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A characterization theorem for operators on white noise functionals

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Abstract. A W-transform of an operator on white noise functionals is introduced and then characterizations for operators on white noise functionals are given in terms of their W-transforms. A simple proof of the analytic characterization theorem for operator symbol and convergence of operators are also discussed.

1. Introduction.

The concept of the symbol of an operator is of fundamental importance in the theory of operators on white noise functionals. N. Obata [**7**] proved an analytic characterization theorem for symbols of operators on white noise functionals, which is an operator version of the characterization theorem for white noise functionals [**4**], [**8**]. This characterization theorem provides a very useful criterion for checking whether or not an operator on Fock space defined only on the exponential vectors becomes a continuous linear operator on the space of white noise functionals.

The purpose of this paper is threefold: we first define a W-transform of an operator on white noise functionals and then obtain a characterization theorem for operators on white noise functionals in terms of their W-transforms. We next apply our characterization theorem to give a simple proof of the analytic characterization theorem for operator symbols due to Obata [**7**]. We finally give a criterion for the convergence of operators on white noise functionals in terms of their W-transforms.

2. Preliminaries.

Let H be a real separable Hilbert space. Let A be an operator on H such that there exists an orthonormal basis $\{e_j\}_{j\geq 0}$ for H satisfying the conditions:

(1) $Ae_j = \lambda_j e_j,\ j = 0, 1, 2, \ldots,$

(2) $1 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty,$

(3) $\|A^{-1}\|_{HS} = (\sum_{j=0}^{\infty} \lambda_j^{-2})^{1/2} < \infty.$

For each $p \geq 0$, define

$$|\xi|_p = |A^p \xi|_0 = \left(\sum_{j=0}^{\infty} \lambda_j^{2p} \langle \xi, e_j \rangle^2\right)^{1/2}, \qquad \xi \in H,$$

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where $|\cdot|_0$ is the norm on H. Then $E_p \equiv \{\xi \in H; |\xi|_p < \infty\}$ is a real separable Hilbert space with norm $|\cdot|_p$. Let E be the projective limit of $\{E_p; p \geq 0\}$ and E^* the topological dual of E. Then E becomes a nuclear space and we have a Gel'fand triple $E \subset H \subset E^*$, and a continuous inclusion: for each $p \geq 0$,

$$E \subset E_p \subset H \subset E_p^* \subset E^*.$$

We note that the norm of a Hilbert space E_p^* is given by

$$|\xi|_{-p} = |A^{-p}\xi|_0 = \left(\sum_{j=0}^{\infty} \lambda_j^{-2p} \langle \xi, e_j \rangle^2\right)^{1/2}.$$

Let μ be the standard Gaussian measure on E^*, i.e., its characteristic function is given by

$$\int_{E^*} e^{i\langle x, \xi \rangle} \mu(dx) = e^{-1/2|\xi|_0^2}, \qquad \xi \in E,$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $E^* \times E$. Then (E^*, μ) is called the *white noise space*. We denote by (L^2) the complex Hilbert space of μ-square integrable functions on E^*. By the Wiener-Ito decomposition theorem, each $\phi \in (L^2)$ admits an expansion:

$$\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n \rangle, \qquad f_n \in H_{\boldsymbol{C}}^{\hat{\otimes} n}, \tag{2.1}$$

where $H_{\boldsymbol{C}}^{\hat{\otimes} n}$ is the n-fold symmetric tensor product of the complexification of H. Moreover, the (L^2)-norm $\|\phi\|_0$ of ϕ is given by

$$\|\phi\|_0 = \left(\sum_{n=0}^{\infty} n! |f_n|_0^2\right)^{1/2},$$

where the norm on $H_{\boldsymbol{C}}^{\hat{\otimes} n}$ is denoted by the same symbol $|\cdot|_0$.

Let $\Gamma(A)$ be the second quantization operator of A defined by

$$\Gamma(A)\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, A^{\otimes n} f_n \rangle,$$

where $\phi \in (L^2)$ is given by the expansion (2.1). Then we note that $\Gamma(A)$ is a positive self-adjoint operator with Hilbert-Schmidt inverse. For each $p \geq 0$, define

$$\|\phi\|_p = \|\Gamma(A)^p \phi\|_0, \qquad \phi \in (L^2). \tag{2.2}$$

Then $(E_p) \equiv \{\phi \in (L^2); \|\phi\|_p < \infty\}$ is a complex Hilbert space with norm $\|\cdot\|_p$. Let (E) be the projective limit of $\{(E_p); p \geq 0\}$ and $(E)^*$ the topological dual of (E). Then (E) is a nuclear space and we have a Gel'fand triple $(E) \subset (L^2) \subset (E)^*$, and a continuous inclusion: for each $p \geq 0$,

$$(E) \subset (E_p) \subset (L^2) \subset (E_p)^* \subset (E)^*.$$

Elements $\phi \in (E)$ and $\Phi \in (E)^*$ are called a *test white noise functional* and a *generalized white noise functional* (or *Hida distribution*), respectively.

It is known (see [**2**], [**7**]) that for each $\Phi \in (E)^*$, there exists a unique sequence $\{F_n\}_{n \geq 0}$, $F_n \in (E_{\boldsymbol{C}}^{\otimes n})^*_{sym}$ such that

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \qquad \phi \in (E), \tag{2.3}$$

where ϕ is given by the expansion (2.1) and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the canonical bilinear form on $(E)^* \times (E)$. In view of (2.3) we use a formal expression for $\Phi \in (E)^*$:

$$\Phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, F_n \rangle.$$

For each $\xi \in E_{\boldsymbol{C}}$, *an exponential vector* φ_ξ is defined by

$$\varphi_\xi(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle :x^{\otimes n}:, \xi^{\otimes n} \rangle.$$

Then it is well-known that $\{\varphi_\xi ; \xi \in E_{\boldsymbol{C}}\}$ spans a dense subspace of (E).

The *S-transform* $S\Phi$ of $\Phi \in (E)^*$ is a function on $E_{\boldsymbol{C}}$ defined by

$$S\Phi(\xi) = \langle\!\langle \Phi, \varphi_\xi \rangle\!\rangle, \qquad \xi \in E_{\boldsymbol{C}}.$$

We need the characterization theorem for white noise functionals due to Potthoff–Streit [**8**] and Kuo–Potthoff–Streit [**4**] with norm estimate due to Kubo–Kuo [**3**].

Theorem 2.1. *The S-transform $F = S\Phi$ of $\Phi \in (E)^*$ satisfies the following conditions*:

(F1) *For each $\xi, \eta \in E_{\boldsymbol{C}}$, the function $z \mapsto F(z\xi + \eta)$ is an entire function on $\boldsymbol{C}$.*

(F2) *There exist $K > 0$, $a > 0$ and $p \geq 0$ such that*

$$|F(\xi)| \leq K e^{a|\xi|_p^2}, \qquad \xi \in E_{\boldsymbol{C}}.$$

Conversely, assume that a $\boldsymbol{C}$-valued function F defined on $E_{\boldsymbol{C}}$ satisfies the above two conditions. Then there exists a unique $\Phi \in (E)^*$ such that $F = S\Phi$. Moreover, for any $q > p$ with $2ae^2 \|A^{-(q-p)}\|_{HS}^2 < 1$, we have the following norm estimate:

$$\|\Phi\|_{-q} \leq K(1 - 2ae^2 \|A^{-(q-p)}\|_{HS}^2)^{-1/2}.$$

Theorem 2.2. *The S-transform $F = S\phi$ of $\phi \in (E)$ satisfies the following conditions*:

(F1$'$) *For each $\xi, \eta \in E_{\boldsymbol{C}}$, the function $z \mapsto F(z\xi + \eta)$ is an entire function on $\boldsymbol{C}$.*

(F2$'$) *For any $p \geq 0$ and $a > 0$, there exists a constant $K > 0$ such that*

$$|F(\xi)| \leq K e^{a|\xi|_{-p}^2}, \qquad \xi \in E_{\boldsymbol{C}}.$$

Conversely, assume that a $\boldsymbol{C}$-valued function F defined on $E_{\boldsymbol{C}}$ satisfies the above two conditions. Then there exists a unique $\phi \in (E)$ such that $F = S\phi$. Moreover, for any $q \geq 0$ and for $a > 0$ and $p > q$ with $2ae^2 \|A^{-(p-q)}\|_{HS}^2 < 1$, we have the following norm *estimate*:

$$\|\phi\|_q \le K(1 - 2ae^2\|A^{-(p-q)}\|_{HS}^2)^{-1/2}.$$

3. Characterization theorems of operators.

Let $L((E),(E)^*)$ (resp. $L((E),(E)))$ denote the space of all continuous linear operators from (E) into $(E)^*$ (resp. (E)). In this section we shall prove a characterization theorem for an operator $\Xi \in L((E),(E)^*)$ and for an operator $\Xi \in L((E),(E))$.

The W-transform of an operator $\Xi \in L((E),(E)^*)$ is defined to be an $(E)^*$-valued function on $E_{\boldsymbol{C}}$ defined by

$$W\Xi(\xi) = \Xi\varphi_\xi, \qquad \xi \in E_{\boldsymbol{C}}.$$

We first note that the W-transform is injective and that for any $\phi \in (E)$ and $\xi, \eta \in E_{\boldsymbol{C}}$, we have $\langle\!\langle W\Xi(z\xi+\eta), \phi\rangle\!\rangle = S(\Xi^*\phi)(z\xi+\eta)$, $z \in \boldsymbol{C}$, where Ξ^* is the adjoint operator of Ξ, i.e., Ξ^* is the continuous linear operator from (E) into $(E)^*$ such that

$$\langle\!\langle \Xi\phi, \psi\rangle\!\rangle = \langle\!\langle \Xi^*\psi, \phi\rangle\!\rangle, \qquad \phi, \psi \in (E).$$

It then follows from Theorem 2.1 that the function $z \mapsto \langle\!\langle W\Xi(z\xi+\eta), \phi\rangle\!\rangle$ is an entire function on $\boldsymbol{C}$.

We note that there exist $p \ge 0$ and $K > 0$ such that

$$\|\Xi\phi\|_{-p} \le K\|\phi\|_p, \qquad \phi \in (E).$$

Hence we have the following growth condition:

$$\|W\Xi(\xi)\|_{-p} \le Ke^{1/2|\xi|_p^2}, \qquad \xi \in E_{\boldsymbol{C}}.$$

Theorem 3.1. *Let $\Xi \in L((E),(E)^*)$ and $G = W\Xi$. Then the function G satisfies the following conditions*:

(G1) *For each $\xi, \eta \in E_{\boldsymbol{C}}$, the function $z \mapsto G(z\xi+\eta)$ is weakly holomorphic, i.e., for any $\phi \in (E)$ the function $z \mapsto \langle\!\langle G(z\xi+\eta), \phi\rangle\!\rangle$ is an entire function on $\boldsymbol{C}$.*

(G2) *There exist $q \ge 0$, $p \ge 0$, $a > 0$ and $K > 0$ such that*

$$\|G(\xi)\|_{-q} \le Ke^{a|\xi|_p^2}, \qquad \xi \in E_{\boldsymbol{C}}.$$

Conversely, assume that an $(E)^*$-valued function G on $E_{\boldsymbol{C}}$ satisfies the above conditions. Then there exists a unique $\Xi \in L((E),(E)^*)$ such that G is the W-transform of Ξ. Moreover, for any $r > p$ with $2ae^2\|A^{-(r-p)}\|_{HS}^2 < 1$, we have

$$\|\Xi\phi\|_{-q} \le K(1 - 2ae^2\|A^{-(r-p)}\|_{HS}^2)^{-1/2}\|\phi\|_r, \qquad \phi \in (E).$$

Proof. The first assertion was shown above. Now, let G be an $(E)^*$-valued function on $E_{\boldsymbol{C}}$ satisfying (G1) and (G2). The uniqueness part of the second assertion is obvious since the W-transform is injective. To prove the existence of Ξ, fix an arbitrary $\phi \in (E)$. Define a $\boldsymbol{C}$-valued function F_ϕ on $E_{\boldsymbol{C}}$ by

$$F_\phi(\xi) = \langle\!\langle G(\xi), \phi\rangle\!\rangle, \qquad \xi \in E_{\boldsymbol{C}}.$$

Then F_ϕ satisfies (F1) and (F2) in Theorem 2.1: In fact, for any $\xi, \eta \in E_{\boldsymbol{C}}$, the function $F_\phi(z\xi + \eta) = \langle\langle G(z\xi + \eta), \phi \rangle\rangle$ of $z \in \boldsymbol{C}$ is holomorphic on $\boldsymbol{C}$ and we have, for $\xi \in E_{\boldsymbol{C}}$

$$|F_\phi(\xi)| \le \|G(\xi)\|_{-q} \|\phi\|_q \le (K\|\phi\|_q) e^{a|\xi|_p^2}.$$

Hence, by Theorem 2.1, there exists a unique $\Phi_\phi \in (E)^*$ such that

$$S\Phi_\phi(\xi) = F_\phi(\xi) = \langle\langle G(\xi), \phi \rangle\rangle, \qquad \xi \in E_{\boldsymbol{C}}.$$

Moreover, by Theorem 2.1 we have, for any $r > p$ with $2ae^2 \|A^{-(r-p)}\|_{HS}^2 < 1$

$$\|\Phi_\phi\|_{-r} \le K\|\phi\|_q (1 - 2ae^2 \|A^{-(r-p)}\|_{HS}^2)^{-1/2}.$$

This inequality implies that the operator $\phi \mapsto \Phi_\phi$ is a continuous linear operator from (E_q) into $(E_r)^*$. Let Ξ be the adjoint operator of this operator. Then Ξ is a continuous linear operator from (E_r) into $(E_q)^*$, and hence $\Xi \in L((E), (E)^*)$ and $\Xi\varphi_\xi = G(\xi)$. Furthermore, we have the following norm estimate:

$$\|\Xi\phi\|_{-q} \le K(1 - 2ae^2 \|A^{-(r-p)}\|_{HS}^2)^{-1/2} \|\phi\|_r$$

as desired. □

For any $\Xi \in L((E), (E)^*)$, the *symbol* $\hat{\Xi}$ of Ξ is defined by

$$\hat{\Xi}(\xi, \eta) = \langle\langle \Xi\varphi_\xi, \varphi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\boldsymbol{C}}.$$

The next result has been proved in [**7**, p. 91]. We here give a simple proof.

Corollary 3.2. *Suppose that a $\boldsymbol{C}$-valued function F on $E_{\boldsymbol{C}} \times E_{\boldsymbol{C}}$ satisfies the following conditions*:

(S1) *For each ξ, ξ', η and η' in $E_{\boldsymbol{C}}$, the function $(z, w) \mapsto F(z\xi + \xi', w\eta + \eta')$ is an entire function on $\boldsymbol{C} \times \boldsymbol{C}$.*

(S2) *There exist $p \ge 0$, $a > 0$ and $K > 0$ such that*

$$|F(\xi, \eta)| \le K e^{a(|\xi|_p^2 + |\eta|_p^2)}, \qquad \xi, \eta \in E_{\boldsymbol{C}}.$$

Then there exists a unique $\Xi \in L((E), (E)^*)$ such that F is the symbol of Ξ.

Proof. For a fixed $\xi \in E_{\boldsymbol{C}}$, define a $\boldsymbol{C}$-valued function F_ξ on $E_{\boldsymbol{C}}$ by $F_\xi(\eta) = F(\xi, \eta)$, $\eta \in E_{\boldsymbol{C}}$. Then the function F_ξ satisfies (F1) and (F2) in Theorem 2.1: In fact, clearly the function $z \mapsto F_\xi(z\eta + \eta') = F(\xi, z\eta + \eta')$ is holomorphic on $\boldsymbol{C}$, and

$$|F_\xi(\eta)| = |F(\xi, \eta)| \le (K e^{a|\xi|_p^2}) e^{a|\eta|_p^2}.$$

Hence there exists a $\Phi_\xi \in (E)^*$ such that $S\Phi_\xi = F_\xi$. Now, define an $(E)^*$-valued function G on $E_{\boldsymbol{C}}$ by $G(\xi) = \Phi_\xi$, $\xi \in E_{\boldsymbol{C}}$. Then we have

$$SG(\xi)(\eta) = S\Phi_\xi(\eta) = F_\xi(\eta) = F(\xi, \eta), \qquad \xi, \eta \in E_{\boldsymbol{C}}.$$

Moreover, for any $q > p$ such that $2ae^2 \|A^{-(q-p)}\|_{HS}^2 < 1$, we have

$$\|G(\xi)\|_{-q} \le (K e^{a|\xi|_p^2})(1 - 2ae^2 \|A^{-(q-p)}\|_{HS}^2)^{-1/2}. \tag{3.1}$$

Now we shall verify that the function G satisfies (G1) and (G2) in Theorem 3.1. Take any $\xi, \xi' \in E_{\boldsymbol{C}}$. Then clearly the function $F(z\xi + \xi', \eta) = \langle\langle G(z\xi + \xi'), \varphi_\eta \rangle\rangle$ of z is holomorphic on $\boldsymbol{C}$ for each $\eta \in E_{\boldsymbol{C}}$. Hence (G1) is satisfied for all $\phi \in V$, where V is the linear span of $\{\varphi_\eta; \eta \in E_{\boldsymbol{C}}\}$. Since V is dense in (E), for any $\phi \in (E)$, we can choose a sequence $\{\phi_k\}$ in V such that $\phi_k \to \phi$ in (E). Note that

$$|\langle\langle G(z\xi + \xi'), \phi_k - \phi \rangle\rangle| \le K e^{a|z\xi+\xi'|_p^2} (1 - 2ae^2 \|A^{-(q-p)}\|_{HS}^2)^{-1/2} \|\phi_k - \phi\|_q.$$

So, the function $\langle\langle G(z\xi + \xi'), \phi_k \rangle\rangle$ of $z \in \boldsymbol{C}$ converges to $\langle\langle G(z\xi + \xi'), \phi \rangle\rangle$ uniformly on every compact subset of $\boldsymbol{C}$, and hence the function $z \mapsto \langle\langle G(z\xi + \xi'), \phi \rangle\rangle$ is holomorphic on $\boldsymbol{C}$. Moreover by (3.1), (G2) is satisfied. Hence by Theorem 3.1, we obtain a continuous linear operator Ξ from (E) into $(E)^*$ such that

$$\Xi\varphi_\xi = G(\xi), \qquad \xi \in E_{\boldsymbol{C}}. \tag{3.2}$$

But by (3.2), we have, for each $\xi, \eta \in E_{\boldsymbol{C}}$

$$F(\xi, \eta) = SG(\xi)(\eta) = \langle\langle G(\xi), \varphi_\eta \rangle\rangle = \langle\langle \Xi\varphi_\xi, \varphi_\eta \rangle\rangle = \hat{\Xi}(\xi, \eta).$$

This completes the proof. □

REMARK. It can be shown that Corollary 3.2 implies the second assertion of Theorem 3.1.

The W-transform of an operator $\Xi \in L((E), (E))$ is defined to be an (E)-valued function on $E_{\boldsymbol{C}}$ defined by

$$W\Xi(\xi) = \Xi\varphi_\xi, \qquad \xi \in E_{\boldsymbol{C}}.$$

Then for any $\Phi \in (E)^*$ and $\xi, \eta \in E_{\boldsymbol{C}}$, we see that $\langle\langle \Phi, W\Xi(z\xi + \eta) \rangle\rangle = S(\Xi^*\Phi)(z\xi + \eta)$, $z \in \boldsymbol{C}$. Hence $z \mapsto \langle\langle \Phi, W\Xi(z\xi + \eta) \rangle\rangle$ is holomorphic on $\boldsymbol{C}$. Moreover, we note that for each $q \ge 0$, there exist $p \ge 0$ and $K > 0$ such that

$$\|\Xi\phi\|_q \le K\|\phi\|_p, \qquad \phi \in (E).$$

In particular, for all $\xi \in E_{\boldsymbol{C}}$,

$$\|W\Xi(\xi)\|_q \le K e^{1/2|\xi|_p^2}.$$

THEOREM 3.3. *Let $\Xi \in L((E), (E))$ and $G = W\Xi$. Then the function G satisfies the following conditions*:

(G1′) *For each $\xi, \eta \in E_{\boldsymbol{C}}$, the function $z \mapsto G(z\xi + \eta)$ is weakly holomorphic, i.e., for any $\Phi \in (E)^*$, the function $z \mapsto \langle\langle \Phi, G(z\xi + \eta) \rangle\rangle$ is an entire function on $\boldsymbol{C}$.*

(G2′) *For any $q \ge 0$, there exist $p \ge 0$, $a > 0$ and $K > 0$ such that*

$$\|G(\xi)\|_q \le K e^{a|\xi|_p^2}, \qquad \xi \in E_{\boldsymbol{C}}.$$

Conversely, assume that an (E)-valued function G on $E_{\boldsymbol{C}}$ satisfies the above conditions. Then there exists a unique $\Xi \in L((E), (E))$ such that G is the W-transform of Ξ. Moreover, for any $r > p$ with $2ae^2\|A^{-(r-p)}\|_{HS}^2 < 1$,

$$\|\Xi\phi\|_q \le K(1 - 2ae^2\|A^{-(r-p)}\|_{HS}^2)^{-1/2}\|\phi\|_r, \qquad \phi \in (E).$$

Proof. The proof is similar to the proof of Theorem 3.1. So we shall only prove the existence of Ξ. To show this, fix arbitrary $\Phi \in (E)^*$. Define a $\boldsymbol{C}$-valued function F_Φ on $E_{\boldsymbol{C}}$ by

$$F_\Phi(\xi) = \langle\!\langle \Phi, G(\xi) \rangle\!\rangle, \qquad \xi \in E_{\boldsymbol{C}}.$$

Then clearly F_Φ satisfies (F1) and (F2) in Theorem 2.1. Hence, by Theorem 2.1, there exists a unique $\Psi_\Phi \in (E)^*$ such that

$$S\Psi_\Phi(\xi) = \langle\!\langle \Phi, G(\xi) \rangle\!\rangle, \qquad \xi \in E_{\boldsymbol{C}}.$$

Moreover, for any $r > p$ with $2ae^2 \|A^{-(r-p)}\|_{HS}^2 < 1$,

$$\|\Psi_\Phi\|_{-r} \le K\|\Phi\|_{-q}(1 - 2ae^2\|A^{-(r-p)}\|_{HS}^2)^{-1/2}. \tag{3.3}$$

Hence the operator $\Phi \mapsto \Psi_\Phi$ is a continuous linear operator from $(E)^*$ into $(E)^*$. Now, let Ξ be the adjoint of this operator. Then Ξ is the desired operator in $L((E), (E))$. □

The following corollary can be proved by similar arguments of the proof of Corollary 3.2.

Corollary 3.4. *Suppose that a $\boldsymbol{C}$-valued function F on $E_{\boldsymbol{C}} \times E_{\boldsymbol{C}}$ satisfies the following conditions*:

(S1′) *For each ξ, ξ', η and η' in $E_{\boldsymbol{C}}$, the function $(z, w) \mapsto F(z\xi + \xi', w\eta + \eta')$ is an entire function on $\boldsymbol{C} \times \boldsymbol{C}$.*

(S2′) *For any $r \ge 0, a > 0$, there exist $p \ge r$ and $K > 0$ such that*

$$|F(\xi, \eta)| \le Ke^{a(|\xi|_p^2 + |\eta|_{-r}^2)}, \qquad \xi, \eta \in E_{\boldsymbol{C}}.$$

Then there exists a unique $\Xi \in L((E), (E))$ such that F is the symbol of Ξ.

Remark. It can be shown that Corollary 3.4 implies the second assertion of Theorem 3.3.

Example. (1) For $\alpha, \beta \in \boldsymbol{C}$, we define an (E)-valued function $\mathscr{C}_{\alpha,\beta}$ on $E_{\boldsymbol{C}}$ by

$$\mathscr{C}_{\alpha,\beta}(\xi) = e^{\alpha\langle \xi, \xi \rangle} \varphi_{\beta\xi}, \qquad \xi \in E_{\boldsymbol{C}}.$$

Then it is easy to show that this $\mathscr{C}_{\alpha,\beta}$ satisfies (G1′) and (G2′). Hence there exists a unique operator $\mathscr{G}_{\alpha,\beta} \in L((E), (E))$ such that

$$\mathscr{G}_{\alpha,\beta}\varphi_\xi = \mathscr{C}_{\alpha,\beta}(\xi) = e^{\alpha\langle \xi, \xi \rangle} \varphi_{\beta\xi}, \qquad \xi \in E_{\boldsymbol{C}}.$$

This operator $\mathscr{G}_{\alpha,\beta}$ has the following integral representation (see [**1**]):

$$\mathscr{G}_{\alpha,\beta}\phi(x) = \int_{E^*} \phi(\sqrt{2\alpha - \beta^2 + 1}\, y + \beta x)\, d\mu(y), \qquad x \in E_{\boldsymbol{C}}^*.$$

(2) Let Ξ_1 and $\Xi_2 \in L((E), (E)^*)$. Let G_1 and G_2 be the W-transform of Ξ_1 and Ξ_2, respectively. Define an $(E)^*$-valued function G on $E_{\boldsymbol{C}}$ by

$$G(\xi) = G_1(\xi) \diamond G_2(\xi), \qquad \xi \in E_{\boldsymbol{C}},$$

where $\Phi \diamond \Psi$ is the Wick product of Φ and $\Psi \in (E)^*$. It is well-known [**3**] that for any $p \geq 0$, there exists $q \geq p$ such that

$$\|\Phi \diamond \Psi\|_{-q} \leq \|\Phi\|_{-p} \|\Psi\|_{-p}, \qquad \Phi, \Psi \in (E)^*. \tag{3.4}$$

Since G_1 and G_2 satisfy (G2), it follows from (3.4) that there exist $p \geq 0$, $q \geq p$, $K > 0$ and $a > 0$ such that

$$\|G(\xi)\|_{-q} \leq \|G_1(\xi)\|_{-p} \|G_2(\xi)\|_{-p} \leq K e^{a|\xi|_p^2}$$

Hence G again satisfies (G2). Now for ξ, ξ', $\eta \in E_{\boldsymbol{C}}$,

$$\begin{aligned}\langle\langle G(z\xi + \xi'), \varphi_\eta \rangle\rangle &= SG(z\xi + \xi')(\eta) \\ &= S(G_1(z\xi + \xi') \diamond G_2(z\xi + \xi'))(\eta) \\ &= SG_1(z\xi + \xi')(\eta) \cdot SG_2(z\xi + \xi')(\eta).\end{aligned}$$

Hence the function $z \mapsto \langle\langle G(z\xi + \xi'), \varphi_\eta \rangle\rangle$ is entire on $\boldsymbol{C}$ for each $\eta \in E_{\boldsymbol{C}}$. Further we can show that the function $z \mapsto \langle\langle G(z\xi + \xi'), \phi \rangle\rangle$ is entire on $\boldsymbol{C}$ for each $\phi \in (E)$. By Theorem 3.1, there is a unique $\Xi \in L((E), (E)^*)$ such that $G = W\Xi$. We denote Ξ by $\Xi_1 \diamond \Xi_2$ and is called the Wick product of Ξ_1 and Ξ_2. Similarly using Theorem 3.3, we can define the Wick product $\Xi_1 \diamond \Xi_2$ of $\Xi_1, \Xi_2 \in L((E), (E))$.

4. Convergence of operators.

In this section we will find a criterion for the convergence of operators on white noise functionals in terms of their W-transform and symbol.

THEOREM 4.1. *Let $\{\Xi_n\}_{n=1}^\infty$ and Ξ be in $L((E), (E)^*)$. Let $G_n = W\Xi_n$, $n \in \boldsymbol{N}$ and $G = W\Xi$. Then Ξ_n converges to Ξ strongly in $L((E), (E)^*)$ if and only if the following conditions hold*:

(O1) *$G_n(\xi)$ converges to $G(\xi)$ in $(E)^*$ for each $\xi \in E_{\boldsymbol{C}}$.*

(O2) *There exist $q \geq 0$, $p \geq 0$, $K > 0$ and $a > 0$ such that*

$$\|G_n(\xi)\|_{-q} \leq K e^{a|\xi|_p^2}, \qquad \xi \in E_{\boldsymbol{C}}, \quad n \in \boldsymbol{N}.$$

PROOF. Suppose that Ξ_n converges to Ξ strongly in $L((E), (E)^*)$. Then for each $\phi \in (E)$, $\Xi_n \phi$ converges to $\Xi \phi$ in $(E)^*$. Hence (O1) is satisfied. To prove (O2), we put

$$X_{q,k} \equiv \{\phi \in (E); \sup_{n \in \boldsymbol{N}} \|\Xi_n \phi\|_{-q} \leq k\}.$$

Then we have $(E) = \bigcup_{q,k \in \boldsymbol{N}} X_{q,k}$. Since (E) is a Fréchet space, by the Baire's category theorem there exist q and k in $\boldsymbol{N}$ such that $X_{q,k}$ contains an open set of (E). So we can see that there exist $p \in \boldsymbol{N}$ and $\varepsilon > 0$ such that $\{\phi \in (E); \|\phi\|_p < \varepsilon\} \subset X_{q,k}$. Then for any $\phi \in (E)$, we have $\|\Xi_n \phi\|_{-q} \leq k/\varepsilon' \|\phi\|_p$ for all $n \in \boldsymbol{N}$, where $0 < \varepsilon' < \varepsilon$. In particular, we have

$$\|G_n(\xi)\|_{-q} = \|\Xi_n \varphi_\xi\|_{-q} \leq \frac{k}{\varepsilon'} \|\varphi_\xi\|_p \leq \frac{k}{\varepsilon'} e^{1/2|\xi|_p^2}.$$

This completes the proof of the "only if" part.

Conversely, assume that $\{G_n\}$ satisfies the given conditions. Then by (O1), for each $\xi \in E_{\boldsymbol{C}}$ and $\psi \in (E)$,

$$\langle\langle \Xi_n \varphi_\xi, \psi \rangle\rangle \to \langle\langle \Xi \varphi_\xi, \psi \rangle\rangle.$$

Since the linear span of $\{\varphi_\xi; \xi \in E_{\boldsymbol{C}}\}$ is dense in (E), it follows by using (O2) and Theorem 3.1 that for any $\phi, \psi \in (E)$, $\langle\langle \Xi_n \phi, \psi \rangle\rangle$ converges to $\langle\langle \Xi \phi, \psi \rangle\rangle$. This means that for any $\phi \in (E)$, $\Xi_n \phi$ converges to $\Xi \phi$ weakly in $(E)^*$. But the weak convergence of a sequence in $(E)^*$ is equivalent to strong convergence. Therefore for any $\phi \in (E)$, $\Xi_n \phi$ converges to $\Xi \phi$ strongly in $(E)^*$. This completes the proof. □

Corollary 4.2. *Let $\{\Xi_n\}_{n=1}^\infty$ and Ξ be in $L((E),(E)^*)$. Let $F_n = \hat{\Xi}_n$, $n \in \boldsymbol{N}$ and $F = \hat{\Xi}$. Then Ξ_n converges to Ξ strongly in $L((E),(E)^*)$ if and only if the following conditions hold*:

(U1) *For each $\xi, \eta \in E_{\boldsymbol{C}}$, $F_n(\xi, \eta)$ converges to $F(\xi, \eta)$.*

(U2) *There exist $p \geq 0$, $K > 0$ and $a > 0$ such that*

$$|F_n(\xi, \eta)| \leq K e^{a(|\xi|_p^2 + |\eta|_p^2)}, \qquad \xi, \eta \in E_{\boldsymbol{C}}, \ n \in \boldsymbol{N}.$$

Proof. To prove the corollary, it suffices to prove that (O1) and (O2) are equivalent to (U1) and (U2). Clearly (O1) and (O2) imply (U1) and (U2). Now assume that (U1) and (U2) are satisfied. Using (U2), we see that for $\xi, \eta \in E_{\boldsymbol{C}}$ and for $n \in \boldsymbol{N}$,

$$|SG_n(\xi)(\eta)| = |F_n(\xi, \eta)| \leq K e^{a|\xi|_p^2} e^{a|\eta|_p^2}.$$

Hence by Theorem 2.1, we have for $q > p$ with $2ae^2 \|A^{-(q-p)}\|_{HS}^2 < 1$

$$\|G_n(\xi)\|_{-q} \leq K e^{a|\xi|_p^2} (1 - 2ae^2 \|A^{-(q-p)}\|_{HS}^2)^{-1/2}, \qquad \xi \in E_{\boldsymbol{C}}, \ n \in \boldsymbol{N}.$$

On the other hand, using (U1) we can show that for $\xi \in E_{\boldsymbol{C}}$,

$$\langle\langle G_n(\xi), \phi \rangle\rangle \to \langle\langle G(\xi), \phi \rangle\rangle$$

for all $\phi \in (E)$. Hence (O1) and (O2) are satisfied. □

Theorem 4.3. *Let $\{\Xi_n\}_{n=1}^\infty$ and Ξ be in $L((E),(E))$. Let $G_n = W\Xi_n$, $n \in \boldsymbol{N}$ and $G = W\Xi$. Then Ξ_n converges to Ξ strongly in $L((E),(E))$ if and only if the following conditions hold*:

(O1′) *For each $\xi \in E_{\boldsymbol{C}}$, $G_n(\xi)$ converges to $G(\xi)$ in (E).*

(O2′) *For each $q \geq 0$, there exist $p \geq 0$, $K > 0$, $a > 0$ such that*

$$\|G_n(\xi)\|_q \leq K e^{a|\xi|_p^2}, \qquad \xi \in E_{\boldsymbol{C}}, \ n \in \boldsymbol{N}.$$

Proof. Suppose that Ξ_n converges to $\Xi \in L((E),(E))$ strongly in $L((E),(E))$. Then for any $\phi \in (E)$, $\Xi_n \phi$ converges to $\Xi \phi$ strongly in (E). Hence (O1′) is obvious. To prove (O2′), for $q \geq 0$ being given, we put

$$Y_k \equiv \{\phi \in (E); \sup_{n \in \boldsymbol{N}} \|\Xi_n \phi\|_q \leq k\}.$$

Then Y_k is closed and $(E) = \bigcup_{k \in \boldsymbol{N}} Y_k$. Hence by using the similar arguments of the proof of Theorem 4.1, we can prove that (O2$'$) holds.

Conversely, assume that $\{G_n\}$ satisfies the given conditions. Let $q \geq 0$ be given. Then by (O1$'$), we have

$$\lim_{n\to\infty} \|\Xi_n \varphi_\xi - \Xi\varphi_\xi\|_q = 0, \qquad \xi \in E_{\boldsymbol{C}}. \tag{4.1}$$

Hence by using (O2$'$) and Theorem 3.3, we can prove that for any $\phi \in (E)$

$$\lim_{n\to\infty} \|\Xi_n \phi - \Xi\phi\|_q = 0.$$

Hence we complete the proof. □

Corollary 4.4. *Let $\{\Xi_n\}_{n=1}^\infty$ and Ξ be in $L((E),(E))$. Let $F_n = \hat{\Xi}_n$, $n \in \boldsymbol{N}$ and $F = \hat{\Xi}$. Then Ξ_n converges to Ξ strongly in $L((E),(E))$ if and only if the following conditions hold*:

(U1$'$) *For each $\xi, \eta \in E_{\boldsymbol{C}}$, $F_n(\xi,\eta)$ converges to $F(\xi,\eta)$.*

(U2$'$) *For each $q \geq 0$ and $a > 0$, there exist $p \geq q$, $K > 0$ such that*

$$|F_n(\xi,\eta)| \leq K e^{a(|\xi|_p^2 + |\eta|_{-q}^2)}, \qquad \xi,\eta \in E_{\boldsymbol{C}}, \quad n \in \boldsymbol{N}.$$

Proof. By similar arguments of the proof of Corollary 4.2, we can prove that (O1$'$) and (O2$'$) are equivalent to (U1$'$) and (U2$'$). □

Example. (1) Let $T_y \in L((E),(E))$ be such that $T_y\varphi_\xi = e^{\langle y,\xi\rangle}\varphi_\xi$. We will prove that $(T_{\theta y} - I)/\theta$ converges to D_y strongly in $L((E),(E))$ as $\theta \to 0$ using Theorem 4.3.

Put $G_\theta(\xi) = (T_{\theta y}\varphi_\xi - \varphi_\xi)/\theta = \varphi_\xi(e^{\langle y,\xi\rangle\theta} - 1)/\theta$, and $G(\xi) = D_y\varphi_\xi = \langle y,\xi\rangle\varphi_\xi$. Then clearly for each $\xi \in E_{\boldsymbol{C}}$, $G_\theta(\xi)$ converges to $G(\xi)$ in (E). By mean value theorem,

$$\left|\frac{e^{\langle y,\xi\rangle\theta} - 1}{\theta}\right| \leq |\langle y,\xi\rangle e^{\langle y,\xi\rangle\theta_0}| \leq (e^{|y|_{-p}^2})e^{|\xi|_p^2},$$

for $|y|_{-p} < \infty$ and $|\theta_0| \leq |\theta| \leq 1$. Hence for each $q \geq 0$, choose $p \geq q$ with $|y|_{-p} < \infty$. Then we have, for $|\theta| \leq 1$

$$\|G_\theta(\xi)\|_q \leq (e^{|y|_{-p}^2})e^{|\xi|_p^2}\|\varphi_\xi\|_q \leq (e^{|y|_{-p}^2})e^{(1+\lambda_0^{-2(p-q)})|\xi|_p^2}.$$

Thus by Theorem 4.3, $(T_{\theta y} - I)/\theta$ converges to D_y strongly in $L((E),(E))$ as $\theta \to 0$.

(2) Let $\alpha(\theta)$ and $\beta(\theta)$ be differentiable $\boldsymbol{C}$-valued functions defined on $\boldsymbol{R}$ with $\beta(\theta) \neq 0$ for all $\theta \in \boldsymbol{R}$. Then it is known [**1**] that $\{\mathscr{G}_{\alpha(\theta),\beta(\theta)}\}_{\theta\in\boldsymbol{R}}$ is a one-parameter subgroup of $SGL((E)) = \{\mathscr{G}_{\alpha,\beta}; \alpha,\beta \in \boldsymbol{C}, \beta \neq 0\}$ if and only if α and β are given by either

$$\alpha(\theta) = \frac{a}{2b}(e^{2b\theta} - 1) \quad \text{and} \quad \beta(\theta) = e^{b\theta} \text{ for some } a,b \in \boldsymbol{C} \text{ with } b \neq 0,$$

or

$$\alpha(\theta) = a\theta \quad \text{and} \quad \beta(\theta) = 1 \text{ for some } a \in \boldsymbol{C}.$$

For any $a, b \in \boldsymbol{C}$, consider a one-parameter subgroup $\{I_{a,b;\theta}\}_{\theta \in \boldsymbol{R}}$ of $SGL((E))$ defined by

$$I_{a,b;\theta} = \begin{cases} \mathscr{G}_{(a/2b)(e^{2b\theta}-1),\, e^{b\theta}}, & b \neq 0 \\ \mathscr{G}_{a\theta,1}, & b = 0. \end{cases}$$

Now we will show that $I_{a,b;\theta}$ converges to $I_{a,0;\theta}$ strongly in $L((E),(E))$ as $b \to 0$. To see this, fix $a \in \boldsymbol{C}$, $\theta \in \boldsymbol{R}$, and for any b, put $G_b = WI_{a,b;\theta}$. Then for each $q \geq 0$,

$$\|G_b(\xi) - G_0(\xi)\|_q \leq |e^{(a/2b)(e^{2b\theta}-1)\langle \xi,\xi\rangle}|\, \|\varphi_{e^{b\theta}\xi} - \varphi_\xi\|_q + |e^{(a/2b)(e^{2b\theta}-1)\langle \xi,\xi\rangle} - e^{a\theta\langle \xi,\xi\rangle}|\, \|\varphi_\xi\|_q.$$

We note that the map $\xi \mapsto \varphi_\xi$ from $E_{\boldsymbol{C}}$ into (E) is continuous. Hence $\lim_{b\to 0}\|G_b(\xi) - G_0(\xi)\|_q = 0$ for all $q \geq 0$. By mean value theorem on complex variable b, we obtain that for each $q \geq 0$ and for each b with $0 < |b| \leq 1$,

$$\|G_b(\xi)\|_q = |e^{(a/2b)(e^{2b\theta}-1)\langle \xi,\xi\rangle}|\, \|\varphi_{e^{b\theta}\xi}\|_q \leq e^{e^{2|\theta|}(|a\theta|\lambda_0^{-2q}+1/2)|\xi|_q^2}.$$

Therefore, by Theorem 4.3, $I_{a,b;\theta}$ converges to $I_{a,0;\theta}$ strongly in $L((E),(E))$ as $b \to 0$.

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