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The dimensions of self-similar sets

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1. Introduction.

Let ϕ_i be similar contraction mappings in $\boldsymbol{R}^d$ with ratios c_i, $1 \le i \le n$. Hu [5] proved that there exists unique compact set $F \subset \boldsymbol{R}^d$ such that

$$F = \bigcup_{i=1}^{n} \phi_i(F). \tag{1}$$

Further $\dim_H F = \dim_B F = \dim_P F = s$ and F is an s-set where s is such that

$$\sum_{i=1}^{n} c_i^s = 1, \tag{2}$$

if ϕ_i's satisfy the open set condition, i.e. there is a bounded nonempty open set O such that

$$\bigcup_{i=1}^{n} \phi_i(O) \subset O \tag{3}$$

with the left hand is disjoint union. Recently Sc [10] proved that F is an s-set here $\sum_{i=1}^{n} c_i^s = 1$ if and only if ϕ_i's satisfy the open condition.

Now for $\varepsilon > 0$ write

$$\Omega(\varepsilon) = \{\sigma \in S^* \,|\, c_\sigma \le \varepsilon \text{ and } c_{\sigma|(|\sigma|-1)} > \varepsilon\},$$

where $S^* = \bigcup_{i=1}^{\infty}\{1, 2, \dots, n\}^i$ and $c_\sigma = c_{\sigma(1)}c_{\sigma(2)}\cdots c_{\sigma(k)}$ for $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(k)) \in S^*$. And for $\sigma \in S^*$, $|\sigma|$ denotes the length of σ and $\sigma|k = (\sigma(1), \dots, \sigma(k))$ for $k \le |\sigma|$. Let $A \subset \boldsymbol{R}^d$ be a bounded open set with $A \supset F$. It is easy to see that $c_0\varepsilon < c_\sigma \le \varepsilon$ for any $\sigma \in \Omega(\varepsilon)$ where $c_0 = \min_{1 \le i \le n} c_i$. We introduce nonnegative real numbers $\alpha_0(A)$ and $\beta_0(A)$ as follows

$$\alpha_0(A) = \sup\left\{ \alpha \,|\, \underline{\lim}_{\varepsilon \to 0} \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma \in \Omega(\varepsilon)} c_\sigma^{s(1-\alpha)}} = \infty \right\}, \tag{4}$$

$$\beta_0(A) = \sup\left\{ \beta \,|\, \overline{\lim}_{\varepsilon \to 0} \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma \in \Omega(\varepsilon)} c_\sigma^{s(1-\beta)}} = \infty \right\}, \tag{5}$$

where $\phi_\sigma = \phi_{\sigma(1)} \circ \phi_{\sigma(2)} \circ \cdots \circ \phi_{\sigma(k)}$ for $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(k)) \in S^*$ and $\mathrm{m_d}(B)$ is the Lebesgue measure of $B \subset \boldsymbol{R}^d$.

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In this paper we prove

(i) $\alpha_0(A)$ and $\beta_0(A)$ are independent of the choice of A and $\alpha_0(A)=\beta_0(A)$. We denote the common value by α_0.

(ii) $\dim_H F=\dim_B F=\dim_P F=\alpha_0 s$.

(For self-similar set F Fa[4] has proved that its Hausdorff dimension, Box dimension and Packing dimension are equal)

(iii) $\mathscr{H}^{\alpha_0 s}(F)<\infty$ iff $\underline{\lim}_{\varepsilon\to 0}\dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)}c_\sigma^{s(1-\alpha_0)}}<\infty$.

(iv) If $\mathscr{H}^{\alpha_0 s}(F)>0$ then $\underline{\lim}_{\varepsilon\to 0}\dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)}c_\sigma^{s(1-\alpha_0)}}>0$.

(v) We generalize this dimension results into the cases of MW-construction (Ma & Wi [9]) and recurrent sets (De [2], Be [1] and Wen [11]).

2. Dimensions of self-similar set.

It is easy to get the following

PROPOSITION 2.1.

$$\alpha_0(A)=\inf\left\{\alpha\,|\,\underline{\lim}_{\varepsilon\to 0}\frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)}c_\sigma^{s(1-\alpha)}}=0\right\},$$

$$\beta_0(A)=\inf\left\{\beta\,|\,\overline{\lim}_{\varepsilon\to 0}\frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)}c_\sigma^{s(1-\beta)}}=0\right\}.$$

PROPOSITION 2.2. $0\le\alpha_0(A)\le 1$; $0\le\beta_0(A)\le 1$.

PROOF. Note that $\sum_{\sigma\in\Omega(\varepsilon)}c_\sigma^s=1$. Taking $\alpha=0$ then

$$\underline{\lim}_{\varepsilon\to 0}\frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)}c_\sigma^{s}}=\underline{\lim}_{\varepsilon\to 0}\,\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(A)\right)\ge c$$

for some positive constant c. Thus $\alpha_0(A)\ge 0$. On the other hand, taking $\alpha=1$, we have $\underline{\lim}_{\varepsilon\to 0}\dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(A)\right)}{\mathrm{Card}\,\Omega(\varepsilon)}\le c$ for some constant c. Thus $\alpha_0(A)\le 1$.

$0\le\beta_0(A)\le 1$ can be proved by the same method. QED

THEOREM 2.3.

(i) *$\alpha_0(A)$ and $\beta_0(A)$ are independent of the choice of A and $\alpha_0(A)=\beta_0(A)$, denoting the common value by α_0;*

(ii) $\dim_H F=\dim_B F=\dim_P F=\alpha_0 s$;

(iii) $\mathscr{H}^{\alpha_0 s}(F)<\infty$ *iff* $\underline{\lim}_{\varepsilon\to 0}\dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)}c_\sigma^{s(1-\alpha_0)}}<\infty$;

(iv) *If* $\mathscr{H}^{\alpha_0 s}(F) > 0$ *then* $\underline{\lim}_{\varepsilon\to 0} \dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)} c_\sigma^{s(1-\alpha_0)}} > 0.$

PROOF. (i) For $B \subset \boldsymbol{R}^d$ and $\varepsilon > 0$ let

$$B^\varepsilon = \{x \in \boldsymbol{R}^d : \text{there exists y} \in \text{B such that } \rho(x,y) < \varepsilon\}$$

where $\rho(x,y)$ is the Euclidean distance between x and y. Since A is a bounded open set containing set F, there are positive numbers δ_1 and δ_2 such that $F^{\delta_1} \subset A \subset F^{\delta_2}$ which means $\alpha_0(F^{\delta_1}) \le \alpha_0(A) \le \alpha_0(F^{\delta_2})$ and $\beta_0(F^{\delta_1}) \le \beta_0(A) \le \beta_0(F^{\delta_2})$. Thus it suffices to prove $\alpha_0(F^\delta)$ and $\beta_0(F^\delta)$ are independent of the choice of positive number δ and $\alpha_0(F^\delta) = \beta_0(F^\delta)$, which follows from the proof of (ii).

(ii) Fixing $x \in F$ and denoting the diameter of A by $|A|$ we choose subfamily $\Omega^*(\varepsilon)$ from $\Omega(\varepsilon)$ such that

(1) for any different σ, $\tau \in \Omega^*(\varepsilon)$, $\rho(\phi_\sigma(x), \phi_\tau(x)) > 4|A|\varepsilon$;

(2) if $\sigma \in \Omega(\varepsilon)\backslash\Omega^*(\varepsilon)$ there exists $\tau \in \Omega^*(\varepsilon)$ such that $\rho(\phi_\sigma(x), \phi_\tau(x)) \le 4|A|\varepsilon$.

Let $J(\varepsilon) = \mathrm{Card}\,\Omega^*(\varepsilon)$. Thus

$$\bigcup_{\sigma\in\Omega^*(\varepsilon)} B(\phi_\sigma(x), 5|A|\varepsilon) \supset \bigcup_{\sigma\in\Omega(\varepsilon)} B(\phi_\sigma(x), |A|\varepsilon) \supset \bigcup_{\sigma\in\Omega(\varepsilon)} \phi_\sigma(A)$$

where $B(x,r)$ denotes a ball in $\boldsymbol{R}^d$ with center at x and radius r. Thus

$$J(\varepsilon)\mathrm{m_d}B(\phi_\sigma(x), 5|A|\varepsilon) \ge \mathrm{m_d} \bigcup_{\sigma\in\Omega(\varepsilon)} \phi_\sigma(A).$$

Therefore for any nonnegative real number α

$$J(\varepsilon)\varepsilon^{\alpha s} \ge \frac{c|A|^{-d}\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)} c_\sigma^{s(1-\alpha)}}, \tag{6}$$

where c is a positive constant. First we prove $\dim_H F \ge \alpha_0(A)s$. It is clear when $\alpha_0(A) = 0$. Suppose $\alpha_0(A) > 0$ and take $0 < \alpha < \alpha_0(A)$. Thus by the definition of $\alpha_0(A)$ and (6) we can take $\varepsilon_1 > 0$ such that

$$J(\varepsilon_1)\varepsilon_1^{\alpha s} \ge 2c_0^{-\alpha s}. \tag{7}$$

Considering any finite open $c_0\varepsilon_1|A|$-covering $\{V_i\}$ of F, we have

(a) if there exists some V_i such that $|V_i| \ge (c_0\varepsilon_1)^2|A|$ then

$$\sum_i |V_i|^{\alpha s} \ge (c_0\varepsilon_1)^{2\alpha s}|A|^{\alpha s}; \tag{8}$$

(b) otherwise for each $\sigma \in \Omega^*(\varepsilon_1)$ let $\mathscr{V}_\sigma = \{V_i : V_i \cap B(\phi_\sigma(x), \varepsilon_1|A|) \ne \varnothing\}$. Then $\mathscr{V}_\sigma$ is a covering of $\phi_\sigma(F)$ and for any different σ, $\tau \in \Omega^*(\varepsilon_1)$, $\mathscr{V}_\sigma \cap \mathscr{V}_\tau = \varnothing$. Take $\lambda_1 \in \Omega^*(\varepsilon_1)$ such that

$$\sum_{V_i\in\mathscr{V}_{\lambda_1}} |V_i|^{\alpha s} = \min_{\sigma\in\Omega^*(\varepsilon_1)} \sum_{V_i\in\mathscr{V}_\sigma} |V_i|^{\alpha s}.$$

Therefore

$$\begin{aligned}\sum_i |V_i|^{\alpha s} &\geq J(\varepsilon_1) \sum_{V_i \in \mathscr{V}_{\lambda_1}} |V_i|^{\alpha s} \geq 2c_0^{-\alpha s}\varepsilon_1^{-\alpha s} \sum_{V_i \in \mathscr{V}_{\lambda_1}} |V_i|^{\alpha s} \\ &= 2(c_{\lambda_1} c_0^{-1}\varepsilon_1^{-1})^{\alpha s} \sum_{V_i \in \mathscr{V}_{\lambda_1}} |\phi_{\lambda_1}^{-1} V_i|^{\alpha s} \\ &\geq 2 \sum_{V_i \in \mathscr{V}_{\lambda_1}} |\phi_{\lambda_1}^{-1} V_i|^{\alpha s} \end{aligned} \tag{9}$$

by (7).

Since $\mathscr{V}_{\lambda_1}$ is a covering of $\phi_{\lambda_1}(F)$, $\phi_{\lambda_1}^{-1}\mathscr{V}_{\lambda_1} = \{\phi_{\lambda_1}^{-1}(V_i) : V_i \in \mathscr{V}_{\lambda_1}\}$ is a finite open $c_0\varepsilon_1|A|$-covering of F. As above we have

(a′) if there exists $\phi_{\lambda_1}^{-1}(V_i) \in \phi_{\lambda_1}^{-1}\mathscr{V}_{\lambda_1}$ such that $|\phi_{\lambda_1}^{-1}(V_i)| \geq (c_0\varepsilon_1)^2|A|$ then (8) holds by (9);

(b′) otherwise denote $\phi_{\lambda_1}^{-1}\mathscr{V}_{\lambda_1}$ by $\{U_i\}$. Repeating the above step for the covering $\{U_i\}$ of F and noticing that Card$\{V_i\}$ is finite, thus (8) holds after finite steps. Consequently $\dim_H F \geq \alpha s$ which means $\dim_H F \geq \alpha_0(A)s$.

Now taking $\delta_1 > 0$ we prove that $\dim_H F \leq \underline{\dim}_B F \leq \alpha_0(F^{\delta_1})s$. Letting $\alpha > \alpha_0(F^{\delta_1})$ there exists sequence $\varepsilon_n \searrow 0$ such that

$$\frac{\varepsilon_n^{-d}\mathrm{m_d}\left(\bigcup_{\sigma \in \Omega(\varepsilon_n)} \phi_\sigma(F^{\delta_1})\right)}{\sum_{\sigma \in \Omega(\varepsilon_n)} c_\sigma^{s(1-\alpha)}} \leq 1.$$

Thus

$$\begin{aligned} \varepsilon_n^{-d}\mathrm{m_d}\left(\bigcup_{\sigma \in \Omega(\varepsilon_n)} \phi_\sigma(F^{\delta_1})\right) &\leq \sum_{\sigma \in \Omega(\varepsilon_n)} c_\sigma^{s(1-\alpha)} \leq (c_0\varepsilon_n)^{-s\alpha}, \\ (c_0\varepsilon_n)^{d-s\alpha} &\geq c_0^d \mathrm{m_d}\left(\bigcup_{\sigma \in \Omega(\varepsilon_n)} \phi_\sigma(F^{\delta_1})\right) \geq c_0^d \mathrm{m_d}(F^{c_0\varepsilon_n\delta_1}), \\ d - s\alpha &\leq \frac{\log(\mathrm{m_d}(F^{c_0\varepsilon_n\delta_1})c_0^d)}{\log(c_0\varepsilon_n)}, \\ d - s\alpha &\leq \overline{\lim}_{n\to 0} \frac{\log[c_0^d \mathrm{m_d}(F^{c_0\varepsilon_n\delta_1})]}{\log(c_0\varepsilon_n)} \leq \overline{\lim}_{\varepsilon \to 0} \frac{\log[\mathrm{m_d}(F^{c_0\varepsilon\delta_1})]}{\log(c_0\varepsilon\delta_1)}, \end{aligned}$$

which implies $\underline{\dim}_B F \leq s\alpha$ by the Proposition 3.2 of Fa [3]. Therefore $\underline{\dim}_B F \leq s\alpha_0(F^{\delta_1})$.

Repeating the above procedure of proof with $\beta_0(A)$ instead of $\alpha_0(A)$ we can attain $\dim_H F \geq \beta_0(A)s$ and $\dim_H F \leq \overline{\dim}_B F \leq \beta_0(F^{\delta_1})s$ for any given $\delta_1 > 0$. As a result, we get $\dim_H F = \dim_P F = \dim_B F = \alpha_0(F^{\delta_1})s = \beta_0(F^{\delta_1})s$ for any given $\delta_1 > 0$ which indicates $\alpha_0(F^{\delta_1})$ and $\beta_0(F^{\delta_1})$ are independent of the choice of $\delta_1 > 0$ and $\alpha_0(F^{\delta_1}) = \beta_0(F^{\delta_1})$. Furthermore $\alpha_0(A) = \beta_0(A)$ and they are independent of the choice of open set A by (i).

(iii) Now we prove $\mathscr{H}^{\alpha_0 s}(F) < \infty$ iff $\underline{\lim}_{\varepsilon\to 0} \dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma \in \Omega(\varepsilon)} c_\sigma^{s(1-\alpha_0)}} < \infty.$

Suppose that $\underline{\lim}_{\varepsilon\to 0} \dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)} c_\sigma^{s(1-\alpha_0)}} = \infty$. Then we can take $\varepsilon_1 > 0$ such that (7) holds with α_0 instead of α. For any $k \in N$ and for any finite open $(c_0\varepsilon_1)^k|A|$-covering $\{V_i\}$ of F, repeating $k-1$ time steps of proof of the above we can get

$$\sum_i |V_i|^{\alpha_0 s} \geq 2^{k-1} \sum_j |U_j|^{\alpha_0 s},$$

where $\{U_j\}$ is a finite open $c_0\varepsilon_1|A|$-covering of F. According to the same method of (ii) after finite steps, saying l steps, we get

$$\sum_j |U_j|^{\alpha_0 s} \geq 2^l (c_0\varepsilon_1)^{2\alpha_0 s}|A|^{\alpha_0 s},$$

$$\sum_j |V_j|^{\alpha_0 s} \geq 2^{l+k-1} (c_0\varepsilon_1)^{2\alpha_0 s}|A|^{\alpha_0 s},$$

which means $\mathscr{H}^{\alpha_0 s}(F) = \infty$ if letting k tends to ∞.

Suppose $\mathscr{H}^{\alpha_0 s}(F) = \infty$. Thus for any $M > 0$ there exists ε_0 such that for any ε_0-covering $\{V_i\}$ of F

$$\sum_i |V_i|^{\alpha_0 s} > M.$$

On the other hand, for any $\varepsilon > 0$

$$J(\varepsilon)(\varepsilon\delta_1)^d \leq \text{const.}\, \mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)} \phi_\sigma(A)\right),$$

since $\bigcup_{\sigma\in\Omega^*(\varepsilon)} B(\phi_\sigma(x), c_0\varepsilon\delta_1) \subset \bigcup_{\sigma\in\Omega(\varepsilon)} \phi_\sigma(A)$ where δ_1 is such that $F^{\delta_1} \subset A$. Thus

$$J(\varepsilon)\varepsilon^{\alpha_0 s} \leq \text{const.}\, \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)} c_\sigma^{s(1-\alpha_0)}}.$$

Now taking ε such that $10\varepsilon|A| < \varepsilon_0$ and considering the covering $\{B(\phi_\sigma(x), 5|A|\varepsilon), \sigma \in \Omega^*(\varepsilon)\}$ of F which is an ε_0-covering of F we have

$$\sum_{\sigma\in\Omega^*(\varepsilon)} (10|A|\varepsilon)^{\alpha_0 s} = \text{const.}\, J(\varepsilon)\varepsilon^{\alpha_0 s} \geq M.$$

Therefore

$$\frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)} c_\sigma^{s(1-\alpha_0)}} \geq \text{const.}\, M,$$

for $\varepsilon < (10|A|)^{-1}\varepsilon_0$ which indicates

$$\underline{\lim}_{\varepsilon\to 0} \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)} c_\sigma^{s(1-\alpha_0)}} = \infty.$$

(iv) Suppose $\underline{\lim}_{\varepsilon\to 0} \dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)} c_\sigma^{s(1-\alpha_0)}} = 0$. Then for any $h > 0$ there exist sequence $\varepsilon_n \searrow 0$ such that

$$\varepsilon_n^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon_n)}\phi_\sigma(A)\right) < h\sum_{\sigma\in\Omega(\varepsilon_n)} c_\sigma^{s(1-\alpha_0)} \le hc_0^{-\alpha_0 s}\varepsilon_n^{-\alpha_0 s}.$$

We consider the covering $\{B(\phi_\sigma(x), 5\varepsilon_n|A|), \sigma\in\Omega^*(\varepsilon_n)\}$ of F. Since

$$\bigcup_{\sigma\in\Omega^*(\varepsilon_n)} B(\phi_\sigma(x), c_0\varepsilon_n\delta_1) \subset \bigcup_{\sigma\in\Omega(\varepsilon_n)}\phi_\sigma(A)$$

where δ_1 is such that $F^{\delta_1}\subset A$, then

$$J(\varepsilon_n)\mathrm{m_d}(B(\phi_\sigma(x), c_0\varepsilon_n\delta_1)) \le \mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon_n)}\phi_\sigma(A)\right), \tag{10}$$

$$J(\varepsilon_n) \le \mathrm{const.}\,\varepsilon_n^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon_n)}\phi_\sigma(A)\right) \le \mathrm{const.}\,h\varepsilon_n^{-\alpha_0 s}.$$

Therefore we have

$$\sum_{\sigma\in\Omega^*(\varepsilon_n)} |B(\phi_\sigma(x), 5\varepsilon_n|A|)|^{\alpha_0 s} = J(\varepsilon_n)(10|A|\varepsilon_n)^{\alpha_0 s} \le \mathrm{const.}\,h,$$

which indicates $\mathscr{H}^{\alpha_0 s}(F) = 0$. As a result, we get that $\mathscr{H}^{\alpha_0 s}(F) > 0$ implies $\underline{\lim}_{\varepsilon\to 0} \dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)} c_\sigma^{s(1-\alpha_0)}} > 0$. QED

Conjecture: If $\underline{\lim}_{\varepsilon\to 0} \dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)} c_\sigma^{s(1-\alpha_0)}} > 0$ then $\mathscr{H}^{\alpha_0 s}(F) > 0$.

Corollary 2.4. *If ϕ_i's satisfy the open set condition then* $\dim_H F = \dim_B F = \dim_P F = s$.

Proof. Let bounded nonempty open set O make ϕ_i's satisfy the open set condition. Taking $A = O^1$ thus

$$\mathrm{const.} \ge \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(O^1)\right)}{\mathrm{Card}\,\Omega(\varepsilon)} \ge \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(\overline{O})\right)}{\mathrm{Card}\,\Omega(\varepsilon)} \ge \mathrm{const.} > 0,$$

which means $\alpha_0 = 1$. Therefore $\dim_H F = \dim_B F = \dim_P F = s$ by Theorem 2.3. QED

Remark 2.5. If the above Conjecture holds then it is easy to get

(a) F is an $\alpha_0 s$-set iff $0 < \underline{\lim}_{\varepsilon\to 0} \dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)}\phi_\sigma(A)\right)}{\sum_{\sigma\in\Omega(\varepsilon)} c_\sigma^{s(1-\alpha_0)}} < \infty$;

(b) ϕ_i's satisfy the open set condition iff $\underline{\lim}_{\varepsilon\to 0} \dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)} \phi_\sigma(A)\right)}{\operatorname{Card}\Omega(\varepsilon)} > 0$;

(c) $\mathscr{H}^s(F) = 0$ iff $\underline{\lim}_{\varepsilon\to 0} \dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega(\varepsilon)} \phi_\sigma(A)\right)}{\operatorname{Card}\Omega(\varepsilon)} = 0$.

3. Generalization to MW-construction and generalized recurrent set.

Let $A = (a_{ij})_{n\times n}$ be an irreducible 0–1 matrix. $\{\phi_{ij} : a_{ij} = 1\}$ is a family of similar maps in $\boldsymbol{R}^d$ with the ratio c_{ij} for ϕ_{ij}. Let s be such that the spectral radius of $(a_{ij}c_{ij}^s)_{n\times n}$ is 1 where we take $a_{ij}c_{ij}^s = 0$ when $a_{ij} = 0$. Write

$$\Omega_A = \left\{\sigma \in \prod_1^\infty\{1,2,\ldots,n\} : \sigma = (\sigma(1),\sigma(2),\ldots), a_{\sigma(l),\sigma(l+1)} = 1, l \in \boldsymbol{N}\right\},$$

$$\Omega_A^* = \left\{\sigma \in \bigcup_{i=2}^\infty\{1,2,\ldots,n\}^i : \sigma = (\sigma(1),\ldots,\sigma(k)), a_{\sigma(l),\sigma(l+1)} = 1, 1 \le l \le k-1\right\}.$$

There exist unique compact sets $F_1, F_2, \ldots, F_n$ which sometimes is called MW-construction such that

$$F_i = \bigcup_{\{j:a_{ij}=1\}} \phi_{ij}(F_j), \quad 1 \le i \le n. \tag{11}$$

It is well-known that when $\{\phi_{ij} : a_{ij} = 1\}$ satisfy the open condition, i.e. there are nonempty bounded open sets $O_1, O_2, \ldots, O_n$ such that

$$O_i \supset \bigcup_{\{j:a_{ij}=1\}} \phi_{ij}(O_j), \quad 1 \le i \le n,$$

with the right hand being disjoint union, we have

$$\dim_H F_i = \dim_B F_i = \dim_P F_i = s, \quad 1 \le i \le n,$$

and F_i are all s-set.

Furthermore in Li [**6**] we prove that

PROPOSITION 3.1. *$\{\phi_{ij} : a_{ij} = 1\}$ satisfies the open set condition iff F_i is an s-set for some $1 \le i \le n$ where s is given above.*

Now for $1 \le i \le n$ let

$$\alpha_i = \sup\left\{\alpha : \underline{\lim}_{\varepsilon\to 0} \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega_i(\varepsilon)} \phi_\sigma(A_{\sigma(|\sigma|)})\right)}{\sum_{\sigma\in\Omega_i(\varepsilon)} c_\sigma^{s(1-\alpha)}} = \infty\right\} \tag{12}$$

$$\beta_i = \sup\left\{\beta : \overline{\lim}_{\varepsilon\to 0} \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega_i(\varepsilon)} \phi_\sigma(A_{\sigma(|\sigma|)})\right)}{\sum_{\sigma\in\Omega_i(\varepsilon)} c_\sigma^{s(1-\beta)}} = \infty\right\}$$

where $A_i \supset F_i$ are bounded open sets; $|\sigma|$ denotes the length of σ; $\Omega_i(\varepsilon) = \{\sigma \in \Omega_A^* : \sigma(1) = i, c_\sigma \le \varepsilon \text{ and } c_{\sigma|(|\sigma|-1)} > \varepsilon\}$; $c_\sigma = c_{\sigma(1),\sigma(2)}c_{\sigma(2),\sigma(3)}\cdots c_{\sigma(|\sigma|-1),\sigma(|\sigma|)}$; $\phi_\sigma = \phi_{\sigma(1),\sigma(2)} \circ \phi_{\sigma(2),\sigma(3)} \circ \cdots \circ \phi_{\sigma(|\sigma|-1),\sigma(|\sigma|)}$. Write $c_0 = \min_{a_{ij}=1} c_{ij}$.

In usual, we always take some bounded open set A with $A \supset \bigcup_i F_i$ instead of A_i's in (12).

Similarly it is easy to get

PROPOSITION 3.2. (1) $0 \le \alpha_i \le \beta_i \le 1$ *for* $1 \le i \le n$;

(2) *When* $\{\phi_{ij} : a_{ij} = 1\}$ *satisfies the open set condition, we have* $\alpha_i = \beta_i = 1$ *for all* $1 \le i \le n$.

Similar to Theorem 2.3 we have

THEOREM 3.3. (I) *All* α_i *and* β_i *are equal, denoting by* α_0 *the common value. And* $\dim_H F_i = \dim_B F_i = \dim_P F_i = \alpha_0 s$ *for* $1 \le i \le n$.

(II) $\mathscr{H}^{\alpha_0 s}(F_i) < \infty$ *for some* $1 \le i \le n$ *iff* $\underline{\lim}_{\varepsilon \to 0} \dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma \in \Omega_i(\varepsilon)} \phi_\sigma(A_{\sigma(|\sigma|)})\right)}{\sum_{\sigma \in \Omega_i(\varepsilon)} c_\sigma^{s(1-\alpha_0)}} < \infty$ *for some* $1 < i < n$. *And if* $\mathscr{H}^{\alpha_0 s}(F_i) > 0$ *for some* $1 \le i \le n$ *then* $\underline{\lim}_{\varepsilon \to 0} \dfrac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma \in \Omega_i(\varepsilon)} \phi_\sigma(A_{\sigma(|\sigma|)})\right)}{\sum_{\sigma \in \Omega_i(\varepsilon)} c_\sigma^{s(1-\alpha_0)}} > 0$ *for all* $1 \le i \le n$.

PROOF. (I) Without loss of generality we suppose that $\alpha_1 = \min_{1 \le i \le n} \alpha_i$, $\beta_1 = \min_{1 \le i \le n} \beta_i$, $\beta_n = \max_{1 \le i \le n} \beta_i$.

Fix some j, $1 \le j \le n$. First step we prove $\dim_H F_j \ge \alpha_1 s$. Taking $x_i \in F_i$ and writing $\delta = \max_i |F_i|$. We choose the subfamily $\Omega_i^*(\varepsilon)$ from $\Omega_i(\varepsilon)$ such that

(1) for any σ, $\tau \in \Omega_i^*(\varepsilon)$ and $\sigma \ne \tau$

$$\rho(\phi_\sigma(x_{\sigma(|\sigma|)}), \phi_\tau(x_{\tau(|\tau|)})) > 4\delta\varepsilon;$$

(2) if $\sigma \in \Omega_i(\varepsilon) \backslash \Omega_i^*(\varepsilon)$ there exists $\tau \in \Omega_i^*(\varepsilon)$ such that

$$\rho(\phi_\sigma(x_{\sigma(|\sigma|)}), \phi_\tau(x_{\tau(|\tau|)})) \le 4\delta\varepsilon.$$

Let $J_i(\varepsilon) = \operatorname{Card} \Omega_i^*(\varepsilon)$. Thus

$$\bigcup_{\sigma \in \Omega_i^*(\varepsilon)} B(\phi_\sigma(x_{\sigma(|\sigma|)}), 5\delta\varepsilon) \supset \bigcup_{\sigma \in \Omega_i(\varepsilon)} B(\phi_\sigma(x_{\sigma(|\sigma|)}), \delta\varepsilon) \supset \bigcup_{\sigma \in \Omega_i(\varepsilon)} \phi_\sigma(A_{\sigma(|\sigma|)}).$$

Therefore we have

$$J_i(\varepsilon)\mathrm{m_d} B(\phi_\sigma(x_{\sigma(|\sigma|)}), 5\delta\varepsilon) \ge \mathrm{m_d}\left(\bigcup_{\sigma \in \Omega_i(\varepsilon)} \phi_\sigma(A_{\sigma(|\sigma|)})\right),$$

$$J_i(\varepsilon)\varepsilon^{\alpha s} \ge \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma \in \Omega_i(\varepsilon)} \phi_\sigma(A_{\sigma(|\sigma|)})\right)}{\sum_{\sigma \in \Omega_i(\varepsilon)} c_\sigma^{s(1-\alpha)}} \left(\sum_{\sigma \in \Omega_i(\varepsilon)} c_\sigma^{s(1-\alpha)}\right) \delta^{-d} \operatorname{const.} \varepsilon^{\alpha s}.$$

Now let $(m_1, \ldots, m_n)$ be the strictly positive right eigenvector responding to the eigenvalue 1. Then

$$(c_{ij}^s a_{ij})_{n \times n} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}.$$

Therefore

$$\left[\frac{\min m_i}{\max m_i}\right]^2 \le \sum_{\sigma\in\Omega_i(\varepsilon)} c_\sigma^s \le \left[\frac{\max m_i}{\min m_i}\right]^2.$$

In addition

$$1 \le (\varepsilon c_\sigma^{-1})^{\alpha s} \le c_0^{-\alpha s}.$$

Therefore

$$J_i(\varepsilon)\varepsilon^{\alpha s} \ge \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega_i(\varepsilon)} \phi_\sigma(A_{\sigma(|\sigma|)})\right)}{\sum_{\sigma\in\Omega_i(\varepsilon)} c_\sigma^{s(1-\alpha)}}\,\delta^{-d}\,\text{const.} \tag{13}$$

If $\alpha_1 = 0$, it is trival. We assume $\alpha_1 > 0$ and take $0 < \alpha < \alpha_1$. Thus we have

$$\varliminf_{\varepsilon\to 0} J_i(\varepsilon)\varepsilon^{\alpha s} = \infty,$$

by (13) for $1 \le i \le n$. Take $\varepsilon_1 > 0$ such that $J_i(\varepsilon_1)\varepsilon_1^{\alpha s}c_0^{\alpha s} \ge 2$ for all $1 \le i \le n$. Considering the arbitrary finite open $c_0\varepsilon_1\delta$-covering $\{V_i\}$ of F_j, thus

(a) if there exists some V_i with $|V_i| \ge (c_0\varepsilon_1)^2\delta$ then

$$\sum_i |V_i|^{\alpha s} \ge (c_0\varepsilon_1)^{2\alpha s}\delta^{\alpha s}; \tag{14}$$

(b) otherwise we have

$$\sum_i |V_i|^{\alpha s} = \varepsilon_1^{\alpha s}\sum_i |\varepsilon_1^{-1}V_i|^{\alpha s}.$$

For each $\sigma\in\Omega_j^*(\varepsilon_1)$, let $\mathscr{V}_\sigma = \{V_i : V_i\cap B(\phi_\sigma(x_{\sigma(|\sigma|)}), \varepsilon_1\delta) \ne \varnothing\}$. Thus $\mathscr{V}_\sigma$ is a covering of $\phi_\sigma(F_{\sigma(|\sigma|)})$ and for any $\sigma, \tau\in\Omega_j^*(\varepsilon_1)$, $\sigma\ne\tau$,

$$\mathscr{V}_\sigma \cap \mathscr{V}_\tau = \varnothing.$$

Take $\lambda_1\in\Omega_j^*(\varepsilon_1)$ such that

$$\sum_{V_i\in\mathscr{V}_{\lambda_1}} |V_i|^{\alpha s} = \min_{\sigma\in\Omega_j^*(\varepsilon_1)} \sum_{V_i\in\mathscr{V}_\sigma} |V_i|^{\alpha s}.$$

Therefore

$$\begin{aligned}\sum_i |V_i|^{\alpha s} &\ge J(\varepsilon_1)\sum_{V_i\in\mathscr{V}_{\lambda_1}} |V_i|^{\alpha s} \ge J(\varepsilon_1)\varepsilon_1^{\alpha s}\sum_{V_i\in\mathscr{V}_{\lambda_1}} |\varepsilon_1^{-1}V_i|^{\alpha s}\\ &\ge 2c_0^{-\alpha s}\sum_{V_i\in\mathscr{V}_{\lambda_1}} |\varepsilon_1^{-1}V_i|^{\alpha s} = 2(c_{\lambda_1}c_0^{-1}\varepsilon_1^{-1})^{\alpha s}\sum_{V_i\in\mathscr{V}_{\lambda_1}} |\phi_{\lambda_1}^{-1}V_i|^{\alpha s}\\ &\ge 2\sum_{V_i\in\mathscr{V}_{\lambda_1}} |\phi_{\lambda_1}^{-1}V_i|^{\alpha s}.\end{aligned} \tag{15}$$

Since $\mathscr{V}_{\lambda_1}$ is a covering of $\phi_{\lambda_1}(F_{\lambda_1(|\lambda_1|)})$, $\phi_{\lambda_1}^{-1}\mathscr{V}_{\lambda_1}$ is a finite open $c_0\varepsilon_1\delta$-covering of $F_{\lambda_1(|\lambda_1|)}$. Denoting $\phi_{\lambda_1}^{-1}\mathscr{V}_{\lambda_1}$ by $\{u_i\}$ as above we have (a′) if there exists $u_i \in \phi_{\lambda_1}^{-1}\mathscr{V}_{\lambda_1}$ such that $|u_i| \geq (c_0\varepsilon_1)^2\delta$ then (14) holds by (15).

(b′) otherwise repeating the above step and considering Card$\{V_i\}$ finite, thus (14) holds after finite steps. Therefore

$$\dim_H F_j \geq \alpha s$$

for any $0 < \alpha < \alpha_1$ which means

$$\dim_H F_j \geq \alpha_1 s.$$

Similar to the proof of Theorem 2.3 we also get $\alpha_1 s \leq \dim_H F_j \leq \underline{\dim}_B F_j \leq \alpha_1 s$ and $\beta_1 s \leq \dim_H F_j \leq \overline{\dim}_B F_j \leq \beta_1 s$ and $\overline{\dim}_B F_j = \beta_n s$. Thus we complete the proof. In addition it is easy to find that all α_i's and β_i's are equal and independent of the choice of A_i's.

(II) Finally using the same method as those in proof of Theorem 2.3 (III) and (IV) we can complete the proof of (II). QED

Corollary 3.4. *When $\{\phi_{ij} : a_{ij} = 1\}$ satisfies the open set condition, we have for every $0 \leq i \leq n$*

$$\dim_H F_i = \dim_B F_i = \dim_P F_i = s.$$

Conjecture: if

$$\underline{\lim}_{\varepsilon\to 0} \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega_i(\varepsilon)} \phi_\sigma(A_{\sigma(|\sigma|)})\right)}{\sum_{\sigma\in\Omega_i(\varepsilon)} c_\sigma^{s(1-\alpha)}} > 0$$

for some $1 \leq i \leq n$, then $\mathscr{H}^{\alpha s}(F_i) > 0$ for all $1 \leq i \leq n$.

Remark 3.5. (1) Since the recurrent set (Dekking [**2**]) and the generalized recurrent set (Li [**8**]) are all the special cases of MW-construction (Bedford [**1**] & Li [**7**]) the Theorem 3.3 also works there. Thus our Theomem 3.3 actually improves the main results of [**11**] [**12**] which discussed the lower bound of Hausdorff dimension of recurrent sets and self-similar sets.

(2) If the above conjecture is ture, it is easy to get

(a) F_i is an αs-set for some $1 \leq i \leq n$ iff

$$0 < \underline{\lim}_{\varepsilon\to 0} \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega_i(\varepsilon)} \phi_\sigma(A_{\sigma(|\sigma|)})\right)}{\sum_{\sigma\in\Omega_i(\varepsilon)} c_\sigma^{s(1-\alpha)}} < \infty$$

for some $1 \leq i \leq n$.

(b) F_i's satisfy the open set condition iff

$$\underline{\lim}_{\varepsilon\to 0} \frac{\varepsilon^{-d}\mathrm{m_d}\left(\bigcup_{\sigma\in\Omega_i(\varepsilon)} \phi_\sigma(A_{\sigma(|\sigma|)})\right)}{\mathrm{Card}\,\Omega_i(\varepsilon)} > 0$$

for some $1 \leq i \leq n$.

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