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On quadratic extensions of number fields and Iwasawa invariants for basic Z_3-extensions

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Abstract. Let $\boldsymbol{Z}_3$ be the ring of 3-adic integers. For each number field F, let $F_{\infty,3}$ denote the basic $\boldsymbol{Z}_3$-extension over F; let $\lambda_3(F)$ and $\mu_3(F)$ denote respectively the Iwasawa λ- and μ-invariants of $F_{\infty,3}/F$. Here a number field means a finite extension over the rational field $\boldsymbol{Q}$ contained in the complex field $\boldsymbol{C}$; $F \subset \boldsymbol{C}$, $[F:\boldsymbol{Q}] < \infty$. Now let k be a number field. Let $\mathcal{Q}_-$ denote the infinite set of totally imaginary quadratic extensions in $\boldsymbol{C}$ over k (so that $\mathcal{Q}_-$ coincides with the set $\mathcal{Q}^-$ in the text when k is totally real); let $\mathcal{Q}_+$ denote the infinite set of quadratic extensions in $\boldsymbol{C}$ over k in which every infinite place of k splits (so that $\mathcal{Q}_+$ coincides with the set $\mathcal{Q}^+$ in the text when k is totally real). After studying the distribution of certain quadratic extensions over k, that of certain cubic extensions over k, and the relation between the two distributions, this paper proves that, if k is totally real, then a subset of $\{K \in \mathcal{Q}_- \mid \lambda_3(K) = \lambda_3(k),\ \mu_3(K) = \mu_3(k)\}$ has an explicit positive density in $\mathcal{Q}_-$. The paper also proves that a subset of $\{L \in \mathcal{Q}_+ \mid \lambda_3(L) = \mu_3(L) = 0\}$ has an explicit positive density in $\mathcal{Q}_+$ if 3 does not divide the class number of k but is divided by only one prime ideal of k. Some consequences of the above results are added in the last part of the paper.

Let k be a number field, namely, a finite extension over the rational field $\boldsymbol{Q}$ contained in the complex field $\boldsymbol{C}$. Let h denote the class number of k, R the regulator of k, D the absolute value of the discriminant of k, w the number of roots of unity in k, M_∞ the set of infinite places of k, r_1 the number of real places in M_∞, and r_2 the number of imaginary places in M_∞. Let ζ_k denote as usual the Dedekind zeta function of k. For each finite place v of k, let $\mathfrak{p}_v$ denote the prime ideal of k corresponding to v, and let q_v denote the norm of $\mathfrak{p}_v$ in k. Given any number field F containing k, we denote by $v_{F/k}$ or simply by v_F the norm in k of the discriminant of F/k, by $h_{F/k}$ the order of the kernel of the norm map from the ideal class group of F to that of k, by $h_{F,3}$ the order of the maximal elementary abelian 3-group in this kernel, and by $F_{\boldsymbol{v}}$ the completion of F at each place $\boldsymbol{v}$ of F. Take any map α from M_∞ to the set consisting of $\boldsymbol{C}$ and the real field $\boldsymbol{R}$ such that $\alpha(v)$ contains k_v for every $v \in M_\infty$. We then define $\mathcal{Q}_\alpha$ to be the set of quadratic extensions K over k in $\boldsymbol{C}$ such that, for each real place v in M_∞, v is ramified or splits in K according as $\alpha(v) = \boldsymbol{C}$ or $\boldsymbol{R}$. A place v of k is said to be inert in a finte extension K over k if v is finite and if $\mathfrak{p}_v$ remains prime in K. Let S_0, S_1, S_2 be mutually disjoint finite sets of finite places of k, and let $\mathcal{Q}_\alpha^*$ denote the set of number fields in $\mathcal{Q}_\alpha$ in which every place of S_0 is ramified, every place of S_1 is inert, and

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every place of S_2 splits. Let us use the letter X as a real variable, assuming $X > 0$ throughout the following. Now, for any set $\mathscr{F}$ of finite extensions over k in $\boldsymbol{C}$ with degrees bounded, let $\mathscr{F}(X)$ denote the set of number fields F in $\mathscr{F}$ for which $v_{F/k}$ do not exceed X:

$$\mathscr{F}(X) = \{F \in \mathscr{F} \mid v_F \leq X\}.$$

Note that $\mathscr{F}(X)$ is always a finite set: $|\mathscr{F}(X)| < \infty$. In this paper, basing our arguments on some fundamental results and proofs in Datskovsky-Wright [**2**] and Davenport-Heilbronn [**3**], we shall prove, along with

$$\lim_{X\to\infty} \frac{|\mathscr{Q}_\alpha^*(X)|}{X} = \frac{\pi^{r_2} hR}{\zeta_k(2) w\sqrt{D}} \left(\prod_{v\in S_0} \frac{1}{q_v+1}\right)\left(\prod_{v\in S_1\cup S_2} \frac{q_v}{2(q_v+1)}\right)$$

(cf. Lemma 2), the following two results.

Theorem 1. *Let $r(\alpha)$ denote the number of real places v in M_∞ with $\alpha(v) = \boldsymbol{R}$. Then*

$$\lim_{X\to\infty} \frac{1}{|\mathscr{Q}_\alpha^*(X)|} \sum_{F\in\mathscr{Q}_\alpha^*(X)} h_{F,3} = 1 + \frac{1}{3^{r_2+r(\alpha)}}.$$

Theorem 2. *With $r(\alpha)$ the same as in Theorem* 1,

$$\liminf_{X\to\infty} \frac{|\{F \in \mathscr{Q}_\alpha^*(X) \mid 3\nmid h_{F/k}\}|}{|\mathscr{Q}_\alpha(X)|}$$
$$\geq \left(1 - \frac{1}{2\cdot 3^{r_2+r(\alpha)}}\right)\left(\prod_{v\in S_0} \frac{1}{q_v+1}\right)\left(\prod_{v\in S_1\cup S_2} \frac{q_v}{2(q_v+1)}\right).$$

Let p be any prime number and let $\boldsymbol{Z}_p$ denote (the additive group of) the ring of p-adic integers. For each number field F, let $F_{\infty,p}$ denote the basic $\boldsymbol{Z}_p$-extension over F, i.e., the $\boldsymbol{Z}_p$-extension over F defined as the composite of F and the unique $\boldsymbol{Z}_p$-extension over $\boldsymbol{Q}$ in $\boldsymbol{C}$; let $\lambda_p(F)$ deonte the Iwasawa λ-invariant of $F_{\infty,p}$ and $\mu_p(F)$ the Iwasawa μ-invariant of $F_{\infty,p}/F$. We mean by a CM-field a totally imaginary quadratic extension in $\boldsymbol{C}$ over a totally real number field. For each CM-field K, let

$$\lambda_p^-(K) = \lambda_p(K) - \lambda_p(K^+), \quad \mu_p^-(K) = \mu_p(K) - \mu_p(K^+),$$

where K^+ denotes the maximal real subfield of K; $K^+ = K \cap \boldsymbol{R}$. Let $\mathscr{Q}^-$ denote the set of CM-fields K with $K^+ = k$, so that $\mathscr{Q}^-$ is infinite or empty according as k is totally real or not. Let $T(p)$ denote the set of places of k lying above p. Then a well known argument in Iwasawa theory leads us to the inequality

$$\limsup_{X\to\infty} \frac{|\{K \in \mathscr{Q}^-(X) \mid \lambda_p^-(K) = 0\}|}{|\mathscr{Q}^-(X)|} \leq \frac{1}{2^{|T(p)|}} \prod_{v\in T(p)} \frac{q_v+2}{q_v+1} \tag{1}$$

if k is a totally real number field (cf. Proposition 1). On the other hand, it is conjectured that $\mu_p(F) = 0$ for every number field F (cf. Iwasawa [**13**], [**15**], Ferrero-

Washington [**4**], etc.). We shall further see in the case $p = 3$ that Theorem 2, together with well known results in Iwasawa theory, yields the following results which generalize Theorem 3 of [**19**] (cf. also [**11**, §3]).

Theorem 3. *Assume that* k *is totally real. Then*

$$\liminf_{X\to\infty} \frac{|\{K \in \mathcal{Q}^-(X) \mid \lambda_3^-(K) = \mu_3^-(K) = 0\}|}{|\mathcal{Q}^-(X)|} \geq \frac{1}{2^{|T(3)|+1}} \prod_{v\in T(3)} \frac{q_v+2}{q_v+1}.$$

Theorem 4. *Assume that* $3 \nmid h$ *and* $|T(3)| = 1$. *Let* u *be the place of* k *lying above* $3 : T(3) = \{u\}$. *Then*

$$\liminf_{X\to\infty} \frac{|\{F \in \mathcal{Q}_\alpha(X) \mid \lambda_3(F) = \mu_3(F) = 0\}|}{|\mathcal{Q}_\alpha(X)|} \geq \left(2 - \frac{1}{3^{r_2+r(\alpha)}}\right) \frac{q_u+2}{4(q_u+1)}.$$

When furthermore k *is totally real,*

$$\liminf_{X\to\infty} \frac{|\{F \in \mathcal{Q}^+(X) \mid \lambda_3(F) = \mu_3(F) = 0\}|}{|\mathcal{Q}^+(X)|} \geq \left(2 - \frac{1}{3^{[k:\boldsymbol{Q}]}}\right) \frac{q_u+2}{4(q_u+1)},$$

$\mathcal{Q}^+$ *denoting the infinite set of all totally real quadratic extensions over* k *in* $\boldsymbol{C}$.

One easily finds that the right hand side of the second inequality in Theorem 4 is greater than 1/2, while it is conjectured in general that $\lambda_p(F) = \mu_p(F) = 0$ for every totally real number field F (cf. Greenberg [**6**], [**16**, §11.7], etc.).

Some consequences of Theorems 2, 3 and 4 will be given in the last part of the paper.

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1. Let v be any place of k. If v is finite, let o_v denote the ring of $\mathfrak{p}_v$-adic integers in k_v, the completion of k at v. Let Ω_v be a fixed algebraic closure of k_v, understanding that $\Omega_v = \boldsymbol{C}$ if $v \in M_\infty$. We then define B_v to be the set of cyclic extensions Γ over k_v in Ω_v with $[\Gamma : k_v] \leq 3$. Note that B_v is a finite set. In case v is finite, we put for each $\Gamma \in B_v$,

$$b(\Gamma) = 3, \quad c(\Gamma) = \frac{|\Delta|_v}{6}, \qquad \text{if } \Gamma = k_v,$$

$$b(\Gamma) = 1, \quad c(\Gamma) = \frac{|\Delta|_v}{2}, \qquad \text{if } [\Gamma : k_v] = 2,$$

$$b(\Gamma) = 0, \quad c(\Gamma) = \frac{|\Delta|_v}{3}, \qquad \text{if } [\Gamma : k_v] = 3.$$

Here Δ is an element of k_v by which the discriminant of Γ/k_v is generated as a module over o_v, and $|x|_v$ for each $x \in \Omega_v$ denotes the absolute value of x on Ω_v such that $|x|_v = q_v^{-1}$ whenever x is a prime element of k_v. Obviously, $c(\Gamma)$ does not depend on the choice of Δ. In case v is infinite, we put for each $\Gamma \in B_v$,

$$b(\Gamma) = 3, \quad c(\Gamma) = \frac{1}{6}, \qquad \text{if } \Gamma = k_v,$$

$$b(\Gamma) = 1, \quad c(\Gamma) = \frac{1}{2}, \qquad \text{if } [\Gamma : k_v] = 2.$$

Now, let V be any finite set of places of k. By a V-signature, we mean an element of the direct product of B_v for all $v \in V$, which is identified in the usual manner with a map from V to the union of B_v for all $v \in V$. Let γ be any V-signature so that $\gamma = (\gamma(v))_{v \in V}$. We then put

$$b_\gamma = \prod_{v \in V} b(\gamma(v)), \quad c_\gamma = \prod_{v \in V} c(\gamma(v)).$$

A number field F is said to belong to γ or to be of signature γ if F is an extension over k of degree not greater than 3 and if, for each $v \in V$, there exists a place $\boldsymbol{v}$ of F which lies above v and satisfies the following two conditions:

(i) $F_{\boldsymbol{v}}$ is isomorphic to $\gamma(v)$ over k_v,

(ii) $[F_{\boldsymbol{v}} : k_v] \geq [F_{\boldsymbol{w}} : k_v]$ for any place $\boldsymbol{w}$ of F lying above v.

Let $\mathscr{U}_\gamma$ denote the set of number fields F of signature γ such that the ramification index of every place of F for F/k equals 1 or 2, $\mathscr{C}_\gamma$ the set of non-cyclic cubic extensions over k which are number fields in $\mathscr{U}_\gamma$, $\mathscr{C}'_\gamma$ the set of unramified cyclic cubic extensions over k in $\boldsymbol{C}$ of signature γ, and $\mathscr{Q}_\gamma$ the set of quadratic extensions over k in $\boldsymbol{C}$ of signature γ. Then $\mathscr{U}_\gamma$ is clearly the disjoint union of $\mathscr{C}_\gamma$, $\mathscr{C}'_\gamma$ and $\mathscr{Q}_\gamma$. It should be noted here that the definition of $\mathscr{Q}_\gamma$ above is consistent with that of $\mathscr{Q}_\alpha$ in the introduction. Next, let

$$\zeta^V(s) = \zeta_k(s) \prod_{v \in V \backslash M_\infty} (1 - q_v^{-s}) \quad \text{for } s \in \boldsymbol{C}.$$

By fundamental properties of ζ_k, ζ^V is a meromorphic function on $\boldsymbol{C}$, which is analytic in $\boldsymbol{C} \backslash \{1\}$, and 1 is a simple pole of ζ^V. Let $\rho(V)$ denote the residue of ζ^V at 1:

$$\rho(V) = \frac{2^{r_1+r_2}\pi^{r_2} hR}{w\sqrt{D}} \prod_{v \in V \backslash M_\infty} \left(1 - \frac{1}{q_v}\right).$$

Let $n_1(V)$ and $n_2(V)$ denote respectively the numbers of real and imaginary infinite places of k not contained in V.

Lemma 1. *As above, let V be a finite set of places of k and γ a V-signature. Then*:

(i) $$\lim_{X \to \infty} \frac{1}{X} \sum_{F \in \mathscr{U}_\gamma(X)} \frac{1}{[F : k]} = \frac{2^{n_1(V)-1}\rho(V)c_\gamma}{3^{n_1(V)} 6^{n_2(V)} \zeta^V(2)} + \frac{\rho(V) b_\gamma c_\gamma}{2^{n_2(V)+1} \zeta^V(2)},$$

(ii) $$\lim_{X \to \infty} \frac{|\mathscr{Q}_\gamma(X)|}{X} = \frac{\rho(V) b_\gamma c_\gamma}{2^{n_2(V)} \zeta^V(2)},$$

(iii) $$\lim_{X \to \infty} \frac{|\mathscr{C}_\gamma(X)|}{X} = \frac{2^{n_1(V)-1}\rho(V)c_\gamma}{3^{n_1(V)-1} 6^{n_2(V)} \zeta^V(2)}.$$

PROOF. The proof of [**2**, Theorem 5.1], in which an "adelic" generalization by Datskovsky-Wright [**1**] of Shintani's work [**22**] actually plays an important part, has essentially shown the assertion (i) as we see in the following. For simplicity, we put

$$\delta = \frac{2^{n_1(V)-1}\rho(V)c_\gamma}{3^{n_1(V)}6^{n_2(V)}\zeta^V(2)} + \frac{\rho(V)b_\gamma c_\gamma}{2^{n_2(V)+1}\zeta^V(2)}.$$

Since (3.10) and Lemma 4.2 of [**2**] imply

$$\limsup_{X\to\infty}\frac{1}{X}\sum_{F\in\mathscr{U}_\gamma(X)}\frac{1}{[F:k]}\le\delta,$$

it suffices to prove

(2) $$\liminf_{X\to\infty}\frac{1}{X}\sum_{F\in\mathscr{U}_\gamma(X)}\frac{1}{[F:k]}\ge\delta.$$

Let Y be a real variable >0 (independent of X). Let M_Y denote the union of M_∞ and the finite set of all finite places v of k with $q_v \le Y$, and let $\mathscr{U}_{\gamma,Y}$ denote the set of finite extensions K over k in $\boldsymbol{C}$ with $[K:k]\le 3$ such that there exists an M_Y-signature to which K and some number field in $\mathscr{U}_\gamma$ belong. Then $\mathscr{U}_{\gamma,Y}\backslash\mathscr{U}_\gamma$ is a set of cubic extensions over k in $\boldsymbol{C}$ in which some place outside M_Y is totally ramified. Hence, in virtue of Lemma 5.1 of [**2**], there exists a positive number C independent of X and Y satisfying

$$\frac{1}{X}\sum_{F\in(\mathscr{U}_{\gamma,Y}\backslash\mathscr{U}_\gamma)(X)}\frac{1}{[F:k]}\le C\sum_{v\notin M_Y}\frac{1}{q_v^2},$$

where v ranges over the finite places outside M_Y. Since

$$\mathscr{U}_\gamma\subseteq\mathscr{U}_{\gamma,Y},\qquad \sum_{v\notin M_Y}\frac{1}{q_v^2}\le\zeta^{M_Y}(2)-1,$$

it follows that

$$\frac{1}{X}\sum_{F\in\mathscr{U}_\gamma(X)}\frac{1}{[F:k]}\ge\frac{1}{X}\sum_{F\in\mathscr{U}_{\gamma,Y}(X)}\frac{1}{[F:k]}-C(\zeta^{M_Y}(2)-1).$$

However, we know from Theorem 4.1, Lemma 3.2 and (3.10) of [**2**] that

$$\lim_{X\to\infty}\frac{1}{X}\sum_{F\in\mathscr{U}_{\gamma,Y}(X)}\frac{1}{[F:k]}$$

exists and that this limit tends to δ as $Y\to\infty$. Therefore, by $\lim_{Y\to\infty}\zeta^{M_Y}(2)=1$, the inequality (2) is proved.

The assertion (ii) follows immediately from (3.9) and Theorem 4.2 of [**2**] (for the proof of this Theorem 4.2, see also Wright [**24**, §5]). In view of

$$|\mathscr{U}_\gamma(X)|=|\mathscr{Q}_\gamma(X)|+|\mathscr{C}_\gamma(X)|+|\mathscr{C}'_\gamma(1)|,$$

we obtain (iii) from (i) and (ii). □

2. As in the introduction, let α be any M_∞-signature and let S_0, S_1, S_2 be mutually disjoint finite sets of finite places of k. For simplicity, we put $S^* = S_0 \cup S_1 \cup S_2$.

LEMMA 2. *Let $\mathcal{Q}_\alpha^*$ denote as before the set of all quadratic extensions over k in $\boldsymbol{C}$ of signature α in which every place of S_0 is ramified, every place of S_1 is inert, and every place of S_2 splits. Then*

$$\lim_{X\to\infty} \frac{|\mathcal{Q}_\alpha^*(X)|}{X} = \frac{\pi^{r_2} hR}{\zeta_k(2) w\sqrt{D}} \left(\prod_{v\in S_0} \frac{1}{q_v+1}\right)\left(\prod_{v\in S_1\cup S_2} \frac{q_v}{2(q_v+1)}\right).$$

PROOF. Let H be the set of $M_\infty \cup S^*$-signatures γ such that $\gamma(v) = \alpha(v)$ for every $v \in M_\infty$, $\gamma(v)$ is a ramified quadratic extension over k_v for every $v \in S_0$, $\gamma(v)$ is the unique unramified quadratic extension over k_v in Ω_v for every $v \in S_1$, and $\gamma(v) = k_v$ for every $v \in S_2$. Obviously,

$$|H| < \infty, \quad n_1(M_\infty \cup S^*) = n_2(M_\infty \cup S^*) = 0.$$

It also follows immediately that $\mathcal{Q}_\alpha^*$ is the disjoint union of $\mathcal{Q}_\gamma$ for all $\gamma \in H$. Therefore, by (ii) of Lemma 1 and by the definitions of b_γ and c_γ, $\gamma \in H$,

$$\begin{aligned}\lim_{X\to\infty} \frac{|\mathcal{Q}_\alpha^*(X)|}{X} &= \sum_{\gamma\in H} \lim_{X\to\infty} \frac{|\mathcal{Q}_\gamma(X)|}{X} = \sum_{\gamma\in H} \frac{\rho(S^*) b_\gamma c_\gamma}{\zeta^{S^*}(2)} \\ &= \frac{\rho(S^*) b_\alpha c_\alpha}{\zeta^{S^*}(2)} \sum_{\gamma\in H} \left(\prod_{v\in S^*} b(\gamma(v)) c(\gamma(v))\right) \\ &= \frac{\rho(S^*) b_\alpha c_\alpha}{\zeta^{S^*}(2)} \prod_{v\in S^*} \left(\sum_{\Gamma\in H(v)} b(\Gamma) c(\Gamma)\right).\end{aligned}$$

Here

$$H(v) = \{\gamma(v) \mid \gamma \in H\} \quad \text{for } v \in S^*.$$

Let us take any $v \in S^*$. For each $\Gamma \in H(v)$, let $d(\Gamma) = |\Delta|_v$ with an element Δ of k_v generating over o_v the discriminant of Γ/k_v, so that $b(\Gamma)c(\Gamma) = d(\Gamma)/2$ by the definitions of $b(\Gamma)$ and $c(\Gamma)$. In the case $v \in S_1 \cup S_2$, since $d(\Gamma) = 1 = |H(v)|$, we obtain

$$\sum_{\Gamma\in H(v)} b(\Gamma) c(\Gamma) = \frac{1}{2}$$

at once. In the case $v \in S_0$, since $H(v)$ coincides with the set of ramified quadratic extensions over k_v in Ω_v, it follows from Theorem 1 of Serre [**21**] or (1.1) of [**2**] that

$$\sum_{\Gamma\in H(v)} b(\Gamma) c(\Gamma) = \frac{1}{2} \sum_{\Gamma\in H(v)} d(\Gamma) = \frac{1}{q_v}.$$

Hence

$$\lim_{X\to\infty} \frac{|\mathcal{Q}_\alpha^*(X)|}{X} = \frac{\rho(S^*) b_\alpha c_\alpha}{\zeta^{S^*}(2) 2^{|S_1|+|S_2|}} \prod_{v\in S_0} \frac{1}{q_v}.$$

On the other hand, we know from the definitions of b_α, c_α, ζ^{S^*}, and $\rho(S^*)$ that

$$b_\alpha c_\alpha = \frac{1}{2^{r_1+r_2}}, \quad \zeta^{S^*}(2) = \zeta_k(2)\prod_{v\in S^*}\left(1-\frac{1}{q_v^2}\right),$$

$$\rho(S^*) = \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{D}}\prod_{v\in S^*}\left(1-\frac{1}{q_v}\right).$$

Lemma 2 is therefore proved. □

REMARK (cf. Nakagawa [**18**, Corollary 5]). In the case $S^* = \emptyset$, $\mathcal{Q}_\alpha^*$ is nothing but $\mathcal{Q}_\alpha$ and Lemma 1 of course implies that

$$\lim_{X\to\infty}\frac{|\mathcal{Q}_\alpha(X)|}{X} = \frac{\pi^{r_2}hR}{\zeta_k(2)w\sqrt{D}}.$$

For any number field F containing k, we denote by $\mathfrak{D}_F$ the discriminant of F/k.

LEMMA 3. *Let K be a non-cyclic cubic extension over k in $\boldsymbol{C}$ in which no place of k is totally ramified. Let L be the Galois closure of K/k in $\boldsymbol{C}$, and K' the unique quadratic extension over k in L. Then L is an unramified cyclic cubic extension over K' and*

$$\mathfrak{D}_{K'} = \mathfrak{D}_K \quad \textit{whence} \quad v_{K'} = v_K.$$

Furthermore, for any place v of k, the following assertions hold:

(i) *v is ramified in K' if and only if some place of K above v is ramified for K/k,*

(ii) *v is inert in K' if and only if the residue degree for K/k of some place of K is equal to* 2,

(iii) *v splits in K' if and only if v is inert in K or splits completely in K.*

PROOF. By the assumption, each place of K ramified for K/k must be unramified in L, so that the ramification index of each place of L for L/k equals 1 or 2. Since L/K' is a cyclic cubic extension, it follows that L is unramified over K'. Hence, v is ramified in K' if some place of K lying above v is ramified for K/k. As the extension L/k is not cyclic, the residue degree for L/k of each finite place of L does not exceed 3. Therefore, v is inert in K' if the residue degree for K/k of some place of K above v is equal to 2. It also follows that, if v is inert in K, then v is unramified in L and hence splits in K'. In the case where v splits completely in K, we easily see that v splits completely in L, which implies that v splits in K'. Thus the assertions (i), (ii), (iii) are proved.

Next, let $\mathfrak{N}$ be the norm for K/k of the discriminant of L/K, i.e., the finite part of the conductor of L/K. We then obtain $\mathfrak{D}_{K'}^3 = \mathfrak{N}\mathfrak{D}_K^2$ from the fact that L is unramified over K'. To prove $\mathfrak{D}_{K'} = \mathfrak{D}_K$ (cf. [**2**, §5]), let us consider the case where v is finite and ramified in K' so that $\mathfrak{p}_v = \tilde{\mathfrak{p}}^2$ with a prime ideal $\tilde{\mathfrak{p}}$ of K'. In such a case, it follows from (i) that there exist distinct prime ideals $\mathfrak{P}_1$, $\mathfrak{P}_2$ of K satisfying $\mathfrak{p}_v = \mathfrak{P}_1\mathfrak{P}_2^2$. Therefore $\mathfrak{P}_1 = \tilde{\mathfrak{P}}^2$ with a prime ideal $\tilde{\mathfrak{P}}$ of L, whence $\tilde{\mathfrak{p}} = \tilde{\mathfrak{P}}\mathfrak{P}_2$. This decomposition shows that $\tilde{\mathfrak{p}}$ splits completely in L. Let ω be an algebraic integer of K' contained in $\tilde{\mathfrak{p}}\backslash\tilde{\mathfrak{p}}^2$, and let σ be the non-trivial element of $\mathrm{Gal}(L/K)$. Noting that $\omega^\sigma \in \tilde{\mathfrak{p}}\backslash\{\omega\}$, let g be the maximal positive integer such that $\omega^\sigma - \omega \in \tilde{\mathfrak{p}}^g$. Then $\mathfrak{p}^g$ equals the $\mathfrak{p}$-part of the conductor of K'/k, namely, the highest power of $\mathfrak{p}$ dividing the conductor of K'/k.

As $\tilde{\mathfrak{P}}$ itself is the $\tilde{\mathfrak{P}}$-part of $\tilde{\mathfrak{p}}$, we also have

$$\omega \in \tilde{\mathfrak{P}} \backslash \tilde{\mathfrak{P}}^2, \quad \omega^\sigma - \omega \in \tilde{\mathfrak{P}}^g \backslash \tilde{\mathfrak{P}}^{g+1}.$$

Therefore $\mathfrak{P}_1^g$ equals the $\mathfrak{P}_1$-part of the conductor of L/K. It further follows that $\mathfrak{p}^g$ equals the norm of $\mathfrak{P}_1^g$ for K/k. Since a prime ideal of k, which is divided by a prime ideal of K ramified in L, is ramified in K', we consequently have $\mathfrak{N} = \mathfrak{D}_{K'}$, i.e., $\mathfrak{D}_{K'} = \mathfrak{D}_K$ (for the classical discussion of the present paragraph, cf. Hasse [7], [**8**] as well as Takagi [**23**]).

Lemma 3 is now completely proved. □

Let $\mathscr{C}_\alpha^*$ denote the set of number fields K in $\mathscr{C}_\alpha$ such that every place of S_0 lies below a place of K ramified for K/k, every place of S_1 lies below a place of K whose residue degree for K/k equals 2, and every place of S_2 is inert or splits completely in K. We then define θ to be the map from $\mathscr{C}_\alpha^*$ to the set of number fields such that for each $K \in \mathscr{C}_\alpha^*$, $\theta(K)$ is the Galois closure of K/k in $\boldsymbol{C}$. Let $\mathscr{A}_F$ denote for each $F \in \mathscr{Q}_\alpha^*$ the set of Galois extensions L over k in $\boldsymbol{C}$, which are unramified cyclic cubic extensions over F, with $\mathrm{Gal}(L/k)$ isomorphic to the symmetric group of degree 3. It is obvious that $\mathscr{A}_{F_1} \cap \mathscr{A}_{F_2} = \varnothing$ for every pair of distinct number fields F_1 and F_2 in $\mathscr{Q}_\alpha^*$.

Lemma 4. *The image of θ coincides with the disjoint union of $\mathscr{A}_F$ for all $F \in \mathscr{Q}_\alpha^*$:*

$$\mathrm{Im}(\theta) = \coprod_{F \in \mathscr{Q}_\alpha^*} \mathscr{A}_F.$$

Proof. Let H^* be the finite set of $M_\infty \cup S^*$-signatures γ such that $\gamma(v) = \alpha(v)$ for every $v \in M_\infty$, $\gamma(v)$ is a ramified quadratic extension over k_v for every $v \in S_0$, $\gamma(v)$ is the unique unramified quadratic extension over k_v in Ω_v for every $v \in S_1$, and $\gamma(v)$ is either k_v or the unique unramified cubic extension over k_v in Ω_v for every $v \in S_2$. It follows that

$$\mathscr{C}_\alpha^* = \coprod_{\gamma \in H^*} \mathscr{C}_\gamma, \quad \mathscr{Q}_\alpha^* = \coprod_{\gamma \in H^*} \mathscr{Q}_\gamma. \tag{3}$$

For any $\gamma \in H^*$, let γ' be the element of H^* such that

$$\begin{aligned} \gamma'(v) &= \gamma(v) \quad \text{for } v \in M_\infty \cup S_0 \cup S_1, \\ &= k_v \qquad \text{for } v \in S_2. \end{aligned}$$

Then, given any $K \in \mathscr{C}_\alpha^*$ and the quadratic extension F over k in $\theta(K)$, we obtain from Lemma 3 and (3) that

$$F \in \mathscr{Q}_{\gamma'} \subseteq \mathscr{Q}_\alpha^*, \quad \theta(K) \in \mathscr{A}_F.$$

On the other hand, take any $L \in \mathscr{A}_F$, with $F \in \mathscr{Q}_\alpha^*$. Then there exists a non-cyclic cubic extension K over k in L, so that $L = KF$. Since L is an unramified cubic extension over F, no place of k is totally ramified in K. Hence Lemma 3, together with (3), shows

$$K \in \mathscr{C}_\alpha^*, \quad L = \theta(K).$$

These complete the proof of Lemma 4. □

Lemma 5. *Let $r(\alpha)$ denote (as in the statement of Theorem* 1) *the number of real infinite places v of k with $\alpha(v) = \boldsymbol{R}$. Then*

$$\lim_{X\to\infty}\frac{|\mathscr{C}_\alpha^*(X)|}{X} = \frac{\pi^{r_2}hR}{2\cdot 3^{r_2+r(\alpha)-1}\zeta_k(2)w\sqrt{D}}\left(\prod_{v\in S_0}\frac{1}{q_v+1}\right)\left(\prod_{v\in S_1\cup S_2}\frac{q_v}{2(q_v+1)}\right).$$

Proof. By Lemma 1 (iii) and (3),

$$\lim_{X\to\infty}\frac{|\mathscr{C}_\alpha^*(X)|}{X} = \sum_{\gamma\in H^*}\lim_{X\to\infty}\frac{|\mathscr{C}_\gamma(X)|}{X} = \sum_{\gamma\in H^*}\frac{3\rho(S^*)c_\gamma}{2\zeta^{S^*}(2)}$$

$$= \frac{3\rho(S^*)c_\alpha}{2\zeta^{S^*}(2)}\sum_{\gamma\in H^*}\left(\prod_{v\in S^*}c(\gamma(v))\right) = \frac{3\rho(S^*)c_\alpha}{2\zeta^{S^*}(2)}\prod_{v\in S^*}\left(\sum_{\Gamma\in H^*(v)}c(\Gamma)\right),$$

where H^* is of course the same as in the proof of Lemma 4 and

$$H^*(v) = \{\gamma(v)\,|\,\gamma\in H^*\}\quad\text{for } v\in S^*.$$

Let us take any $v\in S^*$. It follows that $H^*(v)$ consists of the ramified quadratic extensions over k_v in Ω_v, only of the unramified quadratic extension over k_v in Ω_v, or of k_v and the unramified cubic extension over k_v in Ω_v, according as v belongs to S_0, S_1 or S_2. Hence, by Theorem 1 of [**21**] and the definitions of $c(\Gamma)$, $\Gamma\in H^*(v)$,

$$\sum_{\Gamma\in H^*(v)}c(\Gamma) = \frac{1}{q_v}\qquad\text{if } v\in S_0,$$

$$= \frac{1}{2}\qquad\text{if } v\in S_1,$$

$$= \frac{1}{6}+\frac{1}{3} = \frac{1}{2}\qquad\text{if } v\in S_2.$$

We therefore have

$$\lim_{X\to\infty}\frac{|\mathscr{C}_\alpha^*(X)|}{X} = \frac{3\rho(S^*)c_\alpha}{\zeta^{S^*}(2)2^{|S_1|+|S_2|+1}}\prod_{v\in S_0}\frac{1}{q_v}.$$

This, together with the definitions of c_α, ζ^{S^*} and $\rho(S^*)$, proves the lemma. □

3. We shall prove the theorems stated in the introduction. For each number field K, let $\mathrm{Cl}(K)$ denote the ideal class group of K, h_K the class number of K, and $\mathrm{Cl}(K)_3$ the maximal elementary abelian 3-group in $\mathrm{Cl}(K)$;

$$h_K = |\mathrm{Cl}(K)|,\quad \mathrm{Cl}(K)_3 = \{c\in\mathrm{Cl}(K)\,|\,c^3 = 1\}.$$

Note that $\mathrm{Cl}(K)/\mathrm{Cl}(K)^3$ is isomorphic as a group to $\mathrm{Cl}(K)_3$.

Proof of Theorem 1. Take any $F\in\mathscr{Q}_\alpha^*$ and let σ be the non-trivial element of $\mathrm{Gal}(F/k)$. Then the group ring of $\mathrm{Gal}(F/k) = \{1,\sigma\}$ over the ring of (rational) integers acts on $\mathrm{Cl}(F)$, $\mathrm{Cl}(F)/\mathrm{Cl}(F)^3$, etc. in the obvious manner. We easily see from

$[F:k]=2$ that the homomorphisms

$$\mathrm{Cl}(F)/\mathrm{Cl}(F)^3 \to \mathrm{Cl}(k)/\mathrm{Cl}(k)^3, \quad \mathrm{Cl}(F)_3 \to \mathrm{Cl}(k)_3$$

induced by the norm map $\mathrm{Cl}(F) \to \mathrm{Cl}(k)$ are surjective, the kernel of the former homomorphism is $(\mathrm{Cl}(F)/\mathrm{Cl}(F)^3)^{1-\sigma}$, and $h_{F,3}$ is nothing but the order of the kernel of the latter homomorphism. Therefore,

$$|(\mathrm{Cl}(F)/\mathrm{Cl}(F)^3)^{1-\sigma}| = h_{F,3}. \tag{4}$$

Now, let $\mathfrak{S}_F$ denote the set of subgroups W of $\mathrm{Cl}(F)$ with index 3 satisfying $\mathrm{Cl}(F)^{1+\sigma} \subseteq W$. By class field theory, there exists a bijection $\mathscr{A}_F \to \mathfrak{S}_F$. We further obtain a bijection from $\mathfrak{S}_F$ to the set of subgroups of $(\mathrm{Cl}(F)/\mathrm{Cl}(F)^3)^{1-\sigma}$ with index 3, letting $W^{1-\sigma}\mathrm{Cl}(F)^3/\mathrm{Cl}(F)^3$ correspond to each $W \in \mathfrak{S}_F$. Hence, by (4),

$$|\mathscr{A}_F| = \frac{1}{2}(h_{F,3}-1) \tag{5}$$

as explained in [**2**, §5] (cf. [**3**]).

On the other hand, Lemmas 3 and 4 imply that

$$\theta(\mathscr{C}_\alpha^*(X)) = \coprod_{F \in \mathscr{Q}_\alpha^*(X)} \mathscr{A}_F.$$

It also follows for each $L \in \mathrm{Im}(\theta)$ that the three distinct cubic extensions over k in L form the inverse image of L under θ. Therefore, by (5),

$$|\mathscr{C}_\alpha^*(X)| = \frac{3}{2}\sum_{F \in \mathscr{Q}_\alpha^*(X)} (h_{F,3}-1)$$

or, equivalently,

$$\sum_{F \in \mathscr{Q}_\alpha^*(X)} h_{F,3} = |\mathscr{Q}_\alpha^*(X)| + \frac{2}{3}|\mathscr{C}_\alpha^*(X)|.$$

We can now obtain Theorem 1 from Lemmas 2 and 5. □

Proof of Theorem 2. For simplicity, let

$$\mathscr{G} = \{F \in \mathscr{Q}_\alpha^* \mid 3 \nmid h_{F/k}\} = \{F \in \mathscr{Q}_\alpha^* \mid h_{F,3} = 1\}.$$

It is then clear that

$$\sum_{F \in \mathscr{Q}_\alpha^*(X)} h_{F,3} \geq 3(|\mathscr{Q}_\alpha^*(X)| - |\mathscr{G}(X)|) + |\mathscr{G}(X)|,$$

namely,

$$|\mathscr{G}(X)| \geq \frac{3}{2}|\mathscr{Q}_\alpha^*(X)| - \frac{1}{2}\sum_{F \in \mathscr{Q}_\alpha^*(X)} h_{F,3}.$$

Hence, by Theorem 1,

$$\liminf_{X\to\infty} \frac{|\mathscr{G}(X)|}{|\mathscr{Q}_\alpha^*(X)|} \geq \frac{3}{2} - \frac{1}{2}\lim_{X\to\infty}\frac{1}{|\mathscr{Q}_\alpha^*(X)|}\sum_{F \in \mathscr{Q}_\alpha^*(X)} h_{F,3} = 1 - \frac{1}{2\cdot 3^{r_2+r(\alpha)}}.$$

However, by Lemma 2 (and [**18**, Corollary 5]),

$$\lim_{X\to\infty}\frac{|\mathcal{Q}_\alpha^*(X)|}{|\mathcal{Q}_\alpha(X)|}=\left(\prod_{v\in S_0}\frac{1}{q_v+1}\right)\left(\prod_{v\in S_1\cup S_2}\frac{q_v}{2(q_v+1)}\right). \tag{6}$$

We therefore have

$$\liminf_{X\to\infty}\frac{|\mathcal{G}(X)|}{|\mathcal{Q}_\alpha(X)|}\geq\left(1-\frac{1}{2\cdot 3^{r_2+r(\alpha)}}\right)\left(\prod_{v\in S_0}\frac{1}{q_v+1}\right)\left(\prod_{v\in S_1\cup S_2}\frac{q_v}{2(q_v+1)}\right),$$

which is to be proved. □

Now, let p be a prime number. Assume that k is totally real, i.e., $r_2=0$, and take as α the M_∞-signature such that $\alpha(v)=\boldsymbol{C}$ for all $v\in M_\infty$, i.e., $r(\alpha)=0$. For each subset V of $T(p)$, let $\mathcal{Q}_V^p$ denote the set of CM-fields in $\mathcal{Q}^-$ in which no place of $T(p)\backslash V$ but every place of V splits. Then $\mathcal{Q}_\alpha$ coincides with $\mathcal{Q}^-$ and

$$\mathcal{Q}_V^p=\coprod_{V'}\mathcal{Q}_{V,V'}^p,$$

where V' ranges over the subsets of $T(p)\backslash V$ with $\mathcal{Q}_{V,V'}^p$ denoting the set of CM-fields in $\mathcal{Q}_V^p$ in which every place of V' is ramified and every place of $T(p)\backslash(V\cup V')$ is inert. Hence we see from (6) that

$$\begin{aligned}\lim_{X\to\infty}\frac{|\mathcal{Q}_V^p(X)|}{|\mathcal{Q}^-(X)|}&=\sum_{V'\subseteq T(p)\backslash V}\lim_{X\to\infty}\frac{|\mathcal{Q}_{V,V'}^p(X)|}{|\mathcal{Q}^-(X)|}\\&=\sum_{V'\subseteq T(p)\backslash V}\left(\prod_{v\in V'}\frac{1}{q_v+1}\right)\left(\prod_{v\in T(p)\backslash V'}\frac{q_v}{2(q_v+1)}\right)\\&=\left(\prod_{v\in T(p)\backslash V}\left(\frac{1}{q_v+1}+\frac{q_v}{2(q_v+1)}\right)\right)\left(\prod_{v\in V}\frac{q_v}{2(q_v+1)}\right)\\&=\frac{1}{2^{|T(p)|}}\left(\prod_{v\in V}\frac{q_v}{q_v+1}\right)\left(\prod_{v\in T(p)\backslash V}\frac{q_v+2}{q_v+1}\right).\end{aligned}$$

Next, for each positive integer $m\leq|T(p)|$, let $\mathcal{Q}^{p,m}$ denote the set of CM-fields in $\mathcal{Q}^-$ in which just m places of $T(p)$ split, so that

$$\mathcal{Q}^{p,m}=\coprod_{V}\mathcal{Q}_V^p,$$

with V ranging over the subsets of $T(p)$ such that $|V|=m$. As the discussion of Iwasawa [**16**, §11.7] shows, for any $K\in\mathcal{Q}^-$, $\lambda_p^-(K)$ is at least equal to the number of all places of $T(p)$ which split in K. This fact means that, for any integer $n\leq|T(p)|$,

$$\{K \in \mathscr{Q}^- \mid \lambda_p^-(K) \geq n\} \supseteq \coprod_{m=n}^{|T(p)|} \mathscr{Q}^{p,m}.$$

Thus the following assertion holds:

Proposition 1. *If k is totally real, then for any prime number p and any positive integer $n \leq |T(p)|$,*

$$\liminf_{X\to\infty} \frac{|\{K \in \mathscr{Q}^-(X) \mid \lambda_p^-(K) \geq n\}|}{|\mathscr{Q}^-(X)|} \geq \frac{1}{2^{|T(p)|}} \sum_V \left(\prod_{v \in V} \frac{q_v}{q_v+1}\right)\left(\prod_{v \in T(p)\setminus V} \frac{q_v+2}{q_v+1}\right),$$

where the sum is taken over the subsets V of $T(p)$ with $|V| \geq n$.

It should be added that the inequality (1) in the introduction is an immediate consequence of Proposition 1 since the right hand side of the inequality of the proposition for the case $n = 1$ equals

$$1 - \frac{1}{2^{|T(p)|}} \prod_{v \in T(p)} \frac{q_v+2}{q_v+1}.$$

Furthermore, we immediately obtain from Proposition 1 the following result:

Corollary. *Suppose that k is a real Galois extension over $\boldsymbol{Q}$. Then, for any prime number p and any positive integer $n \leq |T(p)|$,*

$$\liminf_{X\to\infty} \frac{|\{K \in \mathscr{Q}^-(X) \mid \lambda_p^-(K) \geq n\}|}{|\mathscr{Q}^-(X)|} \geq \left(\frac{q+2}{2q+2}\right)^{|T(p)|} \sum_{m=n}^{|T(p)|} \binom{|T(p)|}{m} \left(\frac{q}{q+2}\right)^m$$

where q is the norm in k of a prime ideal of k dividing p.

Proof of Theorem 3. Note that $r_2 = 0$, and take again as α the M_∞-signature such that $r(\alpha) = 0$, i.e., $\mathscr{Q}_\alpha = \mathscr{Q}^-$. For each subset V of $T(3)$, let $\mathscr{R}_V$ denote the set of CM-fields of $\mathscr{Q}^-$ in which every place of V is ramified and every place of $T(3)\setminus V$ is inert: $\mathscr{R}_V = \mathscr{Q}^3_{\varnothing,V}$. Then $\coprod_{V \subseteq T(3)} \mathscr{R}_V$ is nothing but $\mathscr{Q}^3_{\varnothing}$, the set of CM-fields of $\mathscr{Q}^-$ in which no place of $T(3)$ splits. As is well known, a CM-field K in $\mathscr{Q}^-$ satisfies $\lambda_3^-(K) = \mu_3^-(K) = 0$ if and only if 3 neither divides $h_{K/k}$ nor lies below any place of k splitting in K (see, e.g., Friedman [**5**]). Namely, we have

$$\{K \in \mathscr{Q}^- \mid \lambda_3^-(K) = \mu_3^-(K) = 0\} = \coprod_{V \subseteq T(3)} \{K \in \mathscr{R}_V \mid 3 \nmid h_{K/k}\}.$$

This gives, in particular,

$$\liminf_{X\to\infty} \frac{|\{K \in \mathscr{Q}^-(X) \mid \lambda_3^-(K) = \mu_3^-(K) = 0\}|}{|\mathscr{Q}^-(X)|}$$
$$\geq \sum_{V \subseteq T(3)} \liminf_{X\to\infty} \frac{|\{K \in \mathscr{R}_V(X) \mid 3 \nmid h_{K/k}\}|}{|\mathscr{Q}^-(X)|}.$$

Theorem 2 implies however that, for each subset V of $T(3)$,

$$\liminf_{X\to\infty}\frac{|\{K\in\mathscr{R}_V(X)\,|\,3\nmid h_{K/k}\}|}{|\mathscr{Q}^-(X)|}\geq\frac{1}{2}\left(\prod_{v\in V}\frac{1}{q_v+1}\right)\left(\prod_{v\in T(3)\setminus V}\frac{q_v}{2(q_v+1)}\right).$$

Therefore

$$\liminf_{X\to\infty}\frac{|\{K\in\mathscr{Q}^-(X)\,|\,\lambda_3^-(K)=\mu_3^-(K)=0\}|}{|\mathscr{Q}^-(X)|}$$

$$\geq\frac{1}{2}\sum_{V\subseteq T(3)}\left(\prod_{v\in V}\frac{1}{q_v+1}\right)\left(\prod_{v\in T(3)\setminus V}\frac{q_v}{2(q_v+1)}\right)=\frac{1}{2^{|T(3)|+1}}\prod_{v\in T(3)}\frac{q_v+2}{q_v+1},$$

and consequently Theorem 3 is proved. □

Proof of Theorem 4. Let $\mathscr{Q}'_\alpha$ denote the set of number fields of $\mathscr{Q}_\alpha$ in which u does not split, i.e., u is ramified or inert. In view of the assumption of Theorem 4, $3\nmid h_{F/k}$ is equivalent to $3\nmid h_F$ for every $F\in\mathscr{Q}_\alpha$ while it follows from Iwasawa [**12**, II] that $\lambda_3(F)=\mu_3(F)=0$ for every $F\in\mathscr{Q}'_\alpha$ with $3\nmid h_F$ (cf. Iwasawa [**14**, §7.5]). Hence

$$\{F\in\mathscr{Q}_\alpha\,|\,\lambda_3(F)=\mu_3(F)=0\}\supseteq\{F\in\mathscr{Q}'_\alpha\,|\,3\nmid h_{F/k}\}.$$

Therefore, by Theorem 2,

$$\liminf_{X\to\infty}\frac{|\{F\in\mathscr{Q}_\alpha(X)\,|\,\lambda_3(F)=\mu_3(F)=0\}|}{|\mathscr{Q}_\alpha(X)|}$$

$$\geq\left(1-\frac{1}{2\cdot 3^{r_2+r(\alpha)}}\right)\left(\frac{1}{q_u+1}+\frac{q_u}{2(q_u+1)}\right)=\left(2-\frac{1}{3^{r_2+r(\alpha)}}\right)\frac{q_u+2}{4(q_u+1)}.$$

The former part of Theorem 4 is thus verified. When k is totally real, i.e., $r_2=0$, we have

$$\mathscr{Q}_\alpha=\mathscr{Q}^+,\quad r(\alpha)=[k:\boldsymbol{Q}],$$

taking as α the M_∞-signature such that $\alpha(v)=\boldsymbol{R}$ for every $v\in M_\infty$. Hence the latter part of the theorem follows from the former. □

4. For each number field F, let $A(F)$ denote the 3-class group of F and let $F^{(n)}$ denote, for each positive integer n, the intermediate field between F and the basic $\boldsymbol{Z}_3$-extension over F such that $[F^{(n)}:F]=3^n$. Moreover, for any CM-field K, let $A^-(K)$ denote the kernel of the norm map from $A(K)$ to $A(K^+)$.

Assume now that k is totally real. Let K_1 and K_2 be distinct CM-fields such that $K_1^+=K_2^+=k$, and let n be any positive integer. It is then clear that $(K_1^{(n)})^+=(K_2^{(n)})^+=k^{(n)}$. For each $j\in\{1,2\}$, let ι_j denote the natural map $A(K_j^{(n)})\to A(K_1^{(n)}K_2^{(n)})$. As $K_1^{(n)}K_2^{(n)}$ is a quartic bicyclic extension over $k^{(n)}$, we easily see not only that ι_1 and ι_2 are injective but also that

$$A^-(K_1^{(n)}K_2^{(n)})=\iota_1(A^-(K_1^{(n)}))\times\iota_2(A^-(K_2^{(n)})).$$

In particular,

$$|A^-(K_1^{(n)}K_2^{(n)})| = |A^-(K_1^{(n)})| \cdot |A^-(K_2^{(n)})|.$$

On the other hand, Iwasawa's class number formula, i.e., Theorem 11 of [**14**] implies that the three numbers

$$|A^-(K_1^{(n)}K_2^{(n)})|3^{-\lambda_3^-(K_1K_2)n-\mu_3^-(K_1K_2)3^n}, \quad |A^-(K_1^{(n)})|3^{-\lambda_3^-(K_1)n-\mu_3^-(K_1)3^n},$$

$$|A^-(K_2^{(n)})|3^{-\lambda_3^-(K_2)n-\mu_3^-(K_2)3^n}$$

are independent of n if n is sufficiently large. Thus we have

$$\lambda_3^-(K_1K_2) = \lambda_3^-(K_1) + \lambda_3^-(K_2), \quad \mu_3^-(K_1K_2) = \mu_3^-(K_1) + \mu_3^-(K_2)$$

(for more general relations, see e.g., [**9**, §2]). Since Theorem 3 shows that there exist infinitely many CM-fields K with the properties

$$K^+ = k, \quad \lambda_3^-(K) = \mu_3^-(K) = 0,$$

the following result is therefore obtained:

Proposition 2. *If k is totally real, then there exist infinitely many CM-fields L which are quartic bicyclic extensions over k and satisfy $\lambda_3^-(L) = \mu_3^-(L) = 0$.*

Still assuming k to be totally real, let

$$a_k = 2\prod_l l^{m(l)}$$

where l ranges over the prime numbers $\mid 6D$, with $m(l)$ denoting the maximal integer ≥ 0 such that the $l^{m(l)}$-th roots of unity in $\boldsymbol{C}$ belong to some quadratic extension over k. It is known that $a_k\zeta_k(-1)$ is an integer $\neq 0$ (cf. Serre [**20**, Proposition 29]). Now, let p be any prime number and let $\mathscr{B}_p$ denote the set of CM-fields L which are quartic bicyclic extensions over k and satisfy $\lambda_p^-(L) = \mu_p^-(L) = 0$. Then, as the discussion above Proposition 2 suggests, $\mathscr{B}_p$ is an infinite set if and only if there exist infinitely many CM-fields K in $\mathscr{Q}^-$ satisfying $\lambda_p^-(K) = \mu_p^-(K) = 0$. On the other hand, the main result of Naito [**17**] implies that, unless p divides $a_k\zeta_k(-1)$, there exist infinitely many CM-fields K in $\mathscr{Q}^-$ with $\lambda_p^-(K) = \mu_p^-(K) = 0$. Thus follows

Proposition 3 (cf. [**17**]). *If k is totally real, then $\mathscr{B}_p$ is infinite for every prime number p not dividing $a_k\zeta_k(-1)$.*

Remark. Proposition 2 states that $\mathscr{B}_3$ is always infinite whether $a_k\zeta_k(-1)$ is divisible by 3 or not (cf. [**11**, §3]). Proposition 3 for the simplest case where $k = \boldsymbol{Q}$ is given in [**10**]. Meanwhile, there exist infinitely many examples of k with $|\mathscr{B}_2| < \infty$ (cf. [**11**, §2]).

Next, for each number field F, we let t_F denote the number of places of F lying above 3, so that $t_k = |T(3)|$.

Proposition 4. *Assume that k is totally real, $3 \nmid h$, and $t_k = 1$. Then, for any positive integer n, there exist infinitely many totally real number fields F containing k such* *that*

$$[F:k]=2^n, \quad \lambda_3(F)=\mu_3(F)=0.$$

PROOF. It follows from the assumption above and the proof of Theorem 4 that there exist infinitely many number fields F of $\mathscr{Q}^+$ which satisfy $3 \nmid h_F$, $t_F=1$, and so $\lambda_3(F)=\mu_3(F)=0$. Hence, by induction on n, the proposition is proved. □

Since $h_Q=t_Q=1$, Proposition 4 yields

COROLLARY. *For any positive integer n, there exist infinitely many totally real number fields F of degree* 2^n *with* $\lambda_3(F)=\mu_3(F)=0$.

Proposition 4, together with Theorem 2 or 3, also yields

PROPOSITION 5. *Assume that k is totally real,* $3 \nmid h$, *and* $t_k=1$. *Then, for any positive integer n, there exist infinitely many CM-fields K containing k such that*

$$[K:k]=2^n, \quad \lambda_3(K)=\mu_3(K)=0.$$

COROLLARY. *For any positive integer n, there exist infinitely many CM-fields K of degree* 2^n *with* $\lambda_3(K)=\mu_3(K)=0$.

References

[1] B. Datskovsky and D. J. Wright, The adelic zeta function associated to the space of binary cubic forms. II: Local theory, J. Reine Angew. Math. **367** (1986), 27–75.
[2] B. Datskovsky and D. J. Wright, Density of discriminants of cubic extensions, J. Reine Angew. Math. **386** (1988), 116–138.
[3] H. Davenport and H. Heilbronn, On the density of discriminants of cubic fields. II, Proc. Roy. Soc. London Ser. A **322** (1971), 405–420.
[4] B. Ferrero and L. C. Washington, The Iwasawa invariant μ_p vanishes for abelian number fields, Ann. of Math. **109** (1979), 377–395.
[5] E. Friedman, Iwasawa invariants, Math. Ann. **271** (1985), 13–30.
[6] R. Greenberg, On the Iwasawa invariants of totally real number fields, Amer. J. Math. **98** (1976), 263–284.
[7] H. Hasse, Arithmetische Theorie der kubischen Zahlkörper auf klassenkörpertheoretischer Grundlage, Math Z. **31** (1930), 565–582.
[8] H. Hasse, Zahlbericht, 3. Auflage (Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper I, Ia, II), Physica-Verlag, Würzburg-Wien, 1970.
[9] K. Horie, On Iwasawa λ^--invariants of imaginary abelian fields, J. of Number Theory **27** (1987), 238–252.
[10] K. Horie, A note on basic Iwasawa λ-invariants of imaginary quadratic fields, Invent. Math. **88** (1987), 31–38.
[11] K. Horie, On CM-fields with the same maximal real subfield, Acta Arith. **67** (1994), 219–227.
[12] K. Iwasawa, A note on class numbers of algebraic number fields, Abh. Math. Sem. Univ. Hamburg **20** (1956), 257–258.
[13] K. Iwasawa, On some invariants of cyclotomic fields, Amer. J. Math. **80** (1958), 773–783; erratum, ibid., **81** (1959), 280.
[14] K. Iwasawa, On Γ-extensions of algebraic number fields, Bull. Amer. Math. Soc. **65** (1959), 183–226.
[15] K. Iwasawa, On the μ-invariants of $\boldsymbol{Z}_l$-extensions, Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Yasuo Akizuki, Kinokuniya, Tokyo (1973), pp. 1–11.
[16] K. Iwasawa, On $\boldsymbol{Z}_l$-extensions of algebraic number fields, Ann. of Math. **98** (1973), 246–326.
[17] H. Naito, Indivisibility of class numbers of totally imaginary quadratic extensions and their Iwasawa invariants, J. Math. Soc. Japan **43** (1991), 185–194; erratum, ibid., **46** (1994), 725–726.
[18] J. Nakagawa, Orders of quadratic extensions of number fields, Acta Arith. **67** (1994), 229–239.

[19] J. Nakagawa and K. Horie, Elliptic curves with no rational points, Proc. Amer. Math. Soc. **104** (1988), 20–24.
[20] J. P. Serre, Cohomologie des groupes discrets, Prospects in mathematics, Annals of Mathematics Studies, No. 70, Princeton Univ. Press, Princeton, N.J.; Univ. of Tokyo Press, Tokyo, pp. 77–169.
[21] J. P. Serre, Une «formule de masse» pour les extensions totalement ramifiées de degrés donné d'un corps local, C. R. Acad. Sc. Paris A **286** (1978), 1031–1036.
[22] T. Shintani, On Dirichlet series whose coefficients are class numbers of integral binary cubic forms, J. Math. Soc. Japan **24** (1972), 132–188.
[23] T. Takagi, Über eine Theorie des relativ Abel'schen Zahlkörpers, J. Coll. Sci. Univ. Tokyo **41, (9)** (1920), 1–133.
[24] D. J. Wright, Twists of the Iwasawa-Tate zeta function, Math. Z. **200** (1989), 209–231.

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