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Threshold dynamics type approximation schemes for propagating fronts

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Abstract. We study the convergence of general threshold dynamics type approximation schemes to hypersurfaces moving with normal velocity depending on the normal direction and the curvature tensor. We also present results about the asymptotic shape of fronts propagating by threshold dynamics. Our results generalize and extend models introduced in the theories of cellular automaton and motion by mean curvature.

Introduction.

In this paper we study the convergence of general threshold dynamics type approximation schemes to hypersurfaces moving with normal velocity depending on the normal direction and the curvature tensor. These schemes are generalizations and extensions of the threshold dynamics models introduced by Gravner and Griffeath [**GrGr**] to study cellular automaton modeling of excitable media and by Bence, Merriman and Osher [**BMO**] to study the mean curvature evolution.

Cellular automaton models are mathematical models used to understand the transmission of periodic waves through environments such as a network or a tissue. A common feature of many such models is that some threshold level of excitation must occur in a neighborhood of a location to become excited and conduct a pulse. Typical physical systems which exhibit such phenomenology are, among others, neural networks, cardiac muscle, Belousov-Zhabotinsky oscillating chemical reaction, etc.

Interfaces (fronts, hypersurfaces) in $\boldsymbol{R}^N$ evolving with normal velocity

$$V = v(Dn, n), \tag{0.1}$$

where n and Dn are the unit normal vector to the surface and its gradient respectively, arise in geometry, in image processing, in the theory of turbulent flame propagation and combustion, as well as in the study of the asymptotic behavior, as time $t \to \infty$, of general systems describing the evolution in time of some order parameter identifying the different phases of a material or the total (averaged) magnetization of a stochastic system, etc.

Typical examples of interface dynamics appearing in the aforementioned areas are, among others, the anistropic motion with normal velocity

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$$V = -\mathrm{tr}[E(n)Dn] + c(n),$$

where $E \in \mathscr{S}^N$, $\mathscr{S}^N$ being the space of $N \times N$ symmetric matrices, a special case of which is the motion by mean curvature

$$V = -\mathrm{tr}\, Dn = \kappa_1 + \cdots + \kappa_{N-1},$$

where $\kappa_1, \dots, \kappa_{N-1}$ are the principal curvatures of the surface, as well as the curvature-independent motion

$$V = c(n).$$

The main mathematical characteristic of such evolutions is the development of singularities in finite time, independently of the smoothness of the initial surface. A great deal of work has been done during the last few years to interpret the evolution past the singularities. A rather general approach to provide a weak formulation for the motion past singularities, known as the level set approach, was introduced for numerical computations by Osher and Sethian [**OS**]—see, also, Barles [**B1**] for a simple model on flame propagation—and was developed rigorously by Evans and Spruck [**ES**] for mean curvature and by Chen, Giga and Goto [**CGG**] for more general geometric evolutions (see also Barles, Soner, Souganidis [**BSS**], Soner [**Son**], Ishii and Souganidis [**IS**] and Goto [**Go**]). More recently, Barles and Souganidis [**BS**] introduced another equivalent way to study the generalized evolution, which for definiteness we call the direct approach, which is more geometric and is more suitable to study asymptotic problems.

The outcome of all aforementioned work has been the development of a weak notion of evolving fronts called generalized front propagation. The generalized front propagation $\{\Gamma_t\}_{t \ge 0}$, with given normal velocity starting from a surface Γ_0 in $\boldsymbol{R}^N$, is defined for all $t \ge 0$, although it may become extinct in finite time. Moreover, it agrees with the classical differential—geometric motion, as long as the latter exists. The generalized motion may, on the other hand, develop singularities, change topological types and exhibit various other pathologies.

In spite of these peculiarities, the generalized motion $\{\Gamma_t\}_{t \ge 0}$ has been proven to be the right way to extend the classical motion past the singularities. Some of the most definitive results in this direction are about the fact that the generalized evolution (0.1) governs the asymptotic behavior of the solution of semilinear reaction-diffusion equations and systems. The first result in this direction for the Allen-Cahn equation was obtained by Evans, Soner and Souganidis [**ESS**] and later extended by Barles, Soner and Souganidis [**BSS**]. See also Barles and Souganidis [**BS1**] for a number of new and very striking examples.

Another recent striking application of the generalized front propagation is the fact that it governs the macroscopic behavior, for large times and in the context of grain coarsening, of a number of stochastic interacting particle systems like the stochastic Ising model with long-range interactions and general spin flip dynamics (see Katsoulakis and Souganidis [**KS1**, **2**, **3**] as well as [**BS1**] and [**Sou1**, **2**]). Such systems are standard Gibbsian models used in statistical mechanics to describe phase transitions. It turns out that the generalized front propagation not only describes the limiting behavior of such systems but also provides a theoretical justification, from the microscopic point of view, of several phenomenological sharp interface models in phase transitions.

Next we describe the results of this paper in the context of a very simple cellular automaton, the so-called majority voter model, which is described in detail in Griffeath [**Gr**]. The voter model is set on a lattice which represents a population with two possible political choices. Each individual (each cell), from time to time, checks the neighborhood and joins the majority depending on the number of neighbors already belonging to it. This mechanism creates, of course, a moving front. The threshold dynamics here differ from the one's in [**Gr**] in the sense that the "occupied set" may shrink.

To simplify the presentation we consider the whole space $\boldsymbol{R}^N$ instead of the lattice $\boldsymbol{Z}^N$. We then fix a threshold parameter $\theta \in (0,1)$, and we choose a subset $\mathcal{N} \subset \boldsymbol{R}^N$ such that

(0.2) $\quad \mathcal{N}$ is an open bounded neighborhood of the origin with $|\mathcal{N}| = 1$,

where $|A|$ denotes the N-dimensional Lebesgue measure of A, and we define the function $\mu : S^{N-1} \to \boldsymbol{R}$, S^{N-1} being the unit sphere in $\boldsymbol{R}^N$, by

$$\mu(p) = \max\{\lambda \in \boldsymbol{R} : |\{z \in \boldsymbol{R}^N : \langle p, z\rangle \leq -\lambda\} \cap \mathcal{N}| \geq \theta\}, \tag{0.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\boldsymbol{R}^N$.

For each $h > 0$, we define $M_h : \mathcal{M} \to \mathcal{M}$, $\mathcal{M}$ being the set of measurable subsets of $\boldsymbol{R}^N$, by

$$M_h(A) = \{x \in \boldsymbol{R}^N : |(x + h\mathcal{N}) \cap A| \geq \theta h^N\}. \tag{0.4}$$

The meaning of this definition is that, if A is the occupied set at time t, the occupied set $M_h(A)$ at time $t+h$ consists of those points for which the volume of the overlap between $x + h\mathcal{N}$ and A exceeds $\theta|h\mathcal{N}|$.

Next for any $t \geq 0$ and $h > 0$ define the mapping $C_t^h : \mathcal{M} \to \mathcal{M}$ by

$$C_t^h = M_h^{j-1} \quad \text{if } (j-1)h \leq t < jh, \quad \text{with } j \in \boldsymbol{N}, \tag{0.5}$$

where M_h^k denotes the k-times iterate of M_h and M_h^0 is the identity. The two-parameter family $\{C_t^h\}$ describes an approximation for the motion with normal velocity

$$V = \mu(n). \tag{0.6}$$

Indeed let $\Omega_0 \subset \boldsymbol{R}^N$ be open and define

$$\Omega_t^h = C_t^h(\Omega_0).$$

The following theorem is a special case of one of the main results of this paper. Since being precise with its statement will only lead to a far less palatable and readable introduction, here we choose to be a bit imprecise and we denote it in quotes.

"Theorem A" *Let $\{\Gamma_t\}_{t\geq 0}$ be the generalized front propagation of $\Gamma_0 = \partial\Omega_0$ with normal velocity given by* (0.6). *Then, as $h \to 0$,*

$$\partial\Omega_t^h \to \Gamma_t \text{ in the Hausdorff metric.}$$

Next we introduce an approximation for curvature-dependent motions. To this end assume that, in addition to (0.2),

(0.7) $\quad \mathcal{N}$ is symmetric with respect to the origin, i.e., $-\mathcal{N} = \mathcal{N}$, and define, for each $p \in S^{N-1}$, the matrix $E(p) = (E_{kl}(p)) \in \mathscr{S}^N$ and the scalar $B(p) \in \boldsymbol{R}$ by

$$E(p) = \int_{\mathscr{N} \cap p^{\perp}} z \otimes z d\mathscr{H}^{N-1}(z) \tag{0.8}$$

and

$$B(p) = \mathscr{H}^{N-1}(\mathscr{N} \cap p^{\perp}). \tag{0.9}$$

Here and henceforth $p^{\perp}$ denotes the orthogonal complement of the vector p, i.e., $p^{\perp} = \{x \in \boldsymbol{R}^N : \langle x, p\rangle = 0\}$ and $\mathscr{H}^k$ denotes the k-dimensional Hausdorff measure.

Fix $c \in \boldsymbol{R}$ and introduce, for $h > 0$, the map $\tilde{M}_h : \mathscr{M} \to \mathscr{M}$ given by

$$\tilde{M}_h(A) = \{x \in \boldsymbol{R}^N : |(x + \sqrt{h}\mathscr{N}) \cap A| \geq \theta_h h^{N/2}\}, \tag{0.10}$$

where

$$\theta_h = \frac{1}{2} - c\sqrt{h}, \tag{0.11}$$

and define, for each $t \geq 0$ and $h > 0$, the map $\tilde{C}_t^h : \mathscr{M} \to \mathscr{M}$ by

$$\tilde{C}_t^h = \tilde{M}_h^{j-1} \quad \text{if } (j-1)h \leq t < jh, \quad \text{with } j \in \mathbf{N},$$

where the superscript k in $\tilde{M}_h^k$ has the same meaning as in M_h^k above. The two-parameter family $\{\tilde{C}_t^h\}$ describes an approximation for the motion with normal velocity

$$V = -\frac{1}{2} B^{-1}(n) \operatorname{tr}[E(n)Dn] + cB^{-1}(n). \tag{0.12}$$

Indeed let $\Omega_0 \subset \boldsymbol{R}^N$ be open and define

$$\Omega_t^h = \tilde{C}_t^h(\Omega_0).$$

The following theorem is again a special case of one of the main results in the paper. As in "Theorem A" we select to state it here in a somehow imprecise way.

"Theorem B" *Let $\{\Gamma_t\}_{t \geq 0}$ be the generalized front propagation of $\Gamma_0 = \partial\Omega_0$ with normal velocity given by* (0.12). *Then, as $h \to 0$,*

$$\partial\Omega_t^h \to \Gamma_t \text{ in the Hausdorff metric.}$$

An approximation of the type described by "Theorem B" for the special case of the motion by mean curvature was introduced in [**BMO**], which considered the case where, in the definition (0.10), $c = 0$ and the Lebesgue measure of the overlap between A and $x + \sqrt{h}\mathscr{N}$ is replaced by the average value of the characteristic function of A over the Gaussian kernel centered at x. The result of [**BMO**] was rigorously justified by Evans [**E**], Mascarenhas [**M**] and Barles and Georgelin [**BG**]. This was then extended by Ishii [**Is1**] for more general radially symmetric kernels.

We continue presenting, in the same informal way, another result about the asymptotic behavior, as $t \to \infty$, of fronts moving with normal velocity given by either (0.6) or (0.12). To this end, given a continuous function $v \in C(S^{N-1}, (0, \infty))$, we define the Wulff crystal of v by

$$\mathscr{W} = \{x \in \boldsymbol{R}^N : \langle x, p\rangle \leq v(p) \text{ for all } p \in S^{N-1}\}. \tag{0.13}$$

"Theorem C" *Let Ω_0 be a "large" bounded open subset of $\boldsymbol{R}^N$. Let $\{\Gamma_t\}_{t\geq 0}$ be the generalized front propagation of $\partial\Omega_0$ governed by*

$$V = -\mathrm{tr}[E(n)Dn] + v(n), \tag{0.14}$$

where $E \in C(S^{N-1}, \mathcal{S}^N)$ satisfies $E(n) \geq 0$. Then, as $t \to \infty$,

$$t^{-1}\Gamma_t \to \partial\mathcal{W} \text{ in the Hausdorff metric.}$$

When $E = 0$ on S^{N-1}, a discrete version of "Theorem C" was proved in [**GrGr**] and a continuous one by Soravia [**Sor**]. "Theorem C" is also related to a conjecture by Angenent and Gurtin [**AG**], which was proved in [**Son**]. The result of [**Son**] says that, given a uniformly convex Wulff crystal, there exists a particular class of $E \in C(S^{N-1}, \mathcal{S}^N)$ so that the claim of the "Theorem C" holds true. It is, of course, clear from the statement of "Theorem C" above that such a restriction on the choice is not necessary.

The paper is organized as follows: In Section 1 we recall some basic materials. In particular in Subsection 1.1 we recall the definition of the generalized level-set front propagation and summarize a number of facts which are relevant to our analysis. In Subsection 1.2 we recall an abstract formulation, introduced by Barles and Souganidis [**BS2**] to prove convergence of approximations to viscosity solutions of second order pde, which will be used extensively throughout the paper. Section 2 is devoted to curvature-independent motions, i.e., results like "Theorem A". In Section 3 we discuss approximations to curvature dependent motions and we present results like "Theorem B". In Section 4 we discuss about schemes obtained by combinations of two different threshold dynamics. Section 5 is devoted to the asymptotic shapes of propagating fronts obtained by the iteration of the threshold dynamics. Section 6 is devoted to results like "Theorem C" on the asymptotic shapes of propagating fronts for large times. In Section 7 we discuss the asymptotics, similar to Sections 5 and 6, of the threshold dynamics on scaled lattices. Precise references and discussion of the relationship of our results with other works are presented in each section. Part of the results presented in this paper come from the Ph.D. thesis of Pires [**P**] under the supervision of Souganidis.

§1. Preliminaries.

1.1. Generalized front propagation

Here we briefly describe the basic facts about the generalized front propagation defined by the level-set approach. Since we will not be using the direct approach in this paper, we refer to [**BS1**] and [**Sou1**, **2**] for its definition, its applications and relation to the level-set approach.

Although the velocity law of the form (0.1) is of common use, the derivative Dn depends on how n is extended away from the surface, which is inappropriate for our problems. Thus we henceforth use the following description of the velocity law

$$V = v((I - n \otimes n)Dn(I - n \otimes n), n), \tag{1.1}$$

which does not depend on the way n is extended outside the surface, does not loose any information concerning the surface carried by Dn and hence is more natural than (0.1). For a detailed discussion on velocity laws, see [**GiGo**].

In what follows we assume that

$$v \in C(\mathscr{S}^N \times S^{N-1}) \tag{1.2}$$

and

$$\begin{cases} v \text{ is nondecreasing, i.e., for all } p \in S^{N-1} \text{ and all } X, Y \in \mathscr{S}^N, \\ \text{if } X \leq Y \text{ then } v(X,p) \geq v(Y,p). \end{cases} \tag{1.3}$$

Intuitively the monotonicity means the following avoidance inclusion-type property: If $\{A_t\}_{t \geq 0}$, $\{B_t\}_{t \geq 0}$ are two one-parameter families of subsets of $\boldsymbol{R}^N$ with boundaries ∂A_t and ∂B_t moving by (1.1), n denoting the outward unit normal, and if in addition $A_0 \subset B_0$, then $A_t \subset B_t$ for all $t > 0$.

We begin with the classical derivation of the level-set approach. Let $\{\Gamma_t\}_{t \geq 0}$ be a smooth motion with normal velocity v, as in (1.1), let $\{D_t\}_{t \geq 0}$ be a family of smooth open subsets of $\boldsymbol{R}^N$ such that $\Gamma_t = \partial D_t$ and choose n to be the normal vector of Γ_t outward to D_t. Assume that $u : \boldsymbol{R}^N \times [0, \infty) \to \boldsymbol{R}$ is a C^∞ function such that

$$D_t = \{x \in \boldsymbol{R}^N : u(x,t) > 0\}, \Gamma_t = \{x \in \boldsymbol{R}^N : u(x,t) = 0\} \quad \text{and} \quad |Du| \neq 0 \text{ on} \bigcup_{t>0} \Gamma_t \times \{t\}.$$

A straightforward computation—see for example [**ES**]—yields, under the additional assumption that all the smooth level sets of u move with velocity given by (1.1), that u must satisfy the pde

$$u_t + F(D^2u, Du) = 0 \quad \text{in } \boldsymbol{R}^N \times (0, \infty), \tag{1.4}$$

where $F : \mathscr{S}^N \times (\boldsymbol{R}^N \backslash \{0\}) \to \boldsymbol{R}$ is related to v by

$$F(X,p) = -|p| v(-|p|^{-1}(I - \bar{p} \otimes \bar{p}) X (I - \bar{p} \otimes \bar{p}), -\bar{p}). \tag{1.5}$$

Here and below, for all $q \in \boldsymbol{R}^N \backslash \{0\}$,

$$\bar{q} = |q|^{-1} q.$$

Note that the monotonicity assumption (1.3) on v yields that F is degenerate elliptic, i.e., it satisfies, for all $X, Y \in \mathscr{S}^N$ and $p \in \boldsymbol{R}^N \backslash \{0\}$,

$$\text{if } X \leq Y \quad \text{then} \quad F(X,p) \geq F(Y,p). \tag{1.6}$$

To justify and extend the above to the case of not necessarily smooth motions it is necessary to use the notion of viscosity solutions for fully nonlinear elliptic and parabolic, possibly degenerate, partial differential equations, for short pde, introduced by Crandall and Lions. This theory provides the existence and uniqueness of viscosity solution of (1.4) under rather general conditions on F, which are, by the way, satisfied by the F's considered in this paper. We refer to [**ES**], [**CGG**], [**BSS**], [**IS**] and [**Go**] for such results and to the "User's Guide" by Crandall, Ishii and Lions [**CIL**] and the book of Barles [**B2**] for a general overview of the theory of viscosity solutions.

Next we recall the level-set approach to the generalized evolution of hypersurfaces or sets. To this end, let $\mathscr{E}$ denote the collection of triples (Γ, D^+, D^-) of mutually disjoint subsets of $\boldsymbol{R}^N$ such that Γ is closed, D^+ and D^- are open and $\boldsymbol{R}^N = \Gamma \cup D^+ \cup D^-$.

For any $(\Gamma_0, D_0^+, D_0^-) \in \mathscr{E}$ first choose a function $g \in BUC(\boldsymbol{R}^N)$, the space of bounded uniformly continuous functions on $\boldsymbol{R}^N$, such that

$$D_0^+ = \{x \in \boldsymbol{R}^N : g(x) > 0\},\ D_0^- = \{x \in \boldsymbol{R}^N : g(x) < 0\} \quad \text{and} \quad \Gamma_0 = \{x \in \boldsymbol{R}^N : g(x) = 0\},$$

then consider the initial value problem

$$\left\{ \begin{array}{lll} \text{(i)} & u_t + F(D^2u, Du) = 0 & \text{in } \boldsymbol{R}^N \times (0, \infty), \\ \text{(ii)} & u = g & \text{on } \boldsymbol{R}^N \times \{0\}, \end{array} \right. \tag{1.7}$$

where F is defined by (1.5), and let $u \in BUC(\boldsymbol{R}^N \times [0, \infty))$ be the unique viscosity solution of (1.7). Finally, set

$$\Gamma_t = \{x \in \boldsymbol{R}^N : u(x,t) = 0\},\ D_t^+ = \{x \in \boldsymbol{R}^N : u(x,t) > 0\},\ D_t^- = \{x \in \boldsymbol{R}^N : u(x,t) < 0\}.$$

Since F is geometric, i.e., it satisfies, for $\lambda > 0$, $\mu \in \boldsymbol{R}$, and $(X, p) \in \mathscr{S}^N \times \boldsymbol{R}^N \backslash \{0\}$,

$$F(\lambda X + \mu p \otimes p, \lambda p) = F(X, p), \tag{1.8}$$

the collection $\{(\Gamma_t, D_t^+, D_t^-)\}_{t \geq 0} \subset \mathscr{E}$ is determined, independently of the choice of g, by the initial data (Γ_0, D_0^+, D_0^-). (See, for example, [**ES**], [**CGG**], [**IS**], etc.)

Next for each $t \geq 0$ define the mapping $E_t : \mathscr{E} \to \mathscr{E}$ by

$$E_t(\Gamma_0, D_0^+, D_0^-) = (\Gamma_t, D_t^+, D_t^-),$$

and notice that $\{E_t\}_{t \geq 0}$ satisfies the properties: $E_0 = \mathrm{id}_{\mathscr{E}}$, $E_{t+s} = E_t \circ E_s$ for all $t, s \geq 0$—see for example, [**ES**], [**CGG**], [**IS**], and [**Go**].

Definition 1.1. The collection $\{E_t\}_{t \geq 0}$ is called the generalized evolution with normal velocity v. The collection $\{\Gamma_t\}_{t \geq 0}$ of closed sets is called the generalized front propagation of Γ_0 with normal velocity v.

Notice that the generalized front propagation is determined not only by Γ_0 but also by the choice of D_0^+ and D_0^-, which is related to choosing an orientation for the normal to Γ_0. In particular the generalized front propagation differs, in general, if D_0^+ and D_0^- are interchanged or if D_0^+ and D_0^- are replaced by the empty set $\varnothing$ and $D_0^+ \cup D_0^-$, respectively.

For the analysis in this paper it is also important to consider discontinuous solutions to (1.7) (i) with initial data given by characteristic functions. The existence and stability properties of such solutions were studied in detail in Barles, Soner and Souganidis [**BSS**]. An interesting issue regarding generalized evolution of fronts is whether the front develops interior or not. This is related to the uniqueness of discontinuous solutions to the initial value problem for (1.7). We refer to [**BSS**] and [**Sou1**] for this as well as further discussion and examples for which interior develops.

We conclude this subsection introducing some additional notation which helps the presentation of the main results of the paper. To this end, for each $(\Gamma_0, D_0^+, D_0^-) \in \mathscr{E}$

and $t \geq 0$, we define the maps $X_t : \mathscr{F} \to \mathscr{F}$ and $N_t : \mathscr{O} \to \mathscr{O}$, $\mathscr{F}$ and $\mathscr{O}$ being the collections of closed and open subsets of $\boldsymbol{R}^N$ respectively, by

$$X_t(D_0^+ \cup \Gamma_0) = D_t^+ \cup \Gamma_t \quad \text{and} \quad N_t(D_0^+) = D_t^+, \tag{1.9}$$

where $(\Gamma_t, D_t^+, D_t^-) = E_t(\Gamma_0, D_0^+, D_0^-)$. It can be shown that X_t is well-defined. Indeed, the decomposition of any $A \in \mathscr{F}$ into two sets D_0^+ and Γ_0 is not unique in general. However, the set $D_t^+ \cup \Gamma_t$ is independent of the choice of D_0^+ and Γ_0. A similar discussion applies to the mapping N_t. We also call $\{X_t\}_{t \geq 0}$ and $\{N_t\}_{t \geq 0}$ the generalized evolutions with normal velocity v. Hopefully, this notation will not create any confusion in the presentation, although in what follows we use the same expression—generalized evolution with normal velocity v—for the three different collections $\{E_t\}_{t \geq 0}$, $\{X_t\}_{t \geq 0}$, and $\{N_t\}_{t \geq 0}$

It is, of course, immediate that if $A \in \mathscr{O}$, $B \in \mathscr{F}$ and $A \subset B$, then

$$N_t(A) \subset X_t(B).$$

1.2. An abstract formulation

Motivated by the problem of proving convergence of approximation schemes to the viscosity solution of second order, fully nonlinear, possibly degenerate, parabolic pde, Barles and Souganidis introduced in [**BS2**] (see also [**Sou 3**, **4**] for a similar formulation for first order equations) the following abstract formulation.

For each $h \geq 0$, let $G_h : BUC(\boldsymbol{R}^N) \to BUC(\boldsymbol{R}^N)$ be such that for all $u, v \in BUC(\boldsymbol{R}^N)$ and $c \in \boldsymbol{R}$,

$$G_h(u + c) = G_h u + c, \tag{1.10}$$

and

$$\text{if } u \leq v \quad \text{then} \quad G_h u \leq G_h v. \tag{1.11}$$

It follows, from an observation due to Crandall and Tartar [**CT**], that, if (1.10) holds, then (1.11) is equivalent to

$$\|G_h u - G_h v\| \leq \|u - v\|, \tag{1.12}$$

where $\|\varphi\|$ denotes the sup-norm of φ.

Assume also that there exists a continuous function $F : \mathscr{S}^N \times (\boldsymbol{R}^N \backslash \{0\}) \to \boldsymbol{R}$ which is degenerate elliptic, i.e., it satisfies (1.6), such that, for all smooth function φ and all $x \in \boldsymbol{R}^N$, we have

$$\limsup_{h \to 0}{}^* \rho^{-1}(G_h\varphi - \varphi)(x) \leq -F_*(D^2\varphi(x), D\varphi(x)), \tag{1.13}$$

and

$$\liminf_{h \to 0}{}_* \rho^{-1}(G_h\varphi - \varphi)(x) \geq -F^*(D^2\varphi(x), D\varphi(x)). \tag{1.14}$$

Here and henceforth, f^* and f_* denote the upper- and lower-semicontinuous envelopes of the function f, i.e.,

$$f^*(x) = \lim_{r \to 0} \sup\{f(y) : y \in B(x, r)\} \quad \text{and} \quad f_*(x) = \lim_{r \to 0} \inf\{f(y) : y \in B(x, r)\}.$$

(See [**Is2**].) If $\{f^{\varepsilon}\}_{\varepsilon>0}$ is a family of locally bounded functions, following [**BP**], we define the generalized half-relaxed limits $\lim^*$ and $\lim_*$ by

$$\limsup_{\varepsilon\to 0}{}^* \ f(x) = \lim_{r\to 0} \sup\{f^{\varepsilon}(y) : 0 < \varepsilon < r, y \in B(x,r)\}$$

and

$$\lim_{\varepsilon\to 0}{}_* \ f^{\varepsilon}(x) = \lim_{r\to 0} \inf\{f^{\varepsilon}(y) : 0 < \varepsilon < r, y \in B(x,r)\}$$

and finally, if g is a uniformly continuous function, ω_g denotes its modulus of continuity.

Given $T > 0$, a partition $P = \{0 = t_0 < \cdots < t_n = T\}$ of $[0,T]$ with mesh $\|P\| = \max_{1\le i\le n}(t_i - t_{i-1})$ and $g \in BUC(\boldsymbol{R}^N)$ define $u_P : \boldsymbol{R}^N \times [0,T] \to \boldsymbol{R}$ by

$$u_P(\cdot,t) = \begin{cases} G_{t-t_{i-1}}(u_P(\cdot,t_{i-1})) & \text{if } t \in (t_{i-1},t_i], \\ g & \text{if } t = 0. \end{cases} \tag{1.15}$$

We need one more assumption about the way u_P assumes the initial condition g. We assume that

$$\begin{cases} \text{there exists } \omega \in C([0,\infty),[0,\infty)), \text{ independent of } P \text{ and depending} \\ \text{on } g \text{ only through the modulus of continuity of } g, \\ \text{such that } \omega(0) = 0 \text{ and for all } t \in [0,T], \\ \|u_P(\cdot,t) - g\| \le \omega(t). \end{cases} \tag{1.16}$$

It turns out that, if the initial value problem

$$\begin{cases} u_t + F(D^2u, Du) = 0 & \text{in } \boldsymbol{R}^N \times (0,T], \\ u = g & \text{on } \boldsymbol{R}^N \times \{0\} \end{cases} \tag{1.17}$$

has a unique viscosity solution $u \in BUC(\boldsymbol{R}^N \times [0,T])$, then the functions u_P converge to it.

Indeed the following theorem was proved in [**BS2**].

Theorem 1.1. *Assume that* $G_h : BUC(\boldsymbol{R}^N) \to BUC(\boldsymbol{R}^N)$ *satisfies* (1.10), (1.11), (1.13) and (1.14). *For all* $T > 0$, $g \in BUC(\boldsymbol{R}^N)$ *and all partitions* P of $[0,T]$, let $u_P : \boldsymbol{R}^N \times [0,T] \to \boldsymbol{R}$ *be defined by* (1.15) *and assume that, in addition,* (1.16) *is also satisfied. Let* u *be the unique viscosity solution of* (1.17). *Then, as* $\|P\| \to 0$,

$$u_P \to u \quad \textit{uniformly in } \boldsymbol{R}^N \times [0,T].$$

§2. Schemes for Curvature-Independent Motion.

We begin by formulating approximation schemes of the type described in "Theorem A" for motions with curvature-independent normal velocity. To this end, choose $f \in M(\boldsymbol{R}^N)$, $M(\boldsymbol{R}^N)$ being the space of measurable real-valued functions on $\boldsymbol{R}^N$, and a threshold parameter $\theta \in (0,1)$. Throughout this section we assume that

$$f \geq 0 \text{ on } \boldsymbol{R}^N \quad \text{and} \quad \int_{\boldsymbol{R}^N} f(x)\,dx = 1, \tag{2.1}$$

and

$$\text{for each } p \in S^{N-1} \text{ there exists a unique } v(p) \in \boldsymbol{R} \text{ such that} \int_{\langle p,x\rangle \geq v(p)} f\,dx = \theta. \tag{2.2}$$

The assumption that $\int f = 1$ is only made to simplify the presentation. The existence of v in (2.2) is obvious, the only real assumption being its uniqueness. It follows immediately from (2.1) and (2.2) that $v \in C(S^{N-1})$.

We also define the function $F \in C(\boldsymbol{R}^N)$ by

$$F(p) = \begin{cases} -|p|v(-\bar{p}) & \text{if } p \neq 0, \\ 0 & \text{if } p = 0. \end{cases} \tag{2.3}$$

It is then immediate that for all $\varepsilon > 0$ and $p \in \boldsymbol{R}^N \backslash \{0\}$,

$$\int_{\langle p,x\rangle \leq F(p)+\varepsilon} f(x)\,dx > \theta \quad \text{and} \quad \int_{\langle p,x\rangle \leq F(p)-\varepsilon} f(x)\,dx < \theta.$$

We are interested in describing the evolution of sets with normal velocity v. We argue as follows:

For each $h > 0$, define the operators $S_h : L^\infty(\boldsymbol{R}^N) \to L^\infty(\mathbf{R}^N) \cap C(\boldsymbol{R}^N)$ and $M_h : \mathscr{M} \to \mathscr{M}$ by

$$S_h g(x) = h^{-N} \int_{\boldsymbol{R}^N} f(h^{-1}(x-y))g(y)\,dy = \int_{\boldsymbol{R}^N} f(y)g(x-hy)\,dy, \tag{2.4}$$

and

$$M_h(A) = \{x \in \boldsymbol{R}^N : S_h \mathbf{1}_A(x) \geq \theta\}, \tag{2.5}$$

where $\mathbf{1}_A$ denotes the characteristic function of the set A. As mentioned in the Introduction, $M_h(A)$ is the location of A, after time h, when A moves with the threshold dynamics determined by f and θ.

It is worth remarking that, if the set A is a half plane with outward normal vector $p \in S^{N-1}$, e.g., if $A = \{x : \langle x, p\rangle \leq 0\}$, then

$$M_h(A) = \{x \in \boldsymbol{R}^N : \langle p, x\rangle \leq hv(p)\}.$$

Next, for all $t \geq 0$ and $h > 0$ define the mapping $C_t^h : \mathscr{M} \to \mathscr{M}$ by

$$C_t^h = M_h^{j-1}, \quad \text{if } (j-1)h \leq t < jh, \tag{2.6}$$

where M_h^k is the k-th iterate of M_h if $k \in \boldsymbol{N}$ and the identity mapping if $k = 0$. This two-parameter family $\{C_t^h\}$ yields an approximation scheme for the motion with normal velocity v.

Throughout the paper, for each $A \in \mathscr{M}$ and $\varepsilon > 0$, we write

$$A_\varepsilon = \{x \in \boldsymbol{R}^N : \operatorname{dist}(x, A^c) > \varepsilon\} \quad \text{and} \quad A^\varepsilon = \{x \in \boldsymbol{R}^N : \operatorname{dist}(x, A) < \varepsilon\}, \tag{2.7}$$

where $\operatorname{dist}(x, A)$ is the usual distance from x to A.

We have:

Theorem 2.1. *Assume* (2.1) *and* (2.2). *Then for all* $\varepsilon > 0$ *and* $T > 0$, *there exists a* $\delta > 0$ *such that, for all* $A \in \mathscr{M}$, *if* $0 < h < \delta$, *then*

$$N_t(A_\varepsilon) \subset C_t^h(A) \subset N_t(A^\varepsilon) \quad \text{and} \quad X_t(\overline{A_\varepsilon}) \subset C_t^h(A) \subset X_t(\overline{A^\varepsilon}).$$

Theorem 2.1 has the following consequence, from the point of view of the Hausdorff metric.

Theorem 2.2. *Assume* (2.1) *and* (2.2) *and let* K *be a compact subset of* $\boldsymbol{R}^N \times [0, \infty)$ *and* $\varepsilon > 0$. *Then for every closed set* A *there exists* $\delta > 0$ *such that if* $0 < h < \delta$, *then*

$$\left(\bigcup_{t \geq 0} C_t^h(A) \times \{t\} \right) \cap K \subset \bigcup_{t \geq 0} X_t(A) \times \{t\} + B(0, \varepsilon).$$

Similarly for every open set A *there exists* $\delta > 0$ *such that, if* $0 < h < \delta$, *then*

$$\left(\bigcup_{t \geq 0} N_t(A) \times \{t\} \right) \cap K \subset \bigcup_{t \geq 0} C_t^h(A) \times \{t\} + B(0, \varepsilon).$$

Following [**E**] and [**GrGr**], we introduce, for $h > 0$, the operator $G_h : M(\boldsymbol{R}^N) \to M(\boldsymbol{R}^N)$ given by

$$G_h\psi(x) = \sup\{\lambda \in \boldsymbol{R} : S_h \mathbf{1}_{\{\psi \geq \lambda\}}(x) \geq \theta\},$$

where here and henceforth $\{\psi \geq \lambda\}$ is the abbreviated notation for $\{x \in \boldsymbol{R}^N : \psi(x) \geq \lambda\}$.

Following [**BS2**], for $t \in [0, T]$ and $h > 0$, we also introduce the operator $Q_t^h : M(\boldsymbol{R}^N) \to M(\boldsymbol{R}^N)$ given by

$$Q_t^h = G_h^{j-1} \quad \text{if } (j-1) \leq t < jh \quad (j \in \boldsymbol{N}).$$

It is clear (see, for example, [**E**] and [**Is1**]) that

$$G_h\psi(x) = \inf\{\lambda \in \boldsymbol{R} : S_h \mathbf{1}_{\{\psi \geq \lambda\}}(x) < \theta\} = \sup\{\lambda \in \boldsymbol{R} : x \in M_h(\{\psi \geq \lambda\})\}.$$

It is also not hard to check that, if $\lambda = G_h\psi(x)$, then

$$S_h \mathbf{1}_{\{\psi \geq \lambda\}}(x) \geq \theta \quad \text{and} \quad S_h \mathbf{1}_{\{\psi \geq \lambda + \varepsilon\}}(x) < \theta \quad \text{for all } \varepsilon > 0.$$

The above inequalities imply that, for all $\lambda \in \boldsymbol{R}$,

$$G_h\psi(x) \geq \lambda \quad \text{if and only if } x \in M_h(\{\psi \geq \lambda\}).$$

In particular, for all $A \in \mathscr{M}$, we have

$$\mathbf{1}_{M_h(A)} = G_h \mathbf{1}_A \quad \text{and} \quad \mathbf{1}_{C_t^h(A)} = Q_t^h \mathbf{1}_A.$$

The proofs of Theorems 2.1 and 2.2 are based on the following

Theorem 2.3. *Fix* $g \in BUC(\boldsymbol{R}^N)$ *and let* $u \in BUC(\boldsymbol{R}^N \times [0, \infty))$ *be the unique viscosity solution of* (1.7) *with* F *given by* (2.3). *Then, for all* $0 < T < \infty$, *as* $h \to 0$,

$$Q_t^h g(x) \to u(x, t) \quad \textit{uniformly on } \boldsymbol{R}^N \times [0, T].$$

Theorem 2.3 follows from Theorem 1.1 provided we verify its assumptions, which we do next. To this end, we summarize below some of the basic properties of G_h. Since their proof follows along the lines of the analogous statements [**E**] and [**Is1**], we omit them. First of all, if $\rho \in C(\boldsymbol{R})$ is a nondecreasing function, then

$$G_h(\rho \circ \varphi) = \rho \circ (G_h\varphi) \quad \text{for all } \varphi \in M(\boldsymbol{R}^N) \quad \text{and} \quad h > 0. \tag{2.8}$$

Also for any $\varphi, \psi \in M(\boldsymbol{R}^N)$,

$$\text{if } \varphi \le \psi \quad \text{then} \quad G_h\varphi \le G_h\psi. \tag{2.9}$$

It follows that, for all $c \in \boldsymbol{R}$, $y \in \boldsymbol{R}^N$ and $\varphi \in M(\boldsymbol{R}^N)$,

$$G_h(\varphi + c) = G_h\varphi + c, \quad G_hc = c \quad \text{and} \quad G_h\varphi(\cdot + y) = (G_h\varphi)(\cdot + y). \tag{2.10}$$

But then (cf. [**CT**] and the discussion in Section 1.1), for all $h \ge 0$ and $\varphi, \psi \in M(\boldsymbol{R}^N)$,

$$\|G_h\varphi - G_h\psi\| \le \|\varphi - \psi\| \quad \text{and} \quad \|G_h\varphi\| \le \|\varphi\|.$$

Hence, if $\varphi \in BUC(\boldsymbol{R}^N)$,

$$|G_h\varphi(x) - G_h\varphi(y)| \le \omega_\varphi(|x - y|), \tag{2.11}$$

where ω_φ is the modulus of continuity of φ, and hence, G_h maps $BUC(\boldsymbol{R}^N)$ into itself.

We now proceed with the

Proof of Theorem 2.3. 1. Since, for all $h > 0$, the map $G_h : BUC(\boldsymbol{R}^N) \to BUC(\boldsymbol{R}^N)$ (cf. (2.10), (2.11)) satisfies (1.10) and (1.12), we may conclude, using Theorem 1.1, provided we verify (1.13), (1.14) and (1.16).

2. The fact that the generator-type inequalities (1.13) and (1.14) hold is an immediate consequence of

Lemma 2.1. *Let $\varphi \in C^1(\boldsymbol{R}^N)$. Then, for all $z \in \boldsymbol{R}^N$ and $\varepsilon > 0$, there exists $\delta > 0$, such that, for all $x \in B(z, \delta)$ and $h \in (0, \delta]$,*

$$G_h\varphi(x) \le \varphi(x) + (-F(D\varphi(z)) + \varepsilon)h \quad \text{and} \quad G_h\varphi(x) \ge \varphi(x) + (-F(D\varphi(z)) - \varepsilon)h.$$

Proof. 1. Since G_h is translation invariant, we may assume that $z = 0$.

2. Set $p = D\varphi(0)$. We only show that there exists $\delta > 0$ such that, for all $x \in B(0, \delta)$ and $h \in (0, \delta]$,

$$S_h \mathbf{1}_{\{\varphi \ge \varphi(x) + (-F(p)+\varepsilon)h\}}(x) < \theta,$$

which yields the first inequality above. The other inequality follows similarly.

3. Setting $E = \{y \in \boldsymbol{R}^N : \langle p, y \rangle \le F(p) - \varepsilon/2\}$ and noting that $E = \varnothing$ if $p = 0$, we observe that

$$\int_E f(y)\,dy < \theta.$$

4. Choose a sufficiently large $R > 0$ and a sufficiently small $\delta > 0$, so that

$$\int_{B(0,R)^c} f(y)\,dy < \theta - \int_E f(y)\,dy,$$

and, for all $x \in B(0,\delta)$, $h \in (0,\delta]$ and $y \in B(0,R)$,

$$|\varphi(x-hy) - \varphi(x) + h\langle p, y\rangle| \le \frac{\varepsilon h}{2}.$$

5. It follows that, for all $x \in B(0,\delta)$, $h \in (0,\delta]$ and $y \in B(0,R)$,

$$\text{if } \varphi(x-hy) \ge \varphi(x) + (-F(p)+\varepsilon)h \quad \text{then} \quad \langle p, y\rangle \le F(p) - \frac{\varepsilon}{2}.$$

Hence, for all $x \in B(0,\delta)$ and $h \in (0,\delta]$,

$$S_h \mathbf{1}_{\{\varphi \ge \varphi(x)+(-F(p)+\varepsilon)h\}}(x) \le \int_{E\cap B(0,R)} f(y)\,dy + \int_{B(0,R)^c} f(y)\,dy < \theta,$$

which completes the proof. □

3. Continuing with the proof of Theorem 2.3 and in preparation towards proving (1.16) we need the following lemma, in which, as it follows from its proof, we may replace the functions $\pm|x|$ by the functions $\pm\sqrt{|x|^2+1}$.

Lemma 2.2. *There exists a constant $C > 0$ such that, if $\varphi(x) = |x|$ (respectively $\varphi(x) = -|x|$), then for all $x \in \boldsymbol{R}^N$ and $h > 0$,*

$$G_h\varphi(x) \le \varphi(x) + Ch \ (\textit{respectively } G_h\varphi(x) \ge \varphi(x) - Ch)$$

Proof. 1. Since both inequalities are proved similarly, here we only present the proof of the first one.

2. Fix $R > 0$ so that

$$\int_{B(0,R)^c} f(x)\,dx < \theta,$$

and note that, for x, $y \in \boldsymbol{R}^N$ and $h > 0$,

$$\text{if } |x-hy| \ge |x| + Rh \quad \text{then} \quad |y| \ge R.$$

Hence, for all $x \in \boldsymbol{R}^N$ and $h > 0$,

$$S_h \mathbf{1}_{\{\varphi \ge |x|+Rh\}}(x) = \int_{\boldsymbol{R}^N} f(y)\mathbf{1}_{\{\varphi \ge |x|+Rh\}}(x-hy)\,dy \le \int_{B(0,R)^c} f(y)\,dy < \theta,$$

and therefore,

$$G_h\varphi(x) \le \varphi(x) + Rh. \qquad \square$$

4. Next we verify (1.16). To this end, fix $\varepsilon > 0$ and $h > 0$ and observe that there exists a constant $C_\varepsilon > 0$ such that, for all $r \ge 0$, $\omega_g(r) \le \varepsilon + C_\varepsilon r$. We then have, for all $x, y \in \boldsymbol{R}^N$, that

$$-\varepsilon - C_\varepsilon|x-y| + g(y) \le g(x) \le g(y) + \varepsilon + C_\varepsilon|x-y|.$$

Lemma 2.2 yields that for all $x, y \in \boldsymbol{R}^N$,

$$-C_\varepsilon Ch - C_\varepsilon|x-y| - \varepsilon + g(y) \le G_h g(x) \le g(y) + \varepsilon + C_\varepsilon|x-y| + C_\varepsilon Ch.$$

A simple induction now gives for all $(x,t) \in \boldsymbol{R}^N \times [0,\infty)$,

$$-\varepsilon - C_\varepsilon Ct + g(x) \le Q_t^h g(x) \le g(x) + \varepsilon + C_\varepsilon Ct.$$

The function $\omega(r) = \inf\{\varepsilon > 0 : \varepsilon + C_\varepsilon Cr\}$ has the required properties.

5. The proof of Theorem 2.3 is now complete following Theorem 1.1. □

We now continue with the

PROOF OF THEOREM 2.1. 1. Fix $T > 0$, $\varepsilon \in (0,1)$, $A \in \mathscr{M}$, define the function $g : \boldsymbol{R}^N \to \boldsymbol{R}$ by

$$g(x) = \begin{cases} \min\{\mathrm{dist}(x, A^c), 1\} & \text{if } x \in A, \\ -\min\{\mathrm{dist}(x, A), 1\} & \text{if } x \in A^c, \end{cases}$$

and note that $g \in BUC(\boldsymbol{R}^N)$ and

$$\{g > -\varepsilon\} = A^\varepsilon \quad \text{and} \quad \{g \ge \varepsilon\} = \overline{A_\varepsilon}.$$

2. Let $u \in BUC(\boldsymbol{R}^N \times [0,\infty))$ be the unique viscosity solution of (1.7). Then, for all $t \ge 0$, we have

$$N_t(A^\varepsilon) = \{u(\cdot,t) > -\varepsilon\} \quad \text{and} \quad X_t(\overline{A_\varepsilon}) = \{u(\cdot,t) \ge \varepsilon\}.$$

Theorem 2.3 also yields the existence of $\delta > 0$ such that if $0 < h < \delta$ and $0 \le t \le T$, then for all $x \in \boldsymbol{R}^N$,

$$|Q_t^h g(x) - u(x,t)| < \varepsilon/2.$$

3. Fix $0 < h < \delta$ and $0 \le t \le T$. It follows that

$$C_t^h(A) \subset C_t^h(\{g \ge -\varepsilon/2\}) = \{Q_t^h g \ge -\varepsilon/2\} \subset N_t(A^\varepsilon) \subset X_t(\overline{A^\varepsilon}),$$

and

$$C_t^h(A) \supset C_t^h(\{g \ge \varepsilon/2\}) = \{Q_t^h g \ge \varepsilon/2\} \supset X_t(\overline{A_\varepsilon}) \supset N_t(A_\varepsilon),$$

hence the claim. □

To prove Theorem 2.2 we need

LEMMA 2.3. *Let A be a closed subset of $\boldsymbol{R}^N$. Then*

$$\bigcap_{\varepsilon>0} \overline{\bigcup_{t\ge 0} N_t(A^\varepsilon) \times \{t\}} \subset \bigcup_{t \ge 0} X_t(A) \times \{t\}.$$

PROOF. The proof of Theorem 2.1 yields that for some function $u \in C(\boldsymbol{R}^N \times [0,\infty))$ and all $\varepsilon > 0$,

$$N_t(A^\varepsilon) = \{u(\cdot,t) > -\varepsilon\} \quad \text{and} \quad X_t(A) = \{u(\cdot,t) \ge 0\}.$$

Hence, for all $\varepsilon > 0$,

$$\overline{\bigcup_{t\geq 0} N_t(A^\varepsilon) \times \{t\}} \subset \{u \geq -\varepsilon\},$$

and therefore,

$$\bigcap_{\varepsilon>0} \overline{\bigcup_{t\geq 0} N_t(A^\varepsilon) \times \{t\}} \subset \bigcap_{\varepsilon>0} \{u \geq -\varepsilon\} = \bigcap_{t\geq 0} X_t(A) \times \{t\}. \qquad \square$$

We now proceed with the

Proof of Theorem 2.2. 1. Fix a compact subset K of $\boldsymbol{R}^N \times [0, \infty)$ and $\varepsilon > 0$. In view of Theorem 2.1, to prove the first assertion we only need to show that there exists $\delta > 0$ such that

$$\left(\bigcup_{t\geq 0} N_t(A^\delta) \times \{t\} \right) \cap K \subset \bigcup_{t\geq 0} X_t(A) \times \{t\} + B(0, \varepsilon).$$

2. Arguing by contradiction, we assume that for each $n \in \boldsymbol{N}$ there exists $(x_n, t_n) \in K$ such that

$$(x_n, t_n) \in \bigcup_{t\geq 0} N_t(A^{1/n}) \times \{t\} \quad \text{and} \quad (x_n, t_n) \notin \bigcup_{t\geq 0} X_t(A) \times \{t\} + B(0, \varepsilon). \tag{2.12}$$

Note that (2.12) yields that

$$B((x_n, t_n), \varepsilon) \cap \left(\bigcup_{t\geq 0} X_t(A) \times \{t\} \right) = \varnothing. \tag{2.13}$$

3. Since K is compact, we may assume that as $n \to \infty$, $(x_n, t_n) \to (\tilde{x}, \tilde{t})$ for some $(\tilde{x}, \tilde{t}) \in K$, and moreover, that $(x_n, t_n) \in B((\tilde{x}, \tilde{t}), \varepsilon)$ for all $n \in \boldsymbol{N}$. But then, (2.13) yields that

$$(\tilde{x}, \tilde{t}) \notin \bigcup_{t\geq 0} X_t(A) \times \{t\}.$$

On the other hand (2.12) yields that for all $\gamma > 0$

$$(\tilde{x}, \tilde{t}) \in \overline{\bigcup_{t\geq 0} N_t(A^\gamma) \times \{t\}}.$$

These last two statements together with Lemma 2.3 yield a contradiction.

4. The second assertion can be proved similarly. $\square$

§3. Schemes for Anisotropic Mean Curvature Motion.

We formulate here approximation schemes of the type described in "Theorem B" for curvature-dependent motions. To this end, fix $f \in M(\boldsymbol{R}^N)$ such that

$$f(x) \geq 0, \quad f(-x) = f(x) \quad \text{for all } x \in \boldsymbol{R}^N, \quad \text{and} \quad \int_{\boldsymbol{R}^N} f(x)\, dx = 1, \tag{3.1}$$

$$0 < \int_{p^\perp} (1+|x|^2) f(x)\, d\mathscr{H}^{N-1} < \infty \quad \text{for all } p \in S^{N-1}, \tag{3.2}$$

$$\begin{cases} \text{the functions } p \mapsto \int_{p^\perp} f(x)\, d\mathscr{H}^{N-1}(x) \text{ and } p \mapsto \int_{p^\perp} x_i x_j f(x)\, d\mathscr{H}^{N-1}(x), \\ \text{with } i, j \in \{1, \dots, N\}, \text{ are continuous on } S^{N-1}, \end{cases} \tag{3.3}$$

and

$$\int_{\boldsymbol{R}^N} |x|^2 f(x)\, dx < \infty. \tag{3.4}$$

Next we consider collections $\{R(\rho)\}_{0<\rho<1} \subset \boldsymbol{R}$ such that

$$R(\rho) \to \infty \quad \text{and} \quad \sqrt{\rho} R(\rho) \to 0, \quad \text{as } \rho \to 0, \tag{3.5}$$

and functions g: $\boldsymbol{R}^{N-1} \to \boldsymbol{R}$ of the form

$$g(\xi) = a + \langle A\xi, \xi \rangle \quad \text{with } a \in \boldsymbol{R} \quad \text{and} \quad A \in \mathscr{S}^{N-1}. \tag{3.6}$$

Also for any $U \in O(N)$, $O(N)$ denoting the group of $N \times N$ orthogonal matrices, and $f : \boldsymbol{R}^N \to \boldsymbol{R}$ we define $f_U : \boldsymbol{R}^N \to \boldsymbol{R}$ by

$$f_U(x) = f(U^* x).$$

In addition to (3.1)–(3.4), we need to assume that

$$\begin{cases} \text{for all collections } \{R(\rho)\}_{0<\rho<1} \text{ satisfying (3.5) and all} \\ \text{functions } g \text{ of the form (3.6), as } \rho \to 0 \\ \sup_{U \in O(N)} \sup_{0<r<\rho} \left| \int_{B(0,R(\rho))} f_U(\xi, rg(\xi)) g(\xi)\, d\xi - \int_{\boldsymbol{R}^{N-1}} f_U(\xi, 0) g(\xi)\, d\xi \right| \to 0. \end{cases} \tag{3.7}$$

Fix $c \in \boldsymbol{R}$ and define the function v: $\mathscr{S}^N \times S^{N-1} \to \boldsymbol{R}$ by

$$v(X,p) = \left(\int_{p^\perp} f(x)\, d\mathscr{H}^{N-1}(x) \right)^{-1} \left(-\frac{1}{2} \int_{p^\perp} \langle Xx, x \rangle f(x)\, d\mathscr{H}^{N-1}(x) + c \right).$$

Since, for all p, $x \in \boldsymbol{R}^N$, $\langle p \otimes p x, x \rangle = \langle p, x \rangle^2$, it follows that for all $(X,p) \in \mathscr{S}^N \times S^{N-1}$,

$$v((I - p\otimes p) X (I - p \otimes p), p) = v(X,p) \quad \text{and} \quad v(X,-p) = v(X,p).$$

Our goal in this section is to define threshold dynamics-type approximation scheme for hypersurfaces or sets moving with normal velocity v.

As in Section 2, for $h > 0$, we define the operator S_h: $M(\boldsymbol{R}^N) \to M(\boldsymbol{R}^N)$ by

$$S_h \psi(x) = h^{-N/2} \int_{\boldsymbol{R}^N} f(h^{-1/2}(x-y)) \psi(y)\, dy = \int_{\boldsymbol{R}^N} f(y) \psi(x - \sqrt{h} y)\, dy,$$

the mapping M_h: $\mathscr{M} \to \mathscr{M}$ by

$$M_h(A) = \{ x \in \boldsymbol{R}^N : S_h \mathbf{1}_A(x) \geq \theta_h \},$$

where

$$\theta_h = \frac{1}{2} - c\sqrt{h},$$

and, for all $t \geq 0$ and $h > 0$, the mapping $C_t^h : \mathscr{M} \to \mathscr{M}$ by

$$C_t^h = M_h^{j-1} \quad \text{if } (j-1)h \leq t < jh, \quad \text{with } j \in \boldsymbol{N},$$

where as before M_h^k is the k-th iterate of M_h if $k \in \boldsymbol{N}$ and the identity mapping if $k = 0$. This two-parameter family $\{C_t^h\}$ yields an approximation scheme for the motion with normal velocity v.

We have:

Theorem 3.1. *Assume* (3.1)–(3.4) *and* (3.7). *Then for all* $T > 0$ *and* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $h \in (0, \delta)$, $t \in [0, T]$ *and* $A \in \mathscr{M}$, *then*

$$N_t(A_\varepsilon) \subset C_t^h(A) \subset N_t(A^\varepsilon) \quad \text{and} \quad X_t(\overline{A_\varepsilon}) \subset C_t^h(A) \subset X_t(\overline{A^\varepsilon}).$$

Theorem 3.2. *Assume* (3.1)–(3.4) *and* (3.7) *and let* K *be a compact subset of* $\boldsymbol{R}^N \times [0, \infty)$ *and* $\varepsilon > 0$. *Then, for any closed* $A \subset \boldsymbol{R}^N$ *and open* $B \subset \boldsymbol{R}^N$, *there exists* $\delta > 0$ *such that if* $0 < h < \delta$,

$$\left(\bigcup_{t \geq 0} C_t^h(A) \times \{t\} \right) \cap K \subset \bigcup_{t \geq 0} X_t(A) \times \{t\} + B(0, \varepsilon),$$

and

$$\left(\bigcup_{t \geq 0} N_t(B) \times \{t\} \right) \cap K \subset \bigcup_{t \geq 0} C_t^h(B) \times \{t\} + B(0, \varepsilon).$$

We will prove Theorems 3.1 and 3.2 following the same strategy as for Theorems 2.1 and 2.2. To this end, choose h_0 sufficiently small so that

$$1/6 < \theta_h < 5/6 \quad \text{for all } h \in (0, h_0),$$

and, as before, for $h \in (0, h_0)$ and $t \geq 0$, define the operators G_h, Q_t^h: $M(\boldsymbol{R}^N) \to M(\boldsymbol{R}^N)$ by

$$G_h\varphi(x) = \sup\{\lambda \in \boldsymbol{R} : S_h \mathbf{1}_{\{\varphi \geq \lambda\}}(x) \geq \theta_h\},$$

and

$$Q_t^h = G_h^{j-1} \quad \text{if } (j-1)h \leq t < jh, \quad \text{with } j \in \boldsymbol{N}.$$

Next define the function F: $\mathscr{S}^N \times (\boldsymbol{R}^N \backslash \{0\}) \to \boldsymbol{R}$ by

$$\text{(3.8)} \qquad \begin{cases} F(X, p) = -|p| v(-|p|^{-1}(I - \bar{p} \otimes \bar{p}) X (I - \bar{p} \otimes \bar{p}), -\bar{p}) \\ = -\left(\int_{p^\perp} f(x) \, d\mathscr{H}^{N-1}(x) \right)^{-1} \left(\frac{1}{2} \int_{p^\perp} \langle Xx, x \rangle f(x) \, d\mathscr{H}^{N-1} + c|p| \right). \end{cases}$$

It is easy to check that F is degenerate elliptic and geometric, i.e., that it satisfies (1.6) and (1.8). The initial value problem for a function u with level sets moving by normal velocity v is

$$\begin{cases} u_t + F(D^2u, Du) = 0 & \text{in } \boldsymbol{R}^N \times (0, \infty), \\ u = g & \text{on } R^N \times \{0\}. \end{cases} \tag{3.9}$$

It turns out (cf. [**IS**]) that for all $g \in BUC(\boldsymbol{R}^N)$, (3.9) admits a unique viscosity solution $u \in BUC(\boldsymbol{R}^N \times [0, \infty))$.

We have

Theorem 3.3. *Assume* (3.1)–(3.4) *and* (3.7), *fix* $g \in BUC(\boldsymbol{R}^N)$ *and let* $u \in BUC(\boldsymbol{R}^N \times [0, \infty))$ *be the unique viscosity solution of* (3.9) *with F given by* (3.8). *Then, for all* $0 < T < \infty$, *as* $h \to 0$,

$$Q_t^h g(x) \to u(x,t) \quad \textit{uniformly on } \boldsymbol{R}^N \times [0, T].$$

Since Theorems 3.1 and 3.2 follow from Teorem 3.3 exactly as Theorems 2.1 and 2.2 follow from Theorem 2.3, here we only present the proof of Theorem 3.3.

Moreover, since the mapping G_h satisfies (2.8), (2.9), (2.10) and (2.11), as it can be easily checked, the proof of Theorem 3.3 follows, on the basis of Theorem 1.1, exactly the same steps as those in the proof of Theorem 2.3. Below we only state and prove the lemmas which are needed for the proof and refer the reader to the proof of Theorem 2.3 for the rest of the details.

We have

Lemma 3.1. *Let* $\varphi \in C^2(\boldsymbol{R}^N)$, $z \in \boldsymbol{R}^N$ *and* $\varepsilon > 0$, *and assume that* $D\varphi(z) \neq 0$. *There exists* $\delta \in (0, h_0)$ *such that for all* $x \in B(z,\delta)$ *and* $h \in (0, \delta]$,

$$G_h\varphi(x) \le \varphi(x) + (-F(D^2\varphi(z), D\varphi(z)) + \varepsilon)h,$$

and

$$G_h\varphi(x) \ge \varphi(x) + (-F(D^2\varphi(z), D\varphi(z)) - \varepsilon)h.$$

Proof. 1. Since both inequalities are proved similarly, here we present the proof of the first one.

2. Assume, without any loss of generality, that $z = 0$ and fix $a \in \boldsymbol{R}$ such that

$$a > -F(D^2\varphi(0), D\varphi(0)).$$

We need to show that there exists $\delta > 0$ such that for all $x \in B(0,\delta)$ and $h \in (0, \delta]$,

$$S_h \mathbf{1}_{\{\varphi \ge \varphi(x) + ah\}}(x) < \theta_h.$$

3. Fix $\delta_1 > 0$ such that $D\varphi \neq 0$ on $B(0, \delta_1)$, choose a continuous family $\{U(x)\}_{x \in B(0,\delta_1)} \subset O(N)$ such that for all $x \in B(0, \delta_1)$,

$$U(x)\left(\overline{D\varphi(x)}\right) = e_N,$$

where e_N denotes the unit vector in $\boldsymbol{R}^N$ with unity as its N-th component, and note that if $x \in B(0, \delta_1)$, then

$$S_h \mathbf{1}_{\{\varphi \ge \varphi(x) + ah\}}(x) = \int_{\boldsymbol{R}^N} f_{U(x)}(z) \mathbf{1}_{\{\varphi \ge \varphi(x) + ah\}}\left(x - \sqrt{h}U(x)^* z\right) dz.$$

4. The inequality

$$a > -F(D\varphi, D^2\varphi) \quad \text{in } B(0,\delta_1),$$

which is valid if δ_1 is small enough and which we assume hereafter, then reads

$$\frac{1}{2}\int_{\boldsymbol{R}^{N-1}} \langle P^* U(x) D^2\varphi(x) U(x)^* P\xi, \xi\rangle f_{U(x)}(\xi, 0)\, d\xi - a\int_{\boldsymbol{R}^{N-1}} f_{U(x)}(\xi, 0)\, d\xi < -c|D\varphi(x)|,$$

where P denotes the $N \times (N-1)$ matrix, whose (i,j)-th entries are unity if $i = j$ and zero if $i \neq j$.

5. Next choose $\varepsilon > 0$ and $\delta_2 \in (0, \delta_1]$ such that for all $x \in B(0, \delta_2)$,

$$\begin{aligned}\frac{1}{2}\int_{\boldsymbol{R}^{N-1}} &\langle P^* U(0)(D^2\varphi(0) + 3\varepsilon I) U(0)^* P\xi, \xi\rangle f_{U(x)}(\xi, 0)\, d\xi \\ &-(a-\varepsilon)\int_{\boldsymbol{R}^{N-1}} f_{U(x)}(\xi, 0)\, d\xi < -(c + 2\varepsilon)|D\varphi(0)|.\end{aligned} \tag{3.10}$$

6. According to Taylor's theorem, there exists $\gamma > 0$ such that for all $h > 0$, $y \in \boldsymbol{R}^N$ and $x \in B(0, \delta_2)$, if $\sqrt{h}|y| \leq \gamma$, then

$$\begin{aligned}\varphi(x - \sqrt{h}U(x)^* y) &\leq \varphi(x) - \sqrt{h}\langle D\varphi(x), U(x)^* y\rangle + \frac{h}{2}\langle U(x)(D^2\varphi(x) + \varepsilon I) U(x)^* y, y\rangle \\ &\leq \varphi(x) - \sqrt{h}|D\varphi(x)| y_N + Chy_N^2 \\ &\quad + \frac{h}{2}\langle P^* U(x)(D^2\varphi(x) + 2\varepsilon I) U(x)^* P y', y'\rangle,\end{aligned}$$

where $y = (y', y_N) \in \boldsymbol{R}^{N-1} \times \boldsymbol{R}$ and C is some positive constant.

Replacing γ and δ_2 by smaller positive constants if necessary, we deduce that for $y \in B(0, \gamma/\sqrt{h})$ and $x \in B(0, \delta_2)$, if

$$\varphi(x - \sqrt{h}U(x)^* y) \geq \varphi(x) + ah, \tag{3.11}$$

then

$$\begin{aligned}y_N &\leq \frac{\sqrt{h}}{|D\varphi(x)| - C\sqrt{h}y_N}\left(-a + \frac{1}{2}\langle P^* U(x)(D^2\varphi(x) + 2\varepsilon I) U(x) P^* y', y'\rangle\right) \\ &\leq \frac{\sqrt{h}}{|D\varphi(0)|}\left(-a + \varepsilon + \frac{1}{2}\langle P^* U(0)(D^2\varphi(0) + 3\varepsilon I) U(0)^* P y', y'\rangle\right).\end{aligned}$$

Define

$$A_\varepsilon = |\varphi(0)|^{-1} P^* U(0)(D^2\varphi(0) + 3\varepsilon I) U(0)^* P \quad \text{and} \quad a_\varepsilon = (a - \varepsilon)|D\varphi(0)|^{-1}.$$

Then, if (3.11) is satisfied, we have

$$y_N \leq \sqrt{h}\left(-a_\varepsilon + \frac{1}{2}\langle A_\varepsilon y', y'\rangle\right). \tag{3.12}$$

7. Assumption (3.4) yields the existence of a decreasing $\omega \in C([0,\infty),[0,\infty))$ such that $\omega(R) \to 0$ as $R \to \infty$, and

$$\int_{B(0,R)^c} f(y)|y|^2 dy \le \omega(R)^2 \quad \text{for all } R \ge 0.$$

For each $0 < t < 1$, define $R(t) \in (0,\infty)$, by

$$\omega(R(t)) = tR(t)^2, \tag{3.13}$$

and note that the collection $\{R(t)\}_{0<t<1}$ satisfies (3.5). Then choose $\tau \in (0,1)$ such that

$$R(t) \le \gamma/t \quad \text{for all } t \in (0,\tau]. \tag{3.14}$$

8. Write

$$\rho = \sqrt{h}, \quad T(\rho) = B(0,R(\rho)) \times \boldsymbol{R} \subset \boldsymbol{R}^N,$$

and

$$g(\xi) = \left(-a_\varepsilon + \frac{1}{2}\langle A_\varepsilon \xi, \xi\rangle\right) \quad \text{for } \xi \in \boldsymbol{R}^{N-1}.$$

Fix $h \in (0, \min\{h_0, \tau^2\}]$ and $x \in B(0,\delta_2)$ and observe that

$$\begin{aligned} S_h \mathbf{1}_{\{\varphi \ge \varphi(x)+ah\}}(x) &\le \int_{B(0,R(\rho))\cap\{y_N \le \rho g(y')\}} f_{U(x)}(y)\,dy + \int_{B(0,R(\rho))^c} f_{U(x)}(y)\,dy \\ &\le \int_{T(\rho)\cap\{y_N \le \rho g(y')\}} f_{U(x)}(y)\,dy + 2\int_{B(0,R(\rho))^c} f_{U(x)}(y)\,dy, \end{aligned}$$

and

$$\int_{B(0,R(\rho))^c} f_{U(x)}(y)\,dy \le \frac{1}{R(\rho)^2}\int_{B(0,R(\rho))^c} f(y)|y|^2 dy \le \omega(R(\rho))\rho,$$

and also that

$$\frac{1}{2} = \int_{y_N \le 0} f_{U(x)}(y)\,dy \le \int_{T(\rho)\cap\{y_N \le 0\}} f_{U(x)}(y)\,dy + \omega(R(\rho))\rho.$$

9. Next note that

$$\begin{aligned} &\int_{T(\rho)\cap\{y_N \le \rho g(y')\}} f_{U(x)}(y)\,dy - \int_{T(\rho)\cap\{y_N \le 0\}} f_{U(x)}(y)\,dy \\ &= \int_{B(0,R(\rho))} d\xi \int_0^{\rho g(\xi)} f_{U(x)}(\xi, r)\,dr = \int_0^{\rho} dr \int_{B(0,R(\rho))} f_{U(x)}(\xi, rg(\xi))g(\xi)\,d\xi. \end{aligned}$$

It follows from (3.7) that as $\rho \to 0$,

$$\frac{1}{\rho}\left\{\int_{T(\rho)\cap\{y_N \le \rho g(y')\}} f_{U(x)}(y)\,dy - \int_{T(\rho)\cap\{y_N \le 0\}} f_{U(x)}(y)\,dy\right\} \to \int_{\boldsymbol{R}^{N-1}} f_{U(x)}(\xi,0)g(\xi)d\xi,$$

with the convergence uniform in $B(0,\delta_2)$.

Replacing τ by a smaller constant independent of $x \in B(0, \delta_2)$, we may assume that

$$\frac{1}{\rho}\left\{\int_{T(\rho)\cap\{y_N \le \rho g(y')\}} f_{U(x)}(y)\,dy - \int_{T(\rho)\cap\{y_N \le 0\}} f_{U(x)}(y)\,dy\right\} \le \int_{\boldsymbol{R}^{N-1}} f_{U(x)}(\xi, 0) g(\xi)\,d\xi + \varepsilon.$$

Finally, noting that from (3.10),

$$\int_{\boldsymbol{R}^{N-1}} f_{U(x)}(\xi, 0) g(\xi)\,d\xi \le -c - 2\varepsilon,$$

we get

$$S_h \mathbf{1}_{\{\varphi \ge \varphi(x)+ah\}}(x) \le \frac{1}{2} + \rho \int_{\boldsymbol{R}^{N-1}} f_{U(x)}(\xi, 0) g(\xi)\,d\xi + (\varepsilon + 3\omega(R(\rho)))\rho$$

$$\le \frac{1}{2} + \rho(-c - \varepsilon + 3\omega(R(\rho))).$$

Since we may assume that $3\omega(R(\rho)) < \varepsilon$ for all $0 < \rho \le \tau$, we conclude that for some $\delta > 0$ and for all $x \in B(0, \delta)$ and $h \in (0, \delta]$,

$$S_h \mathbf{1}_{\{\varphi \ge \varphi(x)+ah\}}(x) < \theta_h,$$

hence the claim. □

Lemma 3.2. *There exist constants $C > 0$ and $\delta \in (0, h_0)$ such that if $\varphi(x) = \sqrt{|x|^2 + 1}$ (respectively $\varphi(x) = -\sqrt{|x|^2 + 1}$), then for all $x \in \boldsymbol{R}^N$ and $h \in (0, \delta]$,*

$$G_h \varphi(x) \le \varphi(x) + Ch \ (\textit{respectively } G_h \varphi(x) \ge \varphi(x) - Ch).$$

Proof. 1. We only prove the first inequality here, since the proof of the second one is similar.

2. Fix $c_1 > |c|$ and choose, using (3.2) and (3.3), a positive constant C_1 such that for all $U \in O(N)$,

$$\int_{\boldsymbol{R}^{N-1}} f_U(\xi, 0)|\xi|^2 d\xi - \frac{C_1}{2}\int_{\boldsymbol{R}^{N-1}} f_U(\xi, 0)\,d\xi < -c_1. \tag{3.15}$$

3. We first prove, using arguments similar to the ones for Lemma 3.1, that there exists $\delta \in (0, h_0)$ such that for all $h \in (0, \delta]$ and $x \in B(0, \sqrt{h}/\delta)^c$,

$$S_h \mathbf{1}_{\{\varphi \ge \varphi(x)+C_1 h\}}(x) < \theta_h. \tag{3.16}$$

To this end, choose a collection $\{U(p)\}_{p \in S^{N-1}} \subset O(N)$ such that $U(p)p = e_N$, define for $p \in S^{N-1}$, the function $f_p : \boldsymbol{R}^N \to \boldsymbol{R}$, by $f_p(x) = f(U(p)^* x)$, choose $\omega \in C([0, \infty), \boldsymbol{R})$ and define a collection $\{R(\rho)\}_{0<\rho<1} \subset (0, \infty)$ as in the proof of Lemma 3.1.

4. Let $x \in \boldsymbol{R}^N \backslash \{0\}$ and $h \in (0, h_0)$ and observe that if $y \in \boldsymbol{R}^N$ is such that

$$\varphi(x - \sqrt{h} U(\bar{x})^* y) \ge \varphi(x) + C_1 h, \tag{3.17}$$

then

$$|x| y_N \le \sqrt{h}\left(\frac{1}{2}|y|^2 - C_1\sqrt{|x|^2 + 1}\right).$$

If in addition

$$\sqrt{h}\,|y| \le |x|, \tag{3.18}$$

then (3.17) yields

$$y_N \le \left(1 - \frac{\sqrt{h}y_N}{2|x|}\right)^{-1} \frac{\sqrt{h}}{|x|}\left(\frac{1}{2}|y'|^2 - C_1\sqrt{|x|^2+1}\right). \tag{3.19}$$

5. Assume that $|x| \le 1$. If (3.18) holds, then

$$\frac{1}{2} < \left(1 - \frac{\sqrt{h}y_N}{2|x|}\right)^{-1} \le 2,$$

and hence (3.17) and (3.18) imply that

$$y_N \le \frac{\sqrt{h}}{|x|}\left(|y'|^2 - \frac{C_1}{2}\right).$$

Choose $\rho_1 \in (0,1)$ such that

$$\rho R(\rho) \le 1 \quad \text{for all } \rho \in (0, \rho_1],$$

and set $\rho = \sqrt{h}\,|x|^{-1}$ and $g(\xi) = |\xi|^2 - C_1/2$ for $\xi \in \boldsymbol{R}^{N-1}$.

Noting that if $\rho \in (0,\rho_1]$ and $y \in B(0, R(\rho))$, then $\sqrt{h}\,|y|/|x| \le 1$, we find that for $\rho \in (0, \rho_1]$ and $T(\rho) \equiv B(0, R(\rho)) \times \boldsymbol{R} \subset \boldsymbol{R}^N$,

$$\begin{aligned} S_h\mathbf{1}_{\{\varphi \ge \varphi(x)+C_1h\}}(x) &\le \int_{B(0,\,R(\rho))} f_{\bar{x}}(y)\mathbf{1}_{\{\varphi \ge \varphi(x)+C_1h\}}(x - \sqrt{h}U(\bar{x})^*y)\,dy + \rho\omega(R(\rho)) \\ &\le \int_{T(\rho)\cap\{y_N \le \rho g(y')\}} f_{\bar{x}}(y)\,dy + 2\rho\omega(R(\rho)). \end{aligned}$$

Assumption (3.7) and the choice of C_1 yield the existence of a constant $\rho_2 \in (0, \rho_1]$ (see the proof of Lemma 3.1) such that for all $\rho \in (0, \rho_2]$ and $p \in S^{N-1}$,

$$\int_{T(\rho)\cap\{y_N \le \rho g(y')\}} f_p(y)\,dy \le \frac{1}{2} - c_1\rho + \rho\omega(R(\rho)),$$

where we used the fact that

$$\int_{T(\rho)\cap\{y_N \le 0\}} f_p(y)\,dy \le \frac{1}{2} + \rho\omega(R(\rho)).$$

Hence, if $\rho = \sqrt{h}/|x| \in (0, \rho_2]$, then

$$S_h\mathbf{1}_{\{\varphi \ge \varphi(x)+C_1h\}}(x) \le \frac{1}{2} - c_1\rho + 3\rho\omega(R(\rho)).$$

Since $\sqrt{h}/|x| \ge \sqrt{h}$, we can choose $\rho_3 \in (0, \rho_2]$ such that if $0 < \sqrt{h}/|x| \le \rho_3$,

$$S_h\mathbf{1}_{\{\varphi \ge \varphi(x)+C_1h\}}(x) < \frac{1}{2} - |c|h \le \theta_h.$$

6. Now let $|x| \geq 1$. Note that if (3.17) and (3.18) hold, then

$$y_N \leq \left(1 - \frac{\sqrt{h}y_N}{2|x|}\right)^{-1} \frac{\sqrt{h}}{|x|}\left(\frac{1}{2}|y'|^2 - C_1\sqrt{|x|^2+1}\right) \leq \sqrt{h}\left(|y'|^2 - \frac{C_1}{2}\right).$$

Set $\rho = \sqrt{h}$ and observe that if $\rho \in (0, \rho_1]$ and $y \in B(0, R(\rho))$, then by our choice of $\{R(\rho)\}$,

$$|x|^{-1}\sqrt{h}|y| \leq \rho R(\rho) \leq 1.$$

Repeating the computations in Step 5 we conclude that for some $\rho_4 > 0$, if $\sqrt{h} \leq \rho_4$, then

$$S_h \mathbf{1}_{\{\varphi \geq \varphi(x) + C_1 h\}}(x) < \theta_h.$$

Thus, setting $\delta = \min\{\rho_3, \rho_4^2\}$, we have (3.16).

7. Fix $\delta > 0$ and $C_1 > 0$ such that (3.16) holds, and then $x \in \boldsymbol{R}^N$ and $h \in (0, h_0)$ such that $|x| \leq \sqrt{h}/\delta$. We will show that there exists a constant $C_2 > 0$ such that

$$S_h \mathbf{1}_{\{\varphi \geq \varphi(x) + C_2 h\}}(x) < \theta_h.$$

8. Choose $R > 0$ and $C_2 > 0$ such that

$$\int_{B(0,R)^c} f(y)\, dy < \frac{1}{6} \quad \text{and} \quad C_2 \geq R^2 + \frac{2}{\delta}R,$$

and observe that if $y \in \boldsymbol{R}^N$ satisfies

$$\varphi(x - \sqrt{h}y) \geq \varphi(x) + C_2 h,$$

then

$$C_2 h \leq C_2 h\sqrt{|x|^2+1} \leq -2\sqrt{h}\langle x, y\rangle + h|y|^2 \leq \frac{2h}{\delta}|y| + h|y|^2,$$

i.e., $C_2 \leq (2/\delta)|y| + |y|^2$, which implies that $|y| \geq R$.

Therefore, we have

$$S_h \mathbf{1}_{\{\varphi \geq \varphi(x) + C_2 h\}}(x) \leq \int_{B(0,R)^c} f(y)\, dy < \frac{1}{6} < \theta_h.$$

Combining this last inequality with (3.16) concludes the proof of the first inequality. □

We are now in position to prove

Lemma 3.3. *Let $g \in BUC(\boldsymbol{R}^N)$. Then there exist a constant $\delta \in (0, h_0)$ and a continuous function $\omega : [0, \infty) \to [0, \infty)$, with $\omega(0) = 0$, depending only on the modulus of continuity ω_g of g, such that for all $x \in \boldsymbol{R}^N$, $t \geq 0$, and $h \in (0, \delta]$,*

$$|Q_t^h g(x) - g(x)| \leq \omega(t).$$

Proof. The proof is similar to that of Lemma 2.3. The only difference is that we now use Lemma 3.2 in place of Lemma 2.2 and start with the inequality

$$|g(x) - g(y)| \le \varepsilon + C_\varepsilon \varphi(x - y) \quad \text{for all } x, y \in \boldsymbol{R}^N,$$

where $\varepsilon > 0$, $C_\varepsilon > 0$, and $\varphi(x) = \sqrt{|x|^2 + 1} - 1$ and which is valid for all $\varepsilon > 0$ with sufficiently large $C_\varepsilon > 0$. □

Proof of Theorem 3.3. 1. Since the mapping G_h satisfies (2.8), (2.9), (2.10) and (2.11) and since Lemma 3.3 yields (1.16), we may conclude using Lemmas 3.1, 3.2 and 3.3, if we verify (1.13) and (1.14), which, if $D\varphi(x) \ne 0$, are immediate from Lemma 3.1.

2. Next consider (1.13) and (1.14) in the case where $D\varphi(x) = 0$ and assume (see, for example, [**BG**]) that also $D^2\varphi(x) = 0$. As a matter of fact, since the mappings G_h are translation invariant, we may assume that $x = 0$. Moreover, careful inspection of the proof of Proposition 1 of [**BG**] indicates that we may also assume that $\varphi(y) = \alpha|y|^4$ for some $\alpha > 0$.

3. Set $\psi(x) = \sqrt{|x|^2 + 1} - 1$ and note that for all $x \in \boldsymbol{R}^N$, $|x|^4 = ((\psi(x) + 1)^2 - 1)^2$.

4. Lemma 3.2 yields the existence of $\delta > 0$ and $C > 0$ such that for all $x \in B(0, \delta)$ and $0 < h \le \delta$,

$$\begin{aligned} G_h\varphi(x) &= \alpha G_h((\psi + 1)^2 - 1)^2(x) = \alpha((G_h\psi(x) + 1)^2 - 1)^2 \\ &\le \alpha((\psi(x) + Ch + 1)^2 - 1)^2 = \alpha(|x|^2 + Ch)^2 = \varphi(x) + \alpha(2\delta C + C^2h)h. \end{aligned}$$

It follows that

$$\operatorname*{lim^*}_{h\to 0} h^{-1}(G_h\varphi - \varphi)(0) \le 2\alpha\delta C,$$

and letting $\delta \to 0$,

$$\operatorname*{lim^*}_{h\to 0} h^{-1}(G_h\varphi - \varphi)(0) \le 0 = -F_*(0, 0),$$

and hence, (1.13).

5. The fact that

$$\operatorname*{lim_*}_{h\to 0} h^{-1}(G_h\varphi - \varphi)(0) \ge 0 = -F^*(0, 0),$$

follows similarly. □

§4. Mixed schemes.

Here we present a few examples of schemes which can be obtained as a combination of the threshold dynamics discussed in Sections 2 and 3.

To this end, for $i = 1, 2$, let (f_i, θ_i) be a pair of a function and a threshold value, satisfying (2.1) and (2.2) and define $v_i : S^{N-1} \to \boldsymbol{R}$ and for all $h > 0$ the mappings $S_{i,h}$ and $M_{i,h} : \mathscr{M} \to \mathscr{M}$ respectively by

$$\int_{\langle x, p\rangle \ge v_i(p)} f_i(x)\, dx = \theta_i, \quad S_{i,h}\psi(x) = \int_{\boldsymbol{R}^N} f_i(y)\psi(x - hy)\, dy,$$

and

$$M_{i,h}(A) = \{x \in \boldsymbol{R}^N : S_{i,h}\mathbf{1}_A(x) \geq \theta_i\}.$$

The first example is the scheme obtained by interchanging $M_{1,h}$ and $M_{2,h}$ at each interval of length $h > 0$. We thus define $C_t^h : \mathcal{M} \to \mathcal{M}$ by

$$C_t^h = (M_{2,h} \circ M_{1,h})^{j-1} \quad \text{if } (j-1)h \leq t < jh, \quad \text{with } j \in \boldsymbol{N}.$$

Theorem 4.1. *Let $\{X_t\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$ be the generalized evolutions with normal velocity $v_1 + v_2$. For all $T > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < h < \delta$, $0 \leq t \leq T$, and $A \in \mathcal{M}$, then*

$$N_t(A_\varepsilon) \subset C_t^h(A) \subset N_t(A^\varepsilon) \quad \textit{and} \quad X_t(\overline{A_\varepsilon}) \subset C_t^h(A) \subset X_t(\overline{A^\varepsilon}).$$

Assertions analogous to Theorems 2.2 and 2.3 hold for the C_t^h defined above, but we will not discuss them here. Theorem 4.1 is, in principle, a special case of Trotter's product formula in semi-group theory. The proof follows from a straightforward adaptation of the proof in Section 2 and hence we omit it.

Next we consider another example. For $h > 0$ define $M_h : \mathcal{M} \to \mathcal{M}$ by

$$M_h(A) = M_{1,h}(A) \cap M_{2,h}(A),$$

where, for $i = 1, 2$, $M_{i,h}$ are as above. Then define $C_t^h : \mathcal{M} \to \mathcal{M}$ for $t \geq 0$ by $C_t^h = M_h^{j-1}$ if $(j-1)h \leq t < jh$ with $j \in \boldsymbol{N}$.

We have

Theorem 4.2. *Let $\{X_t\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$ be the generalized evolutions with normal velocity* $\min\{v_1, v_2\}$. *For all $T > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < h < \delta$, $0 \leq t \leq T$, and $A \in \mathcal{M}$, then*

$$N_t(A_\varepsilon) \subset C_t^h(A) \subset N_t(A^\varepsilon) \quad \textit{and} \quad X_t(\overline{A_\varepsilon}) \subset C_t^h(A) \subset X_t(\overline{A^\varepsilon}).$$

We remark that assertions analogous to Theorems 2.2 and 2.3 hold for the above C_t^h. We shall not give here the details of the proof of Theorem 4.2 which is again a straightforward adaptation of the arguments in Section 2 and hence we omit it.

If we define

$$M_h(A) = M_{1,h}(A) \cup M_{2,h}(A)$$

and replace the normal velocity $\min\{v_1, v_2\}$ by $\max\{v_1, v_2\}$ in Theorem 4.2, then the resulting assertion is still valid. Moreover, in the above definitions of approximation schemes, if we replace one or both of $M_{i,h}$, $i = 1, 2$, by the operators M_h introduced in Section 3, then the assertions with this replacement togehter with obvious changes of velocity functions still hold valid.

§5. Asymptotics of iterations.

Let f, v, θ, F and for $h > 0$, M_h and G_h be as in Section 2. In the sequel we write M and G for M_1 and G_1 respectively.

Throughout this section assume that

$$v > 0 \quad \text{on } S^{N-1}. \tag{5.1}$$

Our goal here is to show that if a bounded measurable subset A of $\boldsymbol{R}^N$ contains a ball centered at the origin with sufficiently large radius, then the set $M^k(A)$ is asymptotically, as $k \to \infty$, similar to the Wulff crystal $\mathscr{W}$ of "surface energy" v.

Recall that the Wulff crystal $\mathscr{W}$ of "surface energy" v, which is defined by

$$\mathscr{W} = \mathscr{W}_v = \{x \in \boldsymbol{R}^N : \langle x, p\rangle \le v(p) \text{ for all } p \in S^{N-1}\}, \tag{5.2}$$

is a bounded, closed convex subset of $\boldsymbol{R}^N$ having the origin as its interior point.

The concept of the Wulff crystal (shape) is an important one in the study of phase transitions. Equilibrium problems for material that may change phase usually lead to the minimization of functionals involving bulk and surface energies ([**F**], [**FM**], [**T1**]). It is known (see, for example, [**T1**, **2**, **3**], [**F**], and [**FM**] and the references therein) that the Wulff crystal $\mathscr{W}$ is the solution of the Wulff problem, which is about minimizing functionals of form

$$\int_{\partial E} v(n(x))\, d\mathscr{H}^{N-1}(x),$$

among all smooth domains E with a given volume, n being the outward unit normal to its boundary ∂E and v representing the energy density per unit area.

Consider next the function $K : \boldsymbol{R}^N \to \boldsymbol{R}$ given by

$$K(x) = \sup_{p \in S^{N-1}} \{\langle x, p\rangle - v(p)\}. \tag{5.3}$$

It is immediate that K is a convex continuous function and that

$$\mathscr{W} = \{x \in \boldsymbol{R}^N : K(x) \le 0\}, \quad \operatorname{int} \mathscr{W} = \{x \in \boldsymbol{R}^N : K(x) < 0\} \quad \text{and}$$

$$\mathscr{W}^c = \{x \in \boldsymbol{R}^N : K(x) > 0\}.$$

The main result in this section is

Theorem 5.1. *There exists $R > 0$ such that if $A \in \mathscr{M}$ is bounded and contains $B(0, R)$ and if $\varepsilon > 0$, then for a sufficiently large number $J \in \boldsymbol{N}$ and for all $k \in \boldsymbol{N}$ such that $k \ge J$,*

$$\mathscr{W}_\varepsilon \subset k^{-1} M^k(A) \subset \mathscr{W}^\varepsilon.$$

Recall that M denotes M_h with $h = 1$ in the above and in what follows.

Theorem 5.1 is proved as Theorems 2.1 and 3.1 once we establish what plays the role of Theorems 2.3 and 3.3 in this context.

To this end we define for each $k \in \boldsymbol{N}$ the operator $R_k : L^\infty(\boldsymbol{R}^N) \to L^\infty(\boldsymbol{R}^N)$ by

$$R_k \varphi(x) = G^k \varphi(kx).$$

The next theorem corresponds to Theorems 2.3 and 3.3.

Theorem 5.2. *There exists* $R > 0$ *such that if* $A \in \mathcal{M}$ *is bounded and contains* $B(0, R)$, *then as* $k \to \infty$,

$$R_k \mathbf{1}_A(x) \to \begin{cases} 1 & \text{if } x \in \operatorname{int} \mathcal{W}, \\ 0 & \text{if } x \in \mathcal{W}^c, \end{cases}$$

with the convergence uniform outside any neighborhood of $\partial \mathcal{W}$.

In preparation for the proof of Theorem 5.2, we need to consider the stationary first order pde

$$-\langle x, Du \rangle + F(Du) = 0 \quad \text{in } \boldsymbol{R}^N, \tag{5.4}$$

together with the conditions

$$\begin{cases} \text{(i)} & u : \boldsymbol{R}^N \to \{0, 1\}, \\ \text{(ii)} & u_*(0) = 1, \\ \text{(iii)} & u^*(x) = 0 \quad \text{if } |x| \text{ is sufficiently large.} \end{cases} \tag{5.5}$$

Here, as usual, u^* and u_* denote the upper and lower semicontinuous envelopes of u, respectively.

We will prove the following.

Theorem 5.3. *Let* u *be a viscosity supersolution* (*respectively subsolution*) *of* (5.4) *satisfying* (5.5) (i), (ii) (*respectively* (5.5) (i), (iii)). *Then*

$$u = 1 \quad \text{in } \operatorname{int} \mathcal{W} \quad (\text{respectively } u = 0 \quad \text{in } \mathcal{W}^c).$$

Proof. 1. Let K be the function defined by (5.3). Then

$$\begin{cases} \text{(i)} & x \in \operatorname{int} \mathcal{W} & \text{if and only if } K(x) < 0, \\ \text{(ii)} & x \in \partial \mathcal{W} & \text{if and only if } K(x) = 0, \\ \text{(iii)} & x \in \mathcal{W}^c & \text{if and only if } K(x) > 0. \end{cases} \tag{5.6}$$

2. Let u be a supersolution of (5.4), set

$$U = \operatorname{int} \mathcal{W} \quad \text{and} \quad U_0 = U \cap \{u_* = 1\},$$

and observe that $U_0 \neq \emptyset$, U_0 is open, and U is connected.

To conclude it is enough to show that U_0 is closed in U, since then $U = U_0$, hence $u_* = 1$ in U, and therefore $u = 1$ in U.

3. Fix $y \in U \cap \overline{U}_0$ and choose $r > 0$ such that $B(y, 2r) \subset U$. Choose $z \in B(y, r/2) \cap U_0$ and set

$$\varphi(x) = 1 - r^{-1}|x - z|.$$

Note that $B(z, r) \subset U$, $y \in \operatorname{int} B(z, r)$, and

$$\varphi = 0 \leq u_* \quad \text{on } \partial B(z, r) \quad \text{and} \quad \varphi(z) = 1 = u_*(z).$$

Let $\delta = -\max_{B(z,r)} K$. It follows that for all $x \in B(z, r)$ and $p \in S^{N-1}$,

$$\langle x, p \rangle - v(p) \leq -\delta < 0.$$

Furthermore, the definition of F also yields for all $x \in B(z,r)$ and $p \in \boldsymbol{R}^N$,

$$\langle x, p\rangle + F(-p) \leq -\delta|p|.$$

Therefore, if $x \in B(z,r)\backslash\{z\}$, then

$$-\langle x, D\varphi(x)\rangle + F(D\varphi(x)) \leq -\delta|D\varphi(x)| = -r^{-1}\delta.$$

4. If $\min_{B(z,r)}(u_* - \varphi) < 0$, let $\hat{x}$ be a minimum point of $u_* - \varphi$ over $B(z,r)$. Then, $\hat{x} \in \operatorname{int} B(z,r)\backslash\{z\}$ and, since u_* is a viscosity supersolution of (5.4),

$$-\langle \hat{x}, D\varphi(\hat{x})\rangle + F(D\varphi(\hat{x})) \geq 0,$$

which is a contradiction. Hence, $u_* \geq \varphi$ on $B(z,r)$ and, in particular,

$$u_*(x) > 0 \quad \text{if } x \in \operatorname{int} B(z,r).$$

It follows that $z \in U_0$ and therefore U_0 is closed in U.

5. Assume that u is a subsolution of (5.4). It follows from (5.6) (ii) that for each $z \in \partial\mathscr{W}$ there exists $p_z \in S^{N-1}$ such that

$$\langle z, p_z\rangle = v(p_z).$$

Moreover, note also that $\langle x, p_z\rangle \leq v(p_z)$ for all $x \in \mathscr{W}$ and $z \in \partial\mathscr{W}$.

Next for each $z \in \partial\mathscr{W}$ define the half space

$$L_z = \{x \in \boldsymbol{R}^N : \langle x, p_z\rangle \leq v(p_z)\}.$$

The convexity of $\mathscr{W}$ yields that $\mathscr{W} = \bigcap_{z\in\partial\mathscr{W}} L_z$ and hence, $\mathscr{W}^c = \bigcup_{z\in\partial\mathscr{W}} (L_z)^c$.

6. Fix $z \in \partial\mathscr{W}$, choose a $g \in C^1(\boldsymbol{R})$ such that $g = 1$ on $(-\infty, 0]$, $g' < 0$ in $(0,\infty)$ and $g > 0$ in $(0,\infty)$, and define $\varphi \in C^1(\boldsymbol{R}^N)$ by

$$\varphi(x) = g(\langle x, p_z\rangle - v(p_z)).$$

If $x \in (L_z)^c$, then

$$-\langle x, D\varphi(x)\rangle + v(D\varphi(x)) = |g'(\langle x, p_z\rangle - v(p_z))|(\langle x, p_z\rangle - v(p_z)) > 0.$$

7. Fix $R > 0$ such that $u^*(x) = 0$ if $|x| \geq R$, and set $\Omega = (L_z)^c \cap \operatorname{int} B(0,R)$. Since $u^* \leq \varphi$ on $\partial\Omega$, an argument similar to the one in Step 4 yields that $u^* \leq \varphi$ on Ω. Hence, $u = 0$ in Ω. Since R can be taken arbitrarily large, it follows that

$$u(x) = 0 \quad \text{on } (L_z)^c,$$

which implies that $u = 0$ in $\mathscr{W}^c$. □

We need the following lemmas for the proof of Theorems 5.1 and 5.2.

Lemma 5.1. *Let $A \in \mathscr{M}$ be bounded. Then there exists $R > 0$ such that for all $x \in B(0,R)^c$ and $k \in \boldsymbol{N}$,*

$$R_k \mathbf{1}_A(x) = 0.$$

Proof. 1. Let S $(= S_1)$ be the operator defined by

$$S\psi(x) = \int_{\boldsymbol{R}^N} f(y)\psi(x-y)\,dy.$$

Then for all $B \in \mathscr{M}$,

$$G\mathbf{1}_B(x) = 0 \quad \text{if and only if } S\mathbf{1}_B(x) < \theta.$$

2. Choose $L > 0$ so that $A \subset B(0,L)$. Since $\mathbf{1}_A \le \mathbf{1}_{B(0,L)}$ on $\boldsymbol{R}^N$, it follows that for all $k \in \boldsymbol{N}$,

$$R_k\mathbf{1}_A \le R_k\mathbf{1}_{B(0,L)} \quad \text{on } \boldsymbol{R}^N.$$

Select $R \ge L$ so that

$$\int_{B(0,R)^c} f(y)\,dy < \theta.$$

Then for all $x \in B(0, L+R)^c$,

$$S\mathbf{1}_{B(0,L)}(x) = \int_{\boldsymbol{R}^N} f(y)\mathbf{1}_{B(0,L)}(x-y)\,dy \le \int_{B(0,R)^c} f(y)\,dy < \theta,$$

and hence

$$G\mathbf{1}_{B(0,L)} \le \mathbf{1}_{B(0,L+R)} \quad \text{on } \boldsymbol{R}^N.$$

3. A simple induction argument yields that for all $k \in \boldsymbol{N}$,

$$G^k\mathbf{1}_{B(0,L)} \le \mathbf{1}_{B(0,L+kR)} \quad \text{on } \boldsymbol{R}^N,$$

hence

$$R_k\mathbf{1}_{B(0,L)}(x) \le \mathbf{1}_{B(0,L+kR)}(kx) \le \mathbf{1}_{B(0,2R)}(x) \quad \text{on } \boldsymbol{R}^N$$

and therefore, $R_k\mathbf{1}_A = 0$ on $B(0,2R)^c$. □

Lemma 5.2. *There exist $R > 0$ and $\varepsilon > 0$ such that if $A \in \mathscr{M}$ contains the ball $B(0,R)$, then for all $k \in \boldsymbol{N}$,*

$$R_k\mathbf{1}_A = 1 \quad \text{in } B(0,\varepsilon).$$

Proof. 1. Let S be the operator defined in the proof of Lemma 5.1 and fix $\gamma > 0$ such that $\gamma < \min_{|p|=1} v(p)$.

It follows that

$$\min_{|p|=1} \int_{\langle p,y\rangle \ge \gamma} f(y)\,dy > \theta.$$

Moreover, set

$$\delta = \min_{|p|=1} \int_{\langle p,y\rangle \ge \gamma} f(y)\,dy - \theta \ (> 0),$$

choose $R > 0$ so that

$$\int_{B(0,R)^c} f(y)\,dy < \delta,$$

and finally set $\varepsilon = \gamma/3$ and $L = \max\{2R + \varepsilon, \varepsilon^{-1}R^2\}$.

2. If $J \geq L$, then

$$\boldsymbol{S}\mathbf{1}_{B(0,J)} \geq \theta \quad \text{on } B(0, J+\varepsilon). \tag{5.7}$$

Indeed, fix $J \geq L$ and observe that if $x \in B(0,(J+\varepsilon)/2)$ and $y \in B(0,R)$, then

$$|x - y| \leq \frac{J+\varepsilon}{2} + R \leq J,$$

and if $x \in B(0, J+\varepsilon)\backslash B(0,(J+\varepsilon)/2)$, $y \in B(0,R)$, and $\langle \bar{x}, y\rangle \geq \gamma$, then

$$|x-y|^2 = |x|^2 + |y|^2 - 2\langle x, y\rangle \leq (J+\varepsilon)^2 + R^2 - (J+\varepsilon)\gamma \leq J^2 - \varepsilon J + R^2 \leq J^2.$$

Therefore, we have

$$\boldsymbol{S}\mathbf{1}_{B(0,J)} \geq \int_{B(0,R)} f(y)\,dy > 1 - \delta > \theta \quad \text{on } B\left(0, \frac{J+\varepsilon}{2}\right),$$

and for all $x \in B(0, J+\varepsilon)\backslash B(0,(J+\varepsilon)/2)$,

$$\boldsymbol{S}\mathbf{1}_{B(0,J)}(x) \geq \int_{B(0,R)\cap\{\langle \bar{x},y\rangle \geq \gamma\}} f(y)\,dy > \int_{\{\langle \bar{x},y\rangle\geq\gamma\}} f(y)\,dy - \delta \geq \theta.$$

3. It follows from (5.7) that if $J \geq L$, then

$$\boldsymbol{G}\mathbf{1}_{B(0,J)} = 1 \quad \text{on } B(0, J+\varepsilon),$$

and hence

$$\boldsymbol{G}\mathbf{1}_{B(0,J)} \geq \mathbf{1}_{B(0,J+\varepsilon)} \quad \text{on } \boldsymbol{R}^N.$$

A simple induction yields that for all $k \in \boldsymbol{N}$,

$$\boldsymbol{G}^k\mathbf{1}_{B(0,L)} \geq \mathbf{1}_{B(0,L+k\varepsilon)} \quad \text{on } \boldsymbol{R}^N,$$

and hence for all $k \in \boldsymbol{N}$ and $x \in \boldsymbol{R}^N$,

$$\boldsymbol{R}_k\mathbf{1}_{B(0,L)}(x) \geq \mathbf{1}_{B(0,L+k\varepsilon)}(kx) \geq \mathbf{1}_{B(0,\varepsilon)}(x).$$

Finally, if $B(0,L) \subset A$, then $\boldsymbol{R}_k\mathbf{1}_A \geq \boldsymbol{R}_k\mathbf{1}_{B(0,L)} \geq \mathbf{1}_{B(0,\varepsilon)}$ on $\boldsymbol{R}^N$ and hence for all $k \in N$,

$$\boldsymbol{R}_k\mathbf{1}_A = 1 \quad \text{on } B(0,\varepsilon). \qquad \square$$

Lemma 5.3. *Let $A \in \mathscr{M}$. Then $u^+ = \limsup^*_{k\to\infty} \boldsymbol{R}_k\mathbf{1}_A$ (respectively $u^- = \liminf_{*\,k\to\infty} \boldsymbol{R}_k\mathbf{1}_A$) is a viscosity subsolution (respectively a viscosity supersolution) of* (5.4).

Proof. 1. We only show that u^+ is a viscosity subsolution of (5.4) and leave it to the reader to check that u^- is a viscosity supersolution of (5.4).

2. Fix $\varphi \in C^1(\boldsymbol{R}^N)$ and let $\hat{x}$ be a strict maximum point of $u^+ - \varphi$. Without loss of generality we may assume that $D\varphi(\hat{x}) \neq 0$, since otherwise there is nothing to prove, and

that $(u^+ - \varphi)(\hat{x}) = 0$ and $\lim_{|x|\to\infty}\varphi(x) = \infty$. Then,

$$u^+ \le \varphi \quad \text{on } \boldsymbol{R}^N \quad \text{and} \quad \lim_{|x|\to\infty}(u^+ - \varphi)(x) = -\infty.$$

3. Fix any $a > -F(D\varphi(\hat{x}))$ and choose $\delta > 0$ according to Lemma 2.1 such that

$$G_h\varphi(x) \le \varphi(x) + ah \quad \text{for all } x \in B(\hat{x},\delta) \quad \text{and} \quad h \in (0,\delta]. \tag{5.8}$$

Fix any $\varepsilon \in (0,\delta/2]$, set

$$\gamma = -\sup_{B(\hat{x},\varepsilon)^c}(u^+ - \varphi)\ (>0), \quad L_k = \sup_{\boldsymbol{R}^N}(R_k\mathbf{1}_A - \varphi),$$

and

$$I = \left\{k \in \boldsymbol{N} : k \ge 2, L_k + \frac{\varepsilon}{k} \ge L_{k-1}, L_k > -\frac{\gamma}{2}\right\}.$$

It is easily seen that $\# I = \infty$, where $\# I$ denotes the number of elements of I, and that there exists $k_0 \in \boldsymbol{N}$ such that for all $k \ge k_0$,

$$\sup_{B(\hat{x},\varepsilon)^c}(R_k\mathbf{1}_A - \varphi) \le -\frac{\gamma}{2}.$$

4. Fix any $k \in I$ such that $k \ge k_0$. Then,

$$L_k + \frac{\varepsilon}{k} \ge L_{k-1}, \quad L_k > -\frac{\gamma}{2}, \quad \text{and} \quad \sup_{B(\hat{x},\varepsilon)^c}(R_k\mathbf{1}_A - \varphi) \le -\frac{\gamma}{2}.$$

Choose also $\tilde{x} \in B(\hat{x},\varepsilon)$ such that

$$(R_k\mathbf{1}_A - \varphi)(\tilde{x}) > -\frac{\gamma}{2} \quad \text{and} \quad (R_k\mathbf{1}_A - \varphi)(\tilde{x}) \ge L_k - \varepsilon k^{-1},$$

and observe that we may assume by choosing k sufficiently large that

$$(k-1)^{-1} \le \delta \quad \text{and} \quad (k-1)^{-1}k\tilde{x} \in B(\hat{x},\delta).$$

5. Using (5.8) and the fact that

$$L_{k-1} \ge (R_{k-1}g - \varphi) \quad \text{on } \boldsymbol{R}^N,$$

we obtain

$$\begin{aligned}
R_k\mathbf{1}_A(\tilde{x}) &= G_{(k-1)^{-1}} \circ R_{k-1}\mathbf{1}_A((k-1)^{-1}k\tilde{x}) \\
&\le G_{(k-1)^{-1}}\varphi((k-1)^{-1}k\tilde{x}) + L_{k-1} \le \varphi((k-1)^{-1}k\tilde{x}) + (k-1)^{-1}a + L_k + k^{-1}\varepsilon,
\end{aligned}$$

and hence

$$\begin{aligned}
L_k - \frac{\varepsilon}{k} &\le (R_k\mathbf{1}_A - \varphi)(\tilde{x}) \le \varphi((k-1)^{-1}k\tilde{x}) - \varphi(\tilde{x}) + (k-1)^{-1}a + L_k + k^{-1}\varepsilon \\
&\le (k-1)^{-1}\langle\tilde{x}, D\varphi(\tilde{x})\rangle + (k-1)^{-1}a + L_k + k^{-1}\varepsilon + o((k-1)^{-1}),
\end{aligned}$$

where $o(r)$ is a function satisfying $\lim_{r\to 0} o(r)/r = 0$.

Sending $k \to \infty$ along a sequence in I, $\varepsilon \to 0$ and $a \to -F(D\varphi(\hat{x}))$, we conclude that

$$-\langle \hat{x}, D\varphi(\hat{x})\rangle + F(D\varphi(\hat{x})) \le 0. \qquad \square$$

Proof of Theorems 5.1 and 5.2. Theorem 5.2 is an immediate consequence of Lemmas 5.1, 5.2, and 5.3 and Theorem 5.3 and then Theorem 5.1 follows immediately from Theorem 5.2. $\square$

§6. Large times asymptotics.

Consider $v_1 \in C(\mathcal{S}^N \times S^{N-1})$ and $v_2 \in C(S^{N-1})$ and assume that v_1 is monotone, i.e., that it satisfies (1.3), so that the function $v \in C(\mathcal{S}^N \times S^{N-1})$ given by

$$v(X, p) = v_1(X, p) + v_2(p)$$

is also monotone.

In this section we are concerned with the asymptotics, in the limit $t \to \infty$, of the motion of hypersurfaces or sets with normal velocity v. We show that, under appropriate hypotheses on the v_i's, if the initial set is bounded and large enough, the corresponding generalized front propagation is asymptotically similar to the Wulff crystal of the energy function v_2.

Our precise assumptions on v_1 and v_2 are:

$$(6.1) \qquad v_2 > 0 \quad \text{on } S^{N-1},$$

and

$$(6.2) \qquad v_1(\lambda X, p) = \lambda v_1(X, p) \quad \text{for all } \lambda > 0 \quad \text{and} \quad (X, p) \in \mathcal{S}^N \times S^{N-1}.$$

Let $\mathscr{W}$ denote the Wulff crystal of the energy v_2, i.e.,

$$\mathscr{W} = \{x \in \boldsymbol{R}^N : \langle x, p\rangle \le v_2(p) \quad \text{for all } p \in S^{N-1}\}.$$

As noted in Section 5, $\mathscr{W}$ is a bounded, closed convex subset of $\boldsymbol{R}^N$ with the origin as its interior point.

The main result in this section is:

Theorem 6.1. (i) *There exists $R > 0$ such that if $\varepsilon > 0$ and $A \subset \boldsymbol{R}^N$ is bounded and contains $B(0, R)$, then for some $T > 0$ and for all $t \ge T$,*

$$\mathscr{W}_\varepsilon \subset t^{-1} N_t(\operatorname{int} A) \quad \textit{and} \quad t^{-1} X_t(\bar{A}) \subset \mathscr{W}^\varepsilon.$$

(ii) *As $t \to \infty$, $t^{-1}(X_t(\bar{A}) \backslash N_t(\operatorname{int} A)) \to \partial W$ in the Hausdorff metric.*

The initial value problem in the level-set approach corresponding to the motion with normal velocity v is given by

$$(6.3) \qquad \begin{cases} \text{(i)} \quad u_t + F_1(D^2u, Du) + F_2(Du) = 0 & \text{in } \boldsymbol{R}^N \times (0, \infty), \\ \text{(ii)} \quad u = g & \text{on } \boldsymbol{R}^N \times \{0\}, \end{cases}$$

where

$$F_1(X, p) = -|p| v_1(-|p|^{-1}(I - \bar{p} \otimes \bar{p}) X (I - \bar{p} \otimes \bar{p}), -\bar{p}),$$

and

$$F_2(p) = -|p|v_2(-\bar{p}).$$

Below we write

$$F(X,p) = F_1(X,p) + F_2(p),$$

and note that, in view of (6.2), for all $(X,p) \in \mathscr{S}^N \times (\boldsymbol{R}^N \backslash \{0\})$,

$$F_1(X,p) = -v_1(-(I-\bar{p}\otimes\bar{p})X(I-\bar{p}\otimes\bar{p}),-\bar{p}).$$

We continue with a number of observations and lemmas which set the ground for the proof of Theorem 6.1. To this end, note that for each $R > 0$,

$$F_1 \text{ is bounded on } \{X \in \mathscr{S}^N : \|X\| \le R\} \times (\boldsymbol{R}^N \backslash \{0\}), \tag{6.4}$$

and

$$F_1(\lambda^2 X, \lambda p) = \lambda^2 F_1(X,p) \quad \text{for all } \lambda > 0 \quad \text{and} \quad (p,X) \in \mathscr{S}^N \times (\boldsymbol{R}^N \backslash \{0\}). \tag{6.5}$$

Finally, if $u : \boldsymbol{R}^N \times (0,\infty) \to \boldsymbol{R}$, we define

$$\bar{u}(x) = \limsup_{\varepsilon\to 0}\{u(sy,s) : s > \varepsilon^{-1}, y \in B(x,\varepsilon)\}$$

and

$$\underline{u}(x) = \liminf_{\varepsilon\to 0}\{u(sy,s) : s > \varepsilon^{-1}, y \in B(x,\varepsilon)\}.$$

We have

Lemma 6.1. *Let u be a bounded viscosity subsolution* (*respectively supersolution*) *of* (6.3) (i). *Then $\bar{u}$* (*respectively $\underline{u}$*) *is a viscosity subsolution* (*resp. supersolution*) *of*

$$-\langle x, Du\rangle + F_2(Du) = 0 \quad \text{in } \boldsymbol{R}^N.$$

Proof. 1. We only consider here the case where u is a subsolution, the other claim being proved similarly.

2. Set

$$w(x,t) = u(tx,t),$$

and observe that at least formally,

$$\begin{aligned} 0 &\ge u_t(tx,t) + F(D^2u(tx,t), Du(tx,t)) \\ &= w_t(x,t) - t^{-1}\langle x, Dw(x,t)\rangle + F(t^{-2}D^2w(x,t), t^{-1}Dw(x,t)) \\ &= w_t + t^{-1}(-\langle x, Dw\rangle + F_2(Dw)) + t^{-2}F_1(D^2w, Dw). \end{aligned}$$

Indeed, it is easily justified that

$$tw_t - \langle x, Dw\rangle + F_2(Dw) + t^{-1}F_1(D^2w, Dw) \le 0 \quad \text{in } \boldsymbol{R}^N \times (0,\infty) \tag{6.6}$$

holds in the viscosity sense.

3. For $n \in \boldsymbol{N}$ and $t \geq n$, set $g_n(t) = e^{-(1/n)(t-n)}$. Then for all $t \geq n$,

$$|tg_n'(t)| \leq 1. \tag{6.7}$$

4. Now fix $\varphi \in C^2(\boldsymbol{R}^N)$ and assume that $\bar{u} - \varphi$ has a strict maximum at $\hat{x} \in \boldsymbol{R}^N$. Since we may assume that

$$\lim_{|x| \to \infty} \varphi(x) = \infty \quad \text{and} \quad \bar{u}(\hat{x}) = \varphi(\hat{x}),$$

it follows that

$$\bar{u} \leq \varphi \quad \text{on } \boldsymbol{R}^N \quad \text{and} \quad \lim_{|x| \to \infty} (\bar{u} - \varphi) = -\infty.$$

5. Set

$$\varepsilon_n = \sup_{x \in \boldsymbol{R}^N, t \geq n} (w^*(x,t) - \varphi(x)),$$

and observe that since $\bar{u}(\hat{x}) = \varphi(\hat{x})$, $\varepsilon_n \to 0$ as $n \to \infty$.

Set $\delta_n = \varepsilon_n + 1/n$. It is then clear that for $x \in \boldsymbol{R}^N$ and $t = n$,

$$w^*(x,t) - \varphi(x) - \delta_n e^{-(1/n)(t-n)} \leq -\frac{1}{n}.$$

Since

$$\lim_{n \to \infty} \sup_{x \in \boldsymbol{R}^N, t \geq n} (w^*(x,t) - \varphi(x) - \delta_n e^{-(1/n)(t-n)}) = 0,$$

there exists $\alpha_n > 0$ such that

$$\sup_{x \in \boldsymbol{R}^N, t \geq n} (w^*(x,t) - \varphi(x) - \delta_n e^{-(1/n)(t-n)} - \alpha_n t) \geq -\frac{1}{2n}.$$

Note also that

$$\lim_{|x|+t \to \infty} (w^*(x,t) - \varphi(x) - \delta_n e^{-(1/n)(t-n)} - \alpha_n t) = -\infty,$$

and that if $t = n$,

$$w^*(x,t) - \varphi(x) - \delta_n e^{-(1/n)(t-n)} - \alpha_n t < -\frac{1}{n}.$$

It follows from above that the function

$$w^*(x,t) - \varphi(x) - \delta_n e^{-(1/n)(t-n)} - \alpha_n t$$

achieves a maximum over $\boldsymbol{R}^N \times [n, \infty)$ at some $(x_n, t_n) \in \boldsymbol{R}^N \times (n, \infty)$.

Since w^* is a subsolution of (6.6), it follows that there exists a constant $C > 0$ such that

$$\begin{aligned} 0 \geq{} & t_n\left(-\frac{\delta_n}{n} e^{-(1/n)(t_n - n)} + \alpha_n\right) - \langle x_n, D\varphi(x_n)\rangle + F_2(D\varphi(x_n)) \\ & + t_n^{-1} F_1(D^2\varphi(x_n), D\varphi(x_n)) \\ \geq{} & -\delta_n - \langle x_n, D\varphi(x_n)\rangle + F_1(D\varphi(x_n)) - t_n^{-1} C. \end{aligned} \tag{6.8}$$

The choice of α_n yields

$$(6.9)\qquad -\frac{1}{2n} \le w^*(x_n, t_n) - \varphi(x_n) - \delta_n e^{-(1/n)(t_n - n)} - \alpha_n t_n \le \delta_n - \alpha_n t_n.$$

Hence, $\alpha_n t_n \le \delta_n + 1/2n$ and so, $\lim_{n\to\infty} \alpha_n t_n = 0$. Since the set $\{x_n\}$ is clearly bounded, we may assume that $x_n \to \hat{y}$ as $n \to \infty$ for some $\hat{y} \in \boldsymbol{R}^N$. Then we find from (6.9) that $0 \le \bar{u}(\hat{y}) - \varphi(\hat{y})$.

Since $\hat{x}$ is a strict maximum point of $\bar{u} - \varphi$, it follows that $\hat{y} = \hat{x}$. Letting $n \to \infty$ in (6.8), we conclude

$$-\langle \hat{x}, D\varphi(\hat{x})\rangle + F_1(D\varphi(\hat{x})) \le 0. \qquad \square$$

Lemma 6.2. *Let u be an upper semicontinuous viscosity subsolution of* (6.3) (i) *satisfying for some R_0,*

$$(6.10)\qquad 0 \le u \le 1 \quad \text{on } \boldsymbol{R}^N \times [0,\infty) \quad \text{and} \quad u = 0 \quad \text{on } B(0, R_0)^c \times \{0\}.$$

Then there exists $R > 0$ such that for all $(x, t) \in B(0, R)^c \times [1, \infty)$,

$$u(tx, t) = 0.$$

Proof. 1. Let $\eta \in C^\infty(\boldsymbol{R})$ be such that $\eta = 1$ on $(-\infty, 0]$, $\eta' \le 0$ on $\boldsymbol{R}$, and $\eta = 0$ on $[1,\infty)$ and set

$$C_1 = \max_{p\in S^{N-1}} |F_1(-I, p)| \quad \text{and} \quad C_2 = \max_{p\in S^{N-1}} |F_2(p)|.$$

Choose $R \ge R_0$ so that

$$R \ge \frac{C_1}{R_0} + C_2$$

and define $w \in C^\infty(\boldsymbol{R}^N \times [0,\infty))$ by

$$w(x,t) = \eta(|x| - Rt - R_0).$$

2. If $x \ne 0$, we have, with η' denoting the value of η' at $|x| - Rt - R_0$,

$$\begin{aligned} w_t(x,t) + F(D^2 w(x,t), Dw(x,t)) &= -\eta' R + F(\eta'|x|^{-1}(I - \bar{x}\otimes\bar{x}) + \eta''\bar{x}\otimes\bar{x}, \eta'\bar{x}) \\ &= -\eta'(R + |x|^{-1}F_1(-I, -\bar{x}) + F_2(-\bar{x})) \\ &\ge -\eta'(R - C_1 R_0^{-1} - C_2) \ge 0. \end{aligned}$$

It is also obvious that

$$w_t(0,t) + F(D^2 w(0,t), Dw(0,t)) \ge 0 \quad \text{for } t \ge 0.$$

3. Since $w \ge u$ on $\boldsymbol{R}^N \times \{0\}$, the standard comparison results (see, e.g., [**IS**]) yield that $w \ge u$ on $\boldsymbol{R}^N \times [0,\infty)$, which shows that

$$u(x,t) = 0 \quad \text{for } (x,t) \in \boldsymbol{R}^N \times [0,\infty) \quad \text{if } |x| \ge Rt + R_0 + 1.$$

Therefore, noting that if $(x,t) \in B(0,2(R+1))^c \times [1,\infty)$, then

$$2|tx| \geq |x|t + |x| \geq 2(Rt + R_0 + 1),$$

we obtain

$$u(tx,t) = 0 \quad \text{for all } (x,t) \in B(0,2(R+1))^c \times [1,\infty). \qquad \square$$

Lemma 6.3. *There exist constants $R > 0$ and $\varepsilon > 0$ such that if u is a lower semicontinuous viscosity supersolution of* (6.3) (i) *such that*

$$0 \leq u \leq 1 \quad \text{on } \boldsymbol{R}^N \times [0,\infty) \quad \text{and} \quad u = 1 \quad \text{on } B(0,R) \times \{0\}, \tag{6.11}$$

then

$$u(tx,t) = 1 \quad \textit{for all } (x,t) \in B(0,\varepsilon) \times [0,\infty).$$

Proof. 1. Fix $g \in C^\infty(\boldsymbol{R})$ as in the proof of Lemma 6.2 and set

$$C = \max_{p \in S^{N-1}} |F_1(-I,p)|, \quad \gamma = -\max_{S^{N-1}} F_2(\equiv \min_{S^{N-1}} v_2), \quad \text{and} \quad \varepsilon = \gamma/2.$$

Choose $R > 0$ so that $C/R \leq \gamma/2$ and define $w \in C^\infty(\boldsymbol{R}^N \times [0,\infty))$ by

$$w(x,t) = g(|x| - \varepsilon t - R),$$

where η is a function as in the previous proof.

2. It follows that if η' denotes the value $\eta'(|x| - \varepsilon t - R)$,

$$\begin{aligned} w_t(x,t) + F(D^2 w(x,t), Dw(x,t)) &\leq -\eta'(\varepsilon + |x|^{-1} F_1(-I,-\bar{x}) + F_2(-\bar{x})) \\ &\leq -\eta'(\varepsilon + CR^{-1} - \gamma) \leq 0. \end{aligned}$$

3. By comparison we conclude that for all $(x,t) \in \boldsymbol{R}^N \times [0,\infty)$, with $|x| \leq \varepsilon t + R$,

$$u(x,t) \geq \eta(|x| - \varepsilon t - R) = 1.$$

Hence,

$$u(tx,t) = 1 \quad \text{for all } (x,t) \in B(0,\varepsilon) \times [0,\infty). \qquad \square$$

We may now present the

Proof of Theorem 6.1. 1. Fix $R > 0$ to be a constant as in Lemma 6.3, fix a bounded $A \subset \boldsymbol{R}^N$ so that $B(0,R) \subset \operatorname{int} A$ and set

$$\Omega = \bigcup_{t \geq 0} N_t(\operatorname{int} A) \times \{t\} \quad \text{and} \quad u = \mathbf{1}_\Omega.$$

2. It is well-known (see, e.g., [**BSS**]) that u is a lower semicontinuous viscosity supersolution of (6.3) (i) satisfying (6.11). Lemma 6.3 yields the existence of $\delta > 0$ such that

$$\underline{u} = 1 \quad \text{in} \quad \operatorname{int} B(0,\delta).$$

Then, using Lemma 6.1 and Theorem 5.3, we conclude that

$$\underline{u} = 1 \quad \text{in} \quad \operatorname{int} \mathscr{W}.$$

3. Noting that for all $(x,t) \in \boldsymbol{R}^N \times (0,\infty)$,

$$u(tx,t) = \mathbf{1}_{t^{-1}N_t(\operatorname{int} A)}(x),$$

we see from the above that for each $\varepsilon > 0$ there exists $T > 0$ such that for all $t \geq T$,

$$\mathscr{W}_\varepsilon \subset t^{-1}N_t(\operatorname{int} A).$$

4. Next set

$$\Sigma = \bigcup_{t\geq 0} X_t(\bar{A}) \times \{t\} \quad \text{and} \quad u = \mathbf{1}_\Sigma.$$

Then u is an upper semicontinuous viscosity subsolution of (6.3) (i). It follows from Lemma 6.2 that for some $L > 0$,

$$\bar{u} = 0 \quad \text{in } B(0,L)^c.$$

5. We then conclude, using Lemma 6.1 and Theorem 5.3, that $\bar{u} = 0$ in $\mathscr{W}^c$, and furthermore that for each $\varepsilon > 0$ there exists $T > 0$ such that for all $t \geq T$,

$$t^{-1}X_t(\bar{A}) \subset \mathscr{W}^\varepsilon. \qquad \square$$

§7. Asymptotics of threshold dynamics on scaled lattices.

In this section we study the asymptotics of iterations of threshold dynamics on lattices. The threshold dynamics considered are of the type discussed in Section 2 and the result to be established is similar to the one in Section 5. The situation here is, however, a bit restrictive compared to the one in Section 5. Indeed the functions (denoted below by $\mathbf{1}_{\mathscr{N}}$) here corresponding to f in Section 5 are characteristic functions of sets. This restriction is made just to simplify the presentation.

To this end consider sequences $\{\eta_k\}_{k\in\boldsymbol{N}} \subset (0,\infty)$, $\{\mathscr{N}_k\}_{k\in\boldsymbol{N}}$, $\{A_k\}_{k\in\boldsymbol{N}} \subset \boldsymbol{Z}^N$ and $\{\theta_k\}_{k\in\boldsymbol{N}} \subset (0,1)$, a threshold parameter $\theta \in (0,1)$ and a set $\mathscr{N} \in \mathscr{M}$.

If Q denotes the unit cube $[-1/2,1/2]^N \subset \boldsymbol{R}^N$, throughout this section we assume:

$$\eta_k \to 0 \quad \text{as } k \to \infty, \tag{7.1}$$

(7.2) $\#\mathscr{N}_k < \infty$ for all $k \in \boldsymbol{N}$, $|\mathscr{N}| < \infty$, and as $k \to \infty$, $\mathbf{1}_{\eta_k(\mathscr{N}_k+Q)} \to \mathbf{1}_{\mathscr{N}}$ in $L^1(\boldsymbol{R}^N)$,

$$\begin{cases} \text{there exist } \delta_0 > 0 \text{ and } R_0 > 0 \text{ such that} \\ B(0,\delta_0) \cap \eta_k \boldsymbol{Z}^N \subset \eta_k A_k \subset B(0,R_0) \quad \text{for all } k \in \boldsymbol{N}, \end{cases} \tag{7.3}$$

$$\theta_k \to \theta \quad \text{as } k \to \infty, \tag{7.4}$$

$$\begin{cases} \text{for all } p \in S^{N-1} \text{ there exists a unique } v(p) \in \boldsymbol{R} \text{ such that} \\ |\{x \in \boldsymbol{R}^N : \langle x,p\rangle \geq v(p)\} \cap \mathscr{N}| = \theta|\mathscr{N}|, \end{cases} \tag{7.5}$$

and finally,

$$v > 0 \quad \text{on } S^{N-1}. \tag{7.6}$$

Note that the condition (7.5) is exactly the same as (2.2) with f replaced by $|\mathcal{N}|^{-1}\mathbf{1}_{\mathcal{N}}$.

For each $k \in \boldsymbol{N}$ define the mapping M_k of the set of all subsets of $\boldsymbol{Z}^N$ into itself by

$$M_k(A) = \{x \in \boldsymbol{Z}^N : \#((x - \mathcal{N}_k) \cap A) \geq \theta_k \# \mathcal{N}_k\}.$$

Our result here is

Theorem 7.1. *There exists $R > 0$ such that if $\delta_0 \geq R$, then for all $\varepsilon > 0$ there exists $J \in \boldsymbol{N}$ such that for all $n \geq J$ and $k \geq J$,*

$$\mathcal{W}_\varepsilon \cap \frac{\eta_k}{n}\boldsymbol{Z}^N \subset \frac{\eta_k}{n} M_k^n(A_k) \subset \mathcal{W}^\varepsilon,$$

where M_k^n denotes the n-th iterate of M_k and $\mathcal{W}$ is the Wulff crystal defined by (5.2).

The underlying idea here is the following. The threshold dynamics we are considering, which are parametrized by k, evolve subsets of the scaled lattice $\eta_k \boldsymbol{Z}^N$. In this space the initial set $\eta_k A_k$ evolves by the iteration of the mapping

$$A \mapsto \{x \in \eta_k \boldsymbol{Z}^N : \#((x - \eta_k \mathcal{N}_k) \cap A) \geq \theta_k \#(\eta_k \mathcal{N}_k)\}.$$

The resulting set, after n iterations, is exactly the set $\eta_k M_k^n(A_k)$. The asymptotic shape of the evolving set $\eta_k M_k^n(A_k)$ is roughly similar to the Wulff crystal as $n \to \infty$, and the larger k is, the more the shape is similar to the Wulff crystal.

Since the proof of Theorem 7.1 is similar to that of Theorem 5.1, here we only present the outline of the proof.

For each $k \in \boldsymbol{N}$ and $h > 0$ define the operator $G_{k,h}$ on $M(\boldsymbol{R}^N)$ by

$$G_{k,h}\varphi(x) = \sup\{\lambda \in \boldsymbol{R} : \#[(x - h\eta_k \mathcal{N}_k) \cap \{\varphi \geq \lambda\}] \geq \theta_k \# \mathcal{N}_k\},$$

and the function $F \in C(\boldsymbol{R}^N)$ by

$$F(p) = \begin{cases} -|p| v(-\bar{p}) & \text{if } p \neq 0 \\ 0 & \text{if } p = 0. \end{cases}$$

Lemma 7.1. *Let $\varphi \in C^1(\boldsymbol{R}^N)$ and $z \in \boldsymbol{R}^N$. Then for all $\varepsilon > 0$ there exist $J \in \boldsymbol{N}$ and $\delta > 0$ such that for all $k \geq J$, $0 < h \leq \delta$ and $x \in B(z, \delta)$,*

$$G_{k,h}\varphi(x) \leq \varphi(x) + (-F(D\varphi(z)) + \varepsilon)h$$

and

$$G_{k,h}\varphi(x) \geq \varphi(x) + (-F(D\varphi(z)) - \varepsilon)h.$$

Proof. 1. We only present the proof of the first inequality.

2. It is enough to show that for any $a \in \boldsymbol{R}$ such that $a > -F(D\varphi(z))$, there exist $J \in \boldsymbol{N}$ and $\delta > 0$ such that if $k \geq J$, $h \in (0, \delta]$ and $x \in B(z, \delta)$, then

$$\#[(x - h\eta_k \mathcal{N}_k) \cap \{\varphi \geq \varphi(x) + ah\}] < \theta_k \# \mathcal{N}_k.$$

3. Set $p = D\varphi(z)$, fix $a > -F(p)$ and choose $\varepsilon > 0$ such that

$$a \geq -F(p) + 3\varepsilon.$$

In view of (7.2), (7.4) and (7.5), choose $J \in \boldsymbol{N}$ and $\delta > 0$ such that for all $k \geq J$,

$$\#[\{x \in \boldsymbol{R}^N : \langle x, p \rangle \leq F(p) - \varepsilon\} \cap \eta_k \mathcal{N}_k] < (\theta_k - \delta) \# \mathcal{N}_k,$$

and choose $R > 0$ such that for any $k \geq J$,

$$\#[B(0,R)^c \cap \eta_k \mathcal{N}_k] < \delta \# \mathcal{N}_k.$$

4. For each $k \in \boldsymbol{N}$ set

$$\mathcal{N}_k^1 = \mathcal{N}_k \cap B(0, \eta_k^{-1} R) \quad \text{and} \quad \mathcal{N}_k^2 = \mathcal{N}_k \cap B(0, \eta_k^{-1} R)^c.$$

Let $0 < \gamma \leq 1$, $0 < h \leq \gamma$, $x \in B(z, \gamma)$ and

$$y \in (x - h\eta_k \mathcal{N}_k^1) \cap \{\varphi \geq \varphi(x) + ah\},$$

and choose $\zeta \in \mathcal{N}_k^1$ so that $y = x - h\eta_k \zeta$. Then, assuming that γ is sufficiently small enough, by Taylor's theorem we have

$$\begin{aligned} \varphi(x) + ah &\leq \varphi(x - h\eta_k \zeta) \\ &\leq \varphi(x) - h\eta_k \langle D\varphi(x), \zeta \rangle + \varepsilon h \\ &\leq \varphi(x) - h\eta_k \langle p, \zeta \rangle + 2\varepsilon h, \end{aligned}$$

i.e.,

$$h\eta_k \langle p, \zeta \rangle \leq (-a + 2\varepsilon) h.$$

Our choice of ε yields that

$$\eta_k \langle p, \zeta \rangle \leq F(p) - \varepsilon,$$

which shows

$$\begin{aligned} \#[(x - h\eta_k \mathcal{N}_k^1) \cap \{\varphi \geq \varphi(x) + ah\}] &\leq \#[\eta_k \mathcal{N}_k \cap \{x : \langle x, p \rangle \leq F(p) - \varepsilon\}] \\ &< (\theta_k - \delta) \# \mathcal{N}_k. \end{aligned}$$

Thus we have

$$\begin{aligned} &\#[(x - h\eta_k \mathcal{N}_k) \cap \{\varphi \geq \varphi(x) + ah\}] \\ &\quad < (\theta_k - \delta) \# \mathcal{N}_k + \#[(x - h\eta_k \mathcal{N}_k^2) \cap \{\varphi \geq \varphi(x) + ah\}] \\ &\quad < (\theta_k - \delta) \# \mathcal{N}_k + \delta \# \mathcal{N}_k = \theta_k \# \mathcal{N}_k. \end{aligned}$$

□

To continue, we define $u^+, u^- : \boldsymbol{R}^N \to \boldsymbol{R}$ by

$$u^+(x) = \limsup_{r \to 0} \Big\{ \mathbf{1}_{(\eta_k/n) M_k^n(A_k)}(y) : k, n \in \boldsymbol{N}, n \geq r^{-1}, k \geq r^{-1}, y \in B(x,r) \cap \frac{\eta_k}{n} \boldsymbol{Z}^N \Big\},$$

and

$$u^-(x) = \liminf_{r \to 0} \Big\{ \mathbf{1}_{(\eta_k/n) M_k^n(A_k)}(y) : k, n \in \boldsymbol{N}, n \geq r^{-1}, k \geq r^{-1}, y \in B(x,r) \cap \frac{\eta_k}{n} \boldsymbol{Z}^N \Big\}.$$

Lemma 7.2. *The function u^+ (respectively u^-) is a viscosity subsolution (respectively supersolution) of*

$$-\langle x, Du \rangle + F(Du) = 0 \quad \text{in } \boldsymbol{R}^N. \tag{7.7}$$

Proof. 1. We only check here that u^+ is a subsolution of (7.7).

2. Fix $\varphi \in C^1(\boldsymbol{R}^N)$ and assume that $\hat{x} \in \boldsymbol{R}^N$ is such that

$$(u^+ - \varphi)(\hat{x}) = 0 \quad \text{and} \quad (u^+ - \varphi)(x) < 0 \quad \text{for all } x \in \boldsymbol{R}^N \backslash \{0\},$$

and

$$\lim_{|x| \to \infty} (u^+ - \varphi)(x) = -\infty.$$

3. We may assume that $D\varphi(\hat{x}) \neq 0$, since otherwise we are done.

4. Fix any $a > -F(D\varphi(\hat{x}))$ and, according to Lemma 7.1, choose $\delta > 0$ and $J \in \boldsymbol{N}$ such that for all $k \geq J$, $h \in (0, \delta]$ and $x \in B(\hat{x}, \delta)$,

$$G_{k,h}\varphi(x) \leq \varphi(x) + ah. \tag{7.8}$$

Fix any $\varepsilon \in (0, \delta/2]$ and set

$$\gamma = -\sup_{B(\hat{x},\varepsilon)^c} (u^+ - \varphi),$$

and for $k, n \in \boldsymbol{N}$,

$$u_{k,n} = \mathbf{1}_{(\eta_k/n)M_k^n(A_k)}, \quad \boldsymbol{Z}_{k,n} = (\eta_k/n)\boldsymbol{Z}^N, \quad L_{k,n} = \sup_{\boldsymbol{Z}_{k,n}} (u_{k,n} - \varphi),$$

and

$$I_k = \left\{ n \in \boldsymbol{N} : n \geq 2,\ L_{n,k} + \frac{\varepsilon}{n} \geq L_{k,n-1},\ L_{n,k} > -\frac{\gamma}{2} \right\}.$$

It is easily seen that $\# I_k = \infty$ for infinitely many k's and that there exists $J \in \boldsymbol{N}$ such that

$$\sup_{k \geq J, n \geq J} \ \sup_{\boldsymbol{Z}_{k,n} \cap B(\hat{x},\varepsilon)^c} (u_{k,n} - \varphi) \leq -\frac{\gamma}{2}.$$

5. Now choose k, $n \in \boldsymbol{N}$ so that $n \in I_k$, $k \geq J$ and $n \geq J$. It follows that

$$L_{k,n} + \frac{\varepsilon}{n} \geq L_{k,n-1}, L_{k,n} > -\frac{\gamma}{2}, \quad \text{and} \quad \sup_{\boldsymbol{Z}_{k,n} \cap B(\hat{x},\varepsilon)^c} (u_{k,n} - \varphi) \leq -\frac{\gamma}{2}.$$

We conclude arguing exactly as in Steps 4 and 5 of the proof of Lemma 5.3. □

The next two lemmas are proved as Lemmas 5.1 and 5.2.

Lemma 7.3. *For all $R > 0$ there exist $L > 0$ and $J \in \boldsymbol{N}$ such that for all $k \geq J$ and $n \in \boldsymbol{N}$, if $\eta_k A_k \subset B(0, R)$, then*

$$(\eta_k/n) M_k^n(A_k) \subset B(0, L).$$

Lemma 7.4. *There exist $R > 0$, $\varepsilon > 0$ and $J \in \boldsymbol{N}$ such that for all $k \geq J$ and $n \in \boldsymbol{N}$, if $\eta_k A_k \supset B(0, R) \cap \boldsymbol{Z}_{k,1}$, then*

$$(\eta_k/n) M_k^n(A_k) \supset B(0, \varepsilon) \cap \boldsymbol{Z}_{k,n}.$$

Since the proof of Theorem 7.1 is an immediate consequence of Lemmas 7.2, 7.3, and 7.4 and Theorem 5.3, we omit it.

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