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On the structure of the singular set of a complex analytic foliation

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0. Introduction.

A singular foliation on a complex manifold M is determined by an involutive coherent subsheaf E of the tangent sheaf of M. In this paper, we study the problem of local (analytical and topological) triviality of the singular foliation along (a subset of) its singular set $S(E)$. In general, $S(E)$ is an analytic variety, so we examine by stratifying the set.

For stratified subsets or stratified maps, the local topological triviality has been studied by a number of people and it is generally known that if the stratification satisfies the "Whitney condition" or the "Thom condition", then we have the local topological triviality along each stratum (the Isotopy Lemmas of Thom).

We first consider the case of analytical triviality in section 2 of this paper, after reviewing basic definitions and facts on complex analytic singular foliations in section 1. For complex analytic singular foliations we have the fundamental "Tangency Lemma" (Theorem (2.5)), which says that every vector field defining the foliation is "tangential" to the singular set $S(E)$. We discuss and summarize the implications of this lemma, which include the existence of the integral submanifold (leaf) through each point of M (even on $S(E)$) and the local analytical triviality of the foliation along each leaf.

As another application of the Tangency Lemma, we prove, for a complex analytic singular foliation E, the existence of a Whitney stratification of the singular set $S(E)$ so that E induces a non-singular foliation on each of its strata (Theorem (3.4)).

In section 4, we study the local topological triviality along each stratum of a stratification of $S(E)$ as given in Theorem (3.4). This kind of triviality argument can be applied to the case where a stratum consists of (infinitely) many leaves. A. Kabila studied this problem for the case where the codimension of E is one and $S(E)$ is non-singular ([**K**]). We give, for a general singular foliation, a regularity condition and prove the local topological triviality under the condition (Theorem (4.10)).

After the preparation of the manuscript, it was informed to me that a result similar to the last one is indicated in an article of D. Trotman and L. Wilson [**TW**].

In the process of this work, I received many helpful suggestions and advices, especially from T. Suwa. I would like to thank him for answering my questions and for supporting me in various ways. I also thank J.-P. Brasselet, M. Kwieciński, T. Ohmoto, A. Saeki and the referee for helpful conversations and comments.

1. Complex analytic singular foliations.

First of all, we recall some general facts about singular foliations on complex manifolds and fix the notations used in this papar. For further details, see [**BB**] and [**Sw**].

Let M be a (connected) complex manifold of (complex) dimension n, and let $\mathcal{O}_M$, Θ_M and Ω_M be, respectively, the sheaf of holomorphic functions on M, the tangent sheaf and the cotangent sheaf of M.

Now, let E be a coherent subsheaf of Θ_M. Note that, in this case, E is *coherent* if and only if E is locally finitely generated, since Θ_M is locally free. Then the *singular set* of E is defined by

$$S(E) = \{p \in M \mid (\Theta_M/E)_p \text{ is not } (\mathcal{O}_M)_p\text{-free}\},$$

and each point in $S(E)$ is called a *singular point* of E. Concretely, restricting E to a sufficiently small coordinate neighborhood U with coordinates $(z_1, z_2, \dots, z_n)$, we can express E on U as follows:

$$E = (v_1, v_2, \dots, v_r) \qquad \left(v_i = \sum_{j=1}^{n} f_{ij}(z) \frac{\partial}{\partial z_j} \quad (1 \le i \le r) \right), \tag{1.1}$$

where $f_{ij}(z)$ are holomorphic functions defined on U, and r is a non-negative integer. Then the singular set $S(E)$ is paraphrased on U as

$$S(E) \cap U = \{p \in U \mid \operatorname{rank}(f_{ij}(p)) \text{ is not maximal}\}.$$

Next, let us introduce the "*integrability condition*". A coherent subsheaf E of Θ_M is said to be *integrable* (or *involutive*) if for every point p on $M - S(E)$,

$$[E_p, E_p] \subset E_p \tag{1.2}$$

holds ([,] means the *Lie bracket* of smooth vector fields). Moreover, we define the *rank* (we sometimes call it *dimension*) of E to be the rank of locally free sheaf $E|_{M-S(E)}$, and denote it $\operatorname{rank} E$. Using the notation in (1.1), we can rewrite as

$$\operatorname{rank} E = \max_{p \in M} \operatorname{rank}(f_{ij}(p)).$$

In the following definition we define a singular foliation on M in terms of vector fields. Later, we will introduce it again from another viewpoint.

DEFINITION 1.3. A singular foliation on M is a coherent subsheaf E of Θ_M which is integrable.

It is clear that a singular foliation E induces a non-singular foliation on $M - S(E)$.

DEFINITION 1.4. Let E be a singular foliation on M. We say that E is reduced if

$$v \in \Gamma(U, \Theta_M), v|_{U-S(E)} \in \Gamma(U - S(E), E) \Rightarrow v \in \Gamma(U, E)$$

holds for every open set U in M.

REMARK 1.5. We can check the following facts about reduced foliations:
(i) If a singular foliation E is locally free,

$$E \text{ is reduced} \Leftrightarrow \operatorname{codim} S(E) \ge 2.$$

(ii) If E is reduced, then the "integrability condition" holds not only on $M - S(E)$ but on $S(E)$, i.e., (1.2) holds for every point $p \in M$.

Next, as stated above, let us represent singular foliations in terms of holomorphic 1-forms. It is not so difficult to rewrite it from the viewpoint of its "dual", but there are several points which require a little care.

DEFINITION 1.6. Let F be a coherent subsheaf of Ω_M. Then we set

$$S(F) = \{p \in M \mid (\Omega_M/F)_p \text{ is not } (\mathcal{O}_M)_p\text{-free}\},$$

and call it the singular set of F. Each point in $S(F)$ is often called a singular point of F.

DEFINITION 1.7. A coherent subsheaf F of Ω_M is said to be integrable (or involutive) when

$$dF_p \subset \Omega_p \wedge F_p$$

holds for every point $p \in M - S(F)$. Moreover, the rank of F is defined to be the rank of the locally free sheaf $F|_{M-S(F)}$, and denoted rank F.

DEFINITION 1.8. A singular foliation on M is a coherent subsheaf F of Ω_M which is integrable.

DEFINITION 1.9. Let $F(\subset \Omega_M)$ be a singular foliation on M. We say that F is reduced if

$$\omega \in \Gamma(U, \Omega_M), \omega|_{U-S(F)} \in \Gamma(U - S(F), F) \Rightarrow \omega \in \Gamma(U, F)$$

holds for every open set U in M.

Now we discuss the relation between the two definitions, (1.3) and (1.8).

DEFINITION 1.10. For singular foliations $E \subset \Theta_M$ and $F \subset \Omega_M$, we set

$$E^a = \{\omega \in \Omega_M \mid \langle v, \omega \rangle = 0 \quad \text{for all} \quad v \in E\},$$

$$F^a = \{v \in \Theta_M \mid \langle v, \omega \rangle = 0 \quad \text{for all} \quad \omega \in F\},$$

where $\langle\ ,\ \rangle$ means the natural pairing between a vector field and a 1-form. Then $E^a(\subset \Omega_M)$ and $F^a(\subset \Theta_M)$ define reduced singular foliations on M. We call E^a (resp. F^a) the annihilator of E (resp. F).

REMARK 1.11. Note that $S(E^a) \subset S(E)$ and $S(F^a) \subset S(F)$ hold.

DEFINITION 1.12. When $E \subset \Theta_M$ (resp. $F \subset \Omega_M$) is a singular foliation on M, $(E^a)^a$ (resp. $(F^a)^a$) is called the reduction of E (resp. F).

If we use the notations given in (1.10) and (1.12), a singular foliation $E \subset \Theta_M$ (resp. $F \subset \Omega_M$) is reduced if and only if $(E^a)^a = E$ (resp. $(F^a)^a = F$). In this way we can make any singular foliation reduced by taking its reduction. If we consider only reduced foliations, then the two definitions of singular foliation stated above are equivalent, and in this occasion, moreover, there is no difference between the singular set in terms of vecter fields and that in terms of 1-forms.

2. Singular set of a singular foliation.

In this section, we recall some basic properties of the singular set of a singular foliation, and summarize the "tangency lemma" and its consequences which have been studied by P. Baum, D. Cerveau, Y. Mitera, T. Suwa and, for the real case, by T. Nagano, P. Stefan, H.Sussmann, et al. In the preceding section we defined singular foliations from two different aspects, and observed the relations between the two definitions. We have checked that they produce "almost" the same results, so we often express singular foliations only in terms of vector fields. Hereafter, we assume $E(\subset \Theta_M)$ to be a singular foliation on a complex manifold M and set $r = \operatorname{rank} E$.

DEFINITION 2.1. For each point p in M, we set

$$E(p) = \{v(p) \mid v \in E_p\},$$

where $v(p)$ denotes the evaluation of the vector field germ v at p. Note that $E(p)$ is a sub-vector space of the tangent space T_pM.

DEFINITION 2.2. For an integer k with $0 \le k \le r$, we set

$$L^{(k)} = \{p \in M \mid \dim_C E(p) = k\},$$

$$S^{(k)} = \{p \in M \mid \dim_C E(p) \le k\},$$

and set $L^{(-1)} = S^{(-1)} = \emptyset$ for convenience. Clearly we have

$$L^{(k)} = S^{(k)} - S^{(k-1)}, \qquad S^{(k)} = \bigcup_{i=0}^{k} L^{(i)}$$

for $k = 0, 1, 2, \dots, r$.

PROPOSITION 2.3. *$L^{(k)}$ and $S^{(k)}$ are analytic sets for every integer k with $0 \le k \le r$.*

PROOF. If we use the notation in (1.1), $S^{(k)}$ is locally expressed on a small open set U in M as follows:

$$S^{(k)} \cap U = \{z \in U \mid \operatorname{rank}(f_{ij}(z)) \le k\}.$$

All f_{ij} are holomorphic on U, so $S^{(k)}$ is analytic. And besides, we come to the conclusion that $L^{(k)}$ $(= S^{(k)} - S^{(k-1)})$ is analytic because $S^{(k)}$ is analytic and $S^{(k-1)}$ is closed in M. Q.E.D.

By the proposition stated above, we get the *natural filtration* which consists of analytic sets:

$$\begin{array}{ccccccccccccc} S^{(r)} & \supset & S^{(r-1)} & \supset & S^{(r-2)} & \supset & \cdots\cdots & \supset & S^{(1)} & \supset & S^{(0)} & \supset & S^{(-1)}. \\ \| & & \| & & & & & & & & & & \| \\ M & & S(E) & & & & & & & & & & \emptyset \end{array} \tag{2.4}$$

This filtration seems to give us information only about the dimension of the space $E(p)$ at p. In fact, however, if E is integrable at every point $p \in M$ then all $S^{(k)}$ appearing in (2.4) controll the "direction" of $E(p)$ at each point $p \in S^{(k)}$. To be more precise, THEOREM 2.5 (Tangency Lemma). *Suppose $E(\subset \Theta_M)$ is integrable on the whole M. Let k be an integer with $0 \leq k \leq r$ and p a point in $S^{(k)}$. Then we have*

$$E(p) \subset C_p S^{(k)},$$

where $C_p S^{(k)}$ denotes the tangent cone of $S^{(k)}$ at p.

REMARK 2.6. The assumption of theorem (2.5) cannot be dropped. The singular foliation on $\boldsymbol{C}^2 = \{(x, y)\}$ generated by $v_1 = \partial/\partial x$ and $v_2 = x\,\partial/\partial y$ is a counterexample.

This theorem can be showed as a corollary of the main theorems in [**C**]. In this paper, let us indicate that we can get a stronger result than (2.5) when E is reduced.

PROPOSITION 2.7. *Suppose $E(\subset \Theta_M)$ is a reduced foliation and p is a point in M. Let v be a germ in E_p and let $\{\varphi_t = \exp tv\}$ be the local 1-parameter group of transformations induced by v. For all t sufficiently close to 0, we have*

$$(\varphi_t)_* E_p = E_{\varphi_t(p)},$$

where $(\varphi_t)_*$ denotes the differential map of φ_t.

The following proof of this proposition is due to T. Suwa. We first prepare two lemmas in advance. The first one is a property which is easily drawn from the integrability of E.

LEMMA 2.8. *Let v be a germ in E_p and let L_v denote the Lie derivative of v. Then we have*

$$L_v(F_p) \subset F_p,$$

where F is the annihilator of E.

PROOF. Take a germ ω in F_p. For any germ u in E_p, we have

$$\langle u, L_v \omega \rangle = v(\langle u, \omega \rangle) - \langle [v, u], \omega \rangle.$$

We have $\langle u, \omega \rangle = 0$ and $\langle [v, u], \omega \rangle = 0$, since $[v, u] \in E_p$. Hence $\langle u, L_v \omega \rangle = 0$ for any $u \in E_p$, and this implies $L_v \omega \in E_p^a = F_p$. Q.E.D.

LEMMA 2.9. *Suppose that E, F, v and $\{\varphi_t\}$ are as above. For any germ u in E_p and any germ in ω in F_p, we have*

$$\frac{\partial}{\partial t} \langle (\varphi_t)_* u, \omega(\varphi_t(p)) \rangle = \langle (\varphi_t)_* u, L_v \omega(\varphi_t(p)) \rangle.$$

PROOF. Choose a coordinate neighborhood U with coordinates $(z_1, z_2, \ldots, z_n)$ about p such that v, u and ω have representatives on U and that E and F have finite numbers of generators on U. Considering only for t sufficiently close to 0, we may assume that $\varphi_t(p)$ stays in U. Now we write explicitly on U as

$$v = \sum_{i=1}^{n} f_i(z) \frac{\partial}{\partial z_i}, \qquad u = \sum_{i=1}^{n} g_i(z) \frac{\partial}{\partial z_i} \qquad \text{and} \qquad \omega = \sum_{i=1}^{n} h_i(z)\, dz_i,$$

where f_i, g_i and h_i are holomorphic functions on U. Moreover, we set $\varphi_i(t, z) =$

$z_i \circ \varphi_t(z)$ and $\varphi(t,z) = (\varphi_1(t,z), \dots, \varphi_n(t,z))$. Then we have

$$(2.10) \qquad \langle (\varphi_t)_* u, \omega(\varphi_t(z)) \rangle = \sum_{i,j=1}^{n} g_i(z) \frac{\partial \varphi_j(t,z)}{\partial z_i} h_j(\varphi(t,z))$$

and

$$(2.11) \qquad \frac{\partial \varphi_i(t,z)}{\partial t} = f_i(\varphi(t,z))$$

for all z in a small neighborhood around p. Differentialing (2.10) with respect to t and using (2.11), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t} \langle (\varphi_t)_* u, \omega(\varphi_t(z)) \rangle \\
&= \sum_{i,j=1}^{n} g_i(z) \frac{\partial^2 \varphi_j(t,z)}{\partial t \partial z_i} h_j(\varphi(t,z)) + \sum_{i,j,k=1}^{n} g_i(z) \frac{\partial \varphi_j(t,z)}{\partial z_i} \frac{\partial h_j(\varphi(t,z))}{\partial z_k} \frac{\partial \varphi_k(t,z)}{\partial t} \\
&= \sum_{i,j,k=1}^{n} g_i(z) \frac{\partial \varphi_j(t,z)}{\partial z_i} \left\{ \frac{\partial f_k}{\partial z_j} (\varphi(t,z)) h_k(\varphi(t,z)) + f_k(\varphi(t,z)) \frac{\partial h_j}{\partial z_k} (\varphi(t,z)) \right\} \\
&= \langle (\varphi_t)_* u, L_v \omega(\varphi_t(z)) \rangle.
\end{aligned}$$

Q.E.D.

PROOF OF (2.7). We take a coordinate neighborhood U with coordinates $(z_1, z_2, \dots, z_n)$ about p such that v has a representative on U and that E and F have finite numbers of generators on U. In order to prove this proposition, it suffices to show that

$$(2.12) \qquad (\varphi_t)_* E_p \subset E_{\varphi_t(p)}$$

hold for all t sufficiently close to 0. Once we have (2.12), then $(\varphi_t^{-1})_* E_{\varphi_t(p)} = (\varphi_{-t})_* E_{\varphi_t(p)} \subset E_p$ and thus this proposition.

Now we take two sections $u \in \Gamma(U,E)$ and $\omega \in \Gamma(U,F)$ arbitrarily. Using (2.9) repeatedly, we have

$$\frac{\partial^m}{\partial t^m} \langle (\varphi_t)_* u, \omega(\varphi_t(p)) \rangle \bigg|_{t=0} = \langle u, \underbrace{L_v \cdots L_v}_{m\text{-times}} \omega \rangle$$

for all non-negative integer m, and the right-hand side of this equation is equal to zero by (2.8). So we have

$$\langle (\varphi_t)_* u, \omega(\varphi_t(p)) \rangle = 0$$

for all t sufficiently close to 0. This implies (2.12). Q.E.D.

Now let us look back at the tangency lemma (2.5). Take a germ $v \in E_p$ and set $\varphi_t = \exp tv$. Suppose $\varphi_t(p) \notin S^{(k)}$ for some t. Then we have

$$\dim E(p) \le k < \dim E(\varphi_t(p)),$$

which contradicts proposition (2.7). So we have $\varphi_t(p) \in S^{(k)}$ for all t sufficiently close to 0. Hence

$$v(p) = \lim_{t \to 0} \frac{\varphi_t(p) - p}{t}$$

is in the tangent cone $C_p S^{(k)}$ of $S^{(k)}$ at p.

Thus, in the case that E is reduced, theorem (2.5) is easily proved as a corollary of (2.7).

REMARK 2.13. Theorem (2.5) was proved by P. Baum under the hypotheses that E is reduced, $k = 1$ and p is a non-singular point of $S^{(1)}$ (see [**B**]). For the case of real singular foliations, see [**N**], [**Ss**] and [**St**].

Using theorem (2.5) we can prove the following results for a singular foliation E of dimension r on M. For details, we refer to [**MY**].

THEOREM 2.14. *Let k be an integer with $0 \le k \le r$ and $\mathscr{S}^{(k)} = \{X_\alpha\}_{\alpha \in A}$ the natural Whitney stratification of the analytic set $S^{(k)}$. Then for any $\alpha \in A$ and $p \in X_\alpha$, we have $E(p) \subset T_p X_\alpha$. Moreover, E induces a non-singular foliation of dimension k on $X_\alpha - S^{(k-1)}$.*

THEOREM 2.15 (Existence of Integral Submanifolds). *There exist integral submanifolds (whose dimensions are lower than r) also on $S(E)$. To be more precise, there is a family $\mathscr{L}$ of submanifolds of M such that $M = \bigcup_{L \in \mathscr{L}} L$ is a disjoint union and that any $L \in \mathscr{L}$ and $p \in L$, we have $E(p) = T_p L$.*

Each element L in $\mathscr{L}$ is often called a *leaf* of E.

THEOREM 2.16 (Local Analytical Triviality). *Let k be an integer with $0 \le k \le r$ and p a point in $L^{(k)}$ $(= S^{(k)} - S^{(k-1)})$. Then there exist a small polydisk D of dimension $n - k$ transverse to $E(p)$ in $T_p M$, a singular foliation E' on D with $E'(p) = \{0\}$, a neighborhood U of p in M and a submersion $\pi : U \to D$ such that*

$$E|_U \simeq (\pi^*(E'^a))^a.$$

Theorem (2.16) says that the structure of a singular foliation E is locally analytically trivial along the leaf through each point p in M. However the decomposition $M = \bigcup_{L \in \mathscr{L}} L$ does not give a stratification because this is not always locally finite. In the following sections we consider a stratification of $S(E)$ which gives a local triviality of E along each stratum.

3. Stratifications of the singular set.

Let E be a singular foliation on M. Since the singular set $S(E)$ is analytic, we can construct the "natural Whitney stratification" of $S(E)$ (see [**W**]), but this is not enough to achieve our purpose because the dimension of the leaf of E is not always constant on each stratum.

EXAMPLE 3.1. Let f be the holomorphic function on $M = \mathbf{C}^3$ defined by

$$f(x, y, z) = x^2 - y^2(y + z^2),$$

and ω the holomorphic 1-form on C^3 defined by

$$\omega = df = 2x\,dx - y(3y+2z^2)\,dy - 2y^2z\,dz.$$

The coherent subsheaf $F(\subset \Omega_M)$ generated by ω is integrable since $d\omega = ddf = 0$, so F defines a singular foliation on C^3. $E = F^a(\subset \Theta_M)$ is generated by the following three vector fields:

$$(3.2)\qquad \begin{cases} v_1 = y(3y+2z^2)\dfrac{\partial}{\partial x} + 2x\dfrac{\partial}{\partial y} \\ v_2 = 2yz\dfrac{\partial}{\partial y} - (3y+2z^2)\dfrac{\partial}{\partial z} \\ v_3 = y^2z\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial z}. \end{cases}$$

E is reduced, and $\operatorname{rank} E = 2$. By (3.2), $S(E) = S^{(1)} = \{x = yz = y(3y+2z^2) = 0\} = \{x = y = 0\} = \{z\text{-axis}\}$ and $S^{(0)} = \{(0,0,0)\}$. Since $S(E)$ is non-singular, $S(E) = \{z\text{-axis}\}$ is the only stratum of the natural Whitney stratification of $S(E)$, but $\dim E(p)$ is not constant on the stratum.

In the above example, in order to get a Whitney stratification such that the leaf dimension is constant on each stratum, we may separate the bad point $(0,0,0)$ from the z-axis. In this section we prove that there exists a Whitney stratification of $S(E)$ such that $\dim E(p)$ is constant on each stratum.

DEFINITION 3.3. Let $E(\subset \Theta_M)$ be a singular foliation of dimension r on M, and let $\mathscr{S}$ be a stratification of M. We say that $\mathscr{S}$ is adapted to E when, for any stratum $X \in \mathscr{S}$, there is an integer i with $0 \le i \le r$ such that $X \subset L^{(i)}$, i.e., the leaf dimension of E is constant on each stratum $X \in \mathscr{S}$.

THEOREM 3.4. *Let E be a singular foliation of dimension r on M. Then there exists a Whitney stratification $\mathscr{S}$ which satisfies:*

(i) *$\mathscr{S}$ consists of finitely many strata.*

(ii) *$\mathscr{S}$ is adapted to E.*

To prove this theorem, we introduce some notations on analytic sets. The notation and basic facts are due to H. Whitney ([W]).

Let A be an analytic set. We denote by $\operatorname{Sing}(A)$ the singular set of A and denote by $\operatorname{Reg}(A)$ the set of regular (i.e., non-singular) points of A. Moreover we set

$$\Sigma(A) = \operatorname{Sing}(A) \cup \{p \in \operatorname{Reg}(A) \mid \dim_p A < \dim A\},$$

where $\dim_p A$ denotes the dimension of A at each point $p \in \operatorname{Reg}(A)$. For two manifolds X and Y, we define a subset $B(X, Y)$ of X by

$$B(X, Y) = \{p \in X \mid Y \text{ is not Whitney regular over } X \text{ at } p\}.$$

Also for two analytic sets A and A' we set

$$(3.5)\qquad W(A, A') = \Sigma(A) \cup B(A - \Sigma(A),\, A' - \Sigma(A')).$$

$W(A, A')$ is a analytic subset of A whose dimension is lower than $\dim A$.

PROOF OF (3.4). We proceed by downward induction, i.e., we show that if we have already defined $\mathscr{S}$ which satisfies the two conditions in (3.4) on $\bigcup_{i=k+1}^{r} L^{(i)}$ then we can extend it on $\bigcup_{i=k}^{r} L^{(i)}$. At first, $S^{(r-1)}$ is closed in M, so $L^{(r)} = M - S^{(r-1)}$ is a submanifold. Hence we obtain a Whitney stratification $\tilde{\mathscr{S}}^{(r)} = \{L^{(r)}\}$ of $L^{(r)}$, which clearly satisfies the two conditions in (3.4).

Next let k be an integer with $0 \leq k \leq r-1$, and suppose we have already defined a Whitney stratification $\tilde{\mathscr{S}}^{(k+1)}$ of $\bigcup_{i=k+1}^{r} L^{(i)}$ which satisfies the two conditions in (3.4). Let l denote the dimension of $L^{(k)}$ as an analytic set. We define a family of analytic subsets $\{V_i\}_{-1 \leq i \leq l}$ (inductively) as follows:

$$
\text{(3.6)} \qquad \begin{cases} V_l = L^{(k)} \\ V_{l-1} = \operatorname{Clos}_{L^{(k)}}\left(\displaystyle\bigcup_{X \in \tilde{\mathscr{S}}^{(k+1)}} W(V_l, X) \right) \\ \text{For each integer with } -1 \leq i \leq l-2, \\ V_i = \begin{cases} V_{i+1} & (\text{if } \dim V_{i+1} < i+1) \\ \operatorname{Clos}_{L^{(k)}}\left[\left(\displaystyle\bigcup_{j=i}^{l-2} W(V_{i+1}, V_{j+2} - V_{j+1}) \right) \cup \left(\displaystyle\bigcup_{X \in \tilde{\mathscr{S}}^{(k+1)}} W(V_{i+1}, X) \right) \right] \\ & (\text{if } \dim V_{i+1} = i+1), \end{cases} \end{cases}
$$

where $\operatorname{Clos}_{L^{(k)}}(\ \ )$ denotes the closure in $L^{(k)}$. The family $\{V_i\}_{-1 \leq i \leq l}$ is well-defined since

$$
\text{(3.7)} \qquad \dim V_i \leq i
$$

holds for each integer i with $-1 \leq i \leq l$ by the definition of V_i, and all V_i are analytic since $\tilde{\mathscr{S}}^{(k+1)}$ has only a finite number of strata. Note that we also have $V_{-1} = \emptyset$ by (3.7). For each integer i with $-1 \leq i \leq l-1$, moreover, we have $V_i \subset V_{i+1}$ because V_{i+1} is closed in $L^{(k)}$. Thus we obtain a sequence of analytic subsets of $L^{(k)}$:
$0 V_l$

$$
\text{(3.8)} \qquad \begin{array}{ccccccccccccc} V_l & \supset & V_{l-1} & \supset & V_{l-2} & \supset & \cdots\cdots & \supset & V_1 & \supset & V_0 & \supset & V_{-1}. \\ \| & & & & & & & & & & & & \| \\ L^{(k)} & & & & & & & & & & & & \emptyset \end{array}
$$

By (3.8),

$$
L^{(k)} = \bigcup_{i=0}^{l} (V_i - V_{i-1})
$$

turns out to be a disjoint union, so we define a partition of $\bigcup_{i=k}^{r} L^{(i)}$ by

$$
\text{(3.9)} \qquad \tilde{\mathscr{S}}^{(k)} = \tilde{\mathscr{S}}^{(k+1)} \cup \{V_i - V_{i-1} \mid 0 \leq i \leq l,\ V_i - V_{i-1} \neq \emptyset\}.
$$

This partition, in fact, gives a Whitney stratification of $\bigcup_{i=k}^{r} L^{(i)}$ with our two conditions in (3.4). Obviously $\tilde{\mathscr{S}}^{(k)}$ satisfies the two conditions in (3.4) by the definition of $\tilde{\mathscr{S}}^{(k)}$ in (3.9), so all we have to do is to show that $\tilde{\mathscr{S}}^{(k)}$ is a Whitney stratification.

First, let us check that each $V_i - V_{i-1}(\neq \emptyset)$ is a submanifold of M. By (3.5) and (3.6) we have $V_{i-1} \supset \Sigma(V_i)$, hence

$$V_i - V_{i-1} \subset V_i - \Sigma(V_i)$$

holds. Since $V_i - \Sigma(V_i)$ is a submanifold of M and V_{i-1} is closed in V_i, $V_i - V_{i-1}$ is also a submanifold.

Next we check the Whitney regularity of $\tilde{\mathscr{S}}^{(k)}$. Take two strata $X, Y \in \tilde{\mathscr{S}}^{(k)}$ $(X \neq Y)$.

(CASE 1). If $X, Y \in \tilde{\mathscr{S}}^{(k+1)}$, the Whitney regularity between X and Y holds by the inductive assumption.

(CASE 2). If $X \notin \tilde{\mathscr{S}}^{(k+1)}$ and $Y \in \tilde{\mathscr{S}}^{(k+1)}$, $X = V_i - V_{i-1}$ holds for an integer i $(0 \leq i \leq l)$. X is a subset of $S^{(k)}$ since X is contained in $L^{(k)}$, and $S^{(k)}$ is closed in M, so we have $\mathrm{Clos}_M(X) \subset S^{(k)}$. On the other hand $Y \in \tilde{\mathscr{S}}^{(k+1)}$ implies $Y \subset M - S^{(k)}$, hence we have $\mathrm{Clos}_M(X) \cap Y = \emptyset$. Therefore there is no problem about the Whitney regularity of X over Y.

Next, let p be an arbitrary point in $X = V_i - V_{i-1}$. Then we have

$$p \notin W(V_i, Y) \tag{3.10}$$

by $p \in V_i$, $p \notin V_{i-1}$ and the definition of V_{i-1}. (3.5) and (3.10) imply

$$p \notin \Sigma(V_i), \tag{3.11}$$

$$p \notin B(V_i - \Sigma(V_i), Y). \tag{3.12}$$

We can rewrite (3.11) as

$$p \in V_i - \Sigma(V_i), \tag{3.13}$$

so it turns out that Y is Whitney regular at p over $V_i - \Sigma(V_i)$ by (3.12) and (3.13). Since $V_i - V_{i-1}$ is a submanifold of $V_i - \Sigma(V_i)$, we also find Y to be Whitney regular at p over $V_i - V_{i-1}(= X)$. This implies the Whitney regularity of Y over X.

(CASE 3) If $X, Y \notin \tilde{\mathscr{S}}^{(k+1)}$, we can take two integers i, j $(0 \leq i, j \leq l)$ such that $X = V_i - V_{i-1}$ and $Y = V_j - V_{j-1}$. We may assume $i < j$. First, Y is Whitney regular over X because (3.6) says that V_{i-1} contains all points in V_i at which $Y(= V_j - V_{j-1})$ is not Whitney regular over $V_i - \Sigma(V_i)$, so those points cannot remain on $V_i - V_{i-1}(= X)$. In order to check the Whitney regularity of X over Y, it suffices to show

$$\mathrm{Clos}_M(X) \cap Y = \emptyset. \tag{3.14}$$

Since V_i is closed in $L^{(k)}$ by (3.6), we have

$$\mathrm{Clos}_{L^{(k)}}(X) \subset \mathrm{Clos}_{L^{(k)}}(V_i) = V_i \subset V_{j-1}.$$

Moreover using $Y = V_j - V_{j-1}$ and $Y \subset L^{(k)}$ yields

$$\begin{aligned} &\mathrm{Clos}_{L^{(k)}}(X) \cap Y = \emptyset, \\ &(M - L^{(k)}) \cap Y = \emptyset, \end{aligned} \tag{3.15}$$

and on the other hand we can rewrite $\mathrm{Clos}_M(X)$ as

$$\begin{aligned}\mathrm{Clos}_M(X) &= (\mathrm{Clos}_M(X)\cap L^{(k)})\cup(\mathrm{Clos}_M(X)\cap(M-L^{(k)}))\\ &= \mathrm{Clos}_{L^{(k)}}(X)\cup(\mathrm{Clos}_M(X)\cap(M-L^{(k)})).\end{aligned}\tag{3.16}$$

Then (3.14) is an immediate consequence of (3.15) and (3.16). Q.E.D.

Now we give some examples of singular foliations and observe the stratifications we mentioned in theorem (3.4).

Example 3.17. Let f be the holomorphic function on $M = \boldsymbol{C}^3$ defined by

$$f(x,y,z) = z(x^2-y^3),$$

and ω the holomorphic 1-form on $\boldsymbol{C}^3$ defined by

$$\omega = df = 2xz\,dx - 3y^2z\,dy + (x^2-y^3)\,dz.$$

The coherent subsheaf $F(\subset\Omega_M)$ generated by ω is integrable since $d\omega = ddf = 0$, so F defines a singular foliation on $\boldsymbol{C}^3$. $E = F^a(\subset\Theta_M)$ is generated by the following two vector fields:

$$\begin{cases} v_1 = 3y^2\dfrac{\partial}{\partial x} + 2x\dfrac{\partial}{\partial y}\\[2mm] v_2 = 3x\dfrac{\partial}{\partial x} + 2y\dfrac{\partial}{\partial y} - 6z\dfrac{\partial}{\partial z}.\end{cases}\tag{3.18}$$

E is reduced, and $\operatorname{rank} E = 2$. By (3.18), $S(E) = S^{(1)} = \{xz = yz = x^2-y^3 = 0\} = \{x = y = 0\}\cup\{z = x^2-y^3 = 0\}$ and $S^{(0)} = \{(0,0,0)\}$. In this case, a Whitney stratification $\mathscr{S}$ of M defined by

$$\mathscr{S} = \{M - S(E), X_1, X_2, \{0\}\}\qquad\begin{pmatrix} X_1 = \{x = y = 0\} - \{0\}\\ X_2 = \{z = x^2-y^3 = 0\} - \{0\}\end{pmatrix}$$

meets the requirements in theorem (3.4).

The following example tells us that a Whitney stratification on M which satisfies (3.4) cannot generally be obtained by adding some strata to the natural Whitney stratification of $S^{(0)}$.

Example 3.19. Let ω be a holomorphic 1-form on $M = \boldsymbol{C}^3$ defined by

$$\omega = 2xz^2\,dx - 2yz\,dy + y^2\,dz,$$

and $F(\subset\Omega_M)$ a coherent subsheaf generated by ω. $E = F^a(\subset\Theta_M)$ is generated by the following two vector fields:

$$\begin{cases} v_1 = y\dfrac{\partial}{\partial y} + 2z\dfrac{\partial}{\partial z}\\[2mm] v_2 = y\dfrac{\partial}{\partial x} + xz\dfrac{\partial}{\partial y}.\end{cases}\tag{3.20}$$

E is integrable since $[v_1, v_2] = v_2$, so E defines a singular foliation on $\boldsymbol{C}^3$. E is reduced, and $\operatorname{rank} E = 2$. By (3.20), $S(E) = S^{(1)} = \{y = xz = 0\} = \{x\text{-axis}\} \cup \{z\text{-axis}\}$ and $S^{(0)} = \{y = z = 0\} = \{x\text{-axis}\}$. In this situation, a stratification $\mathscr{S}'$ of M defined by

$$\mathscr{S}' = \{M - S(E), S(E) - S^{(0)}, S^{(0)}\}$$

satisfies the two conditions in theorem (3.4), but this is not a Whitney stratification (and E is not trivial along $S^{(0)}$). A Whitney stratification $\mathscr{S}$ of M which meets the requirements in theorem (3.4) is given by

$$\mathscr{S} = \{M - S(E), X_1, X_2, \{0\}\} \qquad \begin{pmatrix} X_1 = \{z\text{-axis}\} - \{0\} \\ X_2 = \{x\text{-axis}\} - \{0\} \end{pmatrix}$$

4. Local topological triviality of singular foliations.

In this section we examine the local topological triviality of singular foliations, which is the main subject in this paper. Let E be a singular foliation on M and $\mathscr{S}$ a stratification of M. For the topological triviality of E along each stratum in $\mathscr{S}$, it is necessary that $\mathscr{S}$ is adapted to E as stated in the preceding section. We consider only stratifications adapted to E hereafter. Note that theorem (3.4) assures that there always exists a stratification adapted to E with a stronger condition (Whitney regularity).

To tell the consequence at first, E is topologically locally trivial along each stratum in $\mathscr{S}$ if $\mathscr{S}$ satisfies the "foliated Verdier condition" which will be mentioned later. We begin this section by recalling basic concepts to give the precise definition of the local topological triviality. For more details, see, for example, [**GWPL**] pp 41–50.

Definition 4.1. Let X be a submanifold of M. A tubular neighborhood of X is a triple (T, π, ρ) which satisfies:

(i) T is a neighborhood of X in M.

(ii) $\pi : T \to X$ is a submersion (with $\pi|_X = id_X$).

(iii) $\rho : T \to \boldsymbol{R}$ is a C^∞-function.

(iv) Let $\pi_X : N_X \to X$ denote the normal bundle of X in M, and let Z denote the image of the zero section of N_X. Then there exist a neighborhood D of Z in N_X and a diffeomorphism $\varphi : D \to T$ such that

$$\begin{tikzcd} D \arrow[rr, "\varphi"] \arrow[dr, "\pi_X|_D"'] & & T \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

commutes.

(v) Let $\tau : N_X \to \boldsymbol{R}$ denote the distance function to Z which is determined by a Riemannian metric induced on N_X. Then

$$\begin{tikzcd} D \arrow[rr, "\varphi"] \arrow[dr, "\tau|_D"'] & & T \arrow[dl, "\rho"] \\ & \boldsymbol{R} & \end{tikzcd}$$

commutes.

DEFINITION 4.2. Let M be a differentiable manifold of dimension n and A a subset of M. For a Whitney stratification $\mathscr{A}$ of A, we set

$$A^i = \bigcup_{\substack{X \in \mathscr{A} \\ \dim X = i}} X \quad (0 \leq i \leq n),$$

i.e., A^i is the union of all strata of dimension i. Each $A^i(\neq \emptyset)$ is an i-dimensional submanifold in M since $\mathscr{A}$ is a Whitney stratification. A family of a tubular neighborhood of each $A^i(\neq \emptyset)$

$$\mathscr{T} = \{(T^i, \pi^i, \rho^i)\}_{0 \leq i \leq n}$$

is called a tubular neighborhood system of $\mathscr{A}$. A tubular neighborhood system $\mathscr{T} = \{(T^i, \pi^i, \rho^i)\}$ of $\mathscr{A}$ is said to be controlled if for all integers i, j $(i < j)$ there exist a neighborhood U^i of A^i in T^i and a neighborhood U^j of A^j in T^j such that

$$\pi^i \circ \pi^j = \pi^i, \tag{4.3}$$

$$\rho^i \circ \pi^j = \rho^i \tag{4.4}$$

hold on $U^i \cap U^j$.

It is generally known that every Whitney stratification admits a controlled tubular neighborhood system.

Now we give the definition of the local topological triviality. In the following we shall fix a Riemannian metric of M.

DEFINITION 4.5. Let M be a complex manifold of dimension n and $E(\subset \Theta_M)$ a singular foliation on M. Also, let X be a submanifold in M and set $l = \dim_{\mathbf{C}} X$. Suppose $X \subset L^{(k)}$, i.e., the leaf dimension of E is constant on X. E is said to be topologically locally trivial along X when for any point $p \in X$ there exist

(T, π, ρ) : a tubular neighborhood of X,

U_p : a sufficiently small neighborhood of p in M,

D : a small neighborhood around 0 in $\mathbf{C}^{n-l}$,

E' : a singular foliation on D,

h : a homeomorphism from U_p onto $(X \cap U_p) \times D$

such that

(i) $h(x) = (x, 0)$ holds for any $x \in X \cap U_p$.

(ii) $h|_{U_p - X}$ transforms the leaves defined by E into the product of $X \cap U_p$ and the leaves defined by E'.

(iii) Let $pr_1 : (X \cap U_p) \times D \to X$ denote the natural projection to the first component, then

$$\begin{tikzcd} U_p \arrow[rr, "h"] \arrow[dr, "\pi|_{U_p}"'] & & (X \cap U_p) \times D \arrow[dl, "pr_1"] \\ & X & \end{tikzcd}$$

commutes.

REMARK 4.6. In this paper we consider only complex analytic foliations, so the trivialization $h : U_p \to (X \cap U_p) \times D$ is in practice a diffeomorphism.

Next, let us introduce the foliated Verdier condition for a stratification of $S(E)$. In order to define the condition, it is necessary to refer to the notion of the distance between two vector subspaces. The distance is generally defined by measuring the angle between V and W.

DEFINITION 4.7. Let V, W be two vector subspaces of a finite-dimensional inner product space. We define the distance between V and W by

$$\delta(W, V) = \sup_{\substack{u \in W^{\perp}-\{0\} \\ v \in V-\{0\}}} \frac{|\langle u, v\rangle|}{\|u\| \cdot \|v\|}$$

where $\|\cdot\|$ denotes the norm induced by the inner product $\langle\ ,\ \rangle$.

Note that $\delta(W, V)$ is not always equal to $\delta(V, W)$. Clearly we have $\delta(W, V) \in [0, 1]$, and we can also express $\delta(W, V)$ as follows:

$$\delta(W, V) = \sup_{v \in V-\{0\}} \inf_{w \in W-\{0\}} \sin\langle\!\langle v, w\rangle\!\rangle,$$

where $\langle\!\langle v, w\rangle\!\rangle$ denotes the angle between v and w.

REMARK 4.8. It is easy to check that $\delta(W, V)$ satisfies the following properties.
(i) $\delta(W, V) = 0 \Leftrightarrow V \subset W$.
(ii) $\dim V = \dim W \Rightarrow \delta(V, W) = \delta(W, V)$.
(iii) $\delta(W, V) = 1 \Leftrightarrow$ there exists a non-zero vector $v \in V$ such that $\langle v, w\rangle = 0$ for all $w \in W$
$\Leftrightarrow V \cap W^{\perp} \neq \{0\}$.
(iv) $\dim W < \dim V \Rightarrow \delta(W, V) = 1$.

Now, we are ready to define the foliated Verdier condition for a stratification adapted to E. In the following we identify tangent spaces of nearby points by parallel translation determined by the Riemannian metric.

DEFINITION 4.9. Let $E(\subset \Theta_M)$ be a singular foliation on M and let X be a submanifold in M such that $X \subset L^{(k)}$, i.e., the leaf dimension of E is constant on X. Let p be a point in X. We say E satisfies the foliated Verdier condition at p over X when there exist a tubular neighborhood (T, π, ρ) of X, a neighborhood U_p around p contained in T, and a real number $\lambda > 0$ such that

$$\delta(E(y), T_p X) \leq \lambda \cdot \rho(y)$$

hold for all $y \in U_p - X$. If E satisfies the foliated Verdier condition over X at every point $p \in X$, then E is simply said to satisfy the foliated Verdier condition over X. Moreover, a stratification $\mathscr{S}$ adapted to E is called a foliated Verdier stratification if E satisfies the foliated Verdier condition over all strata $X \in \mathscr{S}$.

We have the following "isotopy lemma" for singular foliations.

THEOREM 4.10. *Let $E(\subset \Theta_M)$ be a singular foliation of dimension r on M. Suppose $\mathcal{S}$ is a foliated Verdier stratification. Then E is topologically locally trivial along each stratum $X \in \mathcal{S}$.*

We introduce here a lemma for the proof of theorem (4.10). This lemma, which is the most essential part in the proof of (4.10), says that the foliated Verdier condition assures that any continuous vector field on a stratum X can be "lifted" onto each stratum sufficiently close to X.

LEMMA 4.11. *Let $E(\subset \Theta_M)$ be a singular foliation of dimension r on M and let $\mathcal{S}$ be a foliated Verdier stratification. Let X, Y be two strata of $\mathcal{S}$ such that $X \cap \overline{Y} \neq \emptyset$. Given a real continuous vector field $v : X \to TX$ such that $v(x) \neq 0$ for all $x \in X$. Then for any tubular neighborhood (T, π, ρ) of X, we can construct a continuous extension ξ of v on $U \cap Y$ (where U is a sufficiently small neighborhood of X) so that the following conditions are fulfilled:*

(i) $\pi_* \circ \xi = v \circ \pi$ *holds on* $U \cap Y$.

(ii) $\xi(y) \in E(y)$ *hold for all* $y \in U \cap Y$, i.e., ξ *is tangent to the leaves defined by* E *at all point* y *in* $U \cap Y$.

(iii) *Let* $\{\varphi_t = \exp t\xi\}$ *be the local* 1*-parameter group of transformations induced by* ξ. *Then for all* t *sufficiently close to* 0 *and all point* $y \in U \cap Y$, *we have* $\rho(\varphi_t(y)) > 0$, *i.e., the* (*local*) *integral curve through* y *does not meet* X.

In the proof of this lemma, we use some basic facts about linear algebra. Let $\mathscr{V}$ be a finite-dimensional vector space with an inner product $\langle\ ,\ \rangle$, and let V, W be two vector subspaces of $\mathscr{V}$. We set $K = W \cap V^\perp$ and $J = W \cap K^\perp$, and we denote by $pr_V : \mathscr{V} \to V$ the orthogonal projection to V. Then we have the following properties.

(i) $pr_V|_J : J \to V$ is injective.

(ii) $pr_V|_J : J \to V$ is surjective $\Leftrightarrow V \cap W^\perp = \{0\}$.

(iii) $\delta(W, V) = \delta(J, V)$.

PROOF OF (4.11). Set $\dim_{\mathbf{R}} X = 2l$. We will give below how to determine $\xi(y)$ for all point $y \in Y$ around a fixed point $p \in X$. Since we may only consider y sufficiently close to X, we may discuss under the following situation:

$$
\begin{aligned}
&U_p : \text{a coordinate neighborhood around } p,\\
&(x_1, \dots, x_{2n}) : \text{real coordinates on } U_p \text{ such that } x(p) = (0, 0, \dots, 0)\\
&\qquad \text{and } X \cap U_p = \{x_{2l+1} = x_{2l+2} = \cdots = x_{2n} = 0\},\\
&\pi : \quad U_p \to X\\
&\qquad (x_1, x_2, \dots, x_{2n}) \mapsto (x_1, x_2, \dots, x_{2l}, 0, 0, \dots, 0),\\
&\rho : \quad U_p \to \mathbf{R}\\
&\qquad (x_1, x_2, \dots, x_{2n}) \mapsto \sqrt{x_{2l+1}^2 + x_{2l+2}^2 + \cdots + x_{2n}^2}.
\end{aligned}
$$

For any point $y \in U_p$, the vectors $\{(\partial/\partial x_1)_y, (\partial/\partial x_2)_y, \dots, (\partial/\partial x_{2n})_y\}$ form an orthonormal basis of T_yM. We may also assume that there exists $\lambda > 0$ such that

$$\delta(E(y),\ T_pX) \leq \lambda \cdot \rho(y) \tag{4.12}$$

hold for all $y \in U_p \cap Y$. For the sake of simplicity, we put $q = \pi(y)$.

Let ψ_y be the linear map from T_qX to T_yM defined by

$$\psi_y\left(\left(\frac{\partial}{\partial x_i}\right)_q\right) = \left(\frac{\partial}{\partial x_i}\right)_y \qquad (i = 1, 2, \dots, 2l),$$

i.e., $\psi_y(u)$ is the parallel displacement of u. Then the vector field η on $U_p \cap Y$ determined by $\eta(y) = \psi_y(v(q))$ is clearly a continuous extension of v and satisfies the first and third conditions in (4.11), but does not meet the requirement of the second condition. So we will modify η so that $\eta(y)$ is tangent to the leaf defined by E at each y.

We also define two sub-vector spaces of T_yM by

$$\begin{aligned} K(y) &= E(y) \cap \ker(\pi_*|_{T_yM})(= E(y) \cap (\psi_y(T_qX))^{\perp}) \\ J(y) &= E(y) \cap K(y)^{\perp}. \end{aligned}$$

By (4.12), we have

$$\delta(E(y),\ T_qX) < \frac{1}{2} \tag{4.13}$$

for all y sufficiently close to X, so we may assume (4.13) hold for all $y \in U_p \cap Y$. Then we have $T_qX \cap E(y)^{\perp} = \{0\}$ from (4.8), thus $\pi_*|_{J(y)} : J(y) \to T_qX$ is a linear isomorphism. We define here a linear map $L_y : T_qX \to T_yM$, which gives the modification for η, by

$$L_y = (\pi_*|_{J(y)})^{-1} - \psi_y.$$

Then we have

$$\begin{aligned} \delta(E(y), T_qX) &= \delta(J(y), T_qX) \\ &= \delta(T_qX, J(y)) \\ &= \sup_{w \in J(y)-\{0\}} \inf_{u \in T_qX-\{0\}} \sin\langle\!\langle w, u \rangle\!\rangle \\ &= \sup_{w \in J(y)-\{0\}} \sin\langle\!\langle w, \pi_*(w) \rangle\!\rangle \\ &= \sup_{w \in J(y)-\{0\}} \tan\langle\!\langle w, \pi_*(w) \rangle\!\rangle \cdot \cos\langle\!\langle w, \pi_*(w) \rangle\!\rangle \\ &= \sup_{w \in J(y)-\{0\}} \frac{\|w - \psi_y(\pi_*(w))\|}{\|\psi_y(\pi_*(w))\|} \cdot \cos\langle\!\langle w, \pi_*(w) \rangle\!\rangle \\ &= \sup_{w \in J(y)-\{0\}} \frac{\|L_y(\pi_*(w))\|}{\|\pi_*(w)\|} \cdot \cos\langle\!\langle w, \pi_*(w) \rangle\!\rangle. \end{aligned} \tag{4.14}$$

Now we define the vector field $\xi(y)$, which we are asking for, by

$$\xi(y) = \psi_y(v(q)) + L_y(v(q)) \quad (= (\pi_*|_{J(y)})^{-1}(v(q)))$$

for all $y \in U_p \cap Y$. It is clear that ξ satisfies (i) and (ii) in (4.11), so let us check (iii) and that ξ is a continuous extension of v.

Take a sufficiently small $\varepsilon > 0$ and set $I = (-\varepsilon, \varepsilon)$. Suppose that there exists $y \in U_p \cap Y$ such that $\rho(\varphi_t(y)) = 0$ for some $t \in I$. Since $\rho(\varphi_0(y)) = \rho(y) > 0$, we can take a real number $t_0 \neq 0$ such that $\rho(\varphi_t(y)) > 0$ for all $t \in (-|t_0|, |t_0|)$ and $\rho(\varphi_{t_0}(y)) = 0$. We may assume $t_0 > 0$. For the sake of simplicity, we put $f(t) = \rho(\varphi_t(y))$.

Set $\varphi_t(y) = (y_1(t), y_2(t), \dots, y_{2n}(t))$. Since $\varphi_t = \exp t\xi$, we have

$$\xi(\varphi_t(y)) = \sum_{i=1}^{2n} y_i'(t) \frac{\partial}{\partial x_i}.$$

Then for all $t \in (0, t_0)$ we obtain

$$\begin{aligned}
\left|\frac{df}{dt}(t)\right| &= \left|\sum_{i=1}^{2n} \frac{\partial \rho}{\partial x_i}(\varphi_t(y)) \cdot \frac{dy_i}{dt}(t)\right| \\
&= \left|\sum_{i=2l+1}^{2n} \frac{y_i(t)}{\sqrt{y_{2l+1}(t)^2 + \cdots + y_{2n}(t)^2}} \cdot y_i'(t)\right| \\
&= \left|\sum_{i=2l+1}^{2n} \frac{y_i(t)}{\rho(\varphi_t(y))} \cdot y_i'(t)\right| \\
&\leq \sqrt{\sum_{i=2l+1}^{2n} \left(\frac{y_i(t)}{\rho(\varphi_t(y))}\right)^2} \cdot \sqrt{\sum_{i=2l+1}^{2n} y_i'(t)^2} \\
&= 1 \cdot \|L_{\varphi_t(y)}(v(\pi(\varphi_t(y))))\| \\
&= \|L_{\varphi_t(y)}(v(q_t))\|,
\end{aligned} \tag{4.15}$$

where $q_t = \pi(\varphi_t(y))$. On the other hand, (4.14) tells us that for all y we have

$$\delta(E(y), T_q X) \geq \frac{\|L_y(v(q))\|}{\|v(q)\|} \cdot \cos\langle\!\langle (\pi_*|_{J(y)})^{-1}(v(q)), v(q) \rangle\!\rangle, \tag{4.16}$$

and the foliated Verdier condition implies

$$\langle\!\langle (\pi_*|_{J(y)})^{-1}(v(q)), v(q) \rangle\!\rangle \to 0 \quad (\text{as } \rho(y) \to 0).$$

Moreover we may assume U_p is relatively compact, thus

$$M_0 = \sup_{x \in X \cap U_p} \|v(x)\| < +\infty.$$

Hence (4.16) implies that

$$\begin{aligned}
\|L_y(v(q))\| &\leq \delta(E(y), T_q X) \cdot \frac{\|v(q)\|}{\cos\langle\!\langle (\pi_*|_{J(y)})^{-1}(v(q)), v(q) \rangle\!\rangle} \\
&\leq 2M_0 \cdot \delta(E(y), T_q X) \\
&\leq 2\lambda M_0 \cdot \rho(y)
\end{aligned} \tag{4.17}$$

hold for all $y \in U_p \cap Y$. By (4.15) and (4.17), we obtain

$$\left|\frac{df}{dt}(t)\right| \leq 2\lambda M_0 \cdot \rho(\varphi_t(y)) = \exists\lambda_0 \cdot f(t)$$

for all $t \in (0, t_0)$. Thus we have $-\lambda_0 \cdot f(t) \leq (df/dt)(t)$. Integrating the both sides from 0 to t, it turns out that

$$\rho(y) \cdot e^{-\lambda_0 t} \leq f(t)$$

hold for all $t \in (0, t_0)$. This contradicts $f(t_0) = 0$, thus (iii) in (4.11) holds.

Next let us check the continuity of ξ constructed above. E induces a non-singular foliation on Y because $\mathscr{S}$ is adapted to E. Hence it is clear that ξ is continuous on Y by the way of construction. The fact that ξ is a continuous extension of v is an immediate consequence of (4.17). Q.E.D.

PROOF OF (4.10). Let $\mathscr{S}$ be a foliated Verdier stratification. Look upon all strata in $\mathscr{S}$ as real differentiable submanifolds and take a controlled tubular neighborhood system $\mathscr{T} = \{(T^{2i}, \pi^{2i}, \rho^{2i})\}$ of $\mathscr{S}$. Recall that M^{2i} denotes the union of all strata of dimension $2i$ and each $(T^{2i}, \pi^{2i}, \rho^{2i})$ is a tubular neighborhood of M^{2i}. Choose a stratum $X \in \mathscr{S}$ arbitrarily, and set $\dim_{\mathbf{R}} X = 2l$.

By the definition of the local topological triviality, it suffices to show that for every point $p \in X$ E is trivial along X on a sufficiently small neighborhood U of p. This is a local assertion at p, so we may assume

$$\begin{aligned} U &: \text{a coordinate neighborhood of } p \text{ contained in } T^{2l}, \\ (x_1, \ldots, x_{2n}) &: \text{real coordinates on } U \text{ such that } x(p) = (0, 0, \ldots, 0) \\ &\quad \text{and } X \cap U = \{x_{2l+1} = x_{2l+2} = \cdots = x_{2n} = 0\}. \end{aligned}$$

Hereafter we argue only on U. By shrinking all T^{2i} $(l \leq i < n)$, we can take (sufficiently small) closed disk bundles $F^{2i} \subset T^{2i}$ so that

$$\text{(4.18)} \qquad \mathrm{Clos}_U(F^{2i}) - F^{2i} \subset \bigcup_{l \leq j \leq i-1} (M^{2j} \cap U).$$

In order to get a local trivialization $h : U \to (U \cap X) \times D$, it is sufficient to integrate continuous vector fields $\xi_1, \xi_2, \ldots, \xi_{2l}$ on U such that for every j $(1 \leq j \leq 2l)$ we have

(4.19) (i) $(\pi^{2l})_* \circ \xi_j = \dfrac{\partial}{\partial x_j} \circ \pi^{2l}$ holds on U.

(ii) $\xi_j(y) \in E(y)$ hold for all $y \in U - X$, i.e., ξ_j is tangent to the leaves defined by E at all point y in $U - X$.

(iii) For any point $y \in U$, the integral curve of ξ_j through y stays in the stratum including y.

We do this work by constructing $\xi_1^{(2i)}, \xi_2^{(2i)}, \ldots, \xi_{2l}^{(2i)}$ on each $M^{2i} \cap U$ successively for $i = l, l+1, \ldots, n$.

We define $\xi_1^{(2l)}, \xi_2^{(2l)}, \dots, \xi_{2l}^{(2l)}$ on $M^{2l} \cap U$ $(= X \cap U = \{x_{2l+1} = x_{2l+2} = \cdots = x_{2n} = 0\})$ to be $\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_{2l}$ respectively. It is clear that $\xi_1^{(2l)}, \xi_2^{(2l)}, \dots, \xi_{2l}^{(2l)}$ satisfy the three conditions in (4.19).

Suppose we have already constructed all $\xi_1^{(2i)}, \xi_2^{(2i)}, \dots, \xi_{2l}^{(2i)}$ which satisfy (i)–(iii) in (4.19) on each $M^{2i} \cap U$ for $l \le i \le a-1$. In order to define $\xi_1^{(2a)}, \xi_2^{(2a)}, \dots, \xi_{2l}^{(2a)}$ on $M^{2a} \cap U$, we construct them on each $T^{2i} \cap M^{2a} \cap U$ $(l \le i \le a-1)$ and glue the pieces together by means of a partition of unity on $M^{2a} \cap U$.

We may assume that each T^{2i} $(l \le i \le a-1)$ is so small that we could apply lemma (4.11) to $M^{2i} \cap U$ and $T^{2i} \cap U$. Note that $\xi_j^{(2i)} \neq 0$ on $M^{2i} \cap U$ for all i with $l \le i \le a-1$ and for all j with $1 \le j \le 2l$ because $(\pi^{2l})_* \circ \xi_j^{(2i)} = \partial/\partial x_j \circ \pi^{2l}$ holds on $M^{2i} \cap U$ by the inductive assumption. Thus, applying lemma (4.11), we obtain continuous extensions η_j^{2i} of $\xi_j^{(2i)}$ on $T^{2i} \cap M^{2a} \cap U$ which satisfy (i)–(iii) in (4.11) for all i, j with $l \le i \le a-1$, $1 \le j \le 2l$.

Now we define $\xi_j^{(2a)}$ on $M^{2a} \cap U$ as follows. First, we set

$$Q^{2(a-1)} = T^{2(a-1)} \cap M^{2a} \cap U,$$

$$Q^{2i} = \left(T^{2i} - \bigcup_{m=i+1}^{a-1} \mathrm{Clos}_U(F^{2m}) \right) \cap M^{2a} \cap U \text{ (for } i = a-2, a-3, \dots, l+1, l).$$

Note that $\{Q^{2i}\}_{l \le i \le a-1}$ is an open covering of $M^{2a} \cap U$ by (4.18). Then glue all η_j^{2i} on Q^{2i} together by means of a partition of unity on $M^{2a} \cap U$ subordinate to $\{Q^{2i}\}$, and define $\xi_j^{(2a)}$ to be the resulting vecter field. Let us check below that $\xi_j^{(2a)}$ meets the three requirements in (4.19) for each j.

At first, we will show that all η_j^{2k} $(l \le k \le a-1)$ satisfy (i). The following equation holds on Q^{2k}:

$$\begin{aligned}(\pi^{2l})_* \circ \eta_j^{2k} &= (\pi^{2l} \circ \pi^{2k})_* \circ \eta_j^{2k} = (\pi^{2l})_* \circ (\pi^{2k})_* \circ \eta_j^{2k} = (\pi^{2l})_* \circ \xi_j^{(2k)} \circ \pi^{2k} \\ &= \xi_j^{(2l)} \circ \pi^{2l} \circ \pi^{2k} = \frac{\partial}{\partial x_j} \circ \pi^{2l},\end{aligned}$$

thus η_j^{2k} fulfills (i). It is obvious that all η_j^{2k} satisfy (ii). For (iii), it suffices to show that if $y \in \mathrm{Int}(F^{2i}) \cap M^{2a} \cap U$ then the integral curve of $\xi_j^{(2a)}$ through y does not meet $M^{2i} \cap U$ (for every integer i with $l \le i \le a-1$). Since F^{2i} does not intersect Q^{2k} with $l \le k \le i-1$ by the definition of Q^{2k}, $\xi_j^{(2a)}$ has been determined on $\mathrm{Int}(F^{2i}) \cap M^{2a} \cap U$ using only η_j^{2k} with $i \le k \le a-1$. Let $\{(\psi_j^{2k})_t = \exp t\eta_j^{2k}\}$ and $\{(\varphi_j^{(2k)})_t = \exp t\xi_j^{(2k)}\}$ denote the local 1-parameter groups of transformations respectively. By the construction of η_j^{2i}, $(\psi_j^{2i})_t(y)$ does not meet M^{2i}. For each integer k with $i < k \le a-1$, we have $\rho^{2i} = \rho^{2i} \circ \pi^{2k}$ by (4.4). Hence

$$\rho^{2i} \circ (\psi_j^{2k})_t(y) = \rho^{2i} \circ \pi^{2k} \circ (\psi_j^{2k})_t(y) = \rho^{2i} \circ (\varphi_j^{(2k)})_t(y) > 0$$

hold for all t sufficiently close to 0 by the inductive assumption. This implies that $(\psi_j^{2k})_t(y)$ does not meet M^{2i} for all t sufficiently close to 0, thus (iii) holds.

This completes the induction and the proof of this theorem. Q.E.D.

Let us close this paper by giving some examples about foliated Verdier stratifications.

EXAMPLE 4.20. Let v_1, v_2 be holomorphic vector fields on $M = \boldsymbol{C}^3$ defined by

$$(4.21)\qquad \begin{cases} v_1 = y\dfrac{\partial}{\partial x} - xyz\dfrac{\partial}{\partial y} + xy^2\dfrac{\partial}{\partial z} \\ \\ v_2 = z\dfrac{\partial}{\partial x} - xz^2\dfrac{\partial}{\partial y} + xyz\dfrac{\partial}{\partial z}. \end{cases}$$

Let $E(\subset \Theta_M)$ be the coherent subsheaf generated by v_1, v_2. E is integrable since $[v_1, v_2] = xyv_1 + xzv_2$, so E defines a singular foliation on $\boldsymbol{C}^3$. The rank of E is one, and by (4.21), $S(E) = S^{(0)} = \{y = z = 0\} = \{x\text{-axis}\}$. Set $X = \{x\text{-axis}\}$ and $Y = \boldsymbol{C}^3 - \{x\text{-axis}\}$, then $\mathscr{S} = \{X, Y\}$ gives a foliated Verdier stratification of $\boldsymbol{C}^3$. Therefore E is topologically locally trivial along X by (4.10).

EXAMPLE 4.22. Let v_1, v_2, v_3 be holomorphic vector fields on $M = \boldsymbol{C}^3$ defined by

$$(4.23)\qquad \begin{cases} v_1 = 3y^2\dfrac{\partial}{\partial x} + 2x\dfrac{\partial}{\partial y} \\ \\ v_2 = (x^2 - y^3)\dfrac{\partial}{\partial y} + 3y^2 z\dfrac{\partial}{\partial z} \\ \\ v_3 = (x^2 - y^3)\dfrac{\partial}{\partial x} - 2xz\dfrac{\partial}{\partial z}. \end{cases}$$

Let $E(\subset \Theta_M)$ be the coherent subsheaf generated by v_1, v_2, v_3. We can easily check that E is integrable (at every point of $\boldsymbol{C}^3$), so E defines a singular foliation on $\boldsymbol{C}^3$. The rank of E is two, and by (4.23), $S(E) = S^{(1)} = \{xz = yz = x^2 - y^3 = 0\} = \{x = y = 0\} \cup \{z = x^2 - y^3 = 0\}$ and $S^{(0)} = \{x = y = 0\}$. Set $X_1 = \{x = y = 0\} - \{0\}$ and $X_2 = \{z = x^2 - y^3 = 0\} - \{0\}$, then $\mathscr{S} = \{\boldsymbol{C}^3 - S(E), X_1, X_2, \{0\}\}$ gives a foliated Verdier stratification of $\boldsymbol{C}^3$. Therefore E is topologically locally trivial along each stratum of $\mathscr{S}$ by (4.10).

EXAMPLE 4.24. Let us recall the singular foliation E on $\boldsymbol{C}^3$ given in (3.19). If we take the stratification $\mathscr{S}' = \{\boldsymbol{C}^3 - S(E), S(E) - S^{(0)}, S^{(0)}\}$, the structure of E is not trivial at 0 along the strarum $S^{(0)}$. So it is necessary to separate the bad point 0 from $S^{(0)}$ to obtain the local triviality along each stratum. The stratification $\mathscr{S} = \{\boldsymbol{C}^3 - S(E), X_1, X_2, \{0\}\}$ is adapted to E, but this is not a foliated Verdier stratification (E does not satisfy the foliated Verdier condition at $p = (x, 0, 0)$ $(x \neq 0)$ over $X_2 = \{x\text{-axis}\} - \{0\}$. See the directions of the leaves through $(x, 0, z) \in \boldsymbol{C}^3$ for $z \in \boldsymbol{C}$ sufficiently close to 0). We cannot take a foliated Verdier stratification for this type of singular foliations.

References

[B] P. Baum, Structure of foliation singularities, Adv. in Math. **15**, pp 361–374, 1975.

[BB] P. Baum and R. Bott, Singularities of holomorphic foliations, J. of Diff. Geom. **7**, pp 279–342, 1972.

[C] D. Cerveau, Distributions involutives singulières, Ann. Inst. Fourier **29**, pp 261–294, 1979.

[GWPL] C. G. Gibson, K. Wirthmüller, A. A. du Plessis and E. J. N. Looijenga, Topological stability of smooth mappings, Lecture Notes in Mathematics 552, Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[K] A. Kabila, Formes integrables a singularites lisses, Thèse, Université de Dijon, 1983.

[MY] Y. Mitera and J. Yoshizaki, The local analytical triviality of a complex analytic singular foliation, preprint.

[N] T. Nagano, Linear differential systems with singularities and application to transitive Lie algebras, J. Math. Soc. Japan 18, 1966.

[Ss] H. J. Sussmann, Orbits of families of vector fields and integrability of distributions, Trans. Amer. Math. Soc. **180**, pp 171–188, 1973.

[St] P. Stefan, Accessible sets, orbits, and foliations with singularities, Proc. London Math. Soc. **29**, pp 699–713, 1974.

[Sw] T. Suwa, Unfoldings of complex analytic foliations with singularities, Japanese J. of Math. **9**, pp 181–206, 1983.

[TW] D. J. A. Trotman and L. C. Wilson, Stratifications and finite determinacy, Prépublicaitions 94–9, Université de Provence.

[W] H. Whitney, Tangents to an analytic variety, Ann. of Math. **81**, pp 496–549, 1965.

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