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On Macaulayfication of certain quasi-projective schemes

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1. Introduction.

Let X be a Noetherian scheme. A birational proper morphism $Y \to X$ of schemes is said to be a *Macaulayfication* of X if Y is a Cohen-Macaulay scheme. This notion was introduced by Faltings [**8**] and he established that there exists a Macaulayfication of a quasi-projective scheme over a Noetherian ring possessing a dualizing complex if its non-Cohen-Macaulay locus is of dimension 0 or 1. Of course, a desingularization is a Macaulayfication and Hironaka gave a desingularization of arbitrary algebraic variety over a field of characteristic 0. But Faltings' method to construct a Macaulayfication is independent of the characteristic of a scheme. Furthermore, several authors are interested in a Macaulayfication.

For example, Goto and Schenzel independently showed the converse of Faltings' result in a sense. Let A be a Noetherian local ring possessing a dualizing complex, hence its non-Cohen-Macaulay locus is closed, and assume that $\dim A/\mathfrak{p} = \dim A$ for any associated prime ideal $\mathfrak{p}$ of A. Then the non-Cohen-Macaulay locus of A consists of only the maximal ideal if and only if A is a generalized Cohen-Macaulay ring but not a Cohen-Macaulay ring [**16**]. When this is the case, Faltings [**8**, Satz 2] showed that there exists a parameter ideal $\mathfrak{q}$ of A such that the blowing-up $\operatorname{Proj} A[\mathfrak{q}t]$ of $\operatorname{Spec} A$ with center $\mathfrak{q}$ is Cohen-Macaulay, where t denotes an indeterminate. Conversely, Goto [**9**] proved that if there is a parameter ideal $\mathfrak{q}$ of A such that $\operatorname{Proj} A[\mathfrak{q}t]$ is Cohen-Macaulay, then A is a generalized Cohen-Macaulay ring. Moreover, he showed that A is Buchsbaum if and only if $\operatorname{Proj} A[\mathfrak{q}t]$ is Cohen-Macaulay for every parameter ideal $\mathfrak{q}$ of A: see also [**20**].

Brodmann [**3**] also studied the blowing-up of a generalized Cohen-Macaulay ring with center a parameter ideal. Furthermore, he constructed Macaulayfications in a quite different way from Faltings. Let A be a Noetherian local ring possessing a dualizing complex. We let $d = \dim A$ and s be the dimension of its non-Cohen-Macaulay locus. If $s = 0$, then Brodmann [**4**, Proposition 2.13] gave an ideal $\mathfrak{b}$ of height $d-1$ such that $\operatorname{Proj} A[\mathfrak{b}t]$ is Cohen-Macaulay. If $s = 1$, then Faltings' Macaulayfication [**8**, Satz 3] of $\operatorname{Spec} A$ consists of two consecutive blowing-ups $Y \to X \to \operatorname{Spec} A$ where the center of the first blowing-up is an ideal of height $d-1$. In this case, Brodmann gave two other Macaulayfications of $\operatorname{Spec} A$: the first one [**1**] is the composite of a blowing-up $X \to \operatorname{Spec} A$ with center an ideal of height $d-1$ and a finite morphism $Y \to X$; the second one [**4**, Corollary 3.11] consists of two consecutive blowing-ups $Y \to X \to \operatorname{Spec} A$ where the center of the first blowing-up is an ideal of height $d-2$.

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In this article, we are interested in a Macaulayfication of the Noetherian scheme whose non-Cohen-Macaulay locus is of dimension 2. Let A be a Noetherian ring possessing a dualizing complex and X a quasi-projective scheme over A. Then X has a dualizing complex with codimension function v. Furthermore the non-Cohen-Macaulay locus V of X is closed. We define a function $u : X \to \boldsymbol{Z}$ to be $u(p) = v(p) + \dim \overline{\{p\}}$. We will establish the following theorem:

THEOREM 1.1. *If* $\dim V \leq 2$ *and* u *is locally constant on* V, *then* X *has a Macaulayfication.*

If $\dim V \leq 1$, then u is always locally constant on V. Therefore, this theorem contains Faltings' result. Furthermore, we note if X is a projective scheme over a Gorenstein local ring, then u is constant on X.

We agree that A denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$ except for Section 6. Assume that $d = \dim A > 0$. We refer the reader to [**11**], [**12**], [**15**], and [**21**] for unexplained terminology.

2. Preliminaries.

In this section, we state some definitions and properties of a local cohomology and an ideal transform. Let $\mathfrak{b}$ be an ideal of A.

DEFINITION 2.1. The local cohomology functor $H_{\mathfrak{b}}^p(-)$ and the ideal transform functor $D_{\mathfrak{b}}^p(-)$ with respect to $\mathfrak{b}$ are defined to be

$$H_{\mathfrak{b}}^p(-) = \operatorname{inj}\lim_m \operatorname{Ext}_A^p(A/\mathfrak{b}^m, -) \quad \text{and} \quad D_{\mathfrak{b}}^p(-) = \operatorname{inj}\lim_m \operatorname{Ext}_A^p(\mathfrak{b}^m, -),$$

respectively.

For an A-module M, there exist an exact sequence

$$0 \to H_{\mathfrak{b}}^0(M) \to M \xrightarrow{i} D_{\mathfrak{b}}^0(M) \to H_{\mathfrak{b}}^1(M) \to 0 \tag{2.1.1}$$

and isomorphisms

$$D_{\mathfrak{b}}^p(M) \cong H_{\mathfrak{b}}^{p+1}(M) \quad \text{for all } p > 0.$$

They induce that

$$H_{\mathfrak{b}}^p D_{\mathfrak{b}}^0(M) = \begin{cases} 0, & p = 0, 1; \\ H_{\mathfrak{b}}^p(M), & \text{otherwise.} \end{cases} \tag{2.1.2}$$

If $\mathfrak{b}$ contains an M-regular element a, then we can regard $D_{\mathfrak{b}}^0(M)$ as a submodule of the localization M_a with respect to a and i is the inclusion.

It is well-known that $H_{\mathfrak{b}}^p(-)$ is naturally isomorphic to the direct limit of Koszul cohomology. In particular, let $\mathfrak{b} = (f_1, \ldots, f_h)$ and M be an A-module. Then

$$H_{\mathfrak{b}}^h(M) = \operatorname{inj}\lim_m M/(f_1^m, \ldots, f_h^m)M \quad \text{and} \quad H_{\mathfrak{b}}^0(M) = \bigcap_{i=1}^h 0 \underset{M}{:} \langle f_i \rangle,$$

where $0 : \langle f_i \rangle$ denotes $\bigcup_{m=1}^{\infty} 0 : f_i^m$. Furthermore, let $A \to B$ be a ring homomorphism. Then there exists a natural isomorphism $H_{\mathfrak{b}}^p(M) \cong H_{\mathfrak{b}B}^p(M)$ for a B-module M.

The following lemma is frequently used in this article.

LEMMA 2.2 (Brodmann [**2**]). *Let* $\mathfrak{b} = (f_1, \dots, f_h)$ *and* $\mathfrak{c} = (f_1, \dots, f_{h-1})$ *be two ideals. Then there exists a natural long exact sequence*

$$\cdots \to [H_{\mathfrak{c}}^{p-1}(-)]_{f_h} \to H_{\mathfrak{b}}^p(-) \to H_{\mathfrak{c}}^p(-) \to [H_{\mathfrak{c}}^p(-)]_{f_h} \to \cdots.$$

Next we state on the annihilator of local cohomology modules.

DEFINITION 2.3. For any finitely generated A-module M, we define an ideal $\mathfrak{a}_A(M)$ to be

$$\mathfrak{a}_A(M) = \prod_{p=0}^{\dim M - 1} \operatorname{ann} H_{\mathfrak{m}}^p(M).$$

We note that a finitely generated A-module M is Cohen-Macaulay if and only if $\mathfrak{a}_A(M) = A$, and that M is generalized Cohen-Macaulay if and only if $\mathfrak{a}_A(M)$ is an $\mathfrak{m}$-primary ideal. The notion of $\mathfrak{a}_A(-)$ plays a key role in this article. In fact, Schenzel [**17**] showed that $V(\mathfrak{a}_A(A))$ coincides with the non-Cohen-Macaulay locus of A if it possesses a dualizing complex and is equidimensional. He also gave the following lemma [**17**, **18**]:

LEMMA 2.4. *Let M be a finitely generated A-module and $x_1, \dots, x_n$ a system of parameters for M. Then* $(x_1, \dots, x_{i-1})M : x_i \subseteq (x_1, \dots, x_{i-1})M : \mathfrak{a}_A(M)$ *for any* $1 \leq i \leq n$. *In particular, if* $x_i \in \mathfrak{a}_A(M)$, *then the equality holds.*

Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian graded ring where $R_0 = A$. A graded module $M = \bigoplus_n M_n$ is said to be *finitely graded* if $M_n = 0$ for all but finitely many n. The following lemma is an easy consequence of [**7**].

LEMMA 2.5. *Let $\mathfrak{b}$ be a homogeneous ideal of R containing* $R_+ = \bigoplus_{n>0} R_n$ *and M a finitely generated graded R-module. We assume that A possesses a dualizing complex. Let p be the largest integer such that, for all* $q \leq p$, $H_{\mathfrak{b}}^q(M)$ *is finitely graded. Then* $\operatorname{depth} M_{(\mathfrak{p})} \geq p$ *for any closed point $\mathfrak{p}$ of* $\operatorname{Proj} R$, *that is, $\mathfrak{p}$ is a homogeneous prime ideal such that* $\dim R/\mathfrak{p} = 1$ *and* $R_+ \not\subseteq \mathfrak{p}$.

3. A Rees algebra obtained by an ideal transform.

DEFINITION 3.1. A sequence $f_1, \dots, f_h$ of elements of A is said to be a d-sequence on an A-module M if $(f_1, \dots, f_{i-1})M : f_i f_j = (f_1, \dots, f_{i-1})M : f_j$ for any $1 \leq i \leq j \leq h$.

We shall say that $f_1, \dots, f_h$ is an unconditioned strong d-sequence (for short, *u.s.d-sequence*) on M if $f_1^{n_1}, \dots, f_h^{n_h}$ is a d-sequence on M in any order and for arbitrary positive integers $n_1, \dots, n_h$.

The notion of u.s.d-sequences was introduced by Goto and Yamagishi [**10**] to refine arguments on Buchsbaum rings and generalized Cohen-Macaulay rings. Their theory contains Brodmann's study on the Rees algebra with respect to an ideal generated by a pS-sequences [3]. But Brodmann [5] also studied the ideal transform of such a Rees algebra. The purpose of this section is to study an ideal transform of the Rees algebra with respect to an ideal generated by a u.s.d-sequence.

Let $f_0, \dots, f_h$ be a sequence of elements of A where $h \geq 1$ and $\mathfrak{q} = (f_1, \dots, f_h)$.

LEMMA 3.2. *If $f_1, \dots, f_h$ be a d-sequence on A/f_0A, then*

$$[(f_1, \dots, f_k)\mathfrak{q}^n] : f_0 = (f_1, \dots, f_k)[\mathfrak{q}^n : f_0] + 0 : f_0$$

for any $1 \leq k \leq h$ and $n > 0$.

PROOF. It is obvious that the left hand side contains the right one. We shall prove the inverse inclusion by induction on k. Let a be an element of the left hand side.

When $k = 1$, we put $f_0 a = f_1 b$ where $b \in \mathfrak{q}^n$. By using [10, Theorem 1.3], we obtain $b \in (f_0) : f_1 \cap \mathfrak{q}^n \subseteq (f_0)$. If we put $b = f_0 a'$, then $a' \in \mathfrak{q}^n : f_0$ and $f_0(a - f_1 a') = 0$. Thus we get $a \in f_1[\mathfrak{q}^n : f_0] + 0 : f_0$.

When $k > 1$, we put $f_0 a = b + f_k c$ where $b \in (f_1, \dots, f_{k-1})\mathfrak{q}^n$ and $c \in \mathfrak{q}^n$. Then we obtain

$$\begin{aligned} c &\in (f_0, \dots, f_{k-1}) : f_k \cap \mathfrak{q}^n \\ &\subseteq (f_0) + (f_1, \dots, f_{k-1})\mathfrak{q}^{n-1} \end{aligned}$$

by using [10, Theorem 1.3] again. If we put $c = f_0 a' + b'$ where

$$b' \in (f_1, \dots, f_{k-1})\mathfrak{q}^{n-1},$$

then $a' \in \mathfrak{q}^n : f_0$. Thus we get

$$\begin{aligned} a - f_k a' &\in [(f_1, \dots, f_{k-1})\mathfrak{q}^n] : f_0 \\ &= (f_1, \dots, f_{k-1})[\mathfrak{q}^n : f_0] + 0 : f_0 \end{aligned}$$

by induction hypothesis. The proof is completed. □

Let $\bar{\mathfrak{q}} = \mathfrak{q} : \langle f_0 \rangle$. If f_0 is A-regular and $f_1, \dots, f_h$ is a d-sequence on $A/f_0^l A$ for all $l > 0$, then Lemma 3.2 assures us that

$$\mathfrak{q}^{n-1}\bar{\mathfrak{q}} = \bar{\mathfrak{q}}^n = \mathfrak{q}^n : \langle f_0 \rangle \quad \text{for all } n > 0. \tag{3.2.1}$$

Therefore the Rees algebra $\bar{R} = A[\bar{\mathfrak{q}}t]$ is finitely generated over $R = A[\mathfrak{q}t]$. The following is an analogue of [9, Lemma 3.4].

THEOREM 3.3. *Let $B = A[\bar{\mathfrak{q}}/f_h] = \bar{R}_{(f_h t)}$. If f_0 is A-regular and $f_1, \dots, f_h$ is a d-sequence on $A/f_0^l A$ for all $l > 0$, then f_h, $f_1/f_h, \dots, f_{h-1}/f_h$, f_0 is a regular sequence on B.*

PROOF. First we note that $f_1, \dots, f_h$ is a d-sequence on A. In fact, by using Krull's intersection theorem, we obtain

$$\begin{aligned} (f_1, \dots, f_{i-1}) : f_i f_j &= \bigcap_{l=1}^{\infty} (f_0^l, f_1, \dots, f_{i-1}) : f_i f_j \\ &= \bigcap_{l=1}^{\infty} (f_0^l, f_1, \dots, f_{i-1}) : f_j \\ &= (f_1, \dots, f_{i-1}) : f_j \end{aligned}$$

for any $1 \le i \le j \le h$. Next we show that

$$(f_1,\dots,f_{k-1}) : f_k \cap \bar{\mathfrak{q}}^n = (f_1,\dots,f_{k-1})\bar{\mathfrak{q}}^{n-1}, \tag{3.3.1}$$

for any $1 \le k \le h+1$ and $n > 1$, where $f_{h+1} = 1$. If a is an element of the left hand side, then $f_0^l a \in \mathfrak{q}^n$ for a sufficiently large l. By [**10**, Theorem 1.3], we have

$$\begin{aligned} f_0^l a &\in (f_1,\dots,f_{k-1}) : f_k \cap \mathfrak{q}^n \\ &= (f_1,\dots,f_{k-1})\mathfrak{q}^{n-1}. \end{aligned}$$

Lemma 3.2 says

$$a \in [(f_1,\dots,f_{k-1})\mathfrak{q}^{n-1}] : \langle f_0 \rangle = (f_1,\dots,f_{k-1})\bar{\mathfrak{q}}^{n-1}.$$

The inverse inclusion is clear. By (3.3.1) and [**10**, Theorem 1.7], we obtain that

$$f_h, \frac{f_1}{f_h}, \dots, \frac{f_{h-1}}{f_h}$$

is a regular sequence on B.

Finally we shall show that f_0 is regular on $B/(f_h, f_1/f_h, \dots, f_{h-1}/f_h)B$. Let $\alpha \in (f_h, f_1/f_h, \dots, f_{h-1}/f_h)B : f_0$. For a sufficiently large $n > 1$, we may assume $\alpha = a_0/f_h^n$ and

$$f_0 \frac{a_0}{f_h^n} = f_h \frac{a_h}{f_h^n} + \frac{f_1}{f_h}\frac{a_1}{f_h^n} + \cdots + \frac{f_{h-1}}{f_h}\frac{a_{h-1}}{f_h^n}$$

where $a_0, \dots, a_h \in \bar{\mathfrak{q}}^n$. Therefore

$$f_h^{m+1} f_0 a_0 = f_h^m (f_h^2 a_h + f_1 a_1 + \cdots + f_{h-1} a_{h-1})$$

in A for some $m > 0$. Take an integer l such that $f_0^l a_h \in \mathfrak{q}^n$. Then

$$\begin{aligned} f_h^{m+2} f_0^l a_h &\in (f_0^{l+1}, f_1, \dots, f_{h-1}) \cap \mathfrak{q}^{n+m+2} \\ &= (f_0^{l+1}) \cap \mathfrak{q}^{n+m+2} + (f_1,\dots,f_{h-1})\mathfrak{q}^{n+m+1} \\ &\subseteq f_0^{l+1}\bar{\mathfrak{q}}^{n+m+2} + (f_1,\dots,f_{h-1})\mathfrak{q}^{n+m+1}. \end{aligned}$$

If we put

$$f_h^{m+2} f_0^l a_h = f_0^{l+1} b_0 + f_1 b_1 + \cdots + f_{h-1} b_{h-1}$$

where $b_0 \in \bar{\mathfrak{q}}^{n+m+2}$ and $b_1, \dots, b_{h-1} \in \mathfrak{q}^{n+m+1}$, then

$$\begin{aligned} f_h^{m+2} a_h - f_0 b_0 &\in [(f_1,\dots,f_{h-1})\mathfrak{q}^{n+m+1}] : \langle f_0 \rangle \\ &= (f_1,\dots,f_{h-1})\bar{\mathfrak{q}}^{n+m+1}. \end{aligned}$$

Let

$$f_h^{m+2} a_h - f_0 b_0 = f_1 c_1 + \cdots + f_{h-1} c_{h-1}$$

where $c_1, \dots, c_{h-1} \in \bar{\mathfrak{q}}^{n+m+1}$. Then

$$f_0(f_h^{m+1} a_0 - b_0) \in (f_1,\dots,f_{h-1})\mathfrak{q}^{n+m}.$$

Therefore

$$f_h^{m+1}a_0 - b_0 \in (f_1, \dots, f_{h-1})\bar{\mathfrak{q}}^{n+m},$$

that is,

$$\alpha - f_h \frac{b_0}{f_h^{n+m+2}} \in \left(\frac{f_1}{f_h}, \dots, \frac{f_{h-1}}{f_h}\right)B.$$

The proof is completed. □

In the rest of this section, we assume that f_0 is A-regular and that $f_1, \dots, f_h$ is a u.s.d-sequence on $A/f_0^l A$ for all $l > 0$. Let $G = \bigoplus_{n \geq 0} \mathfrak{q}^n/\mathfrak{q}^{n+1}$ and $\bar{G} = \bigoplus_{n \geq 0} \bar{\mathfrak{q}}^n/\bar{\mathfrak{q}}^{n+1}$ be associated graded rings with respect to $\mathfrak{q}$ and $\bar{\mathfrak{q}}$, respectively. We shall compute local cohomology modules of $\bar{G}$ and $\bar{R}$ with respect to $\mathfrak{N} = (f_0, \dots, f_h)R + R_+$.

THEOREM 3.4. *If $p < h+1$, then*

$$[H_{\mathfrak{N}}^p(\bar{G})]_n = 0 \quad \textit{for } n \neq 1-p.$$

Furthermore

$$[H_{\mathfrak{N}}^{h+1}(\bar{G})]_n = 0 \quad \textit{for } n > -h.$$

PROOF. We shall prove that

$$(3.4.1) \qquad [H^p_{(f_0, f_1 t, \dots, f_k t)}(\bar{G})]_n = 0 \quad \text{for } n \neq 1-p$$

if $p < k+1$ by induction on k. It is obvious that f_0 is $\bar{G}$-regular. Therefore $H^0_{(f_0)}(\bar{G}) = 0$.

Suppose $k > 0$. Then $H^p_{(f_0, f_1 t, \dots, f_{k-1} t)}(\bar{G})_{f_k t} = 0$ for $p < k$ by induction hypothesis. By Lemma 2.2, we obtain isomorphisms

$$H^p_{(f_0, f_1 t, \dots, f_k t)}(\bar{G}) \cong H^p_{(f_0, f_1 t, \dots, f_{k-1} t)}(\bar{G}) \quad \text{for } p < k.$$

Therefore (3.4.1) is proved if $p < k$. We also obtain an exact sequence

$$0 \to H^k_{(f_0, f_1 t, \dots, f_k t)}(\bar{G}) \to H^k_{(f_0, f_1 t, \dots, f_{k-1} t)}(\bar{G}) \to H^k_{(f_0, f_1 t, \dots, f_{k-1} t)}(\bar{G})_{f_k t}$$

from Lemma 2.2. Hence $H^k_{(f_0, f_1 t, \dots, f_k t)}(\bar{G})$ is the limit of the direct system $\{K_m\}_{m>0}$ such that

$$K_m = \frac{(f_0^m, (f_1 t)^m, \dots, (f_{k-1} t)^m)\bar{G} : \langle f_k t \rangle}{(f_0^m, (f_1 t)^m, \dots, (f_{k-1} t)^m)\bar{G}}(m(k-1)) \quad \text{for } m > 0$$

and the homomorphism $K_m \to K_{m'}$ is induced from the multiplication of $(f_0 \cdot f_1 t \cdots f_{k-1} t)^{m'-m}$ for any $m' > m$. We shall show that it is the zero map except for degree $1-k$ if m' is sufficiently larger than m.

Let α be a homogeneous element of K_m of degree n and a its representative. That is, $a \in \bar{\mathfrak{q}}^{n+m(k-1)}$ and

$$f_k^l a \in f_0^m \bar{\mathfrak{q}}^{n+m(k-1)+l} + (f_1^m, \dots, f_{k-1}^m)\bar{\mathfrak{q}}^{n+m(k-2)+l} + \bar{\mathfrak{q}}^{n+m(k-1)+l+1}$$

for some $l > 0$. Take an integer $m' > m$ such that $f_0^{m'-m}\bar{\mathfrak{q}} \subseteq \mathfrak{q}$. Then $f_0^{m'-m}\bar{\mathfrak{q}}^n \subseteq \mathfrak{q}^n$ for any $n > 0$ by (3.2.1). By replacing α by its image in $K_{m'}$, we may assume that $a \in \mathfrak{q}^{n+m(k-1)}$ and

$$f_k^l a \in f_0^m \bar{\mathfrak{q}}^{n+m(k-1)+l} + (f_1^m, \dots, f_{k-1}^m)\mathfrak{q}^{n+m(k-2)+l} + \mathfrak{q}^{n+m(k-1)+l+1}.$$

We put $f_k^l a = b + c$ where $b \in f_0^m \bar{\mathfrak{q}}^{n+m(k-1)+l} + (f_1^m, \dots, f_{k-1}^m)\mathfrak{q}^{n+m(k-2)+l}$ and $c \in \mathfrak{q}^{n+m(k-1)+l+1}$. Then, by using [**10**, Theorem 2.6], we obtain

$$\begin{aligned} c &\in (f_0^m, \dots, f_{k-1}^m, f_k^l) \cap \mathfrak{q}^{n+m(k-1)+l+1} \\ &\subseteq f_0^m \bar{\mathfrak{q}}^{n+m(k-1)+l+1} + (f_1^m, \dots, f_{k-1}^m)\mathfrak{q}^{n+m(k-2)+l+1} + f_k^l \mathfrak{q}^{n+m(k-1)+1}. \end{aligned}$$

If we put $c = b' + f_k^l a'$ where $b' \in f_0^m \bar{\mathfrak{q}}^{n+m(k-1)+l+1} + (f_1^m, \dots, f_{k-1}^m)\mathfrak{q}^{n+m(k-2)+l+1}$ and $a' \in \mathfrak{q}^{n+m(k-1)+1}$, then $a - a'$ is also a representative of α. Therefore we may assume that $c = 0$.

By using [**10**, Theorem 2.8], we obtain

$$\begin{aligned} a &\in (f_0^m, \dots, f_{k-1}^m) : f_k \cap \mathfrak{q}^{n+m(k-1)} \\ &= (f_0^m) \cap \mathfrak{q}^{n+m(k-1)} + (f_1^m, \dots, f_{k-1}^m)\mathfrak{q}^{n+m(k-2)} \\ &\quad + \sum_{\substack{I \subseteq \{1,\dots,k-1\} \\ \sharp I \cdot (m-1) \geq n+m(k-1)}} \left\{ \prod_{i \in I} f_i^{m-1} \right\} \{[(f_0^m) + (f_i \mid i \in I)] : f_k\} \\ &\subseteq f_0^m \bar{\mathfrak{q}}^{n+m(k-1)} + (f_1^m, \dots, f_{k-1}^m)\mathfrak{q}^{n+m(k-2)} + \mathfrak{q}^{n+m(k-1)+1} \\ &\quad + \sum_{\substack{I \subseteq \{1,\dots,k-1\} \\ \sharp I \cdot (m-1) = n+m(k-1)}} \left\{ \prod_{i \in I} f_i^{m-1} \right\} \{[(f_0^m) + (f_i \mid i \in I)] : f_k\}. \end{aligned}$$

Here $\sharp I$ denotes the number of elements in I. If $n > 1 - k$, then there is no subset I of $\{1, \dots, k-1\}$ such that $\sharp I \cdot (m-1) = n + m(k-1)$. If $n < 1 - k$, then such I is a proper subset. Let $j \in \{1, \dots, k-1\} \backslash I$ and

$$d \in [(f_0^m) + (f_i \mid i \in I)] : f_k = [(f_0^m) + (f_i \mid i \in I)] : f_j.$$

Then

$$(f_0 \cdots f_{k-1}) \left\{ \prod_{i \in I} f_i^{m-1} \right\} d \in f_0^{m+1} \bar{\mathfrak{q}}^{n+(m+1)(k-1)} + (f_1^{m+1}, \dots, f_{k-1}^{m+1})\mathfrak{q}^{n+(m+1)(k-2)}.$$

In fact, if we put $f_j d = f_0^m e + g$ where $g \in (f_i \mid i \in I)$, then $e \in \bar{\mathfrak{q}}$. Thus the image of α in K_{m+1} is zero if $n \neq 1 - k$.

Put $k = h$. Then

$$[H^p_{\mathfrak{N}}(\bar{G})]_n = [H^p_{(f_0, f_{1^t}, \dots, f_{h^t})}(\bar{G})]_n = 0 \quad \text{for } n \neq 1 - p$$

if $p < h + 1$. The first assertion is proved.

Next we compute $H^{h+1}_{(f_0,f_1t,\dots,f_ht)}(\bar{G})$. It is the limit of the direct system $\{K'_m\}_{m>0}$ such that

$$K'_m = \bar{G}/(f_0^m, (f_1t)^m, \dots, (f_ht)^m)\bar{G}(mh) \quad \text{for } m > 0$$

and the homomorphism $K'_m \to K'_{m'}$ is induced from the multiplication of $(f_0 \cdot f_1t \cdots f_ht)^{m'-m}$ for any $m' > m$. We shall show that it is the zero map for degree $n > -h$ if m' is sufficiently larger than m.

Let α be a homogeneous element of K'_m of degree n and a its representative. That is, $a \in \bar{\mathfrak{q}}^{n+mh}$. If $n > -h$, then

$$(f_0 \cdots f_h)^{m'-m} a \in \mathfrak{q}^{n+m'h} = (f_1^{m'}, \dots, f_h^{m'})\mathfrak{q}^{n+m'(h-1)}$$

for a sufficiently larger m' than m. Thus the image of α in $K'_{m'}$ is zero if $n > -h$. Therefore $[H^{h+1}_{\mathfrak{N}}(\bar{G})]_n = 0$ for $n > -h$. □

By this theorem, we can compute local cohomology of $\bar{R}$.

COROLLARY 3.5. *If $h = 1, 2$, then*

$$H^p_{\mathfrak{N}}(\bar{R}) = 0 \quad \textit{for } p \neq 1, h+2$$

and $H^1_{\mathfrak{N}}(\bar{R}) = [H^1_{\mathfrak{N}}(\bar{R})]_0 = H^1_{(f_0,\dots,f_h)}(A)$.

If $h \geq 3$, then

$$H^p_{\mathfrak{N}}(\bar{R}) = 0 \quad \textit{for } p = 0, 2, 3$$

and $H^1_{\mathfrak{N}}(\bar{R}) = [H^1_{\mathfrak{N}}(\bar{R})]_0 = H^1_{(f_0,\dots,f_h)}(A)$. *Furthermore, if $4 \leq p \leq h+1$, then*

$$[H^p_{\mathfrak{N}}(\bar{R})]_n = \begin{cases} H^{p-1}_{(f_0,\dots,f_h)}(A), & \textit{for } -1 \geq n \geq 3-p; \\ 0, & \textit{otherwise}. \end{cases}$$

PROOF. Passing through the completion, we may assume that A possesses a dualizing complex. Since $H^p_{\mathfrak{N}}(\bar{G})$ is finitely graded for $p < h+1$, $H^p_{\mathfrak{N}}(\bar{R})$ is finitely graded for $p \leq h+1$ [**14**, Proposition 3]. Considering the following two exact sequences

$$0 \to \bar{R}_+ \to \bar{R} \to A \to 0 \quad \text{and} \quad 0 \to \bar{R}_+(1) \to \bar{R} \to \bar{G} \to 0,$$

we obtain the assertion: see the proof of [**5**, Theorem 4.1]. □

Let $S = \bar{R}/R$, that is, $S = \bigoplus_{n>0} \bar{\mathfrak{q}}^n/\mathfrak{q}^n$. The following proposition shall play an important role in the next section.

PROPOSITION 3.6. *If $p < h$, then*

$$[H^p_{\mathfrak{N}}(S)]_n = 0 \quad \textit{for } n \neq 1-p.$$

Moreover,

$$[H^h_{\mathfrak{N}}(S)]_n = 0 \quad \textit{for } n > 1-h.$$

PROOF. In the same way as the proof of Theorem 3.3, we find that $f_1, \dots, f_h$ is a u.s.d-sequence on A. Hence, by using [**10**, Theorem 4.2],

$$[H^p_{(f_1t,\dots,f_ht)}(G)]_n = 0 \quad \text{for } n \neq -p$$

if $p < h$. Furthermore,

$$[H^h_{(f_1 t, \dots, f_h t)}(G)]_n = 0 \quad \text{for } n > -h.$$

By using Lemma 2.2, we obtain

$$[H^p_{\mathfrak{N}}(G)]_n = 0 \quad \text{for } n \neq 1 - p, -p$$

if $p < h$ and

$$[H^p_{\mathfrak{N}}(G)]_n = 0 \quad \text{for } n > 1 - p$$

if $p = h,\ h+1$.

Since $\bar{\mathfrak{q}}^2 = \mathfrak{q}\bar{\mathfrak{q}}$, there exists an exact sequence

$$0 \to S(1) \to G \xrightarrow{\phi} \bar{G} \to S \to 0.$$

Let T be the image of ϕ. We shall show

$$[H^p_{\mathfrak{N}}(S)]_n = [H^p_{\mathfrak{N}}(T)]_n = 0 \quad \text{for } n > 1 - p$$

by induction on $h - p$. If $p > h+1$, then the assertion is obvious. Let $p \leq h+1$. Then following two exact sequences

$$H^p_{\mathfrak{N}}(\bar{G}) \to H^p_{\mathfrak{N}}(S) \to H^{p+1}_{\mathfrak{N}}(T) \to H^{p+1}_{\mathfrak{N}}(\bar{G}),$$

$$H^p_{\mathfrak{N}}(G) \to H^p_{\mathfrak{N}}(T) \to H^{p+1}_{\mathfrak{N}}(S)(1) \to H^{p+1}_{\mathfrak{N}}(G)$$

and the induction hypothesis imply

$$[H^p_{\mathfrak{N}}(S)]_n = [H^p_{\mathfrak{N}}(T)]_n = 0 \quad \text{for } n > 1 - p.$$

In the same way, we can prove that

$$[H^p_{\mathfrak{N}}(S)]_n = [H^p_{\mathfrak{N}}(T)]_n = 0 \quad \text{for } n < 1 - p$$

if $p < h$ by induction on p. □

Finally we show that $\bar{R}$ is an ideal transform of R in a sense.

PROPOSITION 3.7. $\bar{R}_+ = D^0_{(f_0, \dots, f_h)}(R_+)$.

PROOF. We first show that f_0, f_1 is a regular sequence on $\bar{R}_+$. Let $n > 0$. Since f_0 is A-regular, it is also $\bar{\mathfrak{q}}^n$-regular. Let $a \in [f_0\bar{\mathfrak{q}}^n] : f_1 \cap \bar{\mathfrak{q}}^n$. Then $f_0^l a \in \mathfrak{q}^n$ for a sufficiently large l. Since $f_1 a \in (f_0)$, we have $f_0^l a \in (f_0^{l+1}) : f_1 \cap \mathfrak{q}^n \subseteq f_0^{l+1}\bar{\mathfrak{q}}^n$, that is, $a \in f_0\bar{\mathfrak{q}}^n$. Thus we have shown that f_1 is $\bar{R}_+/f_0\bar{R}_+$-regular.

By this and (2.1.1), we obtain

$$D^0_{(f_0, \dots, f_h)}(R_+) \subseteq D^0_{(f_0, \dots, f_h)}(\bar{R}_+) = \bar{R}_+. \tag{3.7.1}$$

Since $\bar{\mathfrak{q}}^n = \mathfrak{q}^{n-1}\bar{\mathfrak{q}}$ for $n \geq 2$, $(f_0^l, f_1, \dots, f_h)\bar{R}_+ \subseteq R_+$ for a sufficiently large l. Hence, we obtain the inverse inclusion of (3.7.1). The proof is completed. □

4. A blowing-up with respect to a certain subsystem of parameters.

In this section, we assume that A possesses a dualizing complex. We fix an integer $s \geq \dim A/\mathfrak{a}_A(A)$. Since $\dim A/\mathfrak{a}_A(M) < \dim M$ for any finitely generated A-module M [**19**, Korollar 2.2.4], there exists a system of parameters $x_1, \ldots, x_d$ for A such that

$$\text{(4.0.1)} \qquad \begin{cases} x_{s+1}, \ldots, x_d \in \mathfrak{a}_A(A); \\ x_i \in \mathfrak{a}_A(A/(x_{i+1}, \ldots, x_d)), \quad \text{for } i \leq s. \end{cases}$$

This notion is a slight improvement of a p-standard system of parameters, which was introduced by Cuong [**6**]. He also gave the statement (1) of Theorem 4.2.

Lemma 4.1. *Let $n_1, \ldots, n_i$ be arbitrary positive integers. Then*

$$\begin{aligned}(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_{k+1}, \ldots, x_d) : x_i^{n_i} \cap (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_k, \ldots, x_d) \\ = (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_{k+1}, \ldots, x_d)\end{aligned}$$

for any $1 \leq i \leq k \leq d$.

Proof. It is obvious that the left hand side contains the right one. Let a be an element of the left hand side and $a = b + x_k c$ where $b \in (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_{k+1}, \ldots, x_d)$. Then

$$\begin{aligned}c &\in (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_{k+1}, \ldots, x_d) : x_i^{n_i} x_k \\ &= (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_{k+1}, \ldots, x_d) : x_k\end{aligned}$$

by Lemma 2.4. Therefore $x_k c$, $a \in (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_{k+1}, \ldots, x_d)$. The proof is completed. □

Let $\mathfrak{q} = (x_{s+1}, \ldots, x_d)$. Lemma 2.2 assures us that $x_{s+1}, \ldots, x_d$ is a u.s.d-sequence on A. Furthermore, we have the following theorem:

Theorem 4.2. (1) *The sequences $x_1^{n_1}, \ldots, x_s^{n_s}, x_{\sigma(s+1)}^{n_{s+1}}, \ldots, x_{\sigma(d)}^{n_d}$ is a d-sequence on A for any positive integers $n_1, \ldots, n_d$ and for any permutation σ on $s+1, \ldots, d$.*

(2) *If $s > 0$, then $x_1^{n_1}, \ldots, x_s^{n_s}$ is a d-sequence on $A/\mathfrak{q}^n$ for any positive integers $n_1, \ldots, n_s$ and n.*

Proof. (1): Let $1 \leq i \leq j \leq d$. We have only to prove that

$$(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}) : x_i^{n_i} x_j^{n_j} = (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}) : x_j^{n_j}$$

for any positive integers $n_1, \ldots, n_d$. If $j > s$, then the both sides are equal to $(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}) : \mathfrak{a}_A(A)$.

Assume that $j \leq s$ and take an element a of the left hand side. By using Lemma 2.4, we get

$$\begin{aligned}a &\in (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_{j+1}, \ldots, x_d) : x_i^{n_i} x_j^{n_j} \\ &= (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_{j+1}, \ldots, x_d) : x_j^{n_j}.\end{aligned}$$

Hence we have

$$x_j^{n_j} a \in (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}) : x_i^{n_i} \cap (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_{j+1}, \ldots, x_d)$$
$$= (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}})$$

by repeating to use Lemma 4.1.

(2): If $n = 1$, then the assertion is proved in the same way as above. Let $1 \leq i \leq j \leq s$ and $n > 1$. Then $x_{s+1}, \ldots, x_d$ is a d-sequence on $A/(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_i^{n_i} x_j^{n_j})$. By using Lemma 3.2, we obtain

$$\begin{aligned}
&[(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}) + \mathfrak{q}^n] : x_i^{n_i} x_j^{n_j} \\
&= (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}) : x_i^{n_i} x_j^{n_j} + \mathfrak{q}^{n-1}[(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_{s+1}, \ldots, x_d) : x_i^{n_i} x_j^{n_j}] \\
&= (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}) : x_j^{n_j} + \mathfrak{q}^{n-1}[(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}, x_{s+1}, \ldots, x_d) : x_j^{n_j}] \\
&\subseteq [(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}) + \mathfrak{q}^n] : x_j^{n_j}.
\end{aligned}$$

Here the second equality follows from the case of $n = 1$. Thus the proof is completed. □

In the same way as the proof of Theorem 3.3, we find that any subsequence of $x_1^{n_1}, \ldots, x_d^{n_d}$ is a d-sequence on A and any subsequence of $x_1^{n_1}, \ldots, x_s^{n_s}$ is a d-sequence on $A/\mathfrak{q}^n$ for arbitrary positive integers $n_1, \ldots, n_d$ and n.

Corollary 4.3. *Fix an integer k such that $1 \leq k \leq d$. Then*

$$H^p_{(x_k, \ldots, x_d)}(A) = \operatorname{inj}\lim_m \frac{(x_k^m, \ldots, x_{k+p-1}^m) : x_{k+p}}{(x_k^m, \ldots, x_{k+p-1}^m)} \quad \text{for } p < d - k + 1.$$

Proof. We shall prove that

$$H^p_{(x_k, \ldots, x_l)}(A) = \operatorname{inj}\lim_m \frac{(x_k^m, \ldots, x_{k+p-1}^m) : x_{k+p}}{(x_k^m, \ldots, x_{k+p-1}^m)} \quad \text{for } p < l - k + 1$$

by induction on $l \geq k$. If $l = k$, then $H^0_{(x_k)}(A) = 0 :_A x_k$.

Suppose $l > k$. Then $x_k, \ldots, x_{l-1}$ is a regular sequence on A_{x_l} because $x_k, \ldots, x_l$ is a d-sequence on A. Hence we obtain isomorphisms

$$H^p_{(x_k, \ldots, x_l)}(A) \cong H^p_{(x_k, \ldots, x_{l-1})}(A) \quad \text{for all } p < l - k$$

and an exact sequence

$$0 \to H^{l-k}_{(x_k, \ldots, x_l)}(A) \to H^{l-k}_{(x_k, \ldots, x_{l-1})}(A) \to H^{l-k}_{(x_k, \ldots, x_{l-1})}(A)_{x_l}$$

by Lemma 2.2. This exact sequence is the direct limit of the exact sequence

$$0 \to \frac{(x_k^m, \ldots, x_{l-1}^m) : x_l}{(x_k^m, \ldots, x_{l-1}^m)} \to A/(x_k^m, \ldots, x_{l-1}^m) \to [A/(x_k^m, \ldots, x_{l-1}^m)]_{x_l}.$$

Thus the proof is completed. □

If $s=0$, then $\operatorname{Proj} A[\mathfrak{q}t] \to \operatorname{Spec} A$ is a Macaulayfication of $\operatorname{Spec} A$: see Theorem 5.1 for details. In the rest of this section, we shall observe $\operatorname{Proj} A[\mathfrak{q}t]$ when $s>0$. Assume that $s>0$ and fix an integer k such that $1 \le k \le s$. We shall compute local cohomology modules of $R = A[\mathfrak{q}t]$ with respect to $(x_k, \dots, x_{s+1})$. Let $\mathfrak{M} = \mathfrak{m}R + R_+$.

THEOREM 4.4. $H^0_{(x_k,\dots,x_{s+1})}(R) = 0 :_A x_k$.

PROOF. Since $x_k, x_{s+1}, \dots, x_d$ is a d-sequence on A, $0 :_A x_k \cap \mathfrak{q}^n = 0$ for $n>0$ by [**10**, Theorem 1.3]. That is,

$$H^0_{(x_k,\dots,x_{s+1})}(\mathfrak{q}^n) = \begin{cases} 0 :_A x_k, & \text{if } n=0; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $H^0_{(x_k,\dots,x_{s+1})}(R) = \bigoplus_{n\ge 0} H^0_{(x_k,\dots,x_{s+1})}(\mathfrak{q}^n) = 0 :_A x_k$. □

Let $C = A[t]/R$, that is, $C = \bigoplus_{n>0} A/\mathfrak{q}^n$.

LEMMA 4.5. *For $k \le l \le s+1$ and $p \le l-k$, the natural homomorphism*

$$\alpha_l^p : H^p_{(x_k,\dots,x_l)}(A[t]) \to H^p_{(x_k,\dots,x_l)}(C)$$

is a monomorphism except for degree 0.

PROOF. We shall work by induction on l. If $l=k$, then $0 :_A x_k \cap \mathfrak{q}^n = 0$ for $n>0$. Therefore

$$\alpha_k^0 : 0 \underset{A[t]}{:} x_k \to \bigoplus_{n>0} \mathfrak{q}^n : x_k/\mathfrak{q}^n$$

is a monomorphism except for degree 0. Let $k < l \le s$. Then $x_k, \dots, x_{l-1}$ is a regular sequence on A_{x_l} and on C_{x_l} by Theorem 4.2. By using Lemma 2.2, we obtain commutative diagrams

$$\begin{array}{ccc} H^p_{(x_k,\dots,x_l)}(A[t]) & \xrightarrow{\sim} & H^p_{(x_k,\dots,x_{l-1})}(A[t]) \\ {\scriptstyle \alpha_l^p}\downarrow & & \downarrow{\scriptstyle \alpha_{l-1}^p} \\ H^p_{(x_k,\dots,x_l)}(C) & \xrightarrow{\sim} & H^p_{(x_k,\dots,x_{l-1})}(C) \end{array} \qquad \text{for } p < l-k$$

and

$$\begin{array}{ccccccc} 0 \longrightarrow & H^{l-k}_{(x_k,\dots,x_l)}(A[t]) & \longrightarrow & H^{l-k}_{(x_k,\dots,x_{l-1})}(A[t]) & \longrightarrow & H^{l-k}_{(x_k,\dots,x_{l-1})}(A[t])_{x_l} \\ & {\scriptstyle \alpha_l^{l-k}}\downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & H^{l-k}_{(x_k,\dots,x_l)}(C) & \longrightarrow & H^{l-k}_{(x_k,\dots,x_{l-1})}(C) & \longrightarrow & H^{l-k}_{(x_k,\dots,x_{l-1})}(C)_{x_l} \end{array}$$

whose rows are exact. Therefore the assertion is true for $p < l-k$ and we find that α_l^{l-k} is the direct limit of

$$\alpha_{l,m} : \frac{(x_k^m, \dots, x_{l-1}^m)A[t] : x_l}{(x_k^m, \dots, x_{l-1}^m)A[t]} \to \bigoplus_{n>0} \frac{[(x_k^m, \dots, x_{l-1}^m) + \mathfrak{q}^n] : x_l}{(x_k^m, \dots, x_{l-1}^m) + \mathfrak{q}^n}.$$

Since $x_l, x_{s+1}, \ldots, x_d$ is a d-sequence on $A/(x_k^m, \ldots, x_{l-1}^m)$,

$$(x_k^m, \ldots, x_{l-1}^m) : x_l \cap [(x_k^m, \ldots, x_{l-1}^m) + \mathfrak{q}^n] = (x_k^m, \ldots, x_{l-1}^m) \quad \text{for } n > 0.$$

Therefore $\alpha_{l,m}$ is a monomorphism except for degree 0 and α_l^{l-k} is also.

If $l = s+1$, then $x_k, \ldots, x_s$ is a regular sequence on $A_{x_{s+1}}$ and $C_{x_{s+1}} = 0$. The assertion is proved in the same way as above. □

Of course, α_{s+1}^p is the zero map in degree 0. Therefore there exists an exact sequence

$$(4.5.1) \qquad 0 \to \operatorname{Coker} \alpha_{s+1}^{p-1} \to H^p_{(x_k,\ldots,x_{s+1})}(R) \to H^p_{(x_k,\ldots,x_{s+1})}(A) \to 0$$

for $0 < p \le s - k + 1$.

THEOREM 4.6. *Let $0 \le q \le s-k$. Then*

$$(x_{k+q}, \ldots, x_d) \operatorname{Coker} \alpha_{s+1}^q = 0$$

and $H^p_{\mathfrak{M}}(\operatorname{Coker} \alpha_{s+1}^q)$ is finitely graded for $p < d - s$.

PROOF. We know that $\operatorname{Coker} \alpha_{s+1}^q = \operatorname{Coker} \alpha_{k+q}^q = \operatorname{inj\,lim}_m \operatorname{Coker} \alpha_{k+q,m}$ and

$$\operatorname{Coker} \alpha_{k+q,m} = \bigoplus_{n>0} \frac{[(x_k^m, \ldots, x_{k+q-1}^m) + \mathfrak{q}^n] : x_{k+q}}{(x_k^m, \ldots, x_{k+q-1}^m) : x_{k+q} + \mathfrak{q}^n}.$$

By using Theorem 4.2 and Lemma 3.2, we obtain

$$(4.6.1) \qquad \begin{aligned}&[(x_k^m, \ldots, x_{k+q-1}^m) + \mathfrak{q}^n] : x_{k+q} \\ &\quad = (x_k^m, \ldots, x_{k+q-1}^m) : x_{k+q} + \mathfrak{q}^{n-1}[(x_k^m, \ldots, x_{k+q-1}^m, x_{s+1}, \ldots, x_d) : x_{k+q}].\end{aligned}$$

Therefore $\operatorname{Coker} \alpha_{k+q,m}$ is annihilated by $(x_{k+q}, \ldots, x_d)$ and $\operatorname{Coker} \alpha_{s+1}^q$ is also.

Next we compute local cohomology modules of $\operatorname{Coker} \alpha_{s+1}^q$. We note that x_{k+q} is a regular element on $A/(x_k^m, \ldots, x_{k+q-1}^m) : x_{k+q}$ and that $x_{s+1}, \ldots, x_d$ is a u.s.d-sequence on $A/(x_k^m, \ldots, x_{k+q-1}^m) : x_{k+q} + (x_{k+q}^l)$ for any $l > 0$: see [**13**, Proposition 2.2]. Therefore, by Proposition 3.6,

$$(4.6.2) \qquad H^p_{(x_{k+q}, x_{s+1}, \ldots, x_d)R + R_+}(\operatorname{Coker} \alpha_{k+q,m}) \text{ is concentrated in degree } 1-p$$

if $p < d - s$. Hence $H^p_{(x_{k+q}, x_{s+1}, \ldots, x_d)R+R_+}(\operatorname{Coker} \alpha_{s+1}^q)$ is also. By the spectral sequence $E_2^{pq} = H^p_{\mathfrak{M}} H^q_{(x_{k+q}, x_{s+1}, \ldots, x_d)R+R_+}(-) \Rightarrow H^{p+q}_{\mathfrak{M}}(-)$, we obtain the second assertion. □

Next we compute $H^{s-k+2}_{(x_k,\ldots,x_{s+1})}(R)$.

THEOREM 4.7. *Let $A_m = A/(x_k^m, \ldots, x_s^m)$ and $\mathfrak{q}_m = \mathfrak{q}A_m$ for any positive integer m. Then*

$$H^{s-k+2}_{(x_k,\ldots,x_{s+1})}(R) = \operatorname*{inj\,lim}_{m,l} A_m[\mathfrak{q}_m t]/x_{s+1}^l A_m[\mathfrak{q}_m t].$$

In particular, $H^p_{\mathfrak{M}} H^{s-k+2}_{(x_k,\ldots,x_{s+1})}(R)$ is finitely graded for $p < d - s$.

PROOF. We consider the exact sequence

$$H^{s-k}_{(x_k,\dots,x_s)}(A[t]) \xrightarrow{\alpha_s^{s-k}} H^{s-k}_{(x_k,\dots,x_s)}(C) \longrightarrow H^{s-k+1}_{(x_k,\dots,x_s)}(R) \longrightarrow H^{s-k+1}_{(x_k,\dots,x_s)}(A[t]) \xrightarrow{\beta} H^{s-k+1}_{(x_k,\dots,x_s)}(C).$$

Since β is the direct limit of

$$A[t]/(x_k^m,\dots,x_s^m)A[t] \to C/(x_k^m,\dots,x_s^m)C,$$

we have $\operatorname{Ker}\beta = \operatorname{inj\,lim}_m A_m[\mathfrak{q}_m t]$. Taking local cohomology modules of a short exact sequence

$$0 \to \operatorname{Coker}\alpha_s^{s-k} \to H^{s-k+1}_{(x_k,\dots,x_s)}(R) \to \operatorname{Ker}\beta \to 0$$

with respect to (x_{s+1}), we obtain

$$\text{(4.7.1)} \qquad H^1_{(x_{s+1})}H^{s-k+1}_{(x_k,\dots,x_s)}(R) = H^1_{(x_{s+1})}(\operatorname{Ker}\beta),$$

because $\operatorname{Coker}\alpha_s^{s-k} = \operatorname{Coker}\alpha_{s+1}^{s-k}$ is annihilated by x_{s+1}. The left hand side of (4.7.1) coincides with $H^{s-k+2}_{(x_k,\dots,x_{s+1})}(R)$ by Lemma 2.2. Thus the first assertion is proved.

Since $x_{s+1},\dots,x_d$ is a u.s.d-sequence on A_m, $H^p_{(x_{s+1},\dots,x_d)R+R_+}(A_m[\mathfrak{q}_m t])$ is concentrated in degree $0 \ge n \ge s-d+2$ if $p \le d-s$: see [10, Theorem 4.1]. From the exact sequence

$$0 \to 0 \underset{A_m}{:} x_{s+1} \to A_m[\mathfrak{q}_m t] \xrightarrow{x_{s+1}^l} A_m[\mathfrak{q}_m t] \to A_m[\mathfrak{q}_m t]/x_{s+1}^l A_m[\mathfrak{q}_m t] \to 0$$

and the spectral sequence $E_2^{pq} = H^p_{\mathfrak{M}}H^q_{(x_{s+1},\dots,x_d)R+R_+}(-) \Rightarrow H^{p+q}_{\mathfrak{M}}(-)$, we find that

$$\text{(4.7.2)} \qquad H^p_{\mathfrak{M}}(A_m[\mathfrak{q}_m t]/x_{s+1}^l A_m[\mathfrak{q}_m t]) \text{ is concentrated in degree } 0 \ge n \ge s-d+2$$

if $p < d-s$. Taking the direct limit of it, we obtain the second assertion. □

Finally we compute local cohomology modules of $B = A[\mathfrak{q}/x_{s+1}] = R_{(x_{s+1}t)}$.

THEOREM 4.8. *Let $\mathfrak{n}$ be a maximal ideal of B. Then*

$$H^p_{\mathfrak{n}}H^q_{(x_k,\dots,x_{s+1})}(B) = 0 \quad \textit{if } q = 0 \textit{ or } p < d-s-1.$$

Furthermore $(x_{k+q-1},\dots,x_{s+1})H^q_{(x_k,\dots,x_{s+1})}(B) = 0$ *for* $q < s-k+2$.

PROOF. Since the blowing-up $\operatorname{Proj} R \to \operatorname{Spec} A$ is a closed map, there exists a homogeneous prime ideal $\mathfrak{p}$ of R such that $x_{s+1}t \notin \mathfrak{p}$, $\dim R/\mathfrak{p} = 1$ and $\mathfrak{n} = [\mathfrak{p}R_{x_{s+1}t}]_0$.

Since x_{s+1} is B-regular, $H^0_{(x_k,\dots,x_{s+1})}(B) = 0$.

Let $1 \le q \le s-k+1$. By applying Lemma 2.5 to (4.6.2), we obtain

$$H^p_{\mathfrak{n}}((\operatorname{Coker}\alpha_{k+q-1,m})_{(x_{s+1}t)}) = 0 \quad \text{for } p < d-s-1.$$

By taking the direct limit of it and using (4.5.1), we have

$$H^p_{\mathfrak{n}}H^q_{(x_k,\dots,x_{s+1})}(B) = 0 \quad \text{for } p < d-s-1.$$

Moreover Theorem 4.6 also assures us $(x_{k+q-1},\dots,x_{s+1})H^q_{(x_k,\dots,x_{s+1})}(B) = 0$.

Next we consider $H^{s-k+2}_{(x_k,\dots,x_{s+1})}(B)$. By applying Lemma 2.5 to (4.7.2) and by taking direct limit, we have

$$H^p_{\mathfrak{n}} H^{s-k+2}_{(x_k,\dots,x_{s+1})}(B) = 0 \quad \text{for } p < d - s - 1.$$

Thus the proof is completed. □

5. Macaulayfications of local rings.

In this section, we shall construct a Macaulayfication of the affine scheme $\operatorname{Spec} A$ if its non-Cohen-Macaulay locus is of dimension 2. Assume that A possesses a dualizing complex and $\dim A/\mathfrak{p} = d$ for any associated prime ideal $\mathfrak{p}$ of A. Then $V(\mathfrak{a}_A(A))$ coincides with the non-Cohen-Macaulay locus of A. We fix an integer $s \geq \dim A/\mathfrak{a}_A(A)$ and let $x_1, \dots, x_d$ be a system of parameters for A satisfying (4.0.1).

First we review Faltings' results [**8**, Sätze 2 and 3]. Let $\mathfrak{q} = (x_{s+1}, \dots, x_d)$, $R = A[\mathfrak{q}t]$ and $X = \operatorname{Proj} R$.

THEOREM 5.1. *With notation as above,*

$$\operatorname{depth} \mathcal{O}_{X,p} \geq d - s \quad \textit{for any closed point } p \textit{ of } X.$$

If $s = 0$ or $A/\mathfrak{q}$ is Cohen-Macaulay, then X is Cohen-Macaulay.

PROOF. Since $x_{s+1}, \dots, x_d$ is a u.s.d-sequence on A, $H^p_{(x_{s+1},\dots,x_d)R+R_+}(R)$ is finitely graded for $p \leq d - s$: see [**10**, Theorem 4.1]. By using Lemma 2.5, we obtain the first assertion.

Furthermore since $\dim \mathcal{O}_{X,p} = d$ for any closed point p of X, X is Cohen-Macaulay if $s = 0$.

Assume that $s > 0$ and $A/\mathfrak{q}$ is Cohen-Macaulay. Then $x_1, \dots, x_s$ is a regular sequence on $A/\mathfrak{q}$. We use theorems in Section 4 as $k = 1$. From (4.6.1), we find that $\operatorname{Coker} \alpha^q_{s+1} = 0$ for all $q \leq s - 1$. That is, $H^p_{\mathfrak{M}} H^q_{(x_1,\dots,x_{s+1})}(R)$ is finitely graded if $p < d - s$ or $q < s + 1$. By the spectral sequence $E_2^{pq} = H^p_{\mathfrak{M}} H^q_{(x_1,\dots,x_{s+1})}(-) \Rightarrow H^{p+q}_{\mathfrak{M}}(-)$, we find that $H^p_{\mathfrak{M}}(R)$ is finitely graded for $p < d + 1$. Lemma 2.5 assures us

$$\operatorname{depth} \mathcal{O}_{X,p} \geq d \quad \text{for any closed point } p \text{ of } X.$$

The proof is completed. □

From now on, we assume that $s > 0$.

Since x_s is A-regular, $\mathfrak{q}$ is a reduction of $\bar{\mathfrak{q}} = \mathfrak{q} : x_s$ by (3.2.1). We put $\bar{R} = A[\bar{\mathfrak{q}}t]$ and $\bar{X} = \operatorname{Proj} \bar{R}$. Then $\bar{X} \to X$ is a finite morphism.

THEOREM 5.2. *With notation as above,*

$$\operatorname{depth} \mathcal{O}_{\bar{X},\bar{p}} \geq d - s + 1 \quad \textit{for any closed point } \bar{p} \textit{ of } \bar{X}.$$

In particular, if $s = 1$, then $\bar{X}$ is Cohen-Macaulay.

PROOF. By Corollary 3.5, $H^p_{(x_s,\dots,x_d)R+R_+}(\bar{R})$ is finitely graded for $p \leq d - s + 1$. By using Lemma 2.5, we obtain the assertion. □

Next we consider an ideal $\mathfrak{b} = \mathfrak{q}^2 + x_s\mathfrak{q} = (x_s, \dots, x_d)\mathfrak{q}$. We put $S = A[\mathfrak{b}t]$ and $Y = \operatorname{Proj} S$. Then Y is the blowing-up of X with center $(x_s, \dots, x_d)\mathcal{O}_X$.

THEOREM 5.3. *With notation as above,*

$$\operatorname{depth} \mathcal{O}_{Y,q} \geq d - s + 1 \quad \textit{for any closed point } q \textit{ of } Y.$$

Furthermore, if $s = 1$ or A is Cohen-Macaulay, then Y is Cohen-Macaulay.

PROOF. Since $(x_s x_{s+1}, \dots, x_s x_d, x_{s+1}^2, \dots, x_d^2)\mathfrak{b}^{d-s-1} = \mathfrak{b}^{d-s}$, we have only to compute the depth of $C_0 = A[\mathfrak{b}/x_s x_{s+1}]$ and $C_1 = A[\mathfrak{b}/x_{s+1}^2]$. If we put $B = A[\mathfrak{q}/x_{s+1}]$, then

$$C_0 = B[x_{s+1}/x_s] \cong B[T]/(x_s T - x_{s+1}) : \langle x_s \rangle,$$
$$C_1 = B[x_s/x_{s+1}] \cong B[T]/(x_{s+1} T - x_s) : \langle x_{s+1} \rangle,$$

where T denotes an indeterminate. We note that B, C_0, C_1 are subrings of the total quotient ring of A because $x_1, \dots, x_d$ are A-regular elements.

First we consider C_0. We regard it as a homomorphic image of $B[T]$. Let $\mathfrak{l}_0$ be a maximal ideal of C_0 and $\mathfrak{n} = \mathfrak{l}_0 \cap B$. Then $\mathfrak{n}$ is a maximal ideal of B because $\operatorname{Spec} C_0 \cup \operatorname{Spec} C_1 \to \operatorname{Spec} B$ is a blowing-up with center $(x_s, x_{s+1})B$, hence a closed map. There exists a polynomial f over B such that $\mathfrak{l}_0 = \mathfrak{n}C_0 + fC_0$ and the leading coefficient of f is not contained in $\mathfrak{n}$.

By Lemma 2.2 and Theorem 4.8, we have, for any $1 \leq k \leq s$,

$$H^p_{\mathfrak{n}B[T]+fB[T]} H^q_{(x_k,\dots,x_{s+1})}(B[T]) = 0 \quad \text{if } p < d - s \text{ or } q = 0. \tag{5.3.1}$$

In fact, the leading coefficient of f is a regular element on $H^{d-s}_{\mathfrak{n}} H^q_{(x_k,\dots,x_{s+1})}(B[T])$ because it acts on the injective envelope of $B/\mathfrak{n}$ as isomorphism. Taking the local cohomology of a short exact sequence

$$0 \to B[T] \xrightarrow{x_s T - x_{s+1}} B[T] \to B[T]/(x_s T - x_{s+1}) \to 0$$

with respect to $(x_k, \dots, x_{s+1}) = (x_k, \dots, x_s, x_s T - x_{s+1})$, we obtain an exact sequence

$$\begin{aligned} 0 \to H^{s-k+1}_{(x_k,\dots,x_{s+1})}(B[T]) &\to H^{s-k+1}_{(x_k,\dots,x_{s+1})}(B[T]/(x_s T - x_{s+1})) \\ &\to H^{s-k+2}_{(x_k,\dots,x_{s+1})}(B[T]) \to H^{s-k+2}_{(x_k,\dots,x_{s+1})}(B[T]) \to 0, \end{aligned}$$

because $(x_s, x_{s+1})H^{s-k+1}_{(x_k,\dots,x_{s+1})}(B) = 0$ by Theorem 4.8. This and (5.3.1) show that

$$H^p_{\mathfrak{n}B[T]+fB[T]} H^{s-k+1}_{(x_k,\dots,x_{s+1})}(B[T]/(x_s T - x_{s+1})) = 0 \quad \text{for } p < d - s.$$

Taking the local cohomology of an exact sequence

$$0 \to \frac{(x_s T - x_{s+1}) : \langle x_s \rangle}{(x_s T - x_{s+1})} \to B[T]/(x_s T - x_{s+1}) \to C_0 \to 0$$

with respect to $(x_k, \dots, x_{s+1})$, we obtain

$$H^{s-k+1}_{(x_k,\dots,x_{s+1})}(C_0) = H^{s-k+1}_{(x_k,\dots,x_{s+1})}(B[T]/(x_s T - x_{s+1})),$$

that is,

$$H^p_{\mathfrak{l}_0} H^{s-k+1}_{(x_k,\dots,x_{s+1})}(C_0) = 0 \quad \text{for } p < d-s. \tag{5.3.2}$$

We note that x_s is C_0-regular. Put $k = s$. Then we have

$$H^p_{\mathfrak{l}_0} H^q_{(x_s,x_{s+1})}(C_0) = 0 \quad \text{if } p < d-s \text{ or } q < 1.$$

By the spectral sequence $E_2^{pq} = H^p_{\mathfrak{l}_0} H^q_{(x_s,x_{s+1})}(-) \Rightarrow H^{p+q}_{\mathfrak{l}_0}(-)$, we obtain

$$H^p_{\mathfrak{l}_0}(C_0) = 0 \quad \text{for } p < d-s+1, \tag{5.3.3}$$

that is, $\operatorname{depth}(C_0)_{\mathfrak{l}_0} \geq d-s+1$.

In the same way, we can show that $\operatorname{depth}(C_1)_{\mathfrak{l}_1} \geq d-s+1$ for any maximal ideal $\mathfrak{l}_1$ of C_1. Thus the first assertion is proved. In particular, Y is Cohen-Macaulay if $s=1$.

Assume that A is Cohen-Macaulay. Using [**8**, Lemma 1] twice, we find that

$$x_{s+1}T_{s+2} - x_{s+2}, \dots, x_{s+1}T_d - x_d, x_sT_{s+1} - x_{s+1}$$

is a regular sequence on $A[T_{s+1}, \dots, T_d]$. Therefore

$$C_0 \cong A[T_{s+1}, \dots, T_d]/(x_{s+1}T_{s+2} - x_{s+2}, \dots, x_{s+1}T_d - x_d, x_sT_{s+1} - x_{s+1})$$

is Cohen-Macaulay. In the same way, we can show that C_1 is Cohen-Macaulay. The proof is completed. □

In the rest of this section, we assume that $s \geq 2$ and let $\bar{\mathfrak{b}} = \mathfrak{b} : \langle x_{s-1} \rangle$.

Lemma 5.4. *For any positive integer n,*

$$\bar{\mathfrak{b}}^n = \mathfrak{b}^n : \langle x_{s-1} \rangle = \mathfrak{q}\mathfrak{b}^{n-1}[(x_s, \dots, x_d) : x_{s-1}] + x_s^n \mathfrak{q}^{n-1}[\mathfrak{q} : x_{s-1}].$$

In particular, $\bar{\mathfrak{b}}^2 = \mathfrak{b}\bar{\mathfrak{b}}$.

Proof. It is sufficient to prove

$$\mathfrak{b}^n : \langle x_{s-1} \rangle \subseteq \mathfrak{q}\mathfrak{b}^{n-1}[(x_s, \dots, x_d) : x_{s-1}] + x_s^n \mathfrak{q}^{n-1}[\mathfrak{q} : x_{s-1}].$$

Take $a \in \mathfrak{b}^n : \langle x_{s-1} \rangle$. Then, by Lemma 2.4, Lemma 3.2 and Theorem 4.2, we have

$$\begin{aligned} a &\in (x_s, \dots, x_d)^{2n} : \langle x_{s-1} \rangle \\ &= (x_s, \dots, x_d)^{2n-1}[(x_s, \dots, x_d) : x_{s-1}] \\ &= [\mathfrak{q}^{2n-1} + x_s\mathfrak{q}^{2n-2} + \dots + (x_s^{2n-1})][(x_s, \dots, x_d) : x_{s-1}] \\ &\subseteq \mathfrak{q}\mathfrak{b}^{n-1}[(x_s, \dots, x_d) : x_{s-1}] + (x_s^n). \end{aligned}$$

If we put $a = b + x_s^n a'$ where $b \in \mathfrak{q}\mathfrak{b}^{n-1}[(x_s, \dots, x_d) : x_{s-1}]$, then $x_s^n a' \in \mathfrak{b}^n : \langle x_{s-1} \rangle$. Since $x_{s-1}^l x_s^n a' \in \mathfrak{b}^n$ for a sufficiently large l, we can put $x_{s-1}^l x_s^n a' = c + x_s^n d$ where $c \in \mathfrak{q}^{2n} + \dots + x_s^{n-1}\mathfrak{q}^{n+1}$ and $d \in \mathfrak{q}^n$. Then $x_{s-1}^l a' - d \in \mathfrak{q}^{n+1} : \langle x_s \rangle = \mathfrak{q}^n[\mathfrak{q} : x_s]$. Hence, $x_{s-1}^l a' \in \mathfrak{q}^n$ and $a' \in \mathfrak{q}^n : \langle x_{s-1} \rangle = \mathfrak{q}^{n-1}[\mathfrak{q} : x_{s-1}]$. The proof is completed. □

Therefore the Rees algebra $\bar{S} = A[\bar{\mathfrak{b}}t]$ is finitely generated over S. Let $\bar{Y} = \operatorname{Proj} \bar{S}$.

Proposition 5.5. $D^0_{(x_{s-1},x_s,x_{s+1})}(S_+) = \bar{S}_+$.

PROOF. First show that x_{s-1}, x_s is an $\bar{S}_+$-regular sequence. Let $n > 0$. It is clear that x_{s-1} is $\bar{\mathfrak{b}}^n$-regular because it is A-regular. Let $a \in (x_{s-1}\bar{\mathfrak{b}}^n : x_s) \cap \bar{\mathfrak{b}}^n$. Then $x_{s-1}^l a \in \mathfrak{b}^n$ for a sufficiently large l. Since $x_s a \in (x_{s-1})$ and $x_s, \dots, x_d$ is a d-sequence on $A/x_{s-1}^{l+1}A$,

$$\begin{aligned} x_{s-1}^l a &\in (x_{s-1}^{l+1}) : x_s \cap \mathfrak{b}^n \\ &\subseteq (x_{s-1}^{l+1}) : x_s \cap (x_{s-1}^{l+1}, x_s, \dots, x_d) \\ &= (x_{s-1}^{l+1}). \end{aligned}$$

Hence $a \in (x_{s-1})$. If we put $a = x_{s-1}a'$, then $a' \in \mathfrak{b}^n : x_{s-1}^{l+1} \subseteq \bar{\mathfrak{b}}^n$, that is, $a \in x_{s+1}\bar{\mathfrak{b}}^n$. Thus we have proved that x_s is $\bar{S}_+/x_{s-1}\bar{S}_+$-regular.

By (2.1.1), we have

$$D^0_{(x_{s-1},x_s,x_{s+1})}(S_+) \subseteq D^0_{(x_{s-1},x_s,x_{s+1})}(\bar{S}_+) = \bar{S}_+. \tag{5.5.1}$$

Since $\mathfrak{q} : x_{s-1} \subseteq \mathfrak{q} : x_s$ by Theorem 4.2, $(x_{s-1}, \dots, x_d)\bar{\mathfrak{b}}^n \subseteq \mathfrak{b}^n$ for all $n > 0$ by Lemma 5.4, that is, $(x_{s-1}, \dots, x_d)\bar{S}_+ \subseteq S_+$. We have shown the inverse inclusion of (5.5.1). □

The following theorem is one of main aims of this section.

THEOREM 5.6. *With notation as above,*

$$\operatorname{depth} \mathcal{O}_{\bar{Y},\bar{q}} \geq d - s + 2 \quad \textit{for any closed point } \bar{q} \textit{ of } \bar{Y}.$$

In particular, if $s = 2$, *then* $\bar{Y}$ *is Cohen-Macaulay.*

PROOF. We have only to compute the depth of

$$\bar{C}_0 = A[\bar{\mathfrak{b}}/x_s x_{s+1}] \quad \text{and} \quad \bar{C}_1 = A[\bar{\mathfrak{b}}/x_{s+1}^2].$$

Proposition 5.5 says that $\bar{C}_i = D^0_{(x_{s-1},x_s,x_{s+1})}(C_i)$ and it is a finitely generated C_i-module for $i = 0, 1$.

Let $\bar{\mathfrak{l}}_i$ be a maximal ideal of $\bar{C}_i$ and $\mathfrak{l}_i = \bar{\mathfrak{l}}_i \cap C_i$. Then $\mathfrak{l}_i$ is a maximal ideal of C_i because $\bar{C}_i$ is integral over C_i. We use (5.3.2) as $k = s - 1$, that is,

$$H^p_{\mathfrak{l}_i} H^2_{(x_{s-1},x_s,x_{s+1})}(C_i) = 0 \quad \text{for } p < d - s. \tag{5.6.1}$$

By using (2.1.2), we obtain

$$H^p_{\mathfrak{l}_i} H^q_{(x_{s-1},x_s,x_{s+1})}(\bar{C}_i) = 0 \quad \text{if } p < d - s \text{ or } q < 2.$$

By the spectral sequence $E_2^{pq} = H^p_{\mathfrak{l}_i} H^q_{(x_{s-1},x_s,x_{s+1})}(-) \Rightarrow H^{p+q}_{\mathfrak{l}_i}(-)$, we find

$$H^p_{\mathfrak{l}_i}(\bar{C}_i) = 0 \quad \text{for } p < d - s + 2, \tag{5.6.2}$$

that is, $\operatorname{depth}(\bar{C}_i)_{\bar{\mathfrak{l}}_i} \geq d - s + 2$. Thus the proof is completed. □

The following corollary shall be used in the next section.

COROLLARY 5.7. *If* $A/(x_s, \dots, x_d)$ *is Cohen-Macaulay, then*

$$\operatorname{depth} \mathcal{O}_{Y,q} \geq d - s + 2 \quad \textit{for any closed point } q \textit{ of } Y.$$

PROOF. It is sufficient to prove $\bar{\mathfrak{b}} = \mathfrak{b}$. Let $a \in \bar{\mathfrak{b}}$ and l be an integer such that $x_{s-1}^l a \in \mathfrak{b}$. Then we have

$$\begin{aligned} a \in (x_s, \ldots, x_d)^2 &: x_{s-1}^l \\ &= (x_s, \ldots, x_d)[(x_s, \ldots, x_d) : x_{s-1}^l] \\ &= (x_s, \ldots, x_d)^2 = \mathfrak{b} + (x_s^2) \end{aligned}$$

by Lemma 3.2. Hence, we may assume that $a \in (x_s^2)$. Let $a = x_s^2 a'$. Since $x_{s-1}^l a \in \mathfrak{b} \subseteq \mathfrak{q}$, $a' \in \mathfrak{q} : x_{s-1}^l x_s^2 = \mathfrak{q} : x_s$ by Theorem 4.2. Hence $a = x_s^2 a' \in x_s \mathfrak{q} \subset \mathfrak{b}$. □

We shall give another Macaulayfication of $\operatorname{Spec} A$ by considering an ideal $\mathfrak{c} = (x_{s-1}, \ldots, x_d)\mathfrak{b}$. Let $Z = \operatorname{Proj} A[\mathfrak{c}t]$, which is the blowing-up of Y with respect to $(x_{s-1}, \ldots, x_d)\mathcal{O}_Y$.

THEOREM 5.8. *With notation as above,*

$$\operatorname{depth} \mathcal{O}_{Z,r} \geq d - s + 2 \quad \textit{for any closed point } r \textit{ of } Z.$$

Furthermore, if $s = 2$ or A is Cohen-Macaulay, then Z is Cohen-Macaulay.

PROOF. Since $(x_{s-1}x_s, x_s^2)\mathfrak{q} + x_{s-1}(x_{s+1}^2, \ldots, x_d^2) + (x_{s+1}^3, \ldots, x_d^3)$ is a reduction of $\mathfrak{c}$, we have only to compute the depth of

$$\begin{aligned} D_0 &= A[\mathfrak{c}/x_{s-1}x_s x_{s+1}] = C_0[x_s/x_{s-1}], \\ D_1 &= A[\mathfrak{c}/x_s^2 x_{s+1}] = C_0[x_{s-1}/x_s], \\ D_2 &= A[\mathfrak{c}/x_{s-1}x_{s+1}^2] = C_1[x_{s+1}/x_{s-1}], \end{aligned}$$

and

$$D_3 = A[\mathfrak{c}/x_{s+1}^3] = C_1[x_{s-1}/x_{s+1}].$$

For $i = 0$ or 1, let $\mathfrak{l}_i$ be a maximal ideal of C_i. By (2.1.1), there exists an exact sequence

$$0 \to C_i \to \bar{C}_i \to H^1_{(x_{s-1}, x_s, x_{s+1})}(C_i) \to 0.$$

By using (5.3.3) and (5.6.2), we obtain

$$H^p_{\mathfrak{l}_i} H^1_{(x_{s-1}, x_s, x_{s+1})}(C_i) = 0 \quad \text{for all } p < d - s.$$

Furthermore, $(x_{s-1}, \ldots, x_d)\bar{C}_i \subseteq C_i$: see the proof of Proposition 5.5. Therefore, by (5.6.1), we have

$$\text{(5.8.1)} \qquad H^p_{\mathfrak{l}_i} H^q_{(x_{s-1}, x_s, x_{s+1})}(C_i) = 0 \quad \text{if } p < d - s \text{ or } q = 0$$

and

$$\text{(5.8.2)} \qquad (x_{s-1}, \ldots, x_d) H^1_{(x_{s-1}, x_s, x_{s+1})}(C_i) = 0.$$

Therefore we can prove

$$\operatorname{depth}(D_i)_{\mathfrak{r}_i} \geq d - s + 2$$

for any maximal ideal $\mathfrak{r}_i$ of D_i and $i = 0, \ldots, 3$ in the same way as Theorem 5.3.

To make sure, we compute the depth of $D_0 \cong C_0[T]/(x_{s-1}T - x_s) : \langle x_{s-1} \rangle$. First we note that $x_{s+1} \in x_s C_0$ and $x_{s+1} \in x_s D_0$. Let $\mathfrak{r}_0$ be a maximal ideal of D_0 and $\mathfrak{l}_0 = \mathfrak{r}_0 \cap C_0$. Then $\mathfrak{l}_0$ is a maximal ideal of C_0 and there exists a polynomial f over C_0 such that $\mathfrak{r}_0 = \mathfrak{l}_0 D_0 + f D_0$ and the leading coefficient of f is not contained in $\mathfrak{l}_0$. We obtain

$$H^p_{\mathfrak{l}_0 C_0[T] + f C_0[T]} H^q_{(x_{s-1}, x_s)}(C_0[T]) = 0 \quad \text{if } p < d - s + 1 \text{ or } q = 0$$

from (5.8.1). Taking the local cohomology of an exact sequence

$$0 \to C_0[T] \xrightarrow{x_{s-1}T - x_s} C_0[T] \to C_0[T]/(x_{s-1}T - x_s) \to 0,$$

we have an exact sequence

$$0 \to H^1_{(x_{s-1}, x_s)}(C_0[T]) \to H^1_{(x_{s-1}, x_s)}(C_0[T]/(x_{s-1}T - x_s))$$
$$\to H^2_{(x_{s-1}, x_s)}(C_0[T]) \to H^2_{(x_{s-1}, x_s)}(C_0[T]) \to 0$$

because of (5.8.2). This says that

$$H^p_{\mathfrak{l}_0 C_0[T] + f C_0[T]} H^1_{(x_{s-1}, x_s)}(C_0[T]/(x_{s-1}T - x_s)) = 0 \quad \text{for } p < d - s + 1.$$

Taking the local cohomology of an exact sequence

$$0 \to \frac{(x_{s-1}T - x_s) : \langle x_{s-1} \rangle}{(x_{s-1}T - x_s)} \to C_0[T]/(x_{s-1}T - x_s) \to D_0 \to 0$$

with respect to (x_{s-1}, x_s), we obtain

$$H^p_{\mathfrak{r}_0} H^1_{(x_{s-1}, x_s)}(D_0) = 0 \quad \text{for } p < d - s + 1.$$

Of course, $H^0_{(x_{s-1}, x_s)}(D_0) = 0$. By the spectral sequence

$$E_2^{pq} = H^p_{\mathfrak{r}_0} H^q_{(x_{s-1}, x_s)}(-) \Rightarrow H^{p+q}_{\mathfrak{r}_0}(-),$$

we get $H^p_{\mathfrak{r}_0}(D_0) = 0$ for any $p < d - s + 2$. That is, $\operatorname{depth}(D_0)_{\mathfrak{r}_0} \geq d - s + 2$.

The last assertion is also proved in the same way as Theorem 5.3. □

6. The proof of Theorem 1.1.

This section is devoted to the proof of Theorem 1.1. Let A be a Noetherian ring possessing a dualizing complex and X a quasi-projective scheme over A. That is, X is a dense open subscheme of $X^* = \operatorname{Proj} R$ where $R = \bigoplus_{n \geq 0} R_n$ is a Noetherian graded ring such that R_0 is a homomorphic image of A and R is generated by R_1 as an R_0-algebra. Let V^* be the non-Cohen-Macaulay locus of X^* and $U^* = X^* \backslash V^*$. Of course $V = V^* \cap X$ is the non-Cohen-Macaulay locus of X. Let $D^\bullet$ be a dualizing complex of R with codimension function v. Assume that X satisfies the assumption of Theorem 1.1.

Without loss of generality, we may assume that

(6.0.1) $\qquad v(\mathfrak{p}) = 0$ for all associated prime ideal $\mathfrak{p}$ of R :

see [**8**, p. 191]. Then the local ring $\mathcal{O}_{X,p}$ of $p \in X$ satisfies the assumption of Section 5, that is, $\dim \mathcal{O}_{X,p}/\mathfrak{p} = \dim \mathcal{O}_{X,p}$ for any associated prime ideal $\mathfrak{p}$ of $\mathcal{O}_{X,p}$. For the sake of completeness, we sketch out the proof. Let $\mathfrak{a}$ be a homogeneous ideal of R such that $V^* = V(\mathfrak{a})$. Then the closed immersion $\operatorname{Proj} R/H^0_{\mathfrak{a}}(R) \to X^*$ is birational as follows. For any minimal prime ideal $\mathfrak{p}$ of R, $\mathfrak{a} \not\subset \mathfrak{p}$ and $H^0_{\mathfrak{a}}(R) \subseteq \mathfrak{p}$ because $R_{\mathfrak{p}}$ is Cohen-Macaulay. Hence the underlying set of $\operatorname{Proj} R/H^0_{\mathfrak{a}}(R)$ coincides with the one of X^*. Furthermore, $f^{-1}(U^*) \to U^*$ is an isomorphism and U^* is dense in X^*. By replacing R by $R/H^0_{\mathfrak{a}}(R)$, we may assume that

(6.0.2) every associated prime ideal of R is minimal.

Next we fix a primary decomposition of (0) in R. For all integer i, let $\mathfrak{q}_i$ be the intersection of all primary component $\mathfrak{q}$ of (0) such that $v(\sqrt{\mathfrak{q}}) = i$. Then $g : \coprod_i \operatorname{Proj} R/\mathfrak{q}_i \to X^*$ is a finite morphism and $g^{-1}(U^*) \to U^*$ is an isomorphism as follows. Note that $\mathfrak{q}_i = R$ for all but finitely many i. Furthermore, for any $\mathfrak{p} \in U^*$, $\mathfrak{p} \supseteq \mathfrak{q}_i$ if and only if $v(\mathfrak{p}) - \dim R_{\mathfrak{p}} = i$ because $R_{\mathfrak{p}}$ is Cohen-Macaulay, hence equidimensional. Therefore U^* is the disjoint union of $U^* \cap V(\mathfrak{q}_i)$. Moreover $R_{(\mathfrak{p})} = [R/\mathfrak{q}_i]_{(\mathfrak{p})}$ if $\mathfrak{p} \in U^* \cap V(\mathfrak{q}_i)$. Because of (6.0.2), $g^{-1}(U^*)$ and U^* are dense in $\operatorname{Proj} R/\mathfrak{q}_i$ and X^*, respectively. Thus $g^{-1}(X) \to X$ is birational proper and the connected components of $g^{-1}(X)$ satisfy the assumption of Theorem 1.1.

Since u is locally constant, $V_i = u^{-1}(i) \cap V$ is closed for any positive integer i. We put $s_i = \dim V_i$. By (6.0.1), we find that $V_1 = \emptyset$, $s_2 \le 0$ and $s_3 \le 1$. Let d be the largest integer such that $V_d \neq \emptyset$ and $s = s_d$. We shall give a closed subscheme W of X such that $V_d = V \cap W$ and $\mathcal{O}_{Y,q}$ is Cohen-Macaulay for all $q \in \pi^{-1}(W)$ where $\pi : Y \to X$ is the blowing-up of X with center W. Let $\mathfrak{a} = \prod_{i>0} \operatorname{ann} H^i(D^\bullet)$, which is finite product. Then it is obvious that $V^* = V(\mathfrak{a})$. Fix a primary decomposition of $\mathfrak{a}$ and let $\mathfrak{a}_d$ be the intersection of all primary component $\mathfrak{q}$ of $\mathfrak{a}$ such that $\sqrt{\mathfrak{q}} \in V_d$. Then we can take homogeneous elements $z_1, \ldots, z_d \in R$ such that

(6.0.3) $$V_i \cap V((z_{d-s_i}, \ldots, z_d)) = \emptyset \quad \text{for } i < d;$$

(6.0.4) $$d(\mathfrak{p}) = d \quad \text{for all minimal prime ideal } \mathfrak{p} \text{ of } R/(z_1, \ldots, z_d) : \langle R_+ \rangle;$$

(6.0.5) $$\begin{cases} z_{s+1}, \ldots, z_d \in \mathfrak{a}_d; \\ z_i \in \prod_{j>d-i} \operatorname{ann} H^j(\operatorname{Hom}(R/(z_{i+1}, \ldots, z_d), D^\bullet)), & \text{for } i \le s \end{cases}$$

in the same way as Section 4. We put

$$\mathfrak{b} = \begin{cases} (z_1, \ldots, z_d), & \text{if } s = 0; \\ (z_1, \ldots, z_d)(z_2, \ldots, z_d), & \text{if } s = 1; \\ (z_1, \ldots, z_d)(z_2, \ldots, z_d)(z_3, \ldots, z_d), & \text{if } s = 2 \end{cases}$$

and prove that $W = V(\mathfrak{b}) \cap X$ satisfies the required properties.

Because of (6.0.3), $V_i \cap W = \emptyset$ for $i < d$. Let $\pi : Y \to X$ be the blowing-up of X with center W, q a closed point of $\pi^{-1}(W)$ and $\mathfrak{p} \subseteq R$ the image of q. Take an element $y \in R_1 \backslash \mathfrak{p}$ and put $x_i = z_i / y^{\deg z_i}$ for all i. Since $(D^\bullet)_{(\mathfrak{p})}$ is a dualizing complex of $R_{(\mathfrak{p})}$, we obtain

$$\begin{cases} x_{s+1}, \ldots, x_d \in \mathfrak{a}_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}); \\ x_i \in \mathfrak{a}_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/(x_{i+1}, \ldots, x_d)), & \text{for } i \le s. \end{cases}$$

from (6.0.5).

When $s=2$, there exist three cases: If $z_1, z_2 \in \mathfrak{p}$, then $x_1, \dots, x_d$ is a system of parameters for $R_{(\mathfrak{p})}$ satisfying (4.0.1) or a regular sequence on the Cohen-Macaulay ring $R_{(\mathfrak{p})}$. Since $\mathfrak{b}_{(\mathfrak{p})} = (x_1, \dots, x_d)(x_2, \dots, x_d)(x_3, \dots, x_d)$, $\mathcal{O}_{Y,q}$ is Cohen-Macaulay by Theorem 5.8.

If $z_2 \in \mathfrak{p}$ but $z_1 \notin \mathfrak{p}$, then $x_2, \dots, x_d$ is a subsystem of parameters for $R_{(\mathfrak{p})}$ satisfying (4.0.1) or a regular sequence on the Cohen-Macaulay ring $R_{(\mathfrak{p})}$. Furthermore $\mathfrak{b}_{(\mathfrak{p})} = (x_2, \dots, x_d)(x_3, \dots, x_d)$ and $R_{(\mathfrak{p})}/(x_2, \dots, x_d)$ is Cohen-Macaulay because $x_1 \in \mathfrak{a}_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/(x_2, \dots, x_d))$ is a unit. Hence $\mathcal{O}_{Y,q}$ is Cohen-Macaulay by Corollary 5.7.

If z_1, $z_2 \notin \mathfrak{p}$, then $x_3, \dots, x_d \in \mathfrak{a}_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})})$ is a subsystem of parameters for $R_{(\mathfrak{p})}$ and $R_{(\mathfrak{p})}/(x_3, \dots, x_d)$ is Cohen-Macaulay. Since $\mathfrak{b}_{(\mathfrak{p})} = (x_3, \dots, x_d)$, $\mathcal{O}_{Y,q}$ is Cohen-Macaulay by Theorem 5.1.

When $s=0$ or 1, we can prove the assertion in the same way as above.

By repeating this procedure, we obtain a Macaulayfication of X. We complete the proof of Theorem 1.1.

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