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A generalized truncation method for multivalued parabolic problems

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Abstract

The generalized truncation method (formerly referred to as the proximal correction method) was recently introduced for the time-discretization of parabolic variational inequalities. The main attraction of the method—which generalizes the truncation method developed by A. Berger for obstacle problems—is the fact that the problems to be solved at each time step are elliptic equations rather than elliptic variational inequalities.

In this paper we apply the new method to a class of problems which includes parabolic variational inequalities as a special case. The convergence results which we obtain in this general context also give rise to new results when applied to the special case of variational inequalities.

We also discuss the applications of our results to several problems that occur in various branches of applied Mathematics.

1. Introduction

Let V and H be Hilbert spaces such that $V \subset H = H^* \subset V^*$ with continuous and dense inclusions. We use $(\cdot, \cdot)$ to denote both the scalar product of H and the duality between V and V^*. The norms of V, H and V^* will be designated, respectively, by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$.

Let $A : V \mapsto V^*$ be a continuous linear map and let g be a (multivalued) maximal monotone function from H to H. This means that g is non-empty subset of $H \times H$ which is monotone in the sense that

$$[v_1, z_1] \in g \quad \text{and} \quad [v_2, z_2] \in g \Longrightarrow (z_2 - z_1, v_2 - v_1) \geq 0. \tag{1.1}$$

and which is not contained in any larger monotone subset of $H \times H$. In the sequel, we will regard the statements $[u, v] \in g$, $u \in g(v)$ and $g(v) \ni u$ as synonymous. $D(g)$ will designate the domain of g, defined as the set $\{v \in H : \exists z \in H \text{ such that } z \in g(v)\}$.

Let $u_0 \in D(g)$ and let $f : [0, T] \mapsto H$ be a strongly measurable function. We are interested in the (numerical) solution of the parabolic problem:

$$\begin{aligned} &u(0) = u_0, \\ &u'(t) + Au(t) + g(u(t)) \ni f(t), \quad \forall\, t \in [0, T]. \end{aligned} \tag{1.2}$$

An important special case of such problems is obtained on letting g be the subgradient $\partial\phi$, defined by the rule

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$$\partial\phi(z) = \{v \in H : \phi(z) - \phi(y) \leq (v, z - y)\ \forall\, y \in H\}, \quad \text{for all } z \in H,$$

where $\phi : H \mapsto (-\infty, \infty]$ is a convex lower-semicontinuous function having nonempty domain $D(\phi) = \{v \in H : \phi(v) < \infty\}$. In this case, (1.2) reduces to the parabolic variational inequality

$$\begin{aligned} &u(0) = u_0 \\ &u'(t) + Au(t) + \partial\phi(u(t)) \ni f(t), \quad \forall\, t \in [0, T]. \end{aligned} \tag{1.3}$$

Parabolic variational inequalities have important applications in the physical and engineering sciences [**2**, **14**, **20**], and have been studied extensively in the Literature, from the point of view of existence of solutions [**9**, **14**, **21**, **22**] and approximation of solutions [**3**, **7**, **15**, **16**, **17**, **19**, **27**].

It is well known that the subgradient $\partial\phi$ is maximal monotone [**23**, **24**]. Therefore, problem (1.3) is a special case of problem (1.2). However, not all problems of the form (1.2) are variational inequalities since there exist maximal monotone operators that are not subgradients of convex functionals [**25**].

One of the simplest methods [**5**, **19**] for the time-discretization of problem (1.2) is the method of Rothe given by

$$\begin{aligned} &U_k^0 = u_0, \\ &U_k^n - U_k^{n-1} + kAU_k^n + kg(U_k^n) \ni kf_k^n, \quad n = 1, 2, \ldots, N, \end{aligned} \tag{1.4}$$

where N is a positive integer, $k = (T/N)$ and $f_k^n \approx f(nk)$.

In using this method we have to solve an elliptic inclusion problem at each time step. The existence of the solutions of such elliptic problems can be established under relatively mild conditions (we will return to this question in section 2). However, the numerical solution of each of the subproblems in (1.4) remains a major problem which may turn out to be as difficult as the original problem (1.2).

An alternative method—the topic of the present paper—is the generalized truncation method which is given by

$$\begin{aligned} &u_k^0 = u_0, \\ &u_k^n - P_{[k]}u_k^{n-1} + kAu_k^n = kf_k^n, \quad n = 1, 2, \ldots, N, \end{aligned} \tag{1.5}$$

where $P_{[k]}$ denotes the resolvent operator $(1 + kg)^{-1}$ which is known [**11**, **12**] to be a single-valued and nonexpansive map from H to H. The subproblems to be solved at each time step are elliptic *equations* which are usually easier to solve than the elliptic inclusion problems occuring in Rothe's scheme (1.4). This method was introduced in [**28**, **29**] within the context of parabolic variational inequalities.

It is known [**30**] that, when applied to obstacle problems, the scheme (1.5) reduces to the well-known truncation method of A. Berger [**6**]. Consequently, it appears that the name 'generalized truncation method' conveys its nature better than the name 'proximal correction method' which was originally suggested for it in [**29**, **30**]. A comprehensive introduction to the method can be found in [**30**].

Our main aim in the present paper is to study the convergence properties and error estimates of the generalized truncation scheme. Similar results for the method have been given in [**28**, **29**, **30**] for the special case of parabolic variational inequalities. However, the results obtained in the present paper also give rise to new results when applied to the problems studied in [**28**, **29**, **30**].

The rest of the paper is divided into three sections. Section 2 contains some basic hypotheses and results on elliptic problems of the form (1.4) which will be used in our work. Section 3 contains convergence results and error estimates, while Section 4 discusses the application of these results to problems that occur in applications.

In order to apply the generalized truncation scheme to a practical problem we require an efficient algorithm for the computation of the values $P_{[k]}v$ of the resolvent operator. If possible, the practical problem should also fit within the framework of the available convergence theory for the method.

It will be seen in Section 4 these two requirements are satisfied for a large class of problems, including such diverse problems as

$$\begin{gathered} u(x,t) \geq 0, \quad \frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) - f(x,t) \geq 0, \\ u(x,t)\left(\frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) - f(x,t)\right) = 0, \end{gathered} \tag{1.6}$$

$$\frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) + |u(x,t)|^{p-2}u(x,t) - f(x,t) = 0, \tag{1.7}$$

$$\frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) + u(x,t)\left(\int_\Omega |u(y,t)|^2\,dy\right)^{p/2-1} - f(x,t) = 0 \tag{1.8}$$

(for $p \geq 2$, $0 \leq t \leq T$ and $x \in \Omega \subset \boldsymbol{R}^n$) with initial condition $u(x,0) = u_0(x)$ and Neumann or Dirichlet boundary conditions.

One of the main attractions of the generalized truncation method is its ability to handle the numerical solution of such diverse problems in a unified context. Specific numerical computations and comparisons with Rothe's method can be found in [**28**, **29**, **30**]. These numerical examples suggest that both methods have the same order of convergence. The method of Rothe generally gives more accurate results than the truncation method. However, Rothe's method is much more difficult to implement, and also tends to require longer execution times.

2. Some remarks on elliptic problems

In the sequel we will assume that there exist $c > 0$ and $M > 0$ such that the elliptic operator A satisfies the two conditions:

$$(Av, v) \geq c\|v\|^2 \quad \forall\, v \in V \tag{2.1}$$

$$|(Av, z)| \leq M\|v\|\,\|z\| \quad \forall\, v, z \in V. \tag{2.2}$$

We shall also assume that the resolvent operators satisfy the conditions

$$P_{[k]}(V) \subset V \text{ and } P_{[k]} \text{ is bounded and hemicontinuous on } V \quad \forall\, k > 0. \tag{2.3}$$

This means that $P_{[k]}$ maps bounded sets in V to bounded sets in V, and that for all v, $z \in V$, the map $\theta \mapsto P_{[k]}(\theta v + (1-\theta)z)$ is continuous from $[0,1]$ to the weak topology of V^*.

In addition, we also assume that A and the resolvent operators are compatible, in the sense that there exist a function $w : V \mapsto H$ and a constant $\beta > 0$ such that

$$(AP_{[k]}v - w(v), v - P_{[k]}v) \geq 0 \quad \text{and} \quad |w(v)| < \beta \qquad \text{for all } k > 0 \text{ and } v \in V. \tag{2.4}$$

If we let g be the subgradient of a convex function and let w be a constant function, then (2.4) reduces to the compatibility conditions employed in [**30**].

The basic properties of the resolvent operator $P_{[k]}$ which will be used in the sequel are contained in the following Lemma whose proof can be found in [**11**, Lemma 1.3].

LEMMA 2.1.

$$|P_{[k]}v - P_{[k]}z| \leq |v - z|, \quad \forall\, v, z \in H, \tag{2.5}$$

$$|P_{[k]}v - P_{[k]}z|^2 \leq (P_{[k]}v - P_{[k]}z, v - z) \quad \forall\, v, z \in H, \tag{2.6}$$

$$|P_{[k]}v - v| \leq k|z|, \quad \forall\, z \in g(v), \tag{2.7}$$

$$\lim_{k \to 0} P_{[k]}v = v, \quad \forall\, v \in \overline{D(g)}. \tag{2.8}$$

Our convergence analysis of the generalized truncation scheme (1.5) will employ the properties of Rothe's scheme (1.4). The following result is on the existence of the solutions to elliptic problems of the type occuring in (1.4).

THEOREM 2.2. *If* (2.1) *and the compatibility conditions* (2.3)–(2.4) *hold then, for every* $b \in H$, *the elliptic problem*

$$Av + g(v) \ni b \tag{2.9}$$

has a unique solution $v \in V$. *Moreover,* v *satisfies the estimate*

$$|Av| \leq 2\beta + |b|. \tag{2.10}$$

PROOF. For every $k > 0$, it follows from (2.1), (2.3) and (2.6) that the operator $A + (1/k)(I - P_{[k]})$ is monotone, hemicontinuous and coercive. Consequently [**21**, Theorem 2.1] there exists a unique $v^{(k)} \in V$ such that

$$Av^{(k)} + \frac{1}{k}(v^{(k)} - P_{[k]}v^{(k)}) = b. \tag{2.11}$$

Rewriting this in the form

$$(Av^{(k)} - AP_{[k]}v^{(k)}) + \frac{1}{k}(v^{(k)} - P_{[k]}v^{(k)}) + (AP_{[k]}v^{(k)} - w(v^{(k)})) = b - w(v^{(k)}),$$

taking scalar products with $v^{(k)} - P_{[k]}v^{(k)}$ and making use of (2.1) and (2.4), we obtain

$$\begin{aligned} kc\|v^{(k)} - P_{[k]}v^{(k)}\|^2 + |v^{(k)} - P_{[k]}v^{(k)}|^2 &\leq k|b - w(v^{(k)})|\,|v^{(k)} - P_{[k]}v^{(k)}| \\ &\leq \frac{k^2}{2}|b - w(v^{(k)})|^2 + \frac{1}{2}|v^{(k)} - P_{[k]}v^{(k)}|^2 \\ &\leq \frac{k^2}{2}(|b| + \beta)^2 + \frac{1}{2}|v^{(k)} - P_{[k]}v^{(k)}|^2 \end{aligned}$$

It follows from this that

$$|v^{(k)} - P_{[k]}v^{(k)}| \leq k(|b| + \beta) \tag{2.12}$$

$$\|v^{(k)} - P_{[k]}v^{(k)}\|^2 \leq \frac{k(|b| + \beta)^2}{2c}. \tag{2.13}$$

Therefore

$$|Av^{(k)}| = \left|b - \frac{1}{k}(v^{(k)} - P_{[k]}v^{(k)})\right| \leq |b| + \frac{1}{k}|v^{(k)} - P_{[k]}v^{(k)}| \leq 2|b| + \beta. \tag{2.14}$$

Equation (2.11) implies that

$$b - Av^{(k)} \in g(P_{[k]}v^{(k)}). \tag{2.15}$$

Using this equation and the analogous equation with k replaced by h, and applying the monotonicity of g, we obtain

$$(Av^{(k)} - Av^{(h)}, P_{[k]}v^{(k)} - P_{[h]}v^{(h)}) \leq 0.$$

Rearranging this, we obtain

$$(Av^{(k)} - Av^{(h)}, v^{(k)} - v^{(h)}) \leq (Av^{(k)} - Av^{(h)}, v^{(k)} - P_{[k]}v^{(k)} - v^{(h)} + P_{[h]}v^{(h)}).$$

Using (2.12) and (2.14) in this inequality, we obtain

$$c\|v^{(k)} - v^{(h)}\|^2 \leq (|Av^{(k)}| + |Av^{(h)}|)(|v^{(k)} - P_{[k]}v^{(k)}| + |v^{(h)} - P_{[h]}v^{(h)}|) \leq G(k + h),$$

where $G = (4|b| + 2\beta)(|b| + \beta)$.

This shows that $v^{(k)}$ is a Cauchy sequence and, therefore that it converges strongly in the norm of V to some $v \in V$ as k tends to zero. It follows from (2.11) that $P_{[k]}v^{(k)}$ also converges strongly to v in the norm of V. Furthermore, since $|(Av^{(k)} - Av, z)| \leq M\|v^{(k)} - v\| \, \|z\|$ for all $z \in V$, we see that $Av^{(k)}$ converges to Av in the weak topology of V^*.

Now, since $|Av^{(k)}|$ is bounded, it contains a subsequence $Av^{(k_n)}$ which converges (to Av) in the weak topology of H as n tends to infinity. The estimate in (2.10) now follows from (2.14) and the fact that

$$|Av| \leq \liminf_{n\to\infty} |Av^{(k_n)}| \leq 2|b| + \beta.$$

Finally, for any $z \in g(y)$, (2.15) and the monotonicity of g shows that $(b - Av^{(k)} - z, P_{[k]}v^{(k)} - y) \geq 0$. Letting k tend to zero, we obtain $(b - Av - z, v - y) \geq 0$. Applying now the the maximality of g, we see that $b - Av \in g(v)$, which proves that v satisfies (2.11). That completes the proof. ∎

REMARK 2.3. If g is the subgradient of a proper lower semi-continuous convex function $\phi : H \mapsto (-\infty, \infty]$, then (2.9) becomes an elliptic variational inequality. The existence of a solution obviously follows from Theorem 2.2. Observe, however, that the classical Lions-Stampacchia Theorem (cf. [**22**] or [**4**, Theorem 3.5]) is not applicable in this situation since ϕ is not lower-semicontinuous in the topology of V.

EXAMPLE 2.4. Let $H = L_2(\Omega)$ and $V = H_0^1(\Omega)$, where Ω is a bounded open set in $\boldsymbol{R}^q$ with smooth boundary $\partial\Omega$, and let A be a partial differential operator of the form

$Av = -\sum_{i=1}^{q}(\partial/\partial x_i)\left(\sum_{j=1}^{q} a_{ij}(\partial v/\partial x_j)\right) + a_{00}v$ where $a_{ij} \in C^2(\bar{\Omega})$ for $0 \leq i,\ j \leq q$ and $\sum_i^q \sum_{j=1}^q a_{ij}x_i x_j \geq c\sum_{i=1}^q x_i^2\ \forall x_1,\ldots,x_q \in \boldsymbol{R}$. Let $\psi_1(x),\ \psi_2(x) \in H^1(\Omega)$ be such that $\psi_1(x) < \psi_2(x)$, for almost all $x \in \Omega$, $\psi_1|_{\partial\Omega} \leq 0 \leq \psi_2|_{\partial\Omega}$, and $A\psi_1$ and $A\psi_2$ are measures, with $(A\psi_1)^+$, $(A\psi_2)^- \in L_2(\Omega)$. Given $-\infty \leq \alpha_1 < 0 < \alpha_2 \leq \infty$, let $g(v)$ be defined almost everywhere in the pointwise manner:

$$g(v(x)) \equiv \begin{cases} \alpha_1 & \text{if } v(x) < \psi_1(x) \\ [\alpha_1, 0] & \text{if } v(x) = \psi_1(x) \\ 0 & \text{if } \psi_1(x) < v(x) < \psi_2(x) \\ [0, \alpha_2] & \text{if } v(x) = \psi_2(x) \\ \alpha_2 & \text{if } v(x) > \psi_2(x). \end{cases}$$

The resolvent operators can be computed from the pointwise expressions

$$P_{[k]}v(x) = \begin{cases} v(x) - k\alpha_1 & \text{if } v(x) < \psi_1(x) + k\alpha_1 \\ v(x) + (\psi_1(x) - v(x))^+ - (v(x) - \psi_2(x))^+ & \text{if } \psi_1(x) + k\alpha_1 \leq v(x) \leq \psi_2(x) + k\alpha_2 \\ v(x) - k\alpha_2 & \text{if } v(x) > \psi_2(x) + k\alpha_2. \end{cases}$$

It is not difficult to verify that (2.3) holds. Furthermore, if we set

$$w(v)(x) = \begin{cases} (A\psi_1)^+ & \text{if } v(x) < \psi_1(x) \\ 0 & \text{if } \psi_1(x) \leq v(x) \leq \psi_2(x) \\ (A\psi_2)^- & \text{if } v(x) > \psi_2(x). \end{cases}$$

Then $|w(v)| \leq |(A\psi_1)^+| + |(A\psi_2)^-|$ and

$$\begin{aligned}(AP_{[k]}v - w(v), v - P_{[k]}v) &= \int_{v(x)<\psi_1(x)} (A\psi_1(x) - (A\psi_1)^+(x))(v(x) - \psi_1(x))\,dx \\ &\quad + \int_{v(x)>\psi_2(x)} (A\psi_2(x) - (A\psi_2)^-(x))(v(x) - \psi_2(x))\,dx \geq 0.\end{aligned}$$

This shows that the compatibility condition (2.4) also holds.

In this context, the estimate (2.10) reduces to the case $p = 2$ of the regularity estimate in Theorem 1.1 of [**9**]. This shows that the compatibility conditions in (2.4) are actually a natural requirement in the study of elliptic problems of the form (2.9). We will see in the next section that such conditions also play an important role in the convergence and error analysis of the truncation scheme (1.5).

3. Convergence results for the generalized truncation method

We shall now begin the study of the convergence of the generalized truncation scheme. All through this section, $L_2(0, T; H)$ will designate the set of all strongly measurable H-valued functions defined on $[0, T]$ for which $\int_0^T |f(t)|^2\,dt < \infty$, $L_\infty(0, T; H)$ will designate the set of strongly measurable essentially bounded H-valued functions defined on $[0, T]$, and an analogous notation will be employed for V-valued and V^*-valued functions. As usual, two measurable functions which coincide almost everywhere will be regarded as identical.

In the sequel, we shall assume that the data u_0 and f of problem (1.2) satisfy the following conditions:

$$\exists v_0 \in H \quad \text{such that } v_0 \in g(u_0), \tag{3.1}$$

$$f, f' \in L_2(0, T; H), \tag{3.2}$$

$$f(0) - Au_0 \in H. \tag{3.3}$$

We observe (cf. [**19**, Theorem 1.3.7]) that if (3.2) holds, then f is a continuous function. Therefore pointwise values of f are well defined, so that (3.3) makes sense. In addition, since $L_2(0, T; V) \subset L_2(0, T; H) \subset L_2(0, T; V^*)$ (because of the continuity of the inclusions $V \subset H \subset V^*$), we also have $f,\ f' \in L_2(0, T; V^*)$.

Let u_k denote the step function assuming the value u_k^n on the interval $((n-1)k, nk]$, where the u_k^n are computed from the truncation scheme (1.5). Let $\tilde{u}_k$ be the piece-wise smooth function defined by the condition $\tilde{u}_k(t) = u_k^n + (t/k - n)(u_k^n - u_k^{n-1})$ for all $(n-1)k < t \le nk$. In the sequel, we will be using analogous notations for other functions f_k, U_k etc.. Obviously, we have $|u_k|^2_{L_2(0,T;H)} = k \sum_{n=1}^N |u_k^n|^2$ and $|\tilde{u}'_k|^2_{L_2(0,T;H)} = k \sum_{n=1}^N |(u_k^n - u_k^{n-1})/k|^2$.

In the sequel, we will always define the f_k^n in (1.4) and (1.5) in the form $f_k^n = (1/k) \int_{(n-1)k}^{nk} f(t)\,dt$.

The following results are well-known [**1**, **13**, **26**] and easy to prove, so their proofs will be omitted.

Lemma 3.1. *For all $k > 0$ we have*

$$|U_k - \tilde{U}_k|_{L_\infty(0,T;H)} \le Gk|\tilde{U}'_k|_{L_\infty(0,T;H)}, \tag{3.4}$$

$$\|f - f_k\|_{L_2(0,T;V^*)} \le G_1 k \|f'\|_{L_2(0,T;V^*)}, \tag{3.5}$$

$$\|\tilde{f}'_k\|_{L_2(0,T;V^*)} \le \|f'\|_{L_2(0,T;V^*)}, \tag{3.6}$$

$$|f_k|_{L_2(0,T;H)} \le |f|_{L_2(0,T;H)}. \tag{3.7}$$

Here, and in all that follows, G, G_1, $G_2, \ldots$ designate generic constants which need not have the same value in any two places. We now prove some apriori estimates for the solution of (1.5).

Proposition 3.1. *Suppose that* (2.1), (2.3), (2.4) *and* (3.1)–(3.3) *hold. Then the piece-wise linear function $\tilde{u}_k$ whose values are obtained from* (1.5) *satisfies the estimate*

$$|\tilde{u}'_k|^2_{L_\infty(0,T;H)} + \|\tilde{u}'_k\|^2_{L_2(0,T;V)} \le G. \tag{3.8}$$

Proof: Since $u_k^1 + kAu_k^1 = P_{[k]}u_0 + kf_k^1$, we have

$$u_k^1 - u_0 + kA(u_k^1 - u_0) = P_{[k]}u_0 - u_0 + k(f(0) - Au_0) + k(f_k^1 - f(0)).$$

Taking scalar products with $u_k^1 - u_0$, we obtain

$$\begin{aligned} |u_k^1 - u_0|^2 + kc\|u_k^1 - u_0\|^2 &\le (P_{[k]}u_0 - u_0 + k(f(0) - Au_0) + k(f_k^1 - f(0)), u_k^1 - u_0) \\ &\le \frac{1}{2}|u_k^1 - u_0|^2 + \frac{1}{2}|P_{[k]}u_0 - u_0 + k(f(0) - Au_0)|^2 \\ &\quad + \frac{kc}{2}\|u_k^1 - u_0\|^2 + \frac{k}{2c}\|f_k^1 - f(0)\|_*^2 \end{aligned}$$

$$\leq \frac{1}{2}|u_k^1 - u_0|^2 + |P_{[k]}u_0 - u_0|^2 + k^2|f(0) - Au_0|^2$$
$$+ \frac{kc}{2}\|u_k^1 - u_0\|^2 + \frac{k}{2c}\|f_k^1 - f(0)\|_*^2.$$

Rearranging and making use of (3.1) and (2.7), we obtain

$$|u_k^1 - u_0|^2 + kc\|u_k^1 - u_0\|^2 \leq 2k^2(|v_0|^2 + |f(0) - Au_0|^2) + \frac{k}{c}\|f_k^1 - f(0)\|_*^2.$$

Now, we have

$$\|f_k^1 - f(0)\|_*^2 = \left\|\frac{1}{k}\int_0^k (f(s) - f(0))\,ds\right\|_*^2 \leq \frac{1}{k}\int_0^k \|f(s) - f(0)\|_*^2\,ds$$
$$= \frac{1}{k}\int_0^k \left\|\int_0^s f'(t)\,dt\right\|_*^2 ds \leq \frac{1}{k}\int_0^k \int_0^s s\|f'(t)\|_*^2\,dt\,ds$$
$$\leq \frac{1}{k}\int_0^k s\int_0^T \|f'(t)\|_*^2\,dt\,ds = \frac{k}{2}\|f'\|_{L_2(0,T;V^*)}^2. \tag{3.9}$$

Consequently, we deduce that

$$\left|\frac{u_k^1 - u_0}{k}\right|^2 + kc\left\|\frac{u_k^1 - u_0}{k}\right\|^2 \leq 2|v_0|^2 + 2|f(0) - Au_0|^2 + \frac{1}{2c}\|f'\|_{L_2(0,T;V^*)}^2 \leq G_1. \tag{3.10}$$

Since $u_k^{n+1} + kAu_k^{n+1} = P_{[k]}u_k^n + kf_k^{n+1}$ and $u_k^n + kAu_k^n = P_{[k]}u_k^{n-1} + kf_k^n$, we have

$$u_k^{n+1} - u_k^n + kA(u_k^{n+1} - u_k^n) = P_{[k]}u_k^n - P_{[k]}u_k^{n-1} + k(f_k^{n+1} - f_k^n).$$

Taking scalar products with $u_k^{n+1} - u_k^n$ and making use of (2.1), we obtain

$$|u_k^{n+1} - u_k^n|^2 + kc\|u_k^{n+1} - u_k^n\|^2 \leq |P_{[k]}u_k^n - P_{[k]}u_k^{n-1}|\,|u_k^{n+1} - u_k^n| + k\|f_k^{n+1} - f_k^n\|_*\|u_k^{n+1} - u_k^n\|$$
$$\leq \frac{1}{2}|P_{[k]}u_k^n - P_{[k]}u_k^{n-1}|^2 + \frac{1}{2}|u_k^{n+1} - u_k^n|^2 + \frac{k}{2c}\|f_k^{n+1} - f_k^n\|_*^2 + \frac{kc}{2}\|u_k^{n+1} - u_k^n\|^2.$$

Therefore, using (2.5) we obtain

$$|u_k^{n+1} - u_k^n|^2 - |u_k^n - u_k^{n-1}|^2 + kc\|u_k^{n+1} - u_k^n\|^2 \leq \frac{k}{c}\|f_k^{n+1} - f_k^n\|_*^2.$$

Adding up, and making use of (3.6) and (3.10), we obtain

$$\left|\frac{u_k^{l+1} - u_k^l}{k}\right|^2 + kc\sum_{n=0}^{l}\left\|\frac{u_k^{n+1} - u_k^n}{k}\right\|^2 \leq \left|\frac{u_k^l - u_0}{k}\right|^2 + kc\left\|\frac{u_k^1 - u_0}{k}\right\|^2 + \frac{k}{c}\sum_{n=1}^{l}\left\|\frac{f_k^{n+1} - f_k^n}{k}\right\|_*^2$$
$$\leq G_1 + \frac{1}{c}\|f'\|_{L_2(0,T;V^*)}^2.$$

Since l is arbitrary, we conclude that $\max_{0\leq n\leq N}|(u_k^{n+1} - u_k^n)/k|^2 \leq G_2$ and $k\sum_{n=0}^{N}\|(u_k^{n+1} - u_k^n)/k\|^2 \leq G_3$. This implies that

$$\max_{0\leq n\leq N}\left|\frac{u_k^{n+1}-u_k^n}{k}\right|^2+k\sum_{n=0}^{N}\left\|\frac{u_k^{n+1}-u_k^n}{k}\right\|^2\leq G$$

which is equivalent to (3.8). That completes the proof. ■

PROPOSITION 3.2. *Let the hypotheses of Proposition* 3.1 *hold. Then the step function* u_k *whose values are obtained from* (1.5) *satisfies the estimates*

$$|u_k - P_{[k]}u_k|_{L_2(0,T;H)} \leq Gk \tag{3.11}$$

$$\|u_k - P_{[k]}u_k\|^2_{L_2(0,T;V)} \leq Gk \tag{3.12}$$

PROOF. Writing (1.5) in the form

$$u_k^n - P_{[k]}u_k^n + k(AP_{[k]}u_k^n - w(u_k^n)) + k(Au_k^n - AP_{[k]}u_k^n) = P_{[k]}u_k^{n-1} - P_{[k]}u_k^n + k(f_k^n - w(u_k^n)),$$

taking scalar products with $u_k^n - P_{[k]}u_k^n$, and using (2.4), we see that

$$\begin{aligned}|u_k^n - P_{[k]}u_k^n|^2 + kc\|u_k^n - P_{[k]}u_k^n\|^2 &\leq |P_{[k]}u_k^{n-1} - P_{[k]}u_k^n + k(f_k^n - w(u_k^n))|\,|u_k^n - P_{[k]}u_k^n|\\ &\leq \frac{1}{2}|u_k^n - P_{[k]}u_k^n|^2 + k^2\left|f_k^n - w(u_k^n) + \frac{P_{[k]}u_k^{n-1} - P_{[k]}u_k^n}{k}\right|^2.\end{aligned}$$

Rearranging, and making use of (2.4), (2.5), and the continuity of the inclusion $V \subset H$, we obtain

$$\begin{aligned}\frac{1}{2}|u_k^n - P_{[k]}u_k^n|^2 + kc\|u_k^n - P_{[k]}u_k^n\|^2 &\leq k^2|f_k^n - w(u_k^n)|^2 + k^2\left|\frac{u_k^n - u_k^{n-1}}{k}\right|^2\\ &\leq G_1k^2\left(|f_k^n|^2 + \beta^2 + \left\|\frac{u_k^n - u_k^{n-1}}{k}\right\|^2\right).\end{aligned} \tag{3.13}$$

This implies that

$$|u_k^n - P_{[k]}u_k^n|^2 \leq G_2k^2\left(|f_k^n|^2 + \beta^2 + \left\|\frac{u_k^n - u_k^{n-1}}{k}\right\|^2\right).$$

Multiplying with k, summing from $n = 0$ to $n = N$, and using (3.7) and (3.8), we obtain

$$|u_k - P_{[k]}u_k|^2_{L_2(0,T;H)} \leq G_2k^2(|f|^2_{L_2(0,T;H)} + \|\tilde{u}_k'\|^2_{L_2(0,T;V)} + T\beta^2) \leq G^2k^2.$$

This proves (3.11).

It also follows from (3.13) that

$$k\|u_k^n - P_{[k]}u_k^n\|^2 \leq G_3k^2\left(|f_k^n|^2 + \beta^2 + \left\|\frac{u_k^n - u_k^{n-1}}{k}\right\|^2\right).$$

Summing both sides from $n = 0$ to $n = N$, and using (3.7) and (3.8), we obtain

$$k\sum_{n=0}^{N}\|u_k^n - P_{[k]}u_k^n\|^2 \le G_3k\left(k\sum_{n=1}^{N}|f_k^n|^2 + kN\beta^2 + k\sum_{n=0}^{N}\left\|\frac{u_k^n - u_k^{n-1}}{k}\right\|^2\right)$$
$$\le G_3k(|f|^2_{L_2(0,T;H)} + T\beta^2 + \|\tilde{u}_k'\|^2_{L_2(0,T;V)}) \le Gk.$$

This proves (3.12) and completes the proof of the Proposition. ■

The next result relates the solution of Rothe's scheme (1.4) to the solution of the generalized truncation scheme (1.5).

THEOREM 3.3. *Suppose that* (2.1)–(2.4) *and* (3.1)–(3.3) *hold. If* U_k *and* u_k *are, respectively, the step functions whose values are obtained from* (1.4) *and* (1.5), *then we have*

$$|u_k - U_k|^2_{L_\infty(0,T;H)} + \|u_k - U_k\|^2_{L_2(0,T;V)} \le Gk. \tag{3.14}$$

PROOF. Since $f_k^n - (U_k^n - U_k^{n-1})/k - AU_k^n \in g(U_k^n)$ and $(u_k^n - P_{[k]}u_k^n)/k \in g(P_{[k]}u_k^n)$, the monotonicity of g implies that

$$(P_{[k]}u_k^n - U_k^n + kAu_k^n - kAU_k^n, P_{[k]}u_k^n - U_k^n) \le (P_{[k]}u_k^{n-1} - U_k^{n-1}, P_{[k]}u_k^n - U_k^n).$$

Rearranging this inequality in the form

$$|P_{[k]}u_k^n - U_k^n|^2 + k(Au_k^n - AU_k^n, u_k^n - U_k^n)$$
$$\le (P_{[k]}u_k^{n-1} - U_k^{n-1}, P_{[k]}u_k^n - U_k^n) + k(Au_k^n - AU_k^n, u_k^n - P_{[k]}u_k^n)$$

and making use of (2.1) and (2.2), we obtain

$$|P_{[k]}u_k^n - U_k^n|^2 + kc\|u_k^n - U_k^n\|^2$$
$$\le |P_{[k]}u_k^{n-1} - U_k^{n-1}|\,|P_{[k]}u_k^n - U_k^n| + Mk\|u_k^n - U_k^n\|\,\|u_k^n - P_{[k]}u_k^n\|.$$

This implies

$$\frac{1}{2}|P_{[k]}u_k^n - U_k^n|^2 - \frac{1}{2}|P_{[k]}u_k^{n-1} - U_k^{n-1}|^2 + \frac{kc}{2}\|u_k^n - U_k^n\|^2 \le \frac{M^2k}{2c}\|u_k^n - P_{[k]}u_k^n\|.$$

Performing a summation and making use of (2.7) and (3.12), we obtain

$$|P_{[k]}u_k^l - U_k^l|^2 + kc\sum_{n=1}^{l}\|u_k^n - U_k^n\|^2 \le \frac{M^2k}{c}\sum_{n=0}^{N}\|u_k^n - P_{[k]}u_k^n\|^2 + |u_0 - P_{[k]}u_0|^2 \le G_1k.$$

Since l is arbitrary, we conclude that $\max_{0\le l\le N}|P_{[k]}u_k^l - U_k^l|^2 \le G_1k$ and $k\sum_{n=0}^{N}\|u_k^n - U_k^n\|^2 \le G_2k$, from which (3.14) follows immediately. ■

This result suggests that the numerical performance of Rothe's scheme and the generalized truncation scheme should be similar. It also shows that to prove the convergence of the solution u_k of the generalized truncation method to the solution u of (1.2), it suffices to prove that the solution U_k of Rothe's scheme converges to u. This will be shown in Theorem 3.5 below. However, before stating and proving this theorem we first establish some basic results for Rothe's scheme (1.4).

PROPOSITION 3.4. *Suppose that* (2.1), (2.2) *and* (3.1)–(3.3) *hold. Let* $\tilde{U}_k$ *be the piecewise linear function constructed from the nodal values of Rothe's scheme* (1.4). *Then* *we have*

$$|\tilde{U}_k'|_{L_\infty(0,T;H)} + \|\tilde{U}_k'\|_{L_2(0,T;V)} \le G, \quad \forall\, k > 0. \tag{3.15}$$

$$|U_k - \tilde{U}_k|_{L_\infty(0,T;H)} \le Gk, \quad \forall\, k > 0. \tag{3.16}$$

PROOF. Using (3.1), the relation $U_k^1 + kAU_k^1 + kg(U_k^1) \ni u_0 + kf_k^1$, and the monotonicity of g, we obtain $(U_k^1 - u_0 + kAU_k^1 - kf_k^1 + kv_0, U_k^1 - u_0) \le 0$. Rearranging this, we obtain

$$\begin{aligned}&|U_k^1 - u_0|^2 + k(AU_k^1 - Au_0, U_k^1 - u_0)\\ &\qquad \le k(f(0) - Au_0 - v_0, U_k^1 - u_0) + k(f_k^1 - f(0), U_k^1 - u_0).\end{aligned}$$

Using (2.1) we obtain

$$\begin{aligned}&|U_k^1 - u_0|^2 + kc\|U_k^1 - u_0\|^2\\ &\quad \le k|U_k^1 - u_0||f(0) - Au_0 - v_0| + k\|U_k^1 - u_0\|\,\|f_k^1 - f(0)\|_*\\ &\quad \le \frac{k^2}{2}|f(0) - Au_0 - v_0|^2 + \frac{1}{2}|U_k^1 - u_0|^2 + \frac{kc}{2}\|U_k^1 - u_0\|^2 + \frac{k}{2c}\|f_k^1 - f(0)\|_*^2.\end{aligned}$$

Therefore, rearranging and making use of (3.9), we conclude that

$$\left|\frac{U_k^1 - u_0}{k}\right|^2 + kc\left\|\frac{U_k^1 - u_0}{k}\right\|^2 \le |f(0) - Au_0 - v_0|^2 + \frac{1}{kc}\|f_k^1 - f(0)\|_*^2 \le G_1. \tag{3.17}$$

On the other hand, if we use (1.4) and the corresponding equation obtained on replacing n with $n-1$ and apply the monotonicity of g we obtain

$$(U_k^{n+1} - U_k^n + kAU_k^{n+1} - kAU_k^n, U_k^{n+1} - U_k^n) \le (U_k^n - U_k^{n-1} + kf_k^{n+1} - kf_k^n, U_k^{n+1} - U_k^n).$$

This implies

$$\begin{aligned}&|U_k^{n+1} - U_k^n|^2 + kc\|U_k^{n+1} - U_k^n\|^2 \le |U_k^n - U_k^{n-1}||U_k^{n+1} - U_k^n| + k\|f_k^{n+1} - f_k^n\|_*\|U_k^{n+1} - U_k^n\|\\ &\quad \le \frac{1}{2}|U_k^n - U_k^{n-1}|^2 + \frac{1}{2}|U_k^{n+1} - U_k^n|^2 + \frac{k}{2c}\|f_k^{n+1} - f_k^n\|_*^2 + \frac{kc}{2}\|U_k^{n+1} - U_k^n\|^2\end{aligned}$$

Therefore

$$|U_k^{n+1} - U_k^n|^2 - |U_k^n - U_k^{n-1}|^2 + kc\|U_k^{n+1} - U_k^n\|^2 \le \frac{k}{c}\|f_k^{n+1} - f_k^n\|_*^2.$$

Adding up and using (3.6) and (3.17), we obtain

$$\begin{aligned}&\left|\frac{U_k^{l+1} - U_k^l}{k}\right|^2 + kc\sum_{n=0}^{l}\left\|\frac{U_k^{n+1} - U_k^n}{k}\right\|^2\\ &\quad \le \left|\frac{U_k^1 - u_0}{k}\right|^2 + kc\left\|\frac{U_k^1 - u_0}{k}\right\|^2 + \frac{k}{c}\sum_{n=1}^{l}\left\|\frac{f_k^{n+1} - f_k^n}{k}\right\|_*^2\\ &\quad \le \left|\frac{U_k^1 - u_0}{k}\right|^2 + kc\left\|\frac{U_k^1 - u_0}{k}\right\|^2 + \frac{1}{c}\|\tilde{f}_k'\|_{L_2(0,T;V^*)}^2\\ &\quad \le G_1(G_2 + \|f'\|_{L_2(0,T;V^*)}^2) \le G_3.\end{aligned}$$

Since l is arbitrary, we conclude that $\max_{0\le l\le N}|(U_k^{l+1}-U_k^l)/k|^2 \le G_4$ and $k\sum_{n=0}^N \|(U_k^{n+1}-U_k^n)/k\|^2 \le G_5$, from which we conclude that

$$\max_{0\le n\le N}\left|\frac{U_k^{n+1}-U_k^n}{k}\right|^2 + k\sum_{n=0}^{N}\left\|\frac{U_k^{n+1}-U_k^n}{k}\right\|^2 \le G.$$

That proves (3.15). The final estimate (3.16) follows from (3.4) and (3.15). ■

We now prove the existence of a solution to (1.2) and the convergence the approximate solutions computed from Rothe's scheme (1.4).

THEOREM 3.5. *Let the hypotheses of Theorem* 3.4 *hold. Then there exists a solution* $u \in L_2(0,T;V)\cap L_\infty(0,T;H)$ *of* (1.2). *Moreover, if* U_k *is the step function whose values are obtained from* (1.4), *then* U_k *converges to* u, *with the error estimate*

$$|U_k-u|^2_{L_\infty(0,T;H)} + \|U_k-u\|^2_{L_2(0,T;V)} \le Gk. \tag{3.18}$$

PROOF. Rothe's scheme (1.4) with step size k can be expressed in the form

$$\tilde U_k'(t) + AU_k(t) + g(U_k(t)) \ni f_k(t). \tag{3.19}$$

Writing the analogous expression with step size h, and invoking the monotonicity of g we obtain the inequality

$$(\tilde U_h'(t) - \tilde U_k'(t) + AU_h(t) - AU_k(t), U_h(t)-U_k(t)) \le (f_h(t)-f_k(t), U_h(t)-U_k(t)).$$

This can be rewritten in the form (with time arguments omitted for the sake of simplicity)

$$\begin{aligned}(\tilde U_h' - \tilde U_k', \tilde U_h - \tilde U_k) &+ (AU_h - AU_k, U_h - U_k)\\ &\le (f_h - f_k, U_h - U_k) + (\tilde U_h' - \tilde U_k', \tilde U_h - U_h - \tilde U_k + U_k).\end{aligned}$$

This implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\tilde U_h - \tilde U_k|^2 + c\|U_h - U_k\|^2\\ &\qquad\le \frac{1}{2c}\|f_h - f_k\|_*^2 + \frac{c}{2}\|U_h - U_k\|^2 + (|\tilde U_h'| + |\tilde U_k'|)(|\tilde U_h - U_h| + |\tilde U_k - U_k|).\end{aligned}$$

Applying (3.15) and (3.16) to the last term in this expression, we obtain

$$\frac{d}{dt}|\tilde U_h - \tilde U_k|^2 + c\|U_h - U_k\|^2 \le \frac{1}{c}\|f_h - f_k\|_*^2 + G_1(h+k).$$

Integrating, we obtain

$$\begin{aligned}&|\tilde U_h(t) - \tilde U_k(t)|^2 + c\int_0^t \|U_h(s) - U_k(s)\|^2\,ds\\ &\qquad\le \frac{1}{c}\int_0^t \|f_h(s) - f_k(s)\|_*^2\,ds + G_1T(h+k)\\ &\qquad\le \frac{1}{c}\|f_h - f_k\|^2_{L_2(0,T;V^*)} + G_1T(h+k)\end{aligned}$$

$$\leq G_2(\|f_h - f\|^2_{L_2(0,T;V^*)} + \|f_k - f\|^2_{L_2(0,T;V^*)} + h + k)$$
$$\leq G_3(h+k).$$

Since t is arbitrary, we see that $|\tilde{U}_h - \tilde{U}_k|^2_{L_\infty(0,T;H)} \leq (G/2)(h+k)$ and $\|U_h - U_k\|^2_{L_2(0,T;V)} \leq (G/2)(h+k)$. By invoking (3.4), we conclude that

$$|U_h - U_k|^2_{L_\infty(0,T;H)} + \|U_h - U_k\|^2_{L_2(0,T;V)} \leq G(h+k). \tag{3.20}$$

If follows from the completeness of $L_\infty(0,T;H) \cap L_2(0,T;V)$ that there exists $u \in L_\infty(0,T;H) \cap L_2(0,T;V)$ such that $|U_k - u|_{L_\infty(0,T;H)} + \|U_k - u\|_{L_2(0,T;V)} \to 0$ as k tends to zero. Letting h tend to zero in (3.20), we obtain (3.18).

We now show that u solves (1.2). It is well known that (3.18) implies that $U_k(t)$ converges strongly to $u(t)$ in V as k tends to zero, for almost every $t \in [0,T]$. Therefore, the continuity of A implies that $Au_k(t)$ converges strongly to $Au(t)$ in V^* as k tends to zero, for almost every $t \in [0,T]$. Also, Proposition 3.4 states that $\tilde{U}_k'$ is bounded in $L_2(0,T;V)$, which implies that it converges weakly in the norm of $L_2(0,T;V)$ to some $\chi \in L_2(0,T;V)$ as k tends to zero. Since $\tilde{U}_k$ converges strongly to u in the norm of $L_2(0,T;V)$, it follows from [**19**, Lemma 1.3.15] that $\chi = u'$. Consequently, $\tilde{U}_k'(t)$ converges weakly to $u'(t)$ in V as k tends to zero, for almost every $t \in [0,T]$. Finally, given any $z \in g(y)$, it follows from (3.19) and the monotonicity of g that

$$(\tilde{U}_k'(t) + AU_k(t) - f_k(t) - z, U_k(t) - y) \geq 0.$$

Letting k tend to zero, we obtain the inequality

$$(u'(t) + Au(t) - f(t) - z, u(t) - y) \geq 0.$$

Therefore, the maximality of g shows that $u'(t) + Au(t) - f(t) \in g(u(t))$, which shows that u satisfies (1.2) for almost every t. That completes the proof. ∎

The convergence of the generalized truncation scheme now follows easily from Theorem 3.3 and Theorem 3.5.

COROLLARY 3.6. *Suppose that the hypotheses of Theorem* 3.3 *hold. Then the step function u_k whose values are obtained from* (1.5) *converges to the solution u of* (1.2), *with the error estimate*:

$$|P_{[k]}u_k - u|^2_{L_\infty(0,T;H)} + \|u_k - u\|^2_{L_2(0,T;V)} \leq Gk.$$

PROOF. Using Theorem 3.4 and Theorem 3.5, we obtain

$$\begin{aligned}|P_{[k]}u_k - u|^2_{L_\infty(0,T;H)} + \|u_k - u\|^2_{L_2(0,T;V)} &\leq 2|P_{[k]}u_k - U_k|^2_{L_\infty(0,T;H)} + 2|U_k - u|^2_{L_\infty(0,T;H)} \\ &\quad + 2\|u_k - U_k\|^2_{L_2(0,T;V)} + 2\|U_k - u\|^2_{L_2(0,T;V)}) \\ &\leq Gk.\end{aligned}$$

That completes the proof. ∎

This result can be proved directly in a manner similar to the proof of Theorem 3.4. However, we preferred the approach followed above since it brings out the relationship between Rothe's scheme and the generalized truncation scheme in a natural manner.

4. Applications of the truncation method

In this section we discuss some classes of problems that occur in applications to which our convergence theory is applicable and for which the relevant resolvent operators are easily computed. Obviously the generalized truncation method is a convenient convergent method for the numerical solution of such problems.

EXAMPLE 4.1. Let $q : [0, \infty) \mapsto [0, \infty)$ be a differentiable function such that $q(0) = 0$ and

$$sq'(s) - q(s) > 0 \quad \forall\, s > 0. \tag{4.1a}$$

It is not difficult to see that this condition implies that, for all $k > 0$, $(1 + kq)^{-1}$ exists as a function mapping $[0, \infty)$ into itself. We make the further assumption that

$$(1 + kq)^{-1}(s) \text{ is continuous in } s \text{ for all } s > 0. \tag{4.1b}$$

Now let $g(0) = 0$ and

$$g(z) = q(|z|)|z|^{-1}z, \quad \forall\, z \in H\backslash\{0\}. \tag{4.2}$$

Then g is maximal monotone since it is the subgradient of the differentiable convex function $\phi(z) = \int_0^{|z|} q(s)\, ds$. For all $v \in V$, we have

$$(v - P_{[k]}v, AP_{[k]}v) = k(g(P_{[k]}v), AP_{[k]}v) = q(|P_{[k]}v|)|P_{[k]}v|^{-1}(P_{[k]}v, AP_{[k]}v) \geq 0.$$

Therefore (2.4) holds with $w(v) \equiv 0$. The fact that $P_{[k]}$ maps V into itself follows from the equation

$$P_{[k]}v = v\{1 + kq(|P_{[k]}v|)|P_{[k]}v|^{-1}\}^{-1}, \quad \forall\, v \in H. \tag{4.3}$$

Furthermore, we have $\|P_{[k]}v\| = \|v\|\{1 + kq(|P_{[k]}v|)|P_{[k]}v|^{-1}\}^{-1} \leq \|v\|$, which shows that $P_{[k]}$ is bounded on V. Finally, the expression

$$P_{[k]}v = v\left\{1 + k\,\frac{q((1 + kq)^{-1}(|v|))}{(1 + kq)^{-1}(|v|)}\right\}^{-1}$$

shows that $P_{[k]}$ is a continuous function from V to V. In particular, $P_{[k]}$ is hemicontinuous. Therefore (2.3) holds.

The computation of $P_{[k]}v$ can be efficiently done with a method described in [**29**], the idea of which is to obtain $|P_{[k]}v|$ from the equation $|P_{[k]}v| + kq(|P_{[k]}v|) = |v|$ and then compute $P_{[k]}v$ from (4.3). Using Newton's method, we obtain the rapidly converging iterative scheme

$$p_0 = |v|, \tag{4.4a}$$

$$p_{m+1} = \{|v| + kp_m q'(p_m) - kq(p_m)\}\{1 + kq'(p_m)\}^{-1}, \tag{4.4b}$$

$$|P_{[k]}v| = \lim_{m\to\infty} p_m. \tag{4.4c}$$

The initial choice in (4.4a) is strongly recommended, for since $\lim_{k\to 0}|P_{[k]}v| = |v|$, it ensures the convergence of the scheme (4.4) whenever k is small enough.

If $p \geq 2$, the function $q(z) = z^{p-1}$ satisfies (4.1a) and corresponds to the the maximal monotone function $g(z) = z|z|^{p-2}$. In this case (1.4) reduces to the nonlinear evolution equation:

$$\begin{aligned} &u(0) = u_0, \\ &u'(t) + Au(t) + |u(t)|^{p-2}u(t) = f(t), \quad \forall\, t \in [0, T]. \end{aligned} \tag{4.5}$$

When the value of the exponent p is either 3, 4 or 5, the resolvent values $P_{[k]}v$ can be computed from the explicit formulae given in [**28**, **29**] instead of the iterative scheme (4.4).

If we take V and H as in Example 2.4 and let $A = -\Delta$, then problem (4.5) reduces to the problem in equation (1.8).

EXAMPLE 4.2. Let H, V and A be as in Example 2.4 and let q satisfy (4.1a). We define $g(v)$ in the pointwise manner

$$g(v(x)) \equiv q(|v(x)|)|v(x)|^{-1}v(x). \tag{4.6}$$

Then for any $v \in V$, we can compute the resolvent values $P_{[k]}v(x)$ in a pointwise manner with the scheme (4.4), interpreting $|\,|$ as the absolute value function in $\boldsymbol{R}$ rather than the norm in H. Equation (4.3) shows that $P_{[k]}v$ vanishes (in the sense of traces) on $\partial\Omega$. Also, for every $v \in V$, we have

$$\frac{\partial P_{[k]}v}{\partial x_i} = \left\{\frac{\partial v}{\partial x_i}\right\}\{1 + kq'(|P_{[k]}v|)\}^{-1},$$

which shows that

$$\|P_{[k]}v\|^2 = \int_\Omega |\nabla P_{[k]}v(x)|^2\,dx \leq \int_\Omega |\nabla v(x)|^2\,dx = \|v\|^2,$$

and hence that $P_{[k]}$ is a bounded map from V to V. The pointwise expression

$$P_{[k]}v(x) = v(x)\left\{1 + k\,\frac{q((1+kq)^{-1}(|v(x)|))}{(1+kq)^{-1}(|v(x)|)}\right\}^{-1}$$

shows that $P_{[k]}$ is continuous from V into V. Finally, we have

$$\begin{aligned}(AP_{[k]}v, v - P_{[k]}v) = k(AP_{[k]}v, g(P_{[k]}v)) &= \int_\Omega q'(|P_{[k]}v|)\left(\sum_{i=1}^{q}\left(\sum_{j=1}^{q} a_{ij}\,\frac{\partial P_{[k]}v}{\partial x_j}\right)\frac{\partial P_{[k]}v}{\partial x_i}\right)dx \\ &\geq 0.\end{aligned}$$

Therefore (2.3) and (2.4) hold with $w(v) \equiv 0$. In this case (1.2) reduces to the problem of finding a function $u(x,t)$ such that $u(x,0) = u_0(x)$ for almost all $x \in \Omega$, $u(\cdot, t) \in V$ for all $t \in [0, T]$, and

$$\frac{\partial u}{\partial t}(x,t) + Au(x,t) + q(|u(x,t)|)|u(x,t)|^{-1}u(x,t) = f(x,t), \quad \forall\, t \in [0,T], \quad \forall\, x \in \Omega.$$

This is a fairly large class of nonlinear parabolic partial differential equations which includes the problem in equation (1.7).

Remark 4.3. Let H, V, A and g be as in Example 2.4. Then it has been shown in Example 2.4 that the compatibility conditions (2.3) and (2.4) hold.

In this case, it is easy to see that problem (1.2) reduces to the problem of finding a function $u(x,t)$ such that $u(x,0) = u_0(x)$ for almost all $x \in \Omega$, $u(\cdot,t) \in V$ for all $t \in [0,T]$, and

$$\frac{\partial u}{\partial t}(x,t) + Au(x,t) + g(u(x,t)) \ni f(x,t), \quad \forall\, t \in [0,T], \quad \forall\, x \in \Omega.$$

If we set $\alpha_1 = -\infty$ and $\alpha_2 = \infty$, then the resolvent operators in (2.16) reduce to the simpler truncation formula $P_{[k]}v(x) = v(x) + (\psi_1(x) - v(x))^+ - (v(x) - \psi_2(x))^+$. In this case (1.2) is called an obstacle problem, and the generalized truncation scheme (1.5) reduces to the time-discretized version of the truncation method due to Berger [**6**, **7**]. Letting $\psi_1 = \psi_2 = 0$, we obtain a problem of the type (1.6).

Remark 4.4. If we perform a finite-element space discretization in the generalized truncation scheme, we obtain a fully discretized scheme for the numerical solution of (1.4). Error estimates have been given in [**8**, **18**] for this scheme when applied to obstacle problems. It would be interesting to see whether these error estimates can be extended to fully discretized version of the generalized truncation method when applied to the non-obstacle type problems described above.

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References

[1] J. Aubin, Approximation of Elliptic Boundary value problems (Wiley Interscience, New York, 1972).
[2] C. Baiocchi, "Disequazioni variazionali", Bolletino dell'Unione Matematica Italiana (5) **18-A** (1981), 173–184.
[3] C. Baiocchi, "Discretization of evolution variational inequalities", In, F. Colombini, A. Marino, L. Modica, S. Spagnolo (eds), Partial Differential Equations and the Calculus of Variations: Volume I (Birkhäuser Boston, Inc. 1989) 59–92.
[4] C. Baiocchi, A. Capelo, Variational and quasivariational inequalities. Applications to free boundary problems (J. Wiley, 1980).
[5] V. Barbu, Non-linear Semigroups and Differential Equations in Banach Spaces (Noordhoff International Publishing, Leyden, The Netherlands, 1976).
[6] A. Berger, "The truncation method for the solution of a class of variational inequalities", RAIRO Ser. Rouge Analyse Numerique **10** (1976) 29–42.
[7] A. Berger, M. Ciment, J. W. C. Rogers, "Numerical solution of a diffusion absorption problem with a free boundary", SIAM J. Num. Anal. **12** (1975) 646–672.
[8] A. Berger, R. Falk, "An error estimate for the truncation method for the solution of parabolic variational inequalities", Math. Comp. **31** (1977), 619–628.
[9] H. Brezis, "Problemes unilatéraux", J. Math. Pures et Appl. **51** (1972) 1–168.
[10] H. Brézis, Opérateur maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert (North Holland, Amsterdam, 1973).
[11] H. Brezis, M. G. Crandall, A. Pazy, "Pertubations of nonlinear maximal monotone sets in Banach space", Comm. Pure and Applied Math. **23** (1970) 123–144.

[12] F. E. Browder, "Nonlinear maximal monotone operators in Banach spaces", Math. Annalen **176** (1968) 88–113.
[13] P. G. Ciarlet, "The finite element method for elliptic problems." North Holland, Amsterdam (1978).
[14] G. Duvaut, J. L. Lions, "Inequalities in Physics and Mechanics," Springer Verlag (1976).
[15] L. Gastaldi, G. Gilardi, "An error estimate for an approximation of a parabolic variational inequality", Boll. Unione Matematica Italiana (6) **1-B** (1982) 501–521.
[16] R. Glowinski, Lectures on Numerical methods for nonlinear variational problems (Tata Institute for Fundamental Research, 1980).
[17] R. Glowinski, J. L. Lions, R. Trémolières, Numerical Analysis of variational inequalities (North Holland, 1981).
[18] C. Johnson, "A convergence estimate for the approximation of a parabolic variational inequality", SIAM J. Numer. Anal. **13** (1976) 599–606.
[19] J. Kacur, Method of Rothe in Evolution Equations (Leipzig: BSB Teubner Verlagsges, 1985).
[20] D. Kinderlehrer, G. Stampacchia, An introduction to Variational Inequalities and their Applications (Academic Press, New York, 1980).
[21] J. L. Lions, "Quelques métodes de résolution des problemés aux limites non linearés (Dunod, Paris, 1969).
[22] J. L. Lions and G. Stampacchia, "Variational inequalities", Comm. Pure and Applied Math. **20** (1967) 493–519.
[23] G. J. Minty, "Monotone (nonlinear) operators in Hilbert space", Duke Math. J. **29** (1973) 341–346.
[24] G. J. Minty, "On the monotonicity of the gradient of a convex function", Pacific J. of Math. **14** (1964) 243–247.
[25] R. T. Rockafellar, Convex Analysis (Princeton University Press, 1970).
[26] R. Temam, Numerical Analysis (D. Reidel publishing Company, Holland, 1973).
[27] R. Trémolières, "Inéquation variationnelles. Existence, approximation, résolution", Doctoral Thesis. Université de Paris VI (1972).
[28] L. U. Uko, Soluzione numerica delle disequazioni variazionali. 'Dottorato di ricerca' thesis, University of Milan (1989).
[29] L. U. Uko, "The proximal correction method for discretization of parabolic variational inequalities", J. Nigerian Math. Soc. **9** (1990) 33–48.
[30] L. U. Uko, "Parabolic variational inequalities and the proximal correction method", Afrika Matematika Série 3 Vol. **3** (1994) 97–111.

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