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On blowing-up of polarized surfaces

By Koji Yokoyama

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Abstract. Let (S, L) be a polarized surface and let $\pi : \tilde{S} \to S$ be the blow-up at r points $p_1, \dots, p_r$ on S. Set $\tilde{L} = \pi^*L - \sum a_i E_i$, where a_i's are positive integers and E_i's are the (-1)-curves over p_i. We consider whether $\tilde{L}$ is ample or not if p_i's are in a general position. The cases of sectional genus two are studied especially precisely.

Introduction.

A polarized surface (S, L) is a pair consisting of a compact complex surface S and an ample line bundle L on S. We consider the following problem: for a polarized surface (S, L), let $p_1, \dots, p_r$ be r points on S, let $\pi : \tilde{S} \to S$ be the blowing-up at these r points, let $a_1, \dots, a_r$ be r positive integers, and let $\tilde{L} := \pi^*L - \sum_{i=1}^r a_i E_i$ a line bundle on $\tilde{S}$, where E_i is the (-1)-curve over p_i for $i = 1, \dots, r$. Then is $\tilde{L}$ ample if $p_1, \dots, p_r$ are in general position?

If the answer is YES, we get a new polarized surface $(\tilde{S}, \tilde{L})$, but this is not always the case. First obviously $(\tilde{L})^2 > 0$, so we must assume $L^2 > \sum_{i=1}^r a_i^2$. In order to apply Nakai's criterion, we should show $\tilde{L} \cdot \tilde{Z} > 0$ for any curve Z on S, where $\tilde{Z}$ is the strict transform of Z on $\tilde{S}$. So, if $\tilde{L} \cdot \tilde{Z} \le 0$, let us call Z(or $\tilde{Z}$) a *bad curve*.

In §1 of this paper we give the following results. See [**K**] and [**Xu**] for related results.

(1) There is a constant c such that $L \cdot Z \le c$ for any bad curve Z. c is computable in terms of $(S, L, a_1, \dots, a_r)$. Thus there are at most finitely many numerical equivalence classes containing a bad curve (c.f. Proposition (1.6)).

(2) In case $a_1 = a_2 = \cdots = a_r = 1, \tilde{L}$ is ample if there is an irreducible reduced member C of $|L|$ with $g(C) > h^1(\mathcal{O}_S)$ (c.f. Theorem (1.8)).

In the latter sections, we study the classification of polarized surfaces (S, L) with sectional genus $g(S, L) = 2$, that is defined by the formula $2g(S, L) - 2 = L(K_S + L)$, where K_S is the canonical bundle of S. The classification of such polarized surfaces are given in [**BLP**] and [**F1**]. But there are some cases which are uncertain to occur, where (S, L) is obtained by blowing-up another polarized surface. Using the above results (1), (2), we determine whether such cases actually occur or not.

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Notation, convention and terminology.

Basically we use the customary notation in algebraic geometry. Throughout this paper a *surface* is a smooth projective algebraic surface defined over the complex number field $\boldsymbol{C}$. The pull-back of a line bundle L on Y by a morphism $f: X \to Y$ is denoted by L_X, or sometimes by L if confusion is impossible or harmless. For a vector bundle $\mathscr{F}$ of rank n on a surface S we denote by $\boldsymbol{P}_S(\mathscr{F})$ or $\boldsymbol{P}(\mathscr{F})$ the $\boldsymbol{P}^{n-1}$-bundle defined by $\mathscr{F}$ and denote the tautological line bundle by $H(\mathscr{F})$ or H.

§1. General case.

(1.1) For a given polarized surface $(S_0, L_0) := (S, L)$, we consider a sequence of blowing-ups $\tilde{S} := S_r \to S_{r-1} \to \cdots \to S_2 \to S_1 \to S_0 := S$. Let $S_i \to S_{i-1}$ be the blowing-up of S_i at $p_i \in S_{i-1}$ and let $L_i := L_{i-1} - a_i E_i$ be a line bundle on S_i for $i = 1, \dots, r$, where a_i is a positive integer and E_i is the (-1)-curve over p_i. Moreover we assume the following.

(1) For each i, the sum of a_j's at points p_j's on E_i is less than a_i.
(2) $(\tilde{L})^2 = L^2 - \sum_{i=1}^r a_i^2 > 0$, where we denote $\tilde{L} := L_r$.

(1.2) Using Nakai's criterion, we can easily see that $\tilde{L}$ is ample if and only if $\tilde{L} \cdot \tilde{Z} > 0$ for any irreducible reduced curve Z on S, where $\tilde{Z}$ denotes the strict transform of Z on $\tilde{S}$.

(1.3) Let m_i be the multiplicity of Z on p_i for $i = 1, \dots, r$. Then $\tilde{L} \cdot \tilde{Z} = L \cdot Z - \sum_{i=1}^r a_i m_i = L \cdot Z - (1/2)\sum_{i=1}^r a_i - \sum_{i=1}^r a_i(m_i - (1/2))$. Schwarz' inequality gives $\tilde{L} \cdot \tilde{Z} \geq L \cdot Z - (1/2)\sum_{i=1}^r a_i - \sqrt{\sum_{i=1}^r a_i^2}\sqrt{\sum_{i=1}^r (m_i - (1/2))^2}$. On the other hand $g(\tilde{Z}) = g(Z) - (1/2)\sum_{i=1}^r m_i(m_i - 1) \geq 0$. So $2g(Z) + (r/4) \geq \sum_{i=1}^r (m_i - (1/2))^2$. Combining them, we get $\tilde{L} \cdot \tilde{Z} \geq L \cdot Z - (1/2)\sum a_i - \sqrt{\sum a_i^2}\sqrt{2g(Z) + (r/4)} = \sqrt{(L \cdot Z - (1/2)\sum a_i)^2} - \sqrt{(2g(Z) + (r/4))\sum a_i^2}$. Therefore if Z is a bad curve, then $(L \cdot Z - (1/2)\sum a_i)^2 - (2g(Z) + (r/4))\sum a_i^2 \leq 0$. Consequently we obtain

(1.4) Lemma. *$(L \cdot Z)^2 - \sum a_i(L \cdot Z) - 2g(Z)\sum a_i^2 + (1/4)((\sum a_i)^2 - r\sum a_i^2) \leq 0$ if Z is a bad curve.*

(1.5) Next we consider the numerical equivalence classes that contain bad curves, which are determined independently of the position of $p_1, \dots, p_r$.

(1.6) Proposition. *For fixed $a_1, a_2, \dots, a_r$, the number of numerical equivalence classes that contain a bad curve is finite.*

Proof. We denote $\sum_{i=1}^r a_i$ by α_1 and $\sum_{i=1}^r a_i^2$ by α_2. For any curve Z, we have the following inequality by using the genus formula together with the Hodge index theorem:

$$(L \cdot Z)^2 - \alpha_1(L \cdot Z) - 2\alpha_2 g(Z) + \frac{1}{4}(\alpha_1^2 - r\alpha_2) \geq \left(1 - \frac{\alpha_2}{L^2}\right)(L \cdot Z)^2 + (-\alpha_1 L - \alpha_2 K_S) \cdot Z + c_1$$

for some constant c_1 independent of Z. On the other hand, $mL - \alpha_1 L - \alpha_2 K_S$ is ample for sufficiently large $m \gg 0$ because L is ample. So $(mL - \alpha_1 L - \alpha_2 K_S)\cdot Z \geq 0$ for any Z. Hence

$$(L\cdot Z)^2 - \alpha_1(L\cdot Z) - 2\alpha_2 g(Z) + \frac{1}{4}(\alpha_1^2 - r\alpha_2) \geq \left(1 - \frac{\alpha_2}{L^2}\right)(L\cdot Z)^2 - m(L\cdot Z) + c_2$$

for some constant c_2 independent of Z. $1 - \alpha_2/L^2$, the coefficient of $(L\cdot Z)^2$, is positive since $(\tilde{L})^2 = L^2 - \alpha_2 > 0$. So $(L\cdot Z)^2 - \alpha_1(L\cdot Z) - 2\alpha_2 g(Z) + (1/4)(\alpha_1^2 - r\alpha_2) > 0$ for sufficiently large $(L\cdot Z)$. By (1.4) these Z cannot be a bad curve, hence $\{L\cdot Z | Z$ is a bad curve$\}$ is bounded, and it follows that the number of numerical equivalence classes containing a bad curve is finite, since L is ample. Therefore the genera of bad curves are bounded, so the multiplicities of them on each p_i is bounded. Hence the number of numerical equivalence classes that contain the strict transform of some bad curve is also finite.

(1.7) Next we give a sufficient condition for $\tilde{L}$ to be ample for r points $p_1, \ldots, p_r$ in a general position, which means that $\tilde{L}$ is ample for $p_1, \ldots, p_r$ outside a Zariski closed proper subset of $S^r := \underbrace{S \times \cdots \times S}_{r \text{ times}}$.

(1.8) Theorem. *We further assume $a_1 = \cdots = a_r = 1$ and there is an irreducible reduced member C of $|L|$ such that $g(C) > h^1(\mathcal{O}_S)$. Then $\tilde{L}$ is ample for r generic points $p_1, \ldots, p_r$ on S.*

Proof: Ampleness is an open condition, so it is sufficient to show the existence of an r-tuple $p_1, \ldots, p_r$ such that $\tilde{L}$ is ample. Let $p_1, \ldots, p_r$ be r generic points on C, and blow-up S at these points. (If C is singular, we replace C with $C \setminus \mathrm{Sing}(C)$ from here). Let Z be a bad curve, and m_i be the multiplicity of Z at p_i. By (1.6) there are only finitely many possibilities for $m := (m_1, \ldots, m_r)$. We fix one such m. By the assumption $h^1(\mathcal{O}_C) > h^1(\mathcal{O}_S)$, we can show that the linear equivalence class of $\sum_{i=1}^r m_i p_i$ is not contained in the image of $\mathrm{Pic}(S) \to \mathrm{Pic}(C)$ for a general choice of $p_1, \ldots, p_r$. Assume that there is a bad curve Z, then $0 \geq \tilde{L}\cdot\tilde{Z} = \tilde{C}\cdot\tilde{Z}$, where $\tilde{C}$, $\tilde{Z}$ are strict transforms of C, Z. So $\tilde{C} \cap \tilde{Z} = \emptyset$, and $0 = (\pi^* Z - \sum_{j=1}^r m_j E_j)_{\tilde{C}}$. Therefore $Z|_C = \sum_{j=1}^r m_j p_j$ in $\mathrm{Pic}(C)$. This contradicts the above choice of $p_1, \ldots, p_r$. So there is no bad curve for a general choice of $p_1, \ldots, p_r$.

§2. Classification of polarized surfaces with sectional genus two.

In the following sections, we will treat polarized surfaces with sectional genus two.

(2.1) Definition. The sectional genus $g(S, L)$ of a polarized surface (S, L) is defined to be $(1/2)L\cdot(K_S + L) + 1$, where K_S is the canonical bundle of S.

(2.2) Definition. A polarized surface (S', L') is called the simple blowing-up of a polarized surface (S, L) at $p_1, \ldots, p_r$, if S' is the blowing-up of S at $p_1, \ldots, p_r$ and $L' = L_{S'} - E_1 - \cdots - E_r$, where E_i is the (-1)-curve over p_i.

(2.3) Remark.

(1) For a simple blowing-up of a polarized surface (S, L), there are at most $L^2 - 1$ points to blow-up because $(L')^2 > 0$.

(2) If there is a curve Z such that $L \cdot Z = 1$, then there exists no simple blowing-up of (S, L) at a point on Z.

(2.4) Definition. A polarized surface (S, L) is a scroll over a curve C if S is a $\boldsymbol{P}^1$-bundle over C and $L \cdot F = 1$ for any fiber F of $S \to C$.

For $g(S, L) \leq 1$, we have a complete classification of (S, L) (c.f. [**F1**]). As for the case $g(S, L) = 2$, we know the following fact (c.f. [**BLP**], [**F2**]).

(2.5) Theorem. *Any polarized surface (S, L) with $g(S, L) = 2$ satisfies one of the following conditions.*

(0) *There is another polarized surface (S_0, L_0) with sectional genus two such that (S, L) is a simple blowing-up of (S_0, L_0) at a point.*

(1) *The canonical bundle K of S is numerically equivalent to L and $L^2 = 1$.*

(1′) *S is a minimal elliptic surface and $KL = L^2 = 1$.*

(2_1) *S is the Jacobian of a curve C of genus two, and L is the class of a translation of the Θ-divisor.*

$(2_{1'})$ *$S \simeq C_1 \times C_2$ for some elliptic curves C_1, C_2 and $L = [F_1 + F_2]$, where F_j is a fiber of $S \to C_j$.*

(2_2) *S is a hyperelliptic surface and $L = [Z + F]$, where F is a fiber of the Albanese fibration $\alpha : S \to \mathrm{Alb}(S)$ and Z is a section of α.*

(2_3) *There is a finite double covering $f : S \to \boldsymbol{P}^2$ branched along a smooth curve of degree six and $L = f^*\mathcal{O}_{\boldsymbol{P}^2}(1)$.*

(2_4) *S is an Enriques surface and its K3-cover $\tilde{S}$ is a finite double covering of $\boldsymbol{P}^1_\sigma \times \boldsymbol{P}^1_\tau$ branched along a smooth member of $|4H_\sigma + 4H_\tau|$. $L_{\tilde{S}}$ is the pull-back of $H_\sigma + H_\tau$.*

(3) *There is a rank two ample vector bundle $\mathscr{F}$ on an elliptic curve C such that $c_1(\mathscr{F}) = 1, (S, L) \simeq (\boldsymbol{P}(\mathscr{F}), 3H(\mathscr{F}) - A_S)$ for some $A \in \mathrm{Pic}(C)$ with $\deg(A) = 1$. $L^2 = 3$.*

(4) *There is a rank two vector bundle $\mathscr{F}$ on an elliptic curve C such that $S \simeq \boldsymbol{P}(\mathscr{F})$ and $L = 2H(\mathscr{F}) + B_S$ for some $B \in \mathrm{Pic}(C)$ with $(c_1(\mathscr{F}), \deg B) = (1, 0)$ or $(0, 1)$. $L^2 = 4$.*

(5) *There is a rank two vector bundle $\mathscr{F}$ on an elliptic curve C together a point p on $\boldsymbol{P} := \boldsymbol{P}(\mathscr{F})$ such that $c_1(\mathscr{F}) = 1, S$ is the blowing-up of $\boldsymbol{P}$ at p and $L = 5H(\mathscr{F})_S - 2A_S - 2E_p$, where A is a line bundle on C with $\deg(A) = 1$ and E_p is the (-1)-curve over p. $L^2 = 1$.*

(6_0) *$S \simeq \boldsymbol{P}^1_\alpha \times \boldsymbol{P}^1_\beta$ and $L = 2H_\alpha + 3H_\beta$. $L^2 = 12$.*

(6_1) *$S \simeq \Sigma_1 = \boldsymbol{P}(\mathscr{F})$ with $\mathscr{F} \simeq [H_\xi] \oplus \mathcal{O}$ on $\boldsymbol{P}^1_\xi$ and $L = 2H(\mathscr{F}) + 2H_\xi$. $L^2 = 12$.*

(6_2) *$S \simeq \Sigma_2 = \boldsymbol{P}(\mathscr{F})$ with $\mathscr{F} \simeq [2H_\xi] \oplus \mathcal{O}$ on $\boldsymbol{P}^1_\xi$ and $L = 2H(\mathscr{F}) + H_\xi$. $L^2 = 12$.*

(7) *$-K$ is ample, $K^2 = 1$ and $L = -2K$. $L^2 = 4$.*

(8) *There is a del Pezzo surface (S'', L'') with $(L'')^2 = 1$ and two points p_1, p_2 on S'' such that S is the blowing-up of S'' at these points and $L = 3L'' - 2E_1 - 2E_2$, where each E_i is the (-1)-curve on p_i for $i = 1, 2$. $L^2 = 1$.*

(9) *(S, L) is a scroll over a curve of genus two.*

A polarized surface of type (0) is obtained as a simple blowing-up of a polarized surface of another type, in fact, one of the types $(2_1), (2_3), (2_4), (3), (4), (6_0), (6_1), (6_2)$, (7) in (2.5) by (2.3). But the existence of a simple blowing-up of each of these types was unknown, for the ampleness may be destroyed by the blowing-up. Similarly, the existence of the types (5) and (8) was unknown too.

We will consider this existence problem for the types $(2_1), (2_3), (2_4)$ in §3, (3), (4), (5) in §4, and $(6_0), (6_1), (6_2), (7), (8)$ in §5.

§3. The case $K \equiv 0$ and $L^2 = 2$.

In this section we consider the existence of simple blowing-ups obtained from the types (2_1), (2_3), (2_4). Since $L^2 = 2$, a polarized surface obtained from these types must be a simple blowing-up at one point. We prove the existence of these polarized surfaces:

(3.1) Theorem. *There is a simple blowing-up of polarized surfaces of the types* $(2_1), (2_3), (2_4)$ *in* (2.5) *at one point. More precisely, for any polarized surface* (S, L) *of these types and for any general point* p *on* S, $\tilde{L} = \pi^* L - E_p$ *is ample on* $\tilde{S}$.

The proof is given in (3.2) and (3.3).

(3.2) Let (S, L) be of the type (2_3) or (2_4). Then $g(S, L) > h^1(\mathcal{O}_S) = 0$, so we can apply (1.8) if there is an irreducible reduced member of $|L|$. This is obvious in case (2_3) by Bertini Theorem.

In case (2_4), we have $\dim|L| = 1$ by Riemann-Roch Theorem and Vanishing Theorem. Hence a general member D of $|L|$ is irreducible or of the form $D = D_1 + D_2, L \cdot D_1 = L \cdot D_2 = 1$ and $D_1^2 \geq 0$, since $L \cdot D = 2$ and L is ample. But $D_1 \cdot D_2 > 0$ since D is connected, so $L \cdot D_1 = 1$ implies $D_1 \cdot D_2 = 1$ and $D_1^2 = 0$, hence $D_2^2 = 0$. We may assume that D_1 is not a fixed component, and then $D_1^2 = 0$ implies that $|D_1|$ has no base point, therefore $D_2 \simeq \boldsymbol{P}^1$ since $D_1 \cdot D_2 = 1$. This contradicts $D_2^2 = 0$. Thus any general member D must be irreducible, as desired.

(3.3) When (S, L) is of the type (2_1), we cannot apply (1.8) since $g(S, L) = h^1(\mathcal{O}_S) = 2$. So we will show directly that there is no bad curve. Since there is no rational curve on S, we have $1 \leq g(\tilde{Z})$ for any bad curve, where we employ the same notation as in §1. Hence we obtain $0 \geq (L \cdot Z)^2 - (L \cdot Z) - 2(g(Z) - 1) \geq (1/2)(L \cdot Z)((L \cdot Z) - 2)$ as in (1.3) and (1.6). Thus we should consider the following two cases:

(a) Z is an elliptic curve and $L \cdot Z = 1$,

(b) Z is a singular curve of genus two and $L \cdot Z = 2$.

We will show that neither of the types really exists.

(3.3.1) In case (a), the inclusion $Z \hookrightarrow S$ is a group homomorphism, so by taking the quotient $Z' = S/Z$ we obtain a fibration $S \to Z'$ whose fiber is isomorphic to Z. On the other hand $C \in |L|$ is a section of this map since $L \cdot Z = 1$. This contradicts $g(C) = 2$.

(3.3.2) In case (b), the resolution Z_0 of Z is an elliptic curve and we have the following commutative diagram induced by $Z_0 \to Z \hookrightarrow S$.

$$\begin{array}{ccc} Z_0 & \xrightarrow{f} & S \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h} \\ \mathrm{Alb}(Z_0) & \xrightarrow[f_0]{} & \mathrm{Alb}(S) \end{array}$$

Then $h \circ f(Z_0) \simeq Z$ is a singular curve, but $f_0 \circ h_0(Z_0) \simeq Z_0$ is a non-singular curve. This is absurd.

Thus both cases (a) and (b) cannot occur, and the proof of (3.1) is completed.

§ 4. The case of scrolls over an elliptic curve.

In this section we will show the following theorem.

(4.1) THEOREM.

(i) *Let* (S, L) *be of the types* (3), (4) *in* (2.5), *and assume that* $\mathscr{F} \not\simeq \mathcal{O}_C \oplus \mathcal{O}_C$. *Let* $p_1, p_2, \ldots, p_r$ *be points on* S *in a general position, where* $r < L^2$. *Then the simple blowing-up of* (S, L) *at these points is actually a polarized surface, i.e.*, $\tilde{L} = \pi^* L - \sum_{i=1}^{r} E_i$ *is ample on* $\tilde{S}$.

(ii) *There is no polarized surface of the type* (5) *in* (2.5).

The rest of this section is devoted to the proof of this theorem. We denote a fiber of $S \to C$ by F.

(4.2) Let (S, L) be of the type (3). Since $h^1(\mathcal{O}_S) = 1, \tilde{L}$ is ample if there is an irreducible reduced member of $|L|$ by (1.8). Since $L \equiv 3H(\mathscr{F}) - F$, any irreducible component Z of a member of $|L|$ is numerically equivalent to one of $H(\mathscr{F}), 2H(\mathscr{F}) - F$ or $3H(\mathscr{F}) - F$. To show the existence of an irreducible reduced member of $|L|$ we will compute the dimension of the complete linear systems that contain such curves. Since $H(\mathscr{F}) - K_S, 3H(\mathscr{F}) - F - K_S$ are ample, we obtain $h^1(H(\mathscr{F})) = h^2(H(\mathscr{F})) = h^1(3H(\mathscr{F}) - F) = h^2(3H(\mathscr{F}) - F) = 0$ by using the Kodaira vanishing theorem. So we have $h^0(H(\mathscr{F})) = 1$ and $h^0(3H(\mathscr{F}) - F) = 2$ by the Riemann-Roch theorem. Hence a generic member of $|L|$ is irreducible and reduced if there are at most finitely many Z with $Z \equiv 2H(\mathscr{F}) - F$. We will show this by constructing explicitly the surface of the type (3) in (2.5).

(4.2.1) We can get the exact sequence $0 \to \mathcal{O}_C \to \mathscr{F} \to \mathcal{O}_C(p) \to 0$, if necessary, by replacing $\mathscr{F}$ with $\mathscr{F} \otimes [G]$ for a suitable line bundle G and a point p on C. Since $\mathrm{Ext}^1(\mathcal{O}_C(p), \mathcal{O}_C) \simeq \boldsymbol{C}$, S is essentially unique. Next we fix the origin o of C and identify C with the Jacobian of it. Then we have the two-fold symmetric product $S^2(C)$ of C defined by the involution $\imath(x, y) := (y, x)$, and we obtain the following commutative diagram:

$$\begin{array}{ccc} C \times C & \xrightarrow{g} & S^2(C) \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h} \\ C & = & C, \end{array}$$

where h and h_0 are defined by $(x, y) \mapsto x + y$. $S^2(C) \to C$ is a $\boldsymbol{P}^1$-bundle over C because $|[x] + [y]| \simeq \boldsymbol{P}^1$ for every $(x, y) \in C \times C$. We set $H := g(\{o\} \times C)$, which is a section of h. Then H is ample since $g^*(H) = (\{o\} \times C) + (C \times \{o\})$ is. Moreover $H^2 = (1/2)((\{o\} \times C) + (C \times \{o\}))^2 = 1$, so the above (S, H) is isomorphic to $(S, H(\mathscr{F}))$ of the type (3) in (2.5). More precisely, $H(\mathscr{F})$ is numerically equivalent to H and there is a translation of C which induces a bijection $\tau : S \to S$ with $\tau^* H = H(\mathscr{F})$.

(4.2.2) For any point q on C, we denote by F_q the fiber of $S \to C$ over q. For any member Z'_q of $|g^*(2H - F_q)|$ and any point x on C, $Z'_q \cdot (\{x\} \times C) = g^*(2H - F_q) \cdot (\{x\} \times C) = 1$. Since Z'_q does not contain $(\{x\} \times C)$, Z'_q meets $(\{x\} \times C)$ at one point. Let us denote the point by y. Then $[y] = [g^*(2H - F_q)]_{\{x\} \times C} = 2[o] - [q - x]$, so $y = x - q$ by Abel's theorem. Hence $Z'_q = \{(x, x - q) \in C \times C\}_{x \in C}$. Now Z'_q is ι-invariant if and only if $2q = o$. In case $q = o$, $g^*(g(Z'_o)) = 2Z'_o$. Hence $|2H - F_q| \neq \emptyset$ if and only if $2q = o$ and $q \neq o$. So there are exactly three points q_1, q_2, q_3 such that $|2H - F_{q_i}| \neq \emptyset$. Consequently there are only three curves that are numerically equivalent to $2H - F$. Hence a general member of $|L|$ is irreducible and reduced, and our assertion is true for the type (3) by (1.8).

(4.3) Let (S, L) be of the type (4) with $(c_1(\mathscr{F}), \deg B) = (1, 0)$. The surface S and ample line bundle $H(\mathscr{F})$ is the same as those in (4.2.1). So we employ the same notation as there. To prove the ampleness of $\tilde{L}$, it is enough to show that there is an irreducible reduced member of $|L|$ as in (4.2). Since $L \equiv 2H$, any irreducible component Z of a member of $|L|$ is one of the following: $Z \equiv H, 2H - F, 2H$, or F. Obviously we have $h^0(F) = 1$. On the other hand $H - K_S$ and $2H - K_S$ are ample. So we have $h^0(H) = 1$ and $h^0(2H) = 3$ by the Kodaira vanishing theorem and the Riemann-Roch theorem. For only three points $q = q_1, q_2, q_3$, there is a member of $|2H - F_q|$ as we have seen in (4.2.2). Since $h^0(L) = 3$, a generic member of $|L|$ is irreducible and reduced. So (1.8) applies.

(4.4) Let (S, L) be of the type (4) with $(c_1(\mathscr{F}), \deg B) = (0, 1)$. If $\mathscr{F} \simeq \mathcal{O}_C \oplus \mathcal{O}_C$, then $S \simeq C \times \boldsymbol{P}^1$ and $C \cdot L = 1$ for any fiber C of the second projection. Hence there is no simple blowing-up of (S, L). So we assume $\mathscr{F} \not\simeq \mathcal{O}_C \oplus \mathcal{O}_C$. We will show that there is an irreducible reduced member of $|L|$. This is enough as in (4.2) and (4.3). As $L \equiv 2H + F$, any numerical equivalence class of an irreducible component Z of a member of $|L|$ is one of $2H + F, 2H, H + F, H$, or F. For $2H + F$ and $H + F$, we can apply the Kodaira vanishing theorem together with the Riemann-Roch theorem, and obtain $h^0(2H + F) = 3$ and $h^0(H + F) = 2$. And obviously $h^0(F) = 1$. In order to study divisors which are numerically equivalent to $2H$ and H, we consider the extension type of $\mathscr{F}$. If necessary, we replace $\mathscr{F}$ by $\mathscr{F} \oplus [G]$ for a suitable line bundle G on C so that we have an exact sequence $0 \to \mathcal{O}_C \to \mathscr{F} \to \mathcal{O}_C(A) \to 0$, where A is a line bundle on C with $\deg(A) = 0$. Then we have

$$\mathrm{Ext}^1(\mathcal{O}_C(A), \mathcal{O}_C) = \begin{cases} \boldsymbol{C} & (A = 0), \\ 0 & (A \neq 0). \end{cases}$$

(4.4.1) When $A = 0, \mathscr{F}$ is indecomposable since $\mathscr{F} \not\simeq \mathcal{O}_C \oplus \mathcal{O}_C$. In this case we have the following:

LEMMA. *For any* $G \in \mathrm{Pic}^0(C)$, *we have*

(a) $h^0(H + G_S) = \begin{cases} 1 & (G = 0), \\ 0 & (G \neq 0), \end{cases}$

(b) $h^0(2H + G_S) = \begin{cases} 1 & (G = 0), \\ 0 & (G \neq 0). \end{cases}$

PROOF. (a) Since $h^i(S, H + G_S) = h^i(C, \mathscr{F} \otimes [G])$, we calculate $h^i(C, \mathscr{F} \otimes [G])$. We have the following exact sequence:

$$0 \to H^0(C, G) \to H^0(C, \mathscr{F} \otimes [G]) \to H^0(C, G)$$

$$\to H^1(C, G) \to H^1(C, \mathscr{F} \otimes [G]) \to H^1(C, G) \to 0$$

If $G = 0$, we have $h^0(C, G) = h^1(C, G) = 1$. So by the above exact sequence $h^0(C, \mathscr{F}) = h^1(C, \mathscr{F}) = 1$ or 2. Suppose that $h^0(C, \mathscr{F}) = 2$. This yields $\mathrm{Bs}|H| = \emptyset$, and $|H|$ defines a morphism $S \to \boldsymbol{P}^1$. Hence we obtain the following commutative diagram:

$$\begin{array}{ccc} S & \longrightarrow & \boldsymbol{P}^1 \times C \\ \downarrow & & \downarrow \\ C & \longrightarrow & C, \end{array}$$

where $\boldsymbol{P}^1 \times C \to C$ is the second projection. This contradicts that $\mathscr{F}$ is indecomposable. Hence $h^0(S, H) = h^1(S, H) = 1$.

If $G \neq 0$, we have $h^0(C, G) = h^1(C, G) = 0$. Hence $h^0(S, H + G_S) = h^1(S, H + G_S) = 0$.

(b) We denote by C the member of $|H|$. For any $G \in \mathrm{Pic}^0(C)$, we have the following exact sequence:

$$0 \to H^0(S, H + G_S) \to H^0(S, 2H + G_S) \to H^0(C, G)$$

$$\to H^1(S, H + G_S) \to H^1(S, 2H + G_S) \to H^1(C, G) \to 0.$$

If $G = 0$, then $h^0(S, 2H) = h^1(S, 2H) = 1$ or 2. Suppose $h^0(S, 2H) = 2$. Then we have $\mathrm{Bs}|2H| = \emptyset$ and we can define the finite morphism $S \to \boldsymbol{P}^1 \times C$ of degree two similarly as above. Now all of the members in $|2H|$ are irreducible and reduced except for $2C$, so the branch locus of the morphism is of the form $\{p\} \times C$ for some point p on $\boldsymbol{P}^1$. This contradicts $[\{p\} \times C] \notin 2 \cdot \mathrm{Pic}(\boldsymbol{P}^1 \times C)$. Hence $h^0(S, 2H) = h^1(S, 2H) = 1$.

When $G \neq 0$, we have $h^i(H + G_S) = h^i(C, G) = 0$ for $i = 1, 2$. Hence $h^0(H + G_S) = h^0(C, G) = 0$.

By this lemma, we easily see that a general member of $|L|$ is irreducible and reduced since $h^0(L) = 3$. So (1.8) applies.

(4.4.2) When $A \neq 0$, we have $\mathscr{F} \simeq \mathscr{O}_C \oplus \mathscr{O}_C(A)$ since $\mathrm{Ext}^1(\mathscr{O}_C(A), \mathscr{O}_C) = 0$. Since $h^0(H + G_S) = h^0(C, \mathscr{O}_C(G) \oplus \mathscr{O}_C(A + G))$, we have

$$h^0(H + G_S) = \begin{cases} 1 & (G = 0, -A), \\ 0 & (\text{otherwise}). \end{cases}$$

And by $h^0(2H+G_S) = h^0(C, \mathcal{O}_C(G) \oplus \mathcal{O}_C(A+G) \oplus \mathcal{O}_C(2A+G))$, we have

$$\text{(1)}\quad \text{if } 2A \neq 0, h^0(2H+G_S) = \begin{cases} 1 & (G = 0, -A, -2A), \\ 0 & (\text{otherwise}), \end{cases}$$

$$\text{(2)}\quad \text{if } 2A = 0, h^0(2H+G_S) = \begin{cases} 2 & (G = 0), \\ 1 & (G = 0, -A), \\ 0 & (\text{ otherwise}). \end{cases}$$

By direct calculation, we see that any general member of $|L|$ is irreducible and reduced. Hence (1.8) applies.

(4.5) Now we should show that the case (5) in (2.5) does not occur. Let $(\boldsymbol{P}(\mathscr{F}), H(\mathscr{F}))$ of the type (5). Similarly as in (4.2.1), $\boldsymbol{P}(\mathscr{F})$ is isomorphic to the 2-fold symmetric product of C. In this case, we will show that there is a morphism $\boldsymbol{P}(\mathscr{F}) \to \boldsymbol{P}^1$ and $L \cdot F' = 2$ for its fiber F'.

We employ the same notation as in (4.2.1). Let $\sigma : C \times C \to C$ be the map such that $\sigma(x, y) = x - y$ and let $g' : C \to \boldsymbol{P}^1$ be the rational map defined by $|2o|$. Since $g'(-x) = g'(x)$, the map $g' \circ \sigma : C \times C \to \boldsymbol{P}^1$ factors through $\boldsymbol{P}(\mathscr{F}) = C \times C/\iota$, and we get the following commutative diagram:

$$\begin{array}{ccc} C \times C & \xrightarrow{g} & \boldsymbol{P}(\mathscr{F}) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ C & \xrightarrow[g']{} & \boldsymbol{P}^1 \end{array}$$

Let F' be the fiber of σ' over $\sigma'(p)$. Then $(5H-2F) \cdot F' = (1/2)g^*(5H-2F) \cdot g^*F' = 2$. Hence $\tilde{L} \cdot \tilde{F}' = 0$ for any p on $\boldsymbol{P}(\mathscr{F})$, where $\tilde{F}'$ is the proper transform of F'. So we have no polarized surface of the type (5) in (2.5).

§ 5. Rational case.

In this section we will show the following

(5.1) Theorem.

(a) *Let (S, L) be of the types $(6_0), (6_1), (6_2), (7)$ in (2.5). Let $p_1, p_2, \ldots, p_r$ be points on S in a general position, where $r < L^2$. Then the simple blowing-up of (S, L) at these points is actually a polarized surface.*

(b) *There exists a polarized surface of the type (8) in (2.5).*

The proof is as follows.

(5.2) It is easy to see that the assertion (a) follows from (1.8).

(5.3) To see the case (8) really occurs, we fix a minimalization $S'' \to \boldsymbol{P}^2$ of S'', which is an eight-points blowing-up of $\boldsymbol{P}^2$. To follow the same notation as in (1.1), we denote (S, L) by $(\tilde{S}, \tilde{L})$ and $(\boldsymbol{P}^2, 9H)$ by (S, L). Let the following sequence of blowing-ups be as in (1.1).

$$\tilde{S} =: S_{10} \to S_9 \to S_8 := S'' \to S_7 \to \cdots \to S_0 := \boldsymbol{P}^2$$

Let $\tilde{L} = L_{10} := 9H - 3E_1 - \cdots - 3E_8 - 2E_9 - 2E_{10}$. For a bad curve Z of degree d, we have $d^2 - 12d - 164 \leq 0$ by (1.4), hence $0 < d \leq 20$. We can compute the degree d and the multiplicity m_i at each p_i of the bad curve by the aid of a computer. (In fact, we enumerate all the multiples $(d, (m_1, \ldots, m_{10}))$ such that $2g(\tilde{Z}) = (d-1)(d-2) - \sum_{i=1}^{10} m_i(m_i - 1) \geq 0$ and $\tilde{L} \cdot \tilde{Z} = 9d - 3\sum_{i=1}^{8} m_i - 2(m_9 + m_{10}) \leq 0$). The result is listed below:

	degree	$(m_1 \cdots\cdots m_{10})$
(1)	1	(1 1 1 0 0 0 0 0 0 0)
(2)	2	(1 1 1 1 1 1 0 0 0 0)
(3-1)	3	(2 1 1 1 1 1 1 1 0 0)
(3-2)	3	(1 1 1 1 1 1 1 1 1 1)
(4)	4	(2 2 2 1 1 1 1 1 1 1)
(5)	5	(2 2 2 2 2 2 1 1 1 1)
(6-1)	6	(3 2 2 2 2 2 2 2 1 1)
(6-2)	6	(2 2 2 2 2 2 2 2 2 2)
(6-3)	6	(2 2 2 2 2 2 2 2 2 1)
(7)	7	(3 3 3 2 2 2 2 2 2 1)
(8)	8	(3 3 3 3 3 3 2 2 2 1)
(9-1)	9	(4 3 3 3 3 3 3 3 2 1)
(9-2)	9	(3 3 3 3 3 3 3 3 3 2)
(10)	10	(4 4 3 3 3 3 3 3 3 3)
(11)	11	(4 4 4 4 4 3 3 3 3 3)
(12-1)	12	(5 4 4 4 4 4 4 3 3 3)
(12-2)	12	(4 4 4 4 4 4 4 4 4 2)
(12-3)	12	(4 4 4 4 4 4 4 4 3 3)
(13)	13	(5 5 5 4 4 4 4 4 3 3)
(14)	14	(5 5 5 5 5 5 4 4 3 3)
(15)	15	(6 5 5 5 5 5 5 5 3 3)
(18)	18	(6 6 6 6 6 6 6 6 5 4)

To show that there is no such curve for any ten points in a general position, we use the following result due to Xu:

(5.3.1) PROPOSITION. *Let $p_1, \ldots, p_r$ be r points on $\boldsymbol{P}^2$ in a general position, and Z be an irreducible reduced curve of degree d with $\mathrm{mult}_{p_i}(Z) = m_i$. Then $d^2 \geq \sum_{i=1}^{r} m_i^2 - m_q$ for any $q \in \{1, \ldots, r\}$ such that $m_q > 0$.*

For the proof, see [**Xu**].

This proposition rules out the above cases except for the cases (3-2), (6-3) and (12-3).

In case (3-2), we have $h^0(3H) = 10$, so $h^0(3H - \sum_{i=1}^{10} E_i) = 0$, hence this case does not occur.

In case (6.3), we have $\tilde{Z} \in |-2K'' - 2E_9 - E_{10}|$, where K'' is the canonical bundle of the Del Pezzo surface $S'' = S_8$. Note that $h^0(-2K'') = 4$ and $|-2K''|$ has no base point by Reider's criterion. Let $\rho : S'' \to \boldsymbol{P}^3$ be the induced morphism. A member of $|-2K'' - 2E_9 - E_{10}|$ corresponds to a (hyper)plane in $\boldsymbol{P}^3$ which is tangent to $\rho(S'')$ at $\rho(p_9)$ and passes $\rho(p_{10})$. Clearly there is no such plane, thus this case is ruled out.

(5.3.2) The case (12-3) is ruled out by the following direct computation. $H^0(12H - 4E_1 - \cdots - 4E_8 - 3E_9 - 3E_{10})$ is a subspace of $H^0(12H) \simeq \boldsymbol{C}^{91}$, satisfying 92 linear equations. By the aid of computer we can show that there is no solution for the ten points $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, $(2,1)$, $(3,2)$, $(-2,3)$, $(-1,5)$, $(1,-2)$, $(-3,2) \in \boldsymbol{P}^2 - H_\infty = \boldsymbol{C}^2$, for example. So there is no curve of this type for any ten points in a general position.

Hence there exists a polarized surface of the type (8) in (2.5).

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Koji Yokoyama

Department of Mathematics,
Tokyo Institute of Technology,
Oh-Okayama, Meguro-ku,
Tokyo 152-8551,
Japan

Current address:

Ministry of Health and Welfare