langle \Theta(z\xi + \xi_1, \eta)u, v\rangle_{\mathscr{H}}, \quad z \in \boldsymbol{C}.$$

Then $g(z)$ is entire holomorphic by Theorem 1.6 and $g(z) = \lim_{n\to\infty} g_n(z)$ by assumption. We shall prove that $g(z)$ is holomorphic on $\boldsymbol{C}$. Let γ be a smooth closed curve in $\boldsymbol{C}$. Since γ is a compact set, by assumption (i) there exists some $M > 0$ such that

$$|g_n(z)| \le C|u|_{\mathscr{H}}|v|_{\mathscr{H}} \exp K(|z\xi + \xi_1|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)}) \le M, \quad z \in \gamma, \quad n = 1, 2, \dots.$$

It then follows from the bounded convergence theorem that

$$0 = \lim_{n\to\infty} \int_\gamma g_n(z)\,dz = \int_\gamma g(z)\,dz.$$

Therefore $g(z)$ is holomorphic by Morera's theorem. Since Θ satisfies the same condition as in (i), by Theorem 1.6 there exists $\Xi \in \mathscr{L}$ such that $\hat{\Xi} = \Theta$. Thus condition (iv) in Theorem 1.8 is satisfied and, consequently, Ξ_n converges to Ξ in $\mathscr{L}$. □

1.6. Quantum stochastic processes.

A one-parameter family of operators $\{\Xi_t\}_{t\in T} \subset \mathscr{L} \equiv \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$, where t runs over an interval $T \subset \boldsymbol{R}$, might be called a *quantum stochastic process* in full generalities. In this paper, following the previous work [**31**] a quantum stochastic process is always assumed to be continuous, i.e., the continuity of $t \mapsto \Xi_t \in \mathscr{L}$. This seemingly strong (in fact, rather weak) assumption is useful to avoid minor technical consideration though weakened trivially in many contexts.

Lemma 1.10. (1) *If $\{\Xi_t\} \subset \mathscr{L}((E)_\beta, (E)_\beta^*)$ is a quantum stochastic process, so is the amplification $\{\Xi_t \otimes I\} \subset \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$.*

(2) *If $\{L_t\} \subset \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ is a quantum stochastic process, so are $\{L_t a_t = L_t(a_t \otimes I)\}$ and $\{a_t^* L_t = (a_t^* \otimes I)L_t\}$.*

Proof. One may check condition (iv) in Theorem 1.8 easily. □

When there is no danger of confusion, the amplification $\Xi_t \otimes I$ is denoted simply by Ξ_t. We use this convention particularly for $\{a_t\}$, $\{a_t^*\}$ and $\{W_t = a_t + a_t^*\}$, where the pair (a_t, a_t^*) or W_t is referred to as the *quantum white noise process*.

Let $\{L_t\} \subset \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ be a quantum stochastic process. It then follows from Theorem 1.8 that for any compact interval $[0, t]$ there exists $p \ge 0$ such that $s \mapsto L_s$ is a continuous map from $[0, t]$ into $\mathscr{L}_p$. Then one may introduce the (Riemannian) integral:

$$\Xi_t = \int_0^t L_s \, ds$$

in an obvious manner. Clearly, $\{\Xi_t\}$ becomes a quantum stochastic process which is differentiable in $\mathscr{L}_p$, hence in $\mathscr{L}$ as well:

$$\frac{d}{dt}\Xi_t = L_t.$$

Moreover, in view of Lemma 1.10 (2) we may define quantum stochastic processes:

$$\int_0^t a_s^* L_s \, ds, \qquad \int_0^t L_s a_s \, ds.$$

These are called quantum stochastic integrals against the creation and the annihilation processes, respectively. In particular,

$$A_t = \int_0^t a_s \, ds, \quad A_t^* = \int_0^t a_s^* \, ds, \quad \Lambda_t = \int_0^t a_s^* a_s \, ds, \tag{1.20}$$

are respectively the *annihilation process*, the *creation process* and the *number process* of Hudson-Parthasarathy [**18**].

2. Wick product of white noise operators.

2.1. Definition.

We start with

Lemma 2.1. *For two operators $\Xi_1, \Xi_2 \in \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ there exists $\Xi \in \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ uniquely determined by*

$$\hat{\Xi}(\xi,\eta) = e^{-\langle\xi,\eta\rangle}\hat{\Xi}_1(\xi,\eta)\hat{\Xi}_2(\xi,\eta), \quad \xi,\eta \in E_{\mathbf{C}}. \tag{2.1}$$

PROOF. For simplicity we denote by $\Theta(\xi,\eta)$ the right hand side of (2.1). It is sufficient to show that Θ satisfies conditions (i) and (ii) in Theorem 1.6. In fact, (i) is obvious. As for (ii) we note that

$$\|\hat{\Xi}_i(\xi,\eta)\|_{OP} \le 2^{\beta}\|\Xi_i\|_{\mathscr{L}_p}\exp\{(1-\beta)2^{(2\beta-1)/(1-\beta)}(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)})\}, \ i=1,2, \tag{2.2}$$

for some $p \ge 0$, see (1.16). Using an obvious inequality $a^2 \le 1 + a^{2/(1-\beta)}$ we have

$$|e^{-\langle\xi,\eta\rangle}| \le \exp\frac{\rho^{2p}}{2}(|\xi|_p^2 + |\eta|_p^2) \le e^{\rho^{2p}}\exp\frac{\rho^{2p}}{2}(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)}). \tag{2.3}$$

Then from (2.2) and (2.3) we obtain

$$\|\Theta(\xi,\eta)\|_{OP} \le C\|\Xi_1\|_{\mathscr{L}_p}\|\Xi_2\|_{\mathscr{L}_p}\exp K(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)}), \tag{2.4}$$

where $C = 2^{2\beta}e^{\rho^{2p}}$ and $K = (1-\beta)2^{\beta/(1-\beta)} + \rho^{2p}/2$. This proves (ii). □

The operator Ξ defined in Lemma 2.1 is denoted by

$$\Xi = \Xi_1 \diamond \Xi_2$$

and is called the *Wick product*. For $\Xi, \Xi_i \in \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ it holds that

$$\Xi \diamond I = I \diamond \Xi = \Xi,$$

$$(\Xi_1 \diamond \Xi_2) \diamond \Xi_3 = \Xi_1 \diamond (\Xi_2 \diamond \Xi_3),$$

$$(\Xi_1 \diamond \Xi_2)^* = \Xi_2^* \diamond \Xi_1^*.$$

Moreover, if $\mathscr{H} = \mathbf{C}$ we have

$$\Xi_1 \diamond \Xi_2 = \Xi_2 \diamond \Xi_1, \quad \Xi_i \in \mathscr{L}((E)_\beta, (E)_\beta^*),$$

that is, equipped with the Wick product $\mathscr{L}((E)_\beta, (E)_\beta^*)$ becomes a commutative algebra. However, the Wick product is not commutative whenever $\dim \mathscr{H} > 1$. In fact,

$$(\Xi_1 \otimes L_1) \diamond (\Xi_2 \otimes L_2) = (\Xi_1 \diamond \Xi_2) \otimes (L_1 L_2), \quad \Xi_i \in \mathscr{L}((E)_\beta, (E)_\beta^*), \ L_i \in \mathscr{L}(\mathscr{H}).$$

PROPOSITION 2.2. *For an operator $\Omega \in \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ the following conditions are equivalent*:

(i) $\Xi \diamond \Omega = \Xi\Omega$ *for any* $\Xi \in \mathscr{L}((E)_\beta^* \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$;

(ii) $\Omega^* \diamond \Xi = \Omega^*\Xi$ *for any* $\Xi \in \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta \otimes \mathscr{H})$;

(iii) *the Fock expansion of Ω contains only annihilation operators, i.e., is of the form*:

$$\Omega = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m}). \tag{2.5}$$

Assume that Ω satisfies one of the above conditions and belongs to $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta \otimes \mathscr{H})$. Then $\Xi \diamond \Omega = \Xi\Omega$ holds for any $\Xi \in \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$.

PROOF. (i) $\Leftrightarrow$ (ii) is obvious by duality.

(i) $\Rightarrow$ (iii) For $\zeta \in E_{\boldsymbol{C}}$ we put

$$D_\zeta = \int_{\boldsymbol{R}} \zeta(t) a_t \, dt.$$

It is known that D_ζ belongs both to $\mathscr{L}((E)_\beta, (E)_\beta)$ and to $\mathscr{L}((E)_\beta^*, (E)_\beta^*)$. Then for $\Xi = D_\zeta \otimes I$, where I is the identity operator on $\mathscr{H}$, we have

$$\Xi \diamond \Omega = \Omega \Xi. \tag{2.6}$$

We shall prove (2.6) by symbols. Since $\hat{\Xi}(\xi, \eta) = e^{\langle \xi, \eta \rangle} \langle \zeta, \xi \rangle I$, we have

$$(\Xi \diamond \Omega)\hat{}(\xi, \eta) = e^{-\langle \xi, \eta \rangle} \hat{\Xi}(\xi, \eta) \hat{\Omega}(\xi, \eta) = \langle \zeta, \xi \rangle \hat{\Omega}(\xi, \eta). \tag{2.7}$$

On the other hand, since

$$\begin{aligned} \langle \widehat{\Omega\Xi}(\xi, \eta) u, v \rangle_{\mathscr{H}} &= \langle\!\langle \Omega \Xi(\phi_\xi \otimes u), \phi_\eta \otimes v \rangle\!\rangle \\ &= \langle\!\langle \Omega(\langle \zeta, \xi \rangle \phi_\xi \otimes u), \phi_\eta \otimes v \rangle\!\rangle \\ &= \langle \zeta, \xi \rangle \langle \hat{\Omega}(\xi, \eta) u, v \rangle_{\mathscr{H}}, \end{aligned}$$

we have

$$\widehat{\Omega\Xi}(\xi, \eta) = \langle \zeta, \xi \rangle \hat{\Omega}(\xi, \eta). \tag{2.8}$$

Then (2.6) follows from (2.7) and (2.8). Now taking the assumption into account, we see that $\Xi\Omega = \Omega\Xi$, i.e., Ω commutes with any $D_\zeta \otimes I$, where ζ runs over $E_{\boldsymbol{C}}$. In case of $\mathscr{H} = \boldsymbol{C}$ an operator commuting with all D_ζ contains no creation operators in its Fock expansion, see [**30**]. This fact admits a straightforward generalization to the case of an arbitrary $\mathscr{H}$ and we obtain the desired assertion.

(iii) $\Rightarrow$ (i) We first note that

$$\Xi_{0,m}(\kappa_{0,m})(\phi_\xi \otimes u) = \phi_\xi \otimes \kappa_{0,m}(\xi^{\otimes m}) u, \quad \xi \in E_{\boldsymbol{C}}, \, u \in \mathscr{H},$$

and hence

$$(\Xi_{0,m}(\kappa_{0,m}))\hat{}(\xi, \eta) = e^{\langle \xi, \eta \rangle} \kappa_{0,m}(\xi^{\otimes m}), \quad \xi, \eta \in E_{\boldsymbol{C}}.$$

We now assume that $\Omega = \sum_{m=0}^\infty \Xi_{0,m}(\kappa_{0,m})$. Then for $\Xi \in \mathscr{L}((E)_\beta^* \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$,

$$\begin{aligned} \langle (\Xi\Omega)\hat{}(\xi, \eta) u, v \rangle_{\mathscr{H}} &= \sum_{m=0}^\infty \langle\!\langle \Xi \Xi_{0,m}(\kappa_{0,m})(\phi_\xi \otimes u), \phi_\eta \otimes v \rangle\!\rangle \\ &= \sum_{m=0}^\infty \langle\!\langle \Xi(\phi_\xi \otimes \kappa_{0,m}(\xi^{\otimes m})) u, \phi_\eta \otimes v \rangle\!\rangle \\ &= \sum_{m=0}^\infty \langle \hat{\Xi}(\xi, \eta) \kappa_{0,m}(\xi^{\otimes m}) u, v \rangle_{\mathscr{H}} \end{aligned}$$

$$= \sum_{m=0}^{\infty} \langle \hat{\Xi}(\xi,\eta) e^{-\langle \xi,\eta \rangle} (\Xi_{0,m}(\kappa_{0,m}))\hat{}(\xi,\eta)u, v\rangle_{\mathscr{H}}$$

$$= \langle \hat{\Xi}(\xi,\eta) e^{-\langle \xi,\eta \rangle} \hat{\Omega}(\xi,\eta)u, v\rangle_{\mathscr{H}}.$$

This implies that $\Xi\Omega = \Xi \diamond \Omega$.

For the last part we need only to repeat similar computation as above keeping in mind that the expansion (2.5) converges in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta \otimes \mathscr{H})$. □

A relevant result appears in Huang-Luo [**17**] where the case of $\mathscr{H} = \boldsymbol{C}$ is in consideration.

Corollary 2.3. *For any* $\Xi \in \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ *it holds that*

$$a_{s_1}^* \cdots a_{s_l}^* \Xi a_{t_1} \cdots a_{t_m} = \Xi \diamond (a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m}) = (a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m}) \diamond \Xi.$$

In particular,

$$a_s^* \Xi = \Xi \diamond a_s^* = a_s^* \diamond \Xi, \quad \Xi a_t = \Xi \diamond a_t = a_t \diamond \Xi,$$

$$a_s \diamond a_t = a_s a_t, \quad a_s^* \diamond a_t = a_s^* a_t, \quad a_s \diamond a_t^* = a_t^* a_s, \quad a_s^* \diamond a_t^* = a_s^* a_t^*, \tag{2.9}$$

where a_t *and* a_t^* *are short hand notations for the amplifications as usual.*

As for topological properties of Wick products we only mention the following

Proposition 2.4. *The Wick product is a separately continuous bilinear map from* $\mathscr{L}((E)_\beta, (E)_\beta^*) \times \mathscr{L}((E)_\beta, (E)_\beta^*)$ *into* $\mathscr{L}((E)_\beta, (E)_\beta^*)$.

Proof. Suppose $\Xi_1, \Xi_2 \in \mathscr{L}((E)_\beta, (E)_\beta^*)$ and put $\Xi = \Xi_1 \diamond \Xi_2$. It follows from (2.4) that

$$|\hat{\Xi}(\xi,\eta)| \le C \|\Xi_1\|_{\mathscr{L}_p} \|\Xi_2\|_{\mathscr{L}_p} \exp K(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)})$$

for some $C \ge 0$ and $K \ge 0$. Then, applying Theorem 1.6, we see that there exist $L > 0$, $0 < M < 1$ and $q > 0$ such that

$$\|\Xi_1 \diamond \Xi_2\|_{\mathscr{L}_{p+q+3}} \le CL(1-M)^{-2} \|\Xi_1\|_{\mathscr{L}_p} \|\Xi_2\|_{\mathscr{L}_p}, \quad \Xi_1, \Xi_2 \in \mathscr{L}_p. \tag{2.10}$$

Suppose Ξ_2 is fixed. Then (2.10) means that $\Xi_1 \mapsto \Xi_1 \diamond \Xi_2$ is a continuous linear map from $\mathscr{L}_p$ into $\mathscr{L}_{p+q+3}$, and hence into $\mathscr{L}$. Since

$$\mathscr{L} \cong ((E)_\beta \otimes (E)_\beta)^* \cong \operatorname*{ind\,lim}_{p\to\infty} ((E)_\beta \otimes (E)_\beta)_{-p} \cong \operatorname*{ind\,lim}_{p\to\infty} \mathscr{L}_p,$$

$\Xi_1 \mapsto \Xi_1 \diamond \Xi_2$ is a continuous linear map from $\mathscr{L}((E)_\beta, (E)_\beta^*)$ into itself. □

Remark. The Wick product of white noise functions has been actively discussed, see [**19**], [**23**] and references therein; see also [**16**] for relevant topics. Recall that each $\Phi \in (E)_\beta^*$ gives rise to a multiplication operator $\Xi_\Phi \in \mathscr{L}((E)_\beta, (E)_\beta^*)$. For two white noise functions $\Phi, \Psi \in (E)_\beta^*$ we denote by $\Phi \diamond \Psi$ the Wick product. It is then easy to see that $\Xi_{\Phi \diamond \Psi} = \Xi_\Phi \diamond \Xi_\Psi$.

2.2. Wick exponential function.

For simplicity we put

$$\Xi^{\diamond n} = \underbrace{\Xi \diamond \dots \diamond \Xi}_{n\,\text{times}}, \quad \Xi^{\diamond 0} = I.$$

Theorem 2.5. *Let $\Xi \in \mathscr{L}((E) \otimes \mathscr{H}, (E)^* \otimes \mathscr{H})$. If $\deg \Xi \le 2/(1-\beta)$, the infinite series*

$$\sum_{n=0}^{\infty} \frac{1}{n!} \Xi^{\diamond n} \tag{2.11}$$

converges in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$. In particular, (2.11) *converges in $\mathscr{L}((E) \otimes \mathscr{H}, (E)^* \otimes \mathscr{H})$ if $\deg \Xi \le 2$.*

Proof. Let S_N denote the N-th partial sum of (2.11). We note that by definition

$$\hat{S}_N(\xi, \eta) = e^{\langle \xi, \eta \rangle} \sum_{n=0}^{N} \frac{1}{n!} (e^{-\langle \xi, \eta \rangle} \hat{\Xi}(\xi, \eta))^n,$$

and hence

$$\lim_{N \to \infty} \hat{S}_N(\xi, \eta) = \exp(\langle \xi, \eta \rangle + e^{-\langle \xi, \eta \rangle} \hat{\Xi}(\xi, \eta)), \quad \xi, \eta \in E_{\mathbf{C}}.$$

Then by Theorem 1.9, S_N converges in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ if and only if there exist some constant numbers $C \ge 0$, $K \ge 0$ and $p \ge 0$ such that

$$\|\hat{S}_N(\xi, \eta)\|_{OP} \le C \exp K(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)}), \quad N = 1, 2, \dots. \tag{2.12}$$

Since the factor $e^{\langle \xi, \eta \rangle}$ does not contribute to the estimate, (2.12) is equivalent to

$$\left\| \sum_{n=0}^{N} \frac{1}{n!} (e^{-\langle \xi, \eta \rangle} \hat{\Xi}(\xi, \eta))^n \right\|_{OP} \le C \exp K(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)}), \quad N = 1, 2, \dots. \tag{2.13}$$

We shall prove that condition (2.13) is satisfied if $d \equiv \deg \Xi \le 2/(1-\beta)$.

Given $\Xi \in \mathscr{L}((E) \otimes \mathscr{H}, (E)^* \otimes \mathscr{H})$ with $d = \deg \Xi < \infty$ we put

$$\Xi = \sum_{l+m \le d} \Xi_{l,m}(\kappa_{l,m}).$$

Choose $p \ge 0$ such that

$$K' = \max_{l+m \le d} \|\kappa_{l,m}\|_{-p} < \infty.$$

In view of

$$\hat{\Xi}(\xi, \eta) = e^{\langle \xi, \eta \rangle} \sum_{l+m \le d} \kappa_{l,m}(\eta^{\otimes l} \otimes \xi^{\otimes m}),$$

we have

$$\begin{aligned}\|e^{-\langle\xi,\eta\rangle}\hat{\Xi}(\xi,\eta)\|_{OP} &\le \sum_{l+m\le d}\|\kappa_{l,m}(\eta^{\otimes l}\otimes\xi^{\otimes m})\|_{OP}\\ &\le \sum_{l+m\le d}\|\kappa_{l,m}\|_{-p}|\eta|_p^l|\xi|_p^m\\ &\le K'\sum_{l+m\le d}|\eta|_p^l|\xi|_p^m. \end{aligned}\tag{2.14}$$

Since

$$\sum_{l+m=k}|\eta|_p^l|\xi|_p^m \le \sum_{l+m=k}(|\eta|_p^{l+m}+|\xi|_p^{l+m}) = (k+1)(|\eta|_p^k+|\xi|_p^k),$$

which follows from an obvious inequality $a^lb^m \le a^{l+m}+b^{l+m}$, $a,b\ge 0$, we see that (2.14) becomes

$$\begin{aligned}\|e^{-\langle\xi,\eta\rangle}\hat{\Xi}(\xi,\eta)\|_{OP} &\le K'\sum_{k=0}^{d}\sum_{l+m=k}|\eta|_p^l|\xi|_p^m\\ &\le K'\sum_{k=0}^{d}(k+1)(|\eta|_p^k+|\xi|_p^k)\\ &\le K'(d+1)\sum_{k=0}^{d}(|\eta|_p^k+|\xi|_p^k). \end{aligned}\tag{2.15}$$

In view of the inequality $1+a+a^2+\cdots+a^d\le 1+d+da^d$, $a\ge 0$, (2.15) becomes

$$\begin{aligned}&\le K'(d+1)(1+d+d|\eta|_p^d+1+d+d|\xi|_p^d)\\ &= 2K'(d+1)^2+K'(d+1)d(|\eta|_p^d+|\xi|_p^d). \end{aligned}\tag{2.16}$$

Since $d\le 2/(1-\beta)$, we have $|\eta|_p^d\le 1+|\eta|_p^{2/(1-\beta)}$. Hence (2.16) becomes

$$\le 2K'(d+1)^2+K'(d+1)d(2+|\eta|_p^{2/(1-\beta)}+|\xi|_p^{2/(1-\beta)}).$$

Therefore,

$$\|e^{-\langle\xi,\eta\rangle}\hat{\Xi}(\xi,\eta)\|_{OP}\le C'+K(|\eta|_p^{2/(1-\beta)}+|\xi|_p^{2/(1-\beta)}),$$

where $C'=2K'(d+1)(2d+1)$ and $K=K'(d+1)d$. Consequently, we obtain

$$\begin{aligned}&\left\|\sum_{n=0}^{N}\frac{1}{n!}\,(e^{-\langle\xi,\eta\rangle}\hat{\Xi}(\xi,\eta))^n\right\|_{OP}\\ &\quad\le \sum_{n=0}^{N}\frac{1}{n!}\|e^{-\langle\xi,\eta\rangle}\hat{\Xi}(\xi,\eta)\|_{OP}^n\\ &\quad\le e^{C'}\exp K(|\eta|_p^{2/(1-\beta)}+|\xi|_p^{2/(1-\beta)}),\end{aligned}$$

as desired. □

Remark. The condition $\deg \varXi \le 2/(1-\beta)$ seems almost best possible to have the convergence in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$. In fact, in case of $\mathscr{H} = \boldsymbol{C}$, it can be proved that (2.12) implies $\deg \varXi \le 2/(1-\beta)$. Letting $N \to \infty$ in (2.12) we have

$$|\exp(e^{-\langle \xi, \eta \rangle} \hat{\varXi}(\xi, \eta))| \le C \exp K(|\eta|_p^{2/(1-\beta)} + |\xi|_p^{2/(1-\beta)}).$$

For simplicity we put

$$\theta(z) = e^{-z\langle \xi, \eta \rangle} \hat{\varXi}(\xi, z\eta), \quad z \in \boldsymbol{C}.$$

Then $F(z) = e^{\theta(z)}$ becomes an entire holomorphic function without zeroes of order $\le 2/(1-\beta)$, i.e.,

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} \le \frac{2}{1-\beta}, \quad M(r) = \max_{|z|=r} |F(z)|.$$

Applying Hadamard's factorization theorem for entire holomorphic functions (see e.g., [**4**]), we see that $\theta(z)$ is a polynomial of degree $\le 2/(1-\beta)$. From

$$\theta(z) = e^{-z\langle \xi, \eta \rangle} \sum_{l,m=0}^{\infty} \varXi_{l,m}(\kappa_{l,m})\hat{}\,(\xi, z\eta) = \sum_{l,m=0}^{\infty} \kappa_{l,m}(\eta^{\otimes l} \otimes \xi^{\otimes m}) z^l$$

we see that $\varXi_{l,m}(\kappa_{l,m}) = 0$ whenever $l > 2/(1-\beta)$. Similarly, $\varXi_{l,m}(\kappa_{l,m}) = 0$ whenever $m > 2/(1-\beta)$, and hence $d \equiv \deg \varXi < \infty$. We shall show that $d \le 2/(1-\beta)$. By definition $\kappa_{l,m} \ne 0$ for some l, m with $l + m = d$. Hence there exist $\xi, \eta \in E_{\boldsymbol{C}}$ such that

$$\omega \equiv \sum_{l+m=d} \kappa_{l,m}(\eta^{\otimes l} \otimes \xi^{\otimes m}) \ne 0.$$

We may assume without loss of generality that $\omega > 0$. Replacing ξ and η in (2.13) with $z\xi$ and $z\eta$, respectively, we obtain

$$\left| \exp\left\{ \sum_{l+m \le d} \kappa_{l,m}(\eta^{\otimes l} \otimes \xi^{\otimes m}) z^{l+m} \right\} \right| \le C \exp K|z|^{2/(1-\beta)}(|\eta|_p^{2/(1-\beta)} + |\xi|_p^{2/(1-\beta)}), \quad z \in \boldsymbol{C},$$

consequently,

$$|\exp\{\omega z^d + P_{d-1}(z)\}| \le C \exp(\omega' |z|^{2/(1-\beta)}), \quad z \in \boldsymbol{C}, \tag{2.17}$$

where $\omega' = K(|\eta|_p^{2/(1-\beta)} + |\xi|_p^{2/(1-\beta)}) > 0$ and $P_{d-1}(z)$ is a polynomial in z of degree at most $d-1$. Then (2.17) holds for any $z \in \boldsymbol{C}$ only when $d \le 2/(1-\beta)$.

The convergent series introduced in Theorem 2.5 is called the *Wick exponential function* of $\varXi$ and is denoted by

$$\text{wexp}\, \varXi = \sum_{n=0}^{\infty} \frac{1}{n!} \varXi^{\diamond n}.$$

Note that $\varXi \mapsto \text{wexp}\, \varXi$ is not continuous. In fact, the Wick exponential is defined only for $\varXi$ with finite degree and such operators do not constitute an open set in $\mathscr{L}((E) \otimes \mathscr{H}, (E)^* \otimes \mathscr{H})$.

The symbol of $\mathrm{wexp}\,\Xi$ was already obtained during the proof of Theorem 2.5:

$$(\mathrm{wexp}\,\Xi)\hat{}(\xi,\eta) = \exp(\langle\xi,\eta\rangle + e^{-\langle\xi,\eta\rangle}\hat{\Xi}(\xi,\eta)). \tag{2.18}$$

Using this one can deduce immediately the following

Lemma 2.6. *For $i = 1,2$ let $\Xi_i \in \mathscr{L}((E)\otimes\mathscr{H}, (E)^*\otimes\mathscr{H})$ with finite degrees. If $\Xi_1 \diamond \Xi_2 = \Xi_2 \diamond \Xi_1$, it holds that*

$$(\mathrm{wexp}\,\Xi_1) \diamond (\mathrm{wexp}\,\Xi_2) = \mathrm{wexp}\,(\Xi_1 + \Xi_2). \tag{2.19}$$

In particular, for any $\Xi \in \mathscr{L}((E)\otimes\mathscr{H}, (E)^*\otimes\mathscr{H})$ with finite degree we have

$$\mathrm{wexp}\,\Xi \diamond \mathrm{wexp}\,(-\Xi) = I.$$

Moreover, with the help of Theorem 1.8 (iv) we can prove the following lemma without difficulty.

Lemma 2.7. *Assume that $\Xi \in \mathscr{L}((E)\otimes\mathscr{H}, (E)^*\otimes\mathscr{H})$ is of finite degree $\le 2/(1-\beta)$. Then $z \mapsto \mathrm{wexp}\,(z\Xi) \in \mathscr{L}((E)_\beta\otimes\mathscr{H}, (E)_\beta^*\otimes\mathscr{H})$ is entire holomorphic and*

$$\frac{d}{dz}\mathrm{wexp}\,(z\Xi) = \Xi \diamond \mathrm{wexp}\,(z\Xi)$$

holds in $\mathscr{L}((E)_\beta\otimes\mathscr{H}, (E)_\beta^*\otimes\mathscr{H})$.

Remark. In their recent paper Cochran-Kuo-Sengupta [**11**] introduced a further generalization of white noise functions. It is plausible that the Wick exponential $\mathrm{wexp}\,\Xi$ converges for *any* $\Xi \in \mathscr{L}((E),(E)^*)$ in a suitably extended space of operators. A further detailed study in this connection has been initiated in [**34**].

2.3. Time-ordered Wick exponential function.

We shall discuss a generalization of Wick exponential function introduced in the previous section.

Theorem 2.8. *Let $\{L_t\}_{t\in T} \subset \mathscr{L}((E)\otimes\mathscr{H}, (E)^*\otimes\mathscr{H})$ be a quantum stochastic process, where $T \subset \boldsymbol{R}$ is an interval containing 0. Assume that $\deg L_t \le 2/(1-\beta)$ for some $0 \le \beta < 1$. Then the infinite series*

$$\Xi_t = I + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, L_{t_1} \diamond L_{t_2} \diamond \cdots \diamond L_{t_n}, \tag{2.20}$$

converges in $\mathscr{L}((E)_\beta\otimes\mathscr{H}, (E)_\beta^*\otimes\mathscr{H})$.

Proof. We may assume that T is a compact interval. We put

$$Y_n = Y_n(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, L_{t_1} \diamond L_{t_2} \diamond \cdots \diamond L_{t_n}.$$

Then by the definition of Wick product we obtain

$$\hat{Y}_n(\xi,\eta) = e^{\langle \xi,\eta\rangle} \int_0^t dt_1 \int_0^{t_1} dt_2$$
$$\cdots \int_0^{t_{n-1}} dt_n \, e^{-\langle \xi,\eta\rangle} \hat{L}_{t_1}(\xi,\eta) e^{-\langle \xi,\eta\rangle} \hat{L}_{t_2}(\xi,\eta)$$
$$\cdots e^{-\langle \xi,\eta\rangle} \hat{L}_{t_n}(\xi,\eta).$$

Consider the Fock expansion:

$$L_t = \sum_{l+m \le d} \Xi_{l,m}(\kappa_{l,m}(t)), \quad \deg L_t \le d < \infty,$$

where $d = \max\{\deg L_t; t \in T\} < \infty$ by assumption. Then we have

$$\hat{L}_t(\xi,\eta) = e^{\langle \xi,\eta\rangle} \sum_{l+m \le d} \kappa_{l,m}(t)(\eta^{\otimes l} \otimes \xi^{\otimes m}).$$

Since $t \mapsto L_t$ is continuous, so is $t \mapsto \kappa_{l,m}(t)$, see Theorem 1.7. Therefore there exist $K' \ge 0$ and $p \ge 0$ such that

$$\|\kappa_{l,m}(t)\|_{-p} \le K', \quad t \in T, \quad l,m = 0,1,2,\dots. \tag{2.21}$$

Then in a similar manner as in the proof of Theorem 2.5, we may find $C_1 \ge 0$ and $K_2 \ge 0$ such that

$$\|e^{-\langle \xi,\eta\rangle} \hat{L}_t(\xi,\eta)\|_{OP} \le C_1 + K_1(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)}), \quad t \in T, \quad \xi,\eta \in E_{\mathbf{C}},$$

where $C_1 = 2K'(d+1)(2d+1)$ and $K_1 = K'(d+1)d$. Thus

$$\|\hat{Y}_n(\xi,\eta)\|_{OP} \le |e^{\langle \xi,\eta\rangle}| \frac{t^n}{n!} \{C_1 + K_2(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)})\}^n,$$

and

$$\sum_{n=0}^{\infty} \|\hat{Y}_n(\xi,\eta)\|_{OP} \le |e^{\langle \xi,\eta\rangle}| \exp t\{C_1 + K_2(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)})\}$$
$$\le C \exp K(|\xi|_p^{2/(1-\beta)} + |\eta|_p^{2/(1-\beta)}) \tag{2.22}$$

for some $C \ge 0$ and $K \ge 0$. It follows from Theorem 1.9 that $\Xi_t \equiv \sum_{n=0}^{\infty} Y_n$ converges in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$. □

The infinite series (2.20) is called the *time-ordered Wick exponential function*. Similarly we may define the *reversed time-ordered Wick exponential function*:

$$I + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \, L_{t_n} \diamond L_{t_{n-1}} \diamond \cdots \diamond L_{t_1},$$

which is the adjoint of (2.20).

If $\{L_t\}$ is a commuting (with respect to the Wick product) family of operators, we have

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \, L_{t_1} \diamond L_{t_2} \diamond \cdots \diamond L_{t_n} = \frac{1}{n!} \left(\int_0^t L_s ds \right)^{\diamond n}.$$

Namely, the time-ordered Wick exponential (2.20) is reduced to the usual Wick exponential function; this case is discussed in [**33**].

3. Application to differential equations.

3.1. Ordinary differential equations.

THEOREM 3.1. *Let* $\{L_t\}$ *be a quantum stochastic process, where* t *runs over an interval* $T \subset \boldsymbol{R}$. *Assume that there exists some* $0 \le \beta < 1$ *such that* $\deg L_t \le 2/(1-\beta)$ *for all* t. *Then the initial value problem*

$$\frac{d\Xi}{dt} = L_t \diamond \Xi, \quad \Xi\big|_{t=0} = \Xi_0 \in \mathscr{L}((E) \otimes \mathscr{H}, (E)^* \otimes \mathscr{H}), \tag{3.1}$$

has a unique solution in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ *which is given by*

$$\Xi_t = \left(I + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \, L_{t_1} \diamond L_{t_2} \diamond \cdots \diamond L_{t_n} \right) \diamond \Xi_0. \tag{3.2}$$

PROOF. During the proof of Theorem 2.8 we have established a local uniform estimate (with respect to t) of the symbol of (3.2), see (2.22). Then for the assertion we need only to prove that the symbol of (3.2) is a unique solution to

$$\frac{d}{dt} \hat{\Xi}_t(\xi,\eta) = e^{-\langle \xi,\eta \rangle} \widehat{L}_t(\xi,\eta) \hat{\Xi}_t(\xi,\eta), \quad \hat{\Xi}_t(\xi,\eta)\big|_{t=0} = \hat{\Xi}_0(\xi,\eta),$$

of which the verification is straightforward. □

REMARK. If we take the initial data Ξ_0 from $\mathscr{L}((E)_\gamma \otimes \mathscr{H}, (E)_\gamma^* \otimes \mathscr{H})$, the solution lies in $\mathscr{L}((E)_\alpha \otimes \mathscr{H}, (E)_\alpha^* \otimes \mathscr{H})$ with $\alpha = \max\{\beta, \gamma\}$. We do not go into this kind of trivial remarks below.

COROLLARY 3.2. *Let* $\{L_t\}$ *and* $\{M_t\}$ *be quantum stochastic processes in* $\mathscr{L}((E) \otimes \mathscr{H}, (E)^* \otimes \mathscr{H})$ *and consider*

$$\frac{d\Xi}{dt} = L_t \diamond \Xi + M_t, \quad \Xi|_{t=0} = \Xi_0. \tag{3.3}$$

Assume that $\{L_t\}$ *is a family of operators commuting with respect to the Wick product and that there exists some* $0 \le \beta < 1$ *such that* $\deg L_t \le 2/(1-\beta)$ *for all* t. *Then the solution to* (3.3) *lies in* $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ *and given by*

$$\Xi_t = \Omega_t \diamond \left(\int_0^t \Omega_s^{\diamond(-1)} \diamond M_s \, ds + \Xi_0 \right),$$

where

$$\Omega_t = \operatorname{wexp} \int_0^t L_s ds, \quad \Omega_t^{\diamond(-1)} = \operatorname{wexp} \left(-\int_0^t L_s ds \right).$$

Proof. We first note that Ω_t is defined due to the fact:

$$\deg \int_0^t L_s \, ds \le \frac{2}{1-\beta},$$

which follows from commutativity of the Fock expansion and the integral. Then translating the initial value problem (3.3) into a differential equation of symbols, we obtain the assertion by the standard argument known as the method of variation of constants. □

Corollary 3.3. *Let $\{L_t\} \subset \mathscr{L}((E) \otimes \mathscr{H}, (E)^* \otimes \mathscr{H})$ be a quantum stochastic process. Assume that $\{L_t\}$ is a family of operators commuting with respect to the Wick product and that there exists some $0 \le \beta < 1$ such that $\deg L_t \le 2/(1-\beta)$ for all t. Then the initial value problem*:

$$\frac{d\Xi}{dt} = L_t \diamond \Xi \quad \Xi|_{t=0} = \Xi_0, \tag{3.4}$$

has a unique solution in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$ which is given by

$$\Xi_t = \left(\operatorname{wexp} \int_0^t L_s ds \right) \diamond \Xi_0.$$

Here are a few examples, some of which have appeared in Huang-Luo [**17**] taking no notice of convergence of Wick products or existence of solutions.

Example 1. Let $\{L_t\} \in \mathscr{L}((E) \otimes \mathscr{H}, (E)^* \otimes \mathscr{H})$ be a quantum stochastic process. Assume that $\deg L_t \le 2/(1-\beta)$ and that the Fock expansion of L_t contains only annihilation operators. Then $L_t \in \mathscr{L}((E) \otimes \mathscr{H}, (E) \otimes \mathscr{H})$ follows automatically. Consider the initial value problem:

$$\frac{d\Xi}{dt} = \Xi L_t, \quad \Xi|_{t=0} = \Xi_0, \tag{3.5}$$

where the right hand side is a usual product. Since $\Xi L_t = \Xi \diamond L_t$ by Proposition 2.2, it follows from Theorem 3.1 that there exists a unique solution in $\mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$. If in addition $\{L_t\}$ is a commuting family with respect to the Wick product, the solution is given by

$$\Xi_t = \Xi_0 \diamond \operatorname{wexp} \int_0^t L_s \, ds.$$

A similar argument is applied to

$$\frac{d\Xi}{dt} = L_t^* \Xi,$$

which is dual to (3.5).

In the following examples we assume that $\mathscr{H} = \boldsymbol{C}$.

Example 2. As a particular case of Example 1 we take $L_t = a_t$. Then one may consider

$$\frac{d\Xi}{dt} = \Xi a_t, \quad \frac{d\Xi}{dt} = a_t^* \Xi,$$

and their linear combination:

$$\frac{d\Xi}{dt} = \omega_1 \Xi a_t + \omega_2 a_t^* \Xi, \quad \omega_1, \omega_2 \in \boldsymbol{C}. \tag{3.6}$$

Equation (3.6) appears in a problem of stochastic limit of an interacting quantum system [**1**]. Since

$$\omega_1 \Xi a_t + \omega_2 a_t^* \Xi = \Xi \diamond (\omega_1 a_t + \omega_2 a_t^*) = (\omega_1 a_t + \omega_2 a_t^*) \diamond \Xi$$

and $\deg(\omega_1 a_t + \omega_2 a_t^*) \le 1$, it follows from Theorem 3.1 that equation (3.6) has a unique solution in $\mathscr{L}((E), (E)^*)$.

Example 3. Since $a_t^* \Xi a_t = \Xi \diamond (a_t^* a_t)$ and $\deg a_t^* a_t = 2$,

$$\frac{d\Xi}{dt} = a_t^* \Xi a_t, \quad \Xi|_{t=0} = \Xi_0, \tag{3.7}$$

admits a unique solution in $\mathscr{L}((E), (E)^*)$ which is expressed as

$$\Xi_t = \Xi_0 \diamond \operatorname{wexp} \Lambda_t, \quad \Lambda_t = \int_0^t a_s^* a_s \, ds,$$

where Λ_t is the number process, see also (1.20).

Example 4. There is no difficulty of discussing equations involving higher powers of quantum white noises such as

$$\frac{d\Xi}{dt} = \Xi a_t^2 + a_t^{*2} \Xi. \tag{3.8}$$

In fact, since $\Xi a_t^2 + a_t^{*2} \Xi_t = \Xi \diamond (a_t^2 + a_t^{*2})$ and $\deg(a_t^2 + a_t^{*2}) = 2$, equation (3.8) has a unique solution in $\mathscr{L}((E), (E)^*)$ and is given by

$$\Xi_t = \Xi_0 \diamond \operatorname{wexp} \int_0^t (a_s^2 + a_s^{*2}) \, ds.$$

3.2. Quantum stochastic differential equations.

Following [**18**] we recall quantum stochastic differential equations of Itô type. For $i = 1, 2, 3, 4$ let $\{L_t^{(i)}\} \subset \mathscr{L}((E) \otimes \mathscr{H}, (E)^* \otimes \mathscr{H})$ be an adapted quantum stochastic process and consider

$$d\Xi = (L_t^{(1)} d\Lambda_t + L_t^{(2)} dA_t + L_t^{(3)} dA_t^* + L_t^{(4)} dt)\Xi, \quad \Xi|_{t=0} = \Xi_0, \tag{3.9}$$

where $\{\Lambda_t\}$, $\{A_t\}$, $\{A_t^*\}$ are defined in (1.20). In fact, equation (3.9) is understood as a formal representation of the integral equation

$$\Xi_t = \Xi_0 + \int_0^t (L_s^{(1)} \Xi_s d\Lambda_s + L_s^{(2)} \Xi_s dA_s + L_s^{(3)} \Xi_s dA_s^* + L_s^{(4)} \Xi_s ds), \tag{3.10}$$

where the integrals are Itô type quantum stochastic integrals of adapted processes. As a result, the solution should be an adapted process. (In short, the role of an infinitesimal increment of the Brownian motion dB_t in the classical Itô theory is played by dA_t, dA_t^*

and $d\Lambda_t$. For a comprehensive account see [**25**], [**35**].) Equation (3.9) is brought into a usual differential equation by means of symbols:

$$\frac{d}{dt}\widehat{\Xi_t}(\xi,\eta) = \xi(t)\eta(t)(L_t^{(1)}\Xi_t)\hat{}(\xi,\eta) + \xi(t)(L_t^{(2)}\Xi_t)\hat{}(\xi,\eta)$$

$$+\eta(t)(L_t^{(3)}\Xi_t)\hat{}(\xi,\eta) + (L_t^{(4)}\Xi_t)\hat{}(\xi,\eta). \tag{3.11}$$

In fact, taking the symbols of the both sides of (3.10) we obtain an integral equation for $\hat{\Xi}_t(\xi,\eta)$, where $\xi,\eta \in E_{\mathbf{C}}$ are fixed. Then (3.11) follows immediately. On the other hand, we can consider the initial value problem:

$$\frac{d\Xi}{dt} = (a_t^* L_t^{(1)} a_t + L_t^{(2)} a_t + a_t^* L_t^{(3)} + L_t^{(4)}) \diamond \Xi, \quad \Xi|_{t=0} = \Xi_0. \tag{3.12}$$

Contrary to (3.9), equation (3.12) is a readily well-posed differential equation for operators. Obviously, in terms of operator symbols (3.12) becomes

$$\frac{d}{dt}\widehat{\Xi_t}(\xi,\eta) = e^{-\langle\xi,\eta\rangle}\{\xi(t)\eta(t)\widehat{L_t^{(1)}}(\xi,\eta) + \xi(t)\widehat{L_t^{(2)}}(\xi,\eta)$$

$$+\eta(t)\widehat{L_t^{(3)}}(\xi,\eta) + \widehat{L_t^{(4)}}(\xi,\eta)\}\widehat{\Xi_t}(\xi,\eta). \tag{3.13}$$

Then equations (3.11) and (3.13) coincide if

$$(L_t^{(i)}\Xi_t)\hat{}(\xi,\eta) = e^{-\langle\xi,\eta\rangle}\widehat{L_t^{(i)}}(\xi,\eta)\widehat{\Xi_t}(\xi,\eta), \quad i = 1,2,3,4,$$

or equivalently if

$$L_t^{(i)}\Xi_t = L_t^{(i)} \diamond \Xi_t, \quad i = 1,2,3,4. \tag{3.14}$$

Then, in view of the results in the previous section (in particular, Example 1) we obtain

Theorem 3.4. *For* $i = 1,2,3,4$ *let* $\{L_t^{(i)}\} \subset \mathscr{L}((E)\otimes\mathscr{H}, (E)^*\otimes\mathscr{H})$ *be an adapted quantum stochastic process. Assume*

(i) $\{L_t^{(i)}\} \subset \mathscr{L}((E)\otimes\mathscr{H}, (E)\otimes\mathscr{H})$;
(ii) $L_t^{(i)}\Xi = L_t^{(i)} \diamond \Xi$ *for any* $\Xi \in \mathscr{L}((E)\otimes\mathscr{H}, (E)^*\otimes\mathscr{H})$;
(iii) *there exists some* $0 \le \beta < 1$ *such that* $\deg L_t^{(i)} \le 2/(1-\beta)$ *for all* t.

Then (3.9) *has a unique solution in* $\mathscr{L}((E)_\beta\otimes\mathscr{H}, (E)_\beta^*\otimes\mathscr{H})$.

The quantum stochastic differential equation with coefficients being adapted (constant) processes defined by $L_t^{(i)} = I\otimes L_i$, $L_i \in \mathscr{L}(\mathscr{H})$, is a typical one first discussed by Hudson-Parthasarathy [**18**]. That $L_t^{(i)}$ satisfies the conditions in Theorem 3.4 illustrates that our approach bears some possibility of generalizing the theory of quantum stochastic differential equations of Itô type.

Appendix.

Throughout this appendix let $\mathfrak{X}$ be a countable Hilbert nuclear space over $\boldsymbol{C}$ or $\boldsymbol{R}$. Then there exists a sequence of Hilbert spaces $\{H_p\}_{p=-\infty}^{\infty}$ such that

$$\cdots \subset H_{p+1} \subset H_p \subset \cdots \subset H_0 \subset \cdots \subset H_{-p} \subset H_{-(p+1)} \subset \cdots,$$

where the natural injection $H_{p+1} \to H_p$ is of Hilbert-Schmidt type for any $p \geq 0$, and

$$\mathfrak{X} \cong \operatorname*{proj\,lim}_{p\to\infty} H_p, \quad \mathfrak{X}^* \cong \operatorname*{ind\,lim}_{p\to\infty} H_{-p}.$$

We denote by $|\cdot|_p$ the norm of H_p. Let $\mathfrak{Y}$ be a Banach space with norm $|\cdot|_{\mathfrak{Y}}$. The topology of $\mathfrak{X} \otimes \mathfrak{Y}$ is given by the seminorms:

$$\|\zeta\|_p = \inf \sum_i |\xi_i|_p |\eta_i|, \tag{A.1}$$

where the infimum is taken over the possible expression of $\zeta = \sum_i \xi_i \otimes \eta_i$, $\xi \in \mathfrak{X}$, $\eta_i \in \mathfrak{Y}$. By the kernel theorem there is a canonical isomorphisms:

$$\mathfrak{X}^* \otimes \mathfrak{Y}^* \cong (\mathfrak{X} \otimes \mathfrak{Y})^* \cong \mathscr{L}(\mathfrak{X}, \mathfrak{Y}^*). \tag{A.2}$$

The topology of $\mathfrak{X}^* \otimes \mathfrak{Y}^*$ is given by the seminorms

$$\|f\|_B = \sup_{\zeta \in B} |\langle f, \zeta \rangle|, \quad f \in \mathfrak{X}^* \otimes \mathfrak{Y}^*,$$

where $B \subset \mathfrak{X} \otimes \mathfrak{Y}$ runs over all bounded subsets. For $f \in \mathfrak{X}^* \otimes \mathfrak{Y}^*$ we put

$$\|f\|_{-p} = \sup\{|\langle f, \zeta \rangle|; \|\zeta\|_p \leq 1\}. \tag{A.3}$$

Note that (A.1) and (A.3) are compatible. By definition for each $f \in \mathfrak{X}^* \otimes \mathfrak{Y}^*$ there exists $p \geq 0$ such that $\|f\|_{-p} < \infty$.

Proposition A.1. *We keep the notations and assumptions as above and let T be a locally compact space. Then for a map $f : T \to \mathfrak{X}^* \otimes \mathfrak{Y}^*$ the following two conditions are equivalent*:

(i) *f is continuous*;

(ii) *for any $t_0 \in \Omega$ there exists $p \geq 0$ such that $\|f(t_0)\|_{-p} < \infty$ and*

$$\lim_{t\to t_0} \|f(t) - f(t_0)\|_{-p} = 0.$$

In that case for any compact subset $T_0 \subset T$ there exists $p \geq 0$ (different from above) such that $f : T_0 \to H_{-p} \otimes_\pi \mathfrak{Y}^*$ is continuous.

Proof. (i) $\Rightarrow$ (ii) Given t_0 we take an open neighborhood $V \subset T$ of t_0 with compact closure. Since f is cotinuous, $f(\overline{V}) \subset \mathfrak{X}^* \otimes \mathfrak{Y}^*$ is compact and hence equicontinuous. Therefore there exist $M \geq 0$ and $p \geq 0$ such that

$$|\langle f(t), \zeta \rangle| \leq M \|\zeta\|_p, \quad \zeta \in \mathfrak{X} \otimes \mathfrak{Y}, \quad t \in \overline{V}.$$

In particular,

$$\|f(t)\|_{-p} \leq M, \quad t \in V. \tag{A.4}$$

With each $\eta \in \mathfrak{Y}$ we associate a function $g_\eta : T \to \mathfrak{X}^*$ by the formula:

$$\langle g_\eta(t), \xi \rangle = \langle f(t), \xi \otimes \eta \rangle, \quad \xi \in \mathfrak{X}.$$

Then for $t \in V$,

$$|\langle g_\eta(t), \xi \rangle| \leq \|f(t)\|_{-p} \|\xi \otimes \eta\|_p = \|f(t)\|_{-p} |\xi|_p |\eta|_{\mathfrak{Y}}$$

and, in view of (A.4) we come to

$$|g_\eta(t)|_{-p} \le \|f(t)\|_{-p}|\eta|_{\mathfrak{Y}} \le M|\eta|_{\mathfrak{Y}}, \quad t \in V. \tag{A.5}$$

Note also that $g_\eta(t) \in H_{-p}$ for $t \in V$, $\eta \in \mathfrak{Y}$.

Let $\{e_j\}_{j=1}^\infty$ be a complete orthonormal basis of H_{p+1}. Then by definition,

$$|g_\eta(t) - g_\eta(t_0)|_{-(p+1)}^2 = \sum_{j=1}^\infty |\langle g_\eta(t) - g_\eta(t_0), e_j\rangle|^2. \tag{A.6}$$

We shall estimate the above sum by dividing into two parts. First in view of (A.5) we obtain

$$|\langle g_\eta(t) - g_\eta(t_0), e_j\rangle| \le |g_\eta(t) - g_\eta(t_0)|_{-p}|e_j|_p \le 2M|\eta|_{\mathfrak{Y}}|e_j|_p, \quad t \in V.$$

Given $\varepsilon > 0$ we choose N such that

$$4M^2 \sum_{j>N} |e_j|_p^2 < \frac{\varepsilon^2}{2},$$

which is possible since $H_{p+1} \to H_p$ is of Hilbert-Schmidt type and $\sum_{j=1}^\infty |e_j|_p^2 < \infty$. Then, (A.6) becomes

$$\begin{aligned} |g_\eta(t) - g_\eta(t_0)|_{-(p+1)}^2 &\le \sum_{j=1}^N |\langle g_\eta(t) - g_\eta(t_0), e_j\rangle|^2 + \frac{\varepsilon^2}{2}|\eta|_{\mathfrak{Y}}^2 \\ &= \sum_{j=1}^N |\langle f(t) - f(t_0), e_j \otimes \eta\rangle|^2 + \frac{\varepsilon^2}{2}|\eta|_{\mathfrak{Y}}^2. \end{aligned} \tag{A.7}$$

Put

$$B = \{e_j \otimes \eta; j = 1, 2, \dots, N, |\eta|_{\mathfrak{Y}} \le 1\}.$$

Obviously, $B \subset \mathfrak{X} \otimes \mathfrak{Y}$ is a bounded subset. Since f is continuous by assumption, there exists an open neighborhood $U \subset \Omega$ of t_0 such that

$$\|f(t) - f(t_0)\|_B < \frac{\varepsilon}{\sqrt{2N}}, \quad t \in U.$$

Then, for $t \in U$ and $1 \le j \le N$ we have

$$|\langle f(t) - f(t_0), e_j \otimes \eta\rangle| \le |\eta|_{\mathfrak{Y}}\|f(t) - f(t_0)\|_B \le \frac{\varepsilon}{\sqrt{2N}}|\eta|_{\mathfrak{Y}}.$$

Thus (A.7) becomes

$$\|g_\eta(t) - g_\eta(t_0)\|_{-(p+1)}^2 \le N \times \left(\frac{\varepsilon}{\sqrt{2N}}|\eta|_{\mathfrak{Y}}\right)^2 + \frac{\varepsilon^2}{2}|\eta|_{\mathfrak{Y}}^2 = \varepsilon^2|\eta|_{\mathfrak{Y}}^2, \quad t \in U \cap V,$$

that is,

$$\|g_\eta(t) - g_\eta(t_0)\|_{-(p+1)} \le \varepsilon|\eta|_{\mathfrak{Y}}, \quad t \in U \cap V. \tag{A.8}$$

Finally we shall prove

$$\|f(t) - f(t_0)\|_{-(p+1)} \le \varepsilon, \quad t \in U \cap V. \tag{A.9}$$

For $\zeta = \sum_i \xi_i \otimes \eta_i \in \mathfrak{X} \otimes_{\mathrm{alg}} \mathfrak{Y}$ it follows from (A.8) that

$$\begin{aligned} |\langle f(t) - f(t_0), \zeta\rangle| &\le \sum_i |\langle f(t) - f(t_0), \xi_i \otimes \eta_i\rangle| \\ &\le \sum_i |\langle g_{\eta_i}(t) - g_{\eta_i}(t_0), \xi_i\rangle| \\ &\le \varepsilon \sum_i |\xi_i|_{p+1} |\eta_i|_{\mathfrak{Y}}. \end{aligned}$$

Taking the infimum over the possible expressions of ζ, we see that

$$|\langle f(t) - f(t_0), \zeta\rangle| \le \varepsilon \|\zeta\|_{p+1}, \quad t \in U \cap V.$$

From this we obtain (A.9).

(ii) $\Rightarrow$ (i) Let $B \subset \mathfrak{X} \otimes \mathfrak{Y}$ be an arbitrary bounded subset. Then we have

$$\begin{aligned} \|f(t) - f(t_0)\|_B &\le \sup_{\zeta \in B} \|f(t) - f(t_0)\|_{-p} \|\zeta\|_p \\ &= \|B\|_p \|f(t) - f(t_0)\|_{-p} \to 0 \end{aligned}$$

as $t \to t_0$ by assumption. This shows that f is continuous at t_0.

The rest of the statement is already clear. □

We now prove the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 1.8.

Proposition A.2. *Let T be a locally compact space. Then for the map $t \mapsto \Xi_t \in \mathscr{L} \equiv \mathscr{L}((E)_\beta \otimes \mathscr{H}, (E)_\beta^* \otimes \mathscr{H})$, $t \in T$, the following two conditions are equivalent*:

(i) *$t \mapsto \Xi_t \in \mathscr{L}$ is continuous*;

(ii) *for each $t_0 \in T$ there exist $p \ge 0$ and an open neighborhood U of t_0 such that*

$$\{\Xi_t; t \in U\} \subset \mathscr{L}_p \quad \text{and} \quad \lim_{t \to t_0} \|\Xi_t - \Xi_{t_0}\|_{\mathscr{L}_p} = 0.$$

Proof. Let $\mathfrak{T}(\mathscr{H})$ be the space of trace class operators on $\mathscr{H}$; then $\mathfrak{T}(\mathscr{H})^* = \mathscr{L}(\mathscr{H})$. Setting $\mathfrak{X} = (E)_\beta \otimes (E)_\beta$ and $\mathfrak{Y} = \mathfrak{T}(\mathscr{H})$, we apply Proposition A.1. For the assertion it is sufficient to show that the norm $\|\cdot\|_{-p}$ used in Proposition A.1 coincides with $\|\cdot\|_{\mathscr{L}_p}$. Note first that

$$\begin{aligned} \|\Xi(\omega)\|_{OP} &= \sup\{|\langle \Xi(\omega), \tau\rangle|; \tau \in \mathfrak{T}(\mathscr{H}), \|\tau\|_{TR} \le 1\} \\ &= \sup\{|\langle \Xi, \omega \otimes \tau\rangle|; \tau \in \mathfrak{T}(\mathscr{H}), \|\tau\|_{TR} \le 1\} \\ &\le \sup\{\|\Xi\|_{-p} \|\omega \otimes \tau\|_p; \tau \in \mathfrak{T}(\mathscr{H}), \|\tau\|_{TR} \le 1\} \\ &= \|\Xi\|_{-p} \|\omega\|_p. \end{aligned}$$

Hence

$$\|\Xi\|_{\mathscr{L}_p} = \sup\{\|\Xi(\omega)\|_{OP}; \omega \in (E)_\beta \otimes (E)_\beta, \|\omega\|_p \le 1\} \le \|\Xi\|_{-p}.$$

We shall prove the converse. Consider $\zeta \in \mathfrak{X} \otimes \mathfrak{Y}$ of the form:

$$\zeta = \sum_i \omega_i \otimes \tau_i \quad \text{(finite sum)}.$$

Then,

$$\langle \Xi, \zeta \rangle = \sum_i \langle \Xi, \omega_i \otimes \tau_i \rangle = \sum_i \langle \Xi(\omega_i), \tau_i \rangle,$$

and

$$|\langle \Xi, \zeta \rangle| \le \sum_i \|\Xi(\omega_i)\|_{OP} \|\tau_i\|_{TR} \le \sum_i \|\Xi\|_{\mathscr{L}_p} \|\omega_i\|_p \|\tau_i\|_{TR}.$$

Using (A.1) we obtain

$$|\langle \Xi, \zeta \rangle| \le \|\Xi\|_{\mathscr{L}_p} \|\zeta\|_p,$$

i.e.,

$$\|\Xi\|_{-p} \le \|\Xi\|_{\mathscr{L}_p}. \qquad \square$$

PROPOSITION A.3. *Let $\{x_n\}$ be a sequence in $\mathfrak{X}^*$ and let $x \in \mathfrak{X}^*$. Then x_n converges to x in $\mathfrak{X}^*$ if and only if there exists $p \ge 0$ such that $\lim_{n\to\infty} |x_n - x|_{-p} = 0$.*

PROOF. Consider $T = \{0, 1, 1/2, 1/3, \ldots\}$ equipped with the relative topology induced from [0, 1]. Set $f(1/n) = x_n$, $f(0) = x$ and apply Proposition A.1. $\square$

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