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Conformally flat 3-manifolds with constant scalar curvature

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Abstract. We classify complete conformally flat three dimensional Riemannian manifolds with constant scalar curvature and constant squared norm of Ricci curvature tensor by applying the Generalized Maximum Principle due to H. Omori.

1. Introduction.

It is interesting to investigate the structure of complete and conformally flat Riemannian manifolds with constant scalar curvature. The class of such manifolds is very wide. In fact, from Theorem due to Yamabe, Trudinger, Aubin and Schoen in [**1**] any compact Riemannian manifold can be deformed into a Riemannian manifold with constant scalar curvature by a conformal transformation. So the metric structure on conformally flat Riemannian manifold M^n will be specified under certain restrictions on the behavior of Ricci curvature tensor of M^n for the classification of them. Throughout this article, let M^n be a connected complete and conformally flat Riemannian manifold of constant scalar curvature r without boundary. The Riemannian product $M^{n-1}(c) \times N^1$ are well known as such examples, where $M^{n-1}(c)$ is $(n-1)$-dimensional manifold of constant sectional curvature c and $N^1 = S^1$ or $\boldsymbol{R}$. The scalar curvature r of $M^{n-1}(c) \times N^1$ is positive (negative, respectively) according as $c > 0$ ($c < 0$, respectively).

In [**3**] (cf. [**2**]), S. T. Goldberg essentially proved that every complete conformally flat Riemannian manifold M^n with positive constant scalar curvature r is a space form if the Ricci curvature tensor of M^n satisfies the inequality

$$\sup \sum_{i,j} R_{ij}^2 < \frac{r^2}{n-1},$$

where $\sum_{i,j} R_{ij}^2$ is the squared norm of the Ricci curvature tensor of M^n. We should remark that the condition of r being positive is essential in the proof of Goldberg's Theorem.

The purpose of this article is to study the metric structure of M^n with (not necessarily positive) constant scalar curvature such that the squared norm of the Ricci

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curvature tensor is constant. The basic tool used here is the Generalized Maximum Principle due to H. Omori:

Generalized Maximum Principle (H. Omori [**5**]). *Let X be an n dimensional complete Riemannian manifold whose sectional curvature is bounded from below. If f is a C^2-function bounded from above on X, then there exists a sequence $\{p_m\}$ of points in X such that*

$$\lim_{m\to\infty} f(p_m) = \sup f, \quad \lim_{m\to\infty} |grad\ f|(p_m) = 0,$$

$$\lim_{m\to\infty} \sup \nabla_l \nabla_l f(p_m) \leq 0 \quad \text{for } l = 1, 2, \ldots, n.$$

Here grad f is the gradient vector field for function f and ∇ is the operator of covariant differentiation on X.

Our Main Theorems in this article are stated as follows:

Main Theorem 1. *Let M^3 be a 3-dimesional complete conformally flat Riemannian manifold with constant scalar curvature and constant squared norm of the Ricci curvature tensor. Then we have*

(1) *If the scalar curvature r is nonnegative, M^3 is either isometric to a space form or else the Riemannian product $M^2(c) \times N^1$ $(c \geq 0)$.*

(2) *If the scalar curvature r is negative, either M^3 is isometric to a space form or else the squared norm of the Ricci curvature tensor of M^3 lies in $(r^2/3, r^2/2]$.*

Remark 1. It is obvious that the 3-dimensional hyperbolic space form $H^3(c)$ is conformally flat and it satisfies $S = r^2/3$, where S is the squared norm of the Ricci curvature tensor of $H^3(c)$. We also know the conformally flat space $H^2(c) \times \boldsymbol{R}$ satisfies $S = r^2/2$. But we do not know whether there exist complete conformally flat Riemannian manifolds with negative constant scalar curvature r and constant squared norm S of the Ricci curvature tensor which satisfies $r^2/3 < S < r^2/2$. From Theorem 3 in the section 3, we propose the following conjecture.

Conjecture. *Let M^3 be a 3-dimensional complete conformally flat Riemannian manifold with negative constant scalar curvature and constant squared norm of the Ricci curvature tensor. Then, M^3 is isometric to a space form or else the Riemannian product $H^2(c) \times N^1$ $(c < 0)$.*

Main Theorem 2. *There is no 3-dimensional compact conformally flat Riemannian manifold M^3 such that the following conditions* (1) *and* (2) *are satisfied*:

(1) *The scalar curvature of M^3 is negative constant.*

(2) *The squared norm of the Ricci curvature tensor of M^3 is constant and the eigenvalues of Ricci curvature tensor are all distinct at every point of M^3.*

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2. Preliminaries.

In this section we shall prepare some local formulas for conformally flat Riemannian manifolds with constant scalar curvature. Let M^n be a conformally flat Riemannian manifold and $e_1, \ldots, e_n$ an orthonormal basis of T_pM^n, where T_pM^n is the tangent space at a point p to M^n. We denote the components of curvature tensor of M^n by R_{ijkl}. Since M^n is conformally flat, we have

$$(2.1)\qquad R_{ijkl} = \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + \delta_{ik}R_{jl} - \delta_{il}R_{jk}) - \frac{r}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$

where R_{ij}'s are the components of Ricci curvature tensor and r the scalar curvature of M^n.

Next we assume that r is constant. Since M is conformally flat, we have

$$(2.2)\qquad \nabla_k R_{ij} = \nabla_j R_{ik},$$

where ∇ is the operator of covariant differentiation on M^n. At each point p, we can choose an orthonormal basis $e_1, \ldots, e_n$ such that $R_{ij} = \lambda_i \delta_{ij}$. We put $S = \sum_{i,j} R_{ij}^2 = \sum_i \lambda_i^2$ and $B = \sum_i (\lambda_i - r/n)^2$. Then $B = S - r^2/n$. By letting $R_{ij,k} := \nabla_k R_{ij}$, $R_{ij,kl} := \nabla_l \nabla_k R_{ij}$, etc., we have the Ricci formulas:

$$(2.3\text{-i})\qquad R_{ij,kl} - R_{ij,lk} = \sum_t R_{tj}R_{tikl} + \sum_t R_{it}R_{tjkl},$$

$$(2.3\text{-ii})\qquad R_{ij,klm} - R_{ij,kml} = \sum_t R_{tj,k}R_{tilm} + \sum_t R_{it,k}R_{tjlm} + \sum_t R_{ij,t}R_{tklm}.$$

Because of $B = S - r^2/n$, the Ricci formula (2.3-i) and (2.1) yield

$$\begin{aligned} \frac{1}{2}\Delta B &= \sum_{i,j,k} R_{ij,k}^2 + \sum_{i,j} R_{ij}\Delta R_{ij} \\ &= \sum_{i,j,k} R_{ij,k}^2 + \sum_{i,j,k,m} R_{ij}R_{mk}R_{mijk} + \sum_{i,j,k,m} R_{ij}R_{im}R_{mkjk} \\ &= \sum_{i,j,k} R_{ij,k}^2 + \frac{1}{n-2}\left(n\sum_i \lambda_i^3 - \frac{2n-1}{n-1}rS + \frac{r^3}{n-1}\right) \end{aligned}$$

and

$$(2.4)\qquad \frac{1}{2}\Delta \sum_{i,j,k} R_{ij,k}^2 = \sum_{i,j,k,l} R_{ij,kl}^2 + \sum_{i,j,k} R_{ij,k}\Delta R_{ij,k}$$

where Δ is the Laplace operator on M^n. By directly calculating ΔR_{ij} and $\Delta R_{ij,k}$ and using Ricci formulas (2.3-i), (2.3-ii), we obtain the equations

$$(2.5)\qquad \Delta R_{ij} = \sum_l R_{ij,ll} = \sum_{l,t} R_{tl}R_{tijl} + \sum_t R_{it}R_{tj}$$

and

$$\varDelta R_{ij,k} = \sum_l R_{ij,kll} = \sum_l \nabla_l R_{ij,kl} \tag{2.6}$$

$$= \sum_l R_{ij,lkl} + \sum_l \nabla_l \left(\sum_t R_{tj} R_{tikl} \right) + \sum_l \nabla_l \left(\sum_t R_{ti} R_{tjkl} \right)$$

$$= \sum_l R_{ij,llk} + \sum_{l,t} R_{tj,l} R_{tikl} + \sum_{l,t} R_{it,l} R_{tjkl}$$

$$+ \sum_{l,t} R_{ij,t} R_{tlkl} + \sum_{l,t} R_{tj,l} R_{tikl} + \sum_{l,t} R_{it,l} R_{tjkl}$$

$$+ \sum_{l,t} R_{tj} R_{tikl,l} + \sum_{l,t} R_{it} R_{tjkl,l}.$$

Hence, by using (2.5) and (2.6), we have

$$\sum_{i,j,k} R_{ij,k} \varDelta R_{ij,k} = 3 \sum_{i,j,k} \lambda_i R_{ij,k}^2 + 5 \sum_{i,j,k,l,t} R_{ij,k} R_{tj,l} R_{tikl} \tag{2.7}$$

$$+ 2 \sum_{i,j,k,l,t} R_{ti} R_{ij,k} R_{tjkl,l} + \sum_{i,j,k,l,t} R_{tl} R_{ij,k} R_{tijl,k},$$

where $R_{ijkl,t} = \nabla_t R_{ijkl}$. Since

$$R_{ijkl,m} = \frac{1}{n-2} (R_{ik,m} \delta_{jl} - R_{il,m} \delta_{jk} + \delta_{ik} R_{jl,m} - \delta_{il} R_{jk,m}) \tag{2.8}$$

in the case of $r =$constant, we also obtain the relations

$$\sum_{i,j,k,l,t} R_{ti} R_{ij,k} R_{tjkl,l} = 0, \tag{2.9}$$

$$\sum_{i,j,k,l,t} R_{tl} R_{ij,k} R_{tijl,k} = \frac{2}{n-2} \sum_{i,j,k} \lambda_i R_{ij,k}^2 - \frac{r}{n-2} \sum_{i,j,k} R_{ij,k}^2 \tag{2.10}$$

and

$$\sum_{i,j,k,l,t} R_{ij,k} R_{tj,l} R_{tikl} = \frac{2}{n-2} \sum_{i,j,k} \lambda_i R_{ij,k}^2 - \frac{r}{(n-1)(n-2)} \sum_{i,j,k} R_{ij,k}^2.$$

Hence, we have finally

$$\frac{1}{2} \varDelta \sum_{i,j,k} R_{ij,k}^2 = \sum_{i,j,k,l} R_{ij,kl}^2 + \frac{3n+2}{n-2} \sum_{i,j,k} \lambda_i R_{ij,k}^2 - \frac{(n+2)r}{(n-1)(n-2)} \sum_{i,j,k} R_{ij,k}^2.$$

Summing up the above computations, we have proved the following Proposition 1:

PROPOSITION 1. *On conformally flat Riemannian manifold M^n with constant scalar curvature r, we have the following formulae* (2.11) *and* (2.12):

$$\frac{1}{2} \varDelta B = \sum_{i,j,k} R_{ij,k}^2 + \frac{1}{n-2} \left(n \sum_i \lambda_i^3 - \frac{2n-1}{n-1} rS + \frac{r^3}{n-1} \right) \tag{2.11}$$

and

$$\frac{1}{2}\Delta\sum_{i,j,k}R_{ij,k}^2=\sum_{i,j,k,l}R_{ij,kl}^2+\frac{3n+2}{n-2}\sum_{i,j,k}\lambda_iR_{ij,k}^2 -\frac{(n+2)r}{(n-1)(n-2)}\sum_{i,j,k}R_{ij,k}^2. \tag{2.12}$$

Remark 2. Using the formula (2.11) and the Generalized Maximum Principle of Omori, we can conclude the result due to Goldberg [**3**] denoted in section 1 and the following result of Tani [**6**]: *Any compact conformally flat Riemannian manifold with constant scalar curvature r and positive Ricci curvature is of constant curvature.*

Now, we consider the case of $n=3$ and $S=$constant. Then (2.11) and (2.12) become

$$\sum_{i,j,k}R_{ij,k}^2=-3\sum_i\lambda_i^3+\frac{5}{2}rS-\frac{1}{2}r^3 \tag{2.13}$$

and

$$\Delta\sum_{i,j,k}R_{ij,k}^2=-3\Delta\sum_i\lambda_i^3 \tag{2.14}$$

respectively. Setting $\mu_i=\lambda_i-r/3$, we have

$$\sum_i\mu_i=0,\quad B=\sum_i\mu_i^2=S-\frac{1}{3}r^2, \tag{2.15}$$

$$\sum_i\lambda_i^3=\sum_i\mu_i^3+rB+\frac{1}{9}r^3, \tag{2.16}$$

$$\sum_i\lambda_i^4=\frac{1}{2}B^2+\frac{4}{3}r\sum_i\mu_i^3+\frac{2}{3}r^2B+\frac{1}{27}r^4 \tag{2.17}$$

and (2.13) can be rewritten as

$$\sum_{i,j,k}R_{ij,k}^2=-3\sum_i\mu_i^3-\frac{1}{2}rB. \tag{2.18}$$

Since

$$\sum_i\lambda_i^3=\sum_{i,j,k}R_{ik}R_{kj}R_{ji},$$

we obtain from (2.16) and (2.5)

$$\begin{aligned}\frac{1}{3}\Delta\sum_i\mu_i^3&=2\sum_{i,j,k}\lambda_iR_{ij,k}^2+\sum_{i,k}\lambda_i^2R_{ii,kk}\\&=2\sum_{i,j,k}\mu_iR_{ij,k}^2+\frac{2}{3}r\sum_{i,j,k}R_{ij,k}^2+\frac{5}{2}r\sum_i\mu_i^3+\frac{1}{2}B^2+\frac{1}{3}r^2B\\&=\frac{1}{2}\left(B^2+r\sum_i\mu_i^3+4\sum_{i,j,k}\mu_iR_{ij,k}^2\right).\end{aligned} \tag{2.19}$$

Therefore we have from (2.18) and (2.19)

$$\Delta \sum_{i,j,k} R_{ij,k}^2 = -\frac{9}{2}\left(B^2 + r\sum_i \mu_i^3 + 4\sum_{i,j,k} \mu_i R_{ij,k}^2\right). \tag{2.20}$$

Moreover we have from (2.12), (2.19), (2.20)

$$\begin{aligned} \sum_{i,j,k,l} R_{ij,kl}^2 &= -11\sum_{i,j,k} \lambda_i R_{ij,k}^2 + \frac{5}{2} r \sum_{i,j,k} R_{ij,k}^2 + \frac{1}{2}\Delta \sum_{i,j,k} R_{ij,k}^2 \\ &= -20\sum_{i,j,k} \mu_i R_{ij,k}^2 - \frac{5}{12} r \sum_{i,j,k} R_{ij,k}^2 - \frac{9}{4} B\left(B - \frac{1}{6} r^2\right). \end{aligned} \tag{2.21}$$

Summing up the above computations, we have proved the following Proposition 2:

PROPOSITION 2. *If the squared norm of Ricci curvature tensor of the conformally flat Riemannian manifold M^3 with constant scalar curvature r is constant, then we have the following three formulae*:

$$\sum_{i,j,k} R_{ij,k}^2 = -3\sum_i \mu_i^3 - \frac{1}{2} rB,$$

$$\Delta \sum_i \mu_i^3 = \frac{3}{2}\left(B^2 + r\sum_i \mu_i^3 + 4\sum_{i,j,k} \mu_i R_{ij,k}^2\right),$$

$$\sum_{i,j,k,l} R_{ij,kl}^2 = -20\sum_{i,j,k} \mu_i R_{ij,k}^2 - \frac{5}{12} r \sum_{i,j,k} R_{ij,k}^2 - \frac{9}{4} B\left(B - \frac{1}{6} r^2\right).$$

3. Theorems and their proofs.

First of all, we consider the case of $r \geq 0$.

THEOREM 1. *In addition to the assumptions in Main Theorem* 1, *if $r \geq 0$, then M^3 is either isometric to a space form or the Riemannian product $M^2 \times N^1$, where M^2 and N^1 are of constant curvature with dimension* 2 *and* 1, *respectively*.

PROOF. Since M^3 is conformally flat and the squared norm of Ricci curvature tensor is constant, we know from (2.1) that the sectional curvature of M^3 is bounded. Applying the Generalized Maximum Principle to the function $B_3 := \sum_i \mu_i^3$, there exists a sequence $\{p_m\}$ of points in M^3 such that

$$\lim_{m\to\infty} B_3(p_m) = \sup B_3, \quad \lim_{m\to\infty} |\text{grad } B_3|(p_m) = 0, \tag{3.1}$$

$$\lim_{m\to\infty} \sup \nabla_l \nabla_l B_3(p_m) \leq 0 \quad \text{for } l = 1, 2, 3. \tag{3.2}$$

From (2.18) and (2.21) it follows that $R_{ij,k}$ and $R_{ij,kl}$ are bounded. Thus, by taking a subsequence if necessary, we see that the sequences $\{\lambda_i(p_m)\}$, $\{\mu_i(p_m)\}$, $\{B_3(p_m)\}$, $\{R_{ij,k}(p_m)\}$ and $\{R_{ij,kl}(p_m)\}$ have their limits λ_i^*, μ_i^*, B_3^*, $R_{ij,k}^*$ and $R_{ij,kl}^*$, respectively.

First we note the following lemma:

Lemma 1.

$$-\frac{B^{3/2}}{\sqrt{6}} \leqq B_3 \leqq \frac{B^{3/2}}{\sqrt{6}}$$

and the equality holds if and only if at least two of μ_i *are equal.*

Proof. Because of $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = B$, by making use of Lagrange multiplier method, it follows that $-B^{3/2}/\sqrt{6} \leqq B_3 \leqq B^{3/2}/\sqrt{6}$ and the equality holds if and only if at least two of μ_i are equal. This finishes the proof.

Now, if $S \le r^2/2$, then we know from the result in [**2**] due to Wu and first-named author that M^3 is either isometric to a space form or Riemannian product $M^2 \times N^1$. Thus, Theorem 1 is true for the case of $S \le r^2/2$.

Next we shall prove that $S > r^2/2$ is impossible. In fact, by (2.15) and the assumptions for r and S, we have $\sum_i R_{ii,k} = 0$ and $\sum_i \mu_i R_{ii,k} = 0$ on M^3. In particular, we get

$$\sum_i R^*_{ii,k} = 0 \quad \text{for all } k \tag{3.3}$$

and

$$\sum_i \mu_i^* R^*_{ii,k} = 0 \quad \text{for all } k. \tag{3.4}$$

$\sum_i \mu_i^{*2} R^*_{ii,k} = 0$ is obtained from (3.1). So we have, together with (3.3) and (3.4), the following system of linear equations:

$$\begin{cases} \sum_i R^*_{ii,k} = 0 \\ \sum_i \mu_i^* R^*_{ii,k} = 0 \\ \sum_i \mu_i^{*2} R^*_{ii,k} = 0 \end{cases} \tag{3.5}$$

for $k = 1, 2$ and 3.

(i) If μ_1^*, μ_2^* and μ_3^* are all distinct, then $R^*_{ii,k} = 0$ for $i, k = 1, 2, 3$. This fact and (2.15) mean

$$\begin{aligned} \sum_{i,j,k} \mu_i^* R^{*2}_{ij,k} &= \frac{1}{3} \sum_{i \ne j \ne k \ne i} (\mu_i^* + \mu_j^* + \mu_k^*) R^{*2}_{ij,k} \\ &\quad + \sum_{i \ne j} (2\mu_i^* + \mu_j^*) R^{*2}_{ii,j} + \sum_i \mu_i^* R^{*2}_{ii,i} \\ &= 0. \end{aligned} \tag{3.6}$$

Therefore (2.21) and (3.6) imply

$$\sum_{i,j,k,l} R^{*2}_{ij,kl} = -\frac{5}{12} r \sum_{i,j,k} R^{*2}_{ij,k} - \frac{9}{4} B\left(B - \frac{1}{6} r^2\right). \tag{3.7}$$

This is a contradiction because $B = S - r^2/3 > r^2/6$ and $r \ge 0$.

(ii) If μ_1^*, μ_2^* and μ_3^* are not distinct with each other, then we have $\mu_1^* = \mu_2^* = \mu_3^*$ or $2\mu_1^* = 2\mu_2^* = -\mu_3^*$.

(ii-1) $\mu_1^* = \mu_2^* = \mu_3^*$ case. In this case, (2.15) implies $\mu_1^* = \mu_2^* = \mu_3^* = 0$ and hence we have $S = r^2/3$.

(ii-2) $2\mu_1^* = 2\mu_2^* = -\mu_3^*$ case. In this case, we have from (2.15)

$$B_3^* = \pm \frac{B^{3/2}}{\sqrt{6}}.$$

If

$$B_3^* = -\frac{B^{3/2}}{\sqrt{6}},$$

then from Lemma 1 we know $B_3 =$constant. Hence the λ_i's are constant. Thus, we have $\Delta_l R_{ij} = 0$. So we see from (2.18)

$$0 = -3B_3^* - \frac{1}{2}rB = B\left(\frac{3\sqrt{B}}{\sqrt{6}} - \frac{1}{2}r\right).$$

This means $B = 0$ or $3\sqrt{B}/\sqrt{6} = r/2$, that is, $S = r^2/3$ or $S = r^2/2$.

If

$$B_3^* = \frac{B^{3/2}}{\sqrt{6}},$$

then we see from (2.18)

$$0 \geq \frac{3}{\sqrt{6}}B^{3/2} + \frac{1}{2}rB = 3B\left(\frac{\sqrt{B}}{\sqrt{6}} + \frac{1}{6}r\right) \geq 0.$$

This means $B = 0$ and hence $S = r^2/3$. Thus, the proof of Theorem 1 is completed. □

Next we consider the case of $r < 0$.

Theorem 2. *In addition to the assumptions in Main Theorem* 1, *if* $r < 0$, *then the squared norm* S *of the Ricci curvature tensor satisfies the following inequality*:

$$\frac{r^2}{3} \leq S \leq \frac{r^2}{2}. \tag{3.9}$$

Proof. By rewriting (2.19) in term of B_3, we have

$$\Delta B_3 = \frac{3}{2}\left(B^2 + rB_3 + 4\sum_{i,j,k}\mu_i R_{ij,k}^2\right). \tag{3.10}$$

Hence, by using (3.2), we have

$$B^2 + rB_3^* + 4\sum_{i,j,k}\mu_i^* R_{ij,k}^{*2} \leq 0. \tag{3.11}$$

(i) If μ_i^*'s are distinct with each other, then by making use of the similar method to the proof of Theorem 1, we have

$$\sum_{i,j,k} \mu_i^* R_{ij,k}^{*2} = 0 \tag{3.12}$$

and hence the inequality

$$B^2 + rB_3^* \leq 0. \tag{3.13}$$

On the other hand, by using (2.18) we know

$$r^2 B + 6rB_3^* \geq 0. \tag{3.14}$$

Thus, (3.13) and (3.14) imply the inequality

$$-\frac{1}{2} r^2 B \leq 3rB_3^* \leq -3B^2.$$

Therefore, the proposed relation

$$\frac{r^2}{3} \leq S \leq \frac{r^2}{2}$$

in Theorem 2 is valid in this case.

(ii) If μ_i^*'s are not distinct with each other, then by the same reasons as in Theorem 1 we have $S = r^2/3$ or

$$B_3^* = \pm \frac{B^{3/2}}{\sqrt{6}}. \tag{3.15}$$

If

$$B_3^* = -\frac{B^{3/2}}{\sqrt{6}},$$

then $\nabla_l R_{ij} = 0$ by the same reason as in the case of (ii-2) of the proof of Theorem 1. Therefore we have from (2.18)

$$0 = -3B_3^* - \frac{1}{2} rB = B\left(\frac{3\sqrt{B}}{\sqrt{6}} - \frac{r}{2}\right). \tag{3.16}$$

Because of $r < 0$, we see $3\sqrt{B}/\sqrt{6} - r/2 > 0$. Hence, $B = 0$, that is, $S = r^2/3$ is valid. It should be remarked that in this case $r < 0$, we can only get $S = r^2/3$ and we can not conclude $S = r^2/2$ from the above equality (3.16).

If

$$B_3^* = \frac{B^{3/2}}{\sqrt{6}},$$

then we see from (2.18)

$$3\frac{B^{3/2}}{\sqrt{6}} + \frac{1}{2} rB = 3B\left(\frac{1}{\sqrt{6}}\sqrt{B} + \frac{r}{6}\right) \leq 0.$$

This means $B \le r^2/6$. Hence we conclude $S \le r^2/2$. This completes the proof of Theorem 2. □

Proof of Main Theorem 1. From Theorem 1, we know that the assertion (1) in Main Theorem 1 is true. When $r < 0$, we have the inequality $r^2/3 \le S \le r^2/2$ from Theorem 2. If $S = r^2/3$, then we know that

$$\sum_{i,j}\left(R_{ij} - \frac{1}{3} r\delta_{ij}\right)^2 = 0.$$

Hence the conformally flat Riemannian manifold M^3 is Einstein space. So we can conclude that M^3 is space form by which the assertion (2) in Main Theorem 1 is also true. This completes the proof of Main Theorem 1. □

In order to prove Main Theorem 2, we will prepare the following Theorem 3.

Theorem 3. *In addition to the assumptions in Main Theorem* 1, *if* $r < 0$, *then we have*

$$\sup B_3 = \frac{B^{3/2}}{\sqrt{6}}.$$

Proof. We use the same notations above. Suppose $B_3^* := \sup B_3 \ne B^{3/2}/\sqrt{6}$, then we shall get a contradiction. To do it, we first prepare the following lemma:

Lemma 2. If $B_3^* \ne B^{3/2}/\sqrt{6}$, *then we have*

(1) μ_1^*, μ_2^*, μ_3^* *are all distinct.*

(2) $0 < B < \dfrac{r^2}{6}$.

(3) $\displaystyle\sum_{i,j,k,l} R_{ij,kl}^{*2} = -\frac{5}{12} r \sum_{i,j,k} R_{ij,k}^{*2} - \frac{9}{4} B\left(B - \frac{r^2}{6}\right)$.

(4) $B_3^* \ge -\dfrac{B^2}{r} > 0$.

Proof. As a consequence of Lemma 1 noted in the proof of Theorem 1, the statement (1) is obtained.

In view of Theorem 2 and its proof, the statement (1) yields the inequality

$$\frac{r^2}{3} < S < \frac{r^2}{2}$$

that is, the relation (2) holds:

$$0 < B < \frac{r^2}{6}. \tag{3.17}$$

The relation (3) is proved as follows:

From the statement (1), we have, by using (3.5),

$$R_{ii,k}^* = 0 \quad \text{for } i, k = 1, 2, 3. \tag{3.18}$$

Hence (2.18) means

$$\sum_{i,j,k} R^{*2}_{ij,k} = 6R^{*2}_{12,3} = -3\sum_i \mu_i^{*3} - \frac{r}{2}B. \tag{3.19}$$

By differentiating (2.18) and using (3.1), we get

$$\sum_{i,j,k} R^*_{ij,k} R^*_{ij,kl} = -\frac{3}{2}\lim_{m\to\infty} \nabla_l B_3(p_m) = 0 \quad \text{for } l = 1,2,3. \tag{3.20}$$

According to (3.18) and (3.20), we infer

$$R^*_{ij,kl} = 0 \quad \text{for } i \neq j \neq k \neq i. \tag{3.21}$$

From the Ricci formula (2.3-i) and (2.1), we have

$$R^*_{ij,kk} - R^*_{kk,ij} = \sum_t R^*_{it} R^*_{tkjk} + \sum_t R^*_{tk} R^*_{tijk} = 0 \quad \text{for } i \neq j \neq k \neq i. \tag{3.22}$$

Thus, (3.21) and (3.22) yield

$$R^*_{ii,jk} = 0 \quad \text{for } i \neq j \neq k \neq i. \tag{3.23}$$

Because of

$$0 = \frac{1}{2}\nabla_l \nabla_k S = \sum_{i,j} R_{ij} R_{ij,kl} + \sum_{i,j} R_{ij,k} R_{ij,l} \tag{3.24}$$

and

$$0 = \nabla_l \nabla_k r = \sum_i R_{ii,kl},$$

we have

$$\sum_i R^*_{ii,kl} = 0 \tag{3.25}$$

and

$$\sum_i \lambda_i^* R^*_{ii,kl} + \sum_{i,j} R^*_{ij,k} R^*_{ij,l} = 0. \tag{3.26}$$

In view of Ricci formula (2.3-i) and (2.1), the relations (3.23), (3.25) and (3.26) imply

$$R^*_{jj,jk} = R^*_{kk,jk} = 0 \quad \text{for } j \neq k. \tag{3.27}$$

Thus, (2.21), (3.6), (3.21), (3.23) and (3.27) yield the relation (3) of this Lemma 2:

$$\begin{aligned} \sum_{i,j,k,l} R^{*2}_{ij,kl} &= 3\sum_{i,l} R^{*2}_{ii,ll} - 2\sum_i R^{*2}_{ii,ii} \\ &= -\frac{5}{12} r \sum_{i,j,k} R^{*2}_{ij,k} - \frac{9}{4} B\left(B - \frac{1}{6}r^2\right). \end{aligned} \tag{3.28}$$

Finally, the inequality (4) is proved as follows:

By using (2.19), we have

$$0 \geq \frac{1}{3} \lim_{m\to\infty} \sup \Delta \sum_i \mu_i^3(p_m) = \lim_{m\to\infty} \sup\left(-\Delta \sum_{i,j,k} R_{ij,k}^2(p_m)\right)$$
$$= \frac{9}{2}\left(B^2 + r\sum_i B_3^* + 4\sum_{i,j,k} \mu_i^* R_{ij,k}^{*2}\right).$$

Since μ_i^*'s are all distinct, (3.18) is also true. Hence,

$$\sum_{i,j,k} \mu_i^* R_{ij,k}^{*2} = 0.$$

Thus the inequality

$$B_3^* \geq -\frac{B^2}{r} > 0 \tag{3.29}$$

holds because $r < 0$. Thus the proof of Lemma 2 is finished.

Now we continue to prove Theorem 3. By taking $k = l$ in (3.25) and (3.26), we have

$$\sum_i R_{ii,ll}^* = 0 \quad \text{for } l = 1, 2, 3 \tag{3.30}$$

and

$$\sum_i \mu_i^* R_{ii,ll}^* = -\sum_{i,j} R_{ij,l}^{*2} = -2R_{12,3}^{*2} \tag{3.31}$$
$$= \frac{1}{6} rB + B_3^* \quad \text{for } l = 1, 2, 3,$$

respectively. On the other hand, the Ricci formula (2.3-i) and (2.1) yield

$$R_{ij,ij}^* - R_{ji,ji}^* = (\mu_i^* - \mu_j^*)\left(\mu_i^* + \mu_j^* + \frac{1}{6}r\right) \quad \text{for } l = 1, 2, 3. \tag{3.32}$$

In view of (3.32), we can solve the rank 5 linear system (3.30) and (3.31) of six equations and six unknowns $R_{ii,ll}^*$. In fact, the solutions are

$$R_{ii,ll}^* = \mu_i^{*2} + \frac{1}{6} r\mu_i^* - \frac{1}{3}B + yg_i g_l. \tag{3.33}$$

Here

$$g_i = \mu_i^{*2} - \frac{B_3^*}{B}\mu_i^* - \frac{1}{3}B \tag{3.34}$$

and y is a quantity which need not to be determined. Hence we get

$$\sum_{i,l} R_{ii,ll}^{*2} = \sum_{i,l}\left(\mu_i^{*2} + \frac{1}{6} r\mu_i^* - \frac{B}{3} + yg_i g_l\right)^2 \tag{3.35}$$
$$= \frac{B}{2}\left(B + \frac{1}{6}r^2\right) + x^2 + rB_3^*$$

and

$$\sum_i R^{*2}_{ii,,ii} = \sum_i \left(\mu_i^{*2} + \frac{1}{6} r\mu_i^* - \frac{1}{3} B + y g_i^2\right)^2 \tag{3.36}$$

$$= \frac{1}{2} B^2 + \frac{1}{36} r^2 B + \frac{B^2}{3} + \frac{1}{2} x^2 + \frac{1}{3} r B_3^* - \frac{2}{3} B^2$$

$$+ 2y \sum_i \mu_i^{*2} g_i^2 + \frac{1}{3} yr \sum_i \mu_i^* g_i^2 - \frac{2}{3} Bx$$

$$= \frac{1}{2} x^2 - \frac{1}{3} x\left(B + \frac{1}{B} r B_3^*\right) + \frac{1}{3} r B_3^* + \frac{1}{6} B^2 + \frac{1}{36} r^2 B,$$

where

$$x = y\left(\frac{B^2}{6} - \frac{B_3^{*2}}{B}\right).$$

Here we used

$$\sum_i \mu_i^{*2} g_i^2 = \frac{B}{6}\left(\frac{B^2}{6} - \frac{B_3^{*2}}{B}\right)$$

and

$$\sum_i \mu_i^* g_i^2 = \frac{B_3^*}{B}\left(\frac{B_3^{*2}}{B} - \frac{B^2}{6}\right).$$

By using (3) in Lemma 2, (3.35) and (3.36), we have

$$\sum_{i,j,k,l} R^{*2}_{ij,kl} = 2x^2 + \frac{2}{3} x\left(B + r\frac{B_3^*}{B}\right) + \frac{3}{2} B\left(B + \frac{1}{6} r^2\right) \tag{3.37}$$

$$- \frac{B^2}{3} - r^2 \frac{B}{18} + 3 r B_3^* - \frac{2}{3} r B_3^*$$

$$= 2x^2 + \frac{2}{3} x\left(B + r\frac{B_3^*}{B}\right) + \frac{7}{6} B\left(B + \frac{1}{6} r^2\right) + \frac{7}{3} r B_3^*$$

$$= -\frac{5}{12} r \sum_{i,j,k} R^{*2}_{ij,k} - \frac{9}{4} B\left(B - \frac{1}{6} r^2\right)$$

$$= \frac{5}{4} r B_3^* - \frac{9}{4} B\left(B - \frac{1}{6} r^2\right) + \frac{5}{24} r^2 B.$$

Therefore, we obtain the quadratic equation of x

$$x^2 + \frac{1}{3} x\left(B + r\frac{B_3^*}{B}\right) + \frac{13}{24} r B_3^* + \frac{41}{24} B^2 - \frac{7}{36} r^2 B = 0 \tag{3.38}$$

of which the solutions are

$$x = -\frac{1}{6}\left(B + r\frac{B_3^*}{B}\right) \pm \frac{1}{6}\sqrt{r^2\left(\frac{B_3^*}{B}\right)^2 - \frac{35}{2} r B_3^* + 7 r^2 B - \frac{121}{2} B^2}. \tag{3.39}$$

Since μ_i^*'s are all distinct as noted in Lemma 2, we can conclude, by using the inequality (4) in Lemma 2

$$\mu_1^* < \mu_2^* < 0 \quad \text{and} \quad \mu_3^* > 0 \tag{3.40}$$

and

$$\frac{1}{6}B < \mu_1^{*2} < \frac{1}{2}B + \frac{\sqrt{2}}{3}\frac{B^{3/2}}{r}, \quad 0 < \mu_2^{*2} < \frac{1}{6}B, \quad \frac{1}{2}B < \mu_3^{*2} < \frac{2}{3}B. \tag{3.41}$$

Here we assumed without loss of generality $\mu_1^* < \mu_2^* < \mu_3^*$ and used the formulas $B_3^* = 3\mu_j^*(\mu_j^{*2} - B/2)$ for $j = 1$, 2 and 3.

On the other hand, by making use of the similar computation as in (2.19), we have from (3.2) the inequality

$$\begin{aligned} 0 &\geq \lim_{m\to\infty} \frac{1}{3} \sup \nabla_l \nabla_l B_3(p_m) \\ &= \sum_i \lambda_i^{*2} R_{ii,ll}^* + 2\sum_{i,j} \lambda_i^* R_{ij,l}^{*2}. \end{aligned}$$

From (3.18) and (3.19), we obtain

$$\begin{aligned} 2\sum_{i,j} \mu_i^* R_{ij,l}^{*2} &= \sum_{i,j} (\mu_i^* + \mu_j^*) R_{ij,l}^{*2} = -2\mu_l^* R_{12,3}^* \\ &= \left(\frac{1}{6} rB + B_3^*\right)\mu_l^*. \end{aligned}$$

Hence, by using (3.30), (3.33) and this relation, we get the inequality

$$\begin{aligned} 0 &\geq \sum_i \lambda_i^{*2} R_{ii,ll}^* + 2\sum_{i,j} \mu_i^* R_{ij,l}^{*2} + \frac{2}{3} r \sum_{i,j} R_{ij,l}^{*2} \\ &= \sum_i \left(\mu_i^{*2} + \frac{2}{3} r\mu_i^*\right)\left(\mu_i^{*2} + \frac{1}{6} r\mu_i^* - \frac{1}{3}B + y g_i g_l\right) \\ &\quad + \left(\frac{1}{6} rB + B_3^*\right)\mu_l^* + \frac{2}{3} r\left(-\frac{1}{6} rB - B_3^*\right) \\ &= \left(\frac{1}{6} rB + B_3^*\right)\mu_l^* + x g_l + \frac{1}{6} B^2 + \frac{1}{6} rB_3^* \\ &= \left(-\frac{3}{B} x\left(\mu_l^{*2} - \frac{2}{3}B\right) + 3\mu_l^{*2} + \frac{1}{2} r\mu_l^* - B\right)\left(\mu_l^{*2} - \frac{1}{6}B\right). \end{aligned} \tag{3.42}$$

Next, we estimate x by using (4) in Lemma 2 and so on.

Lemma 3. *If* $B_3^* \neq B^{3/2}/\sqrt{6}$, *then*

$$x \leq -\frac{1}{6}\left(B + r\frac{B_3^*}{B}\right).$$

Proof. If $x > -(B + r(B_3^*/B))/6$, then, by taking $l = 1$ in (3.42) and using (3.17), (3.29) and (3.41), we have

$$
\begin{aligned}
0 &\geq -\frac{3}{B}x\left(\mu_1^{*2} - \frac{2}{3}B\right) + 3\mu_1^{*2} + \frac{1}{2}r\mu_1^* - B \\
&\geq \frac{3}{2B}\left(\mu_1^{*2} - \frac{2}{3}B\right)\left(B + r\frac{B_3^*}{B}\right) + 3\mu_1^{*2} + \frac{1}{2}r\mu_1^* - B \\
&\geq 3\mu_1^{*2} + \frac{1}{2}r\mu_1^* - B \\
&> -\frac{1}{2}\left(B + \frac{1}{\sqrt{6}}r\sqrt{B}\right) > 0.
\end{aligned}
$$

This is a contradiction. Thus the proof of lemma 3 is finished.

Now, Lemma 3 means that

$$
x = -\frac{1}{6}\left(B + r\frac{B_3^*}{B}\right) - \frac{1}{6}\sqrt{r^2\left(\frac{B_3^*}{B}\right)^2 - \frac{35}{2}rB_3^* + 7Br^2 - \frac{121}{2}B^2}. \tag{3.43}
$$

Finally we estimate B as follows.

Lemma 4. *If* $B_3^* \neq B^{3/2}/\sqrt{6}$, *then*

$$
B < \frac{16}{(47)^2 \cdot 9}r^2.
$$

Proof. By taking $l = 2$ in (3.42) and using the inequality $\mu_2^{*2} - B/6 < 0$ in (3.41), we have

$$
0 \leq -\frac{3}{B}x\left(\mu_2^{*2} - \frac{2}{3}B\right) + 3\mu_2^{*2} + \frac{1}{2}r\mu_2^* - B. \tag{3.44}
$$

First of all, we assert $B < r^2/7$. In fact, since B and r are constant by the assupmtion and $B < r^2/6$ as noted in Lemma 2, there exists a constant $a > 0$ such that $(1+a)B < r^2/6$. If $a \geq 1/6$, then $B < r^2/7$. In this case our assertion holds. If $a < 1/6$, then we have

$$
x = -\frac{1}{6}\left(B + r\frac{B_3^*}{B}\right) - \frac{1}{6}\sqrt{\left((1-6a)B + r\frac{B_3^*}{B}\right)^2 + G}, \tag{3.45}
$$

where G is defined by

$$
G = -(1-6a)^2B^2 - 2(1-6a)rB_3^* - \frac{35}{2}rB_3^* + 7Br^2 - \frac{121}{2}B^2.
$$

According to $rB_3^* \leq -B^2$ of (4) in Lemma 2 and $(1+a)B < r^2/6$ we have

$$
\begin{aligned}
G &\geq -(1-6a)^2B^2 + 2(1-6a)B^2 + \frac{35}{2}B^2 + 42(1+a)B^2 - \frac{121}{2}B^2 \\
&= B^2(42a - 36a^2) > 0.
\end{aligned}
$$

Therefore, we obtain

$$
x \leq -\frac{1}{6}\left(B + r\frac{B_3^*}{B}\right) + \frac{1}{6}\left((1-6a)B + r\frac{B_3^*}{B}\right) = -aB.
$$

Using this inequality, $\mu_2^{*2} - B/6 < 0$ in (3.41) and $\mu_2^* < 0$, (3.44) implies

$$\begin{aligned} 0 &\le 3a\left(\mu_2^{*2} - \frac{2}{3}B\right) + 3\mu_2^{*2} + \frac{1}{2}r\mu_2^* - B \\ &= 3(a+1)\mu_2^{*2} + \frac{1}{2}r\mu_2^* - (2a+1)B \\ &< \frac{1}{2}(a+1)B - \frac{1}{2\sqrt{6}}\sqrt{B} - (2a+1)B. \end{aligned}$$

Hence, we have

$$(1+6a)B < \frac{r^2}{6}.$$

If $6a \ge 1/6$, then our assertion is true. If $6a < 1/6$, then by applying the above procedures for $a_1 = 6a$, we get

$$(1+6^2a)B < \frac{r^2}{6}.$$

If we repeat the above procedure k times so that $6^k a \ge 1/6$, then we have

$$(1+6^k a)B < \frac{r^2}{6}.$$

Thus we obtain $B < r^2/7$, that is, our assertion holds.

Next, by taking account of $B < r^2/7$, we will show $B < r^2/24$ and $B < (4r)^2/(3\cdot 47)^2$ finally. In fact, from (3.43), $rB_3^* \le -B^2$ of (4) in Lemma 2 and $B < r^2/7$, we have

$$x < -\frac{1}{6}\left(B + r\frac{B_3^*}{B}\right) - \frac{1}{6}\left(B - r\frac{B_3^*}{B}\right) = -\frac{B}{3}. \tag{3.46}$$

Hence, by substituting the above inequality (3.46) into (3.44), from $0 < \mu_2^{*2} < B/6$ and $\mu_2^* < 0$, we get

$$\begin{aligned} 0 &\le \left(\mu_2^{*2} - \frac{2}{3}B\right) + 3\mu_2^{*2} + \frac{1}{2}r\mu_2^* - B = 4\mu_2^{*2} + \frac{1}{2}r\mu_2^* - \frac{5}{3}B \\ &< \frac{2}{3}B - \frac{1}{2\sqrt{6}}r\sqrt{B} - \frac{5}{3}B = -\sqrt{B}\left(\sqrt{B} + \frac{r}{2\sqrt{6}}\right). \end{aligned} \tag{3.47}$$

Thus we have

$$B < \frac{r^2}{24}.$$

By applying the above method to $B < r^2/24$, which introduced (3.46) and (3.47), we get

$$B < \frac{16}{(47)^2\cdot 9}r^2.$$

Thus the proof of Lemma 4 is completed.

In view of Lemma 4 and the inequality $rB_3^* \leq -B^2$ of (4) in Lemma 2, we obtain

$$\left(-\frac{35}{2}r - \frac{4r^2}{3\sqrt{B}}\right)B_3^* + \frac{19}{3}r^2B - \frac{121}{2}B^2 + \frac{1}{3}r^2\frac{B_3^{*2}}{B^2}$$
$$= \frac{1}{3}\left(\frac{rB_3^*}{B} - \frac{105}{4}B - 2r\sqrt{B}\right)^2 - \frac{1}{3}\left(\frac{105}{4}B + 2r\sqrt{B}\right)^2 + \frac{19}{3}r^2B - \frac{121}{2}B^2 > 0.$$

Therefore, we have

$$x < -\frac{1}{6}\left(B + r\frac{B_3^*}{B}\right) - \frac{1}{6}\sqrt{\frac{2}{3}\left(r\sqrt{B} + r\frac{B_3^*}{B}\right)^2} \tag{3.48}$$
$$= -\frac{1}{6}B + \frac{1}{3\sqrt{6}}r\sqrt{B} - \frac{1}{6}\left(1 - \frac{2}{\sqrt{6}}\right)r\frac{B_3^*}{B}.$$

Hence, by substituting the above inequality (3.48) into (3.44) and using the inequality $rB_3^* \leq -B^2$ of (4) in Lemma 2, we get finally

$$0 \leq -\frac{3}{B}\left(\mu_2^{*2} - \frac{2}{3}B\right)\left(-\frac{1}{6}B + \frac{1}{3\sqrt{6}}r\sqrt{B} - \frac{1}{6}\left(1 - \frac{2}{\sqrt{6}}\right)r\frac{B_3^*}{B}\right) + 3\mu_2^{*2} + \frac{1}{2}r\mu_2^* - B$$
$$\leq \frac{3}{B}\left(\mu_2^{*2} - \frac{2}{3}B\right)\left(\frac{1}{6}B - \frac{1}{3\sqrt{6}}r\sqrt{B} - \frac{1}{6}\left(1 - \frac{2}{\sqrt{6}}\right)B\right) + 3\mu_2^{*2} + \frac{1}{2}r\mu_2^* - B$$
$$< -\frac{1}{2\sqrt{6}}B + \frac{1}{2}B - \frac{1}{2\sqrt{6}}rB - B < 0.$$

But this is impossible. Thus the proof of Theorem 3 is completed. □

Proof of Main Theorem 2. From Lemma 1 in the proof of Theorem 1, it follows that $B_3^* = B^{3/2}/\sqrt{6}$ if and only if two of μ_i^*'s are equal with each other. Therefore, from Theorem 3 and the assumptions (1) and (2) in Main Theorem 2 we complete the proof of Main Theorem 2. □

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