J. Math. Soc. Japan
Vol. 50, No. 3, 1998

A generalization of Rejection Lemma of Drozd-Kirichenko

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(Received June 17, 1996)
(Revised Sept. 27, 1996)

1. Introduction.

Let R be a complete discrete valuation ring, which is fixed once for all as our base ring. Let K denote the quotient field of R. As for basic terminology such as R-lattice, R-order etc., we mostly follow that of [**CR**]. Let Λ be an R-order in a K-algebra $\tilde{\Lambda} := K\Lambda \simeq K \otimes_R \Lambda$, and let $\operatorname{Ind}\Lambda$ denote the set of isomorphism classes of indecomposable left Λ-lattices. For an overorder Γ of Λ in $\tilde{\Lambda}$, we may naturally consider $\operatorname{Ind}\Gamma$ as a subset of $\operatorname{Ind}\Lambda$. A subset $\mathscr{S}$ of $\operatorname{Ind}\Lambda$ will be called a *rejectable* subset if there is an overorder Γ such that $\mathscr{S} = \operatorname{Ind}\Lambda - \operatorname{Ind}\Gamma$. The map $\Gamma \mapsto (\operatorname{Ind}\Lambda - \operatorname{Ind}\Gamma)$ defines a bijection from the set of all overorders of Λ onto the set of all rejectable subsets of $\operatorname{Ind}\Lambda$. Indeed, the inverse map is given by $\mathscr{S} \mapsto \Lambda(\mathscr{S})$. Here, for any subset $\mathscr{S}$ of $\operatorname{Ind}\Lambda$, $\Lambda(\mathscr{S})$ is defined as the intersection $\bigcap_{L \in \operatorname{Ind}\Lambda - \mathscr{S}} O_l(L)$ of the left multiplier $O_l(L) := \{x \in \tilde{\Lambda} \mid xL \subseteq L\}$.

For any subset $\mathscr{S}$, $\Lambda(\mathscr{S})$ is an R-subalgebra of $\tilde{\Lambda}$ containing Λ, but is not necessarily an R-order of $\tilde{\Lambda}$. A subset $\mathscr{S}$ will be called *cofaithful* if $\Lambda(\mathscr{S})$ is an R-order of $\tilde{\Lambda}$. In particular, a rejectable subset $\mathscr{S}$ is always cofaithful. A subset $\mathscr{S}$ will be called *trivial* if $\Lambda(\mathscr{S}) = \Lambda$. While a subset $\mathscr{S}$ will be called *bounded* if the rational length $l(L) := \operatorname{length}_{\tilde{\Lambda}}(K \otimes_R L)$ is bounded on $\mathscr{S}$. When $\mathscr{S}$ is a singleton set, there is known a criterion (= determinable necessary and sufficient condition) for $\mathscr{S}$ to be rejectable, i.e. Rejection Lemma of Drozd-Kirichenko [**DK**].

1.1 D-K Rejection Lemma *$\mathscr{S} = \{P\}$ is rejectable if and only if P is bijective (= projective and injective) and $P \not\simeq \operatorname{rad} P$.*

1.1.1 Utility of D-K Rejection Lemma was well exhibited in [**DK**] where it was applied for Bass orders, and more generally quasi-Bass orders in semi-simple K-algebras. Further, in [**HN-1**], it was applied for Bass orders in non-semi-simple K-algebras.

1.1.2 Hijikata [**H**] studied also almost Bass orders, which is defined as a Gorenstein order Λ such that $O_l(\operatorname{rad}\Lambda)$ is also Gorenstein. He has shown that very precise results for almost Bass orders (including classification) can be derived by D-K Rejection Lemma, and suggested a possibility to extend Rejection Lemma for $\mathscr{S}$ with more than two points. Notably, the result of [**HN-2**] shows that, excepting for a small number of them (counted in representation type), each local order Λ of finite representation type has a minimal rejectable subset $\mathscr{S}$ consisting of four points of a definite shape, whose $\Lambda(\mathscr{S})$ is the unique minimal local overorder of Λ, and this is the reason, in a sense, why the Λ can be of finite representation type.

1.2. Our Results. In this paper we assume that the ambient K-algebra $\tilde{\Lambda}$ is *semisimple*. For main results, we moreover have to assume that $\mathscr{S}$ is *bounded*.

In 4.6, we give a criterion for $\mathscr{S}$ to be cofaithful, in particular a bounded rejectable subset $\mathscr{S}$ is necessarily finite.

In 5.3, we give a criterion for $\mathscr{S}$ to be trivial.

In 5.5, assuming $\mathscr{S}$ to be finite rejectable, we give an algorithm to describe the Auslander-Reiten quiver $\mathfrak{A}(\Lambda(\mathscr{S}))$ of $\Lambda(\mathscr{S})$ from $\mathfrak{A}(\Lambda)$.

In principle, the above three Theorems together give a criterion for bounded $\mathscr{S}$ to be rejectable (= Rejection Lemma). Because, for any non-trivial $\mathscr{S}$, any minimal non-trivial cofaithful subset $\mathscr{S}'$ of $\mathscr{S}$ is rejectable, which is determined by 4.6 and 5.3. Then, by 5.5, the problem for Λ is reduced to that of the overorder $\Lambda(\mathscr{S}')$.

1.2.1. A remarkable fact is that the criterion for $\mathscr{S}$ to be rejectable depends, as in the case of D-K Rejection Lemma, only on the structure of $\mathscr{S}$, but not on the structure of the whole $\mathfrak{A}(\Lambda)$. To be precise, the information we need is the following.

(1) Structure of $\mathscr{S}$ as valued translation quiver.

(2) Preassignment of the subset $\mathscr{S}_p$ (resp. $\mathscr{S}_i$) consisting of projective (resp. injective) vertices in $\mathfrak{A}(\Lambda)$ contained in $\mathscr{S}$.

1.2.2. One of the basic problems is to determine all minimal finite rejectable subsets, which has an intimate connection with the classification of (the sequence of) orders of finite representation type. We shall describe, in the next 1.3, some of the results toward the above problem, obtained as an application of § 5, supplemented by some additional considerations such as an analogy of Bautista-Brenner Theorem [**BB**] for rejectable subsets.

1.3. EXAMPLES. If $\mathscr{S}$ is minimal finite rejectable, then each of $\mathscr{S}_p$ and $\mathscr{S}_i$ is a singleton set (5.3.2), so that we write as $\mathscr{S}_p = \{P\}$, $\mathscr{S}_i = \{I\}$. In the diagrams below, unspecified arrow $\rightarrow$ has the valuation (1,1).

1.3.1. Assume that $\mathscr{S}$ has at most four points, $\#\mathscr{S} \leq 4$. If $\mathscr{S}$ is minimal rejectable, then it should have one of the following forms.

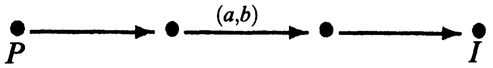
(1) $\underset{P=I}{\bullet}$

(2) $\underset{P}{\bullet} \longrightarrow \underset{I}{\bullet}$

$$\left.\begin{array}{ll} \text{(3-0)} & \underset{P}{\bullet} \xrightarrow{(a,b)} \bullet \longrightarrow \underset{I}{\bullet} \\ \text{(3-0}'\text{)} & \underset{P}{\bullet} \longrightarrow \bullet \xrightarrow{(a,b)} \underset{I}{\bullet} \end{array}\right\} (a,b) = (1,1), (1,2) \text{ or } (2,1)$$

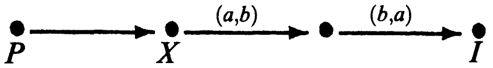
(3-1) $\underset{P}{\bullet} \xrightarrow{(a,b)} \bullet \xrightarrow{(b,a)} \underset{I}{\bullet}$ $P = \tau I, \quad (a,b) = (1,2), (2,1), (1,3)$ or (3.1)

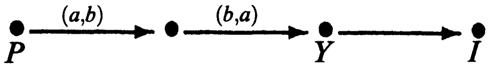
(4-0) 

(4-0′) 

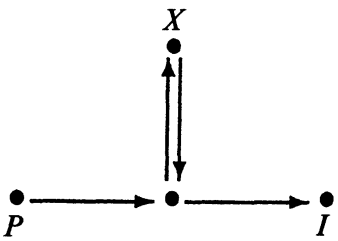
(4-0″) 

$(a,b) = (1,1), (1,2)$ or $(2,1)$

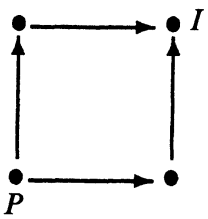
(4-1) 

(4-1′) 

$X = \tau I$, $P = \tau Y$; $(a,b) = (1,2)$ or $(2,1)$

(4-2) 

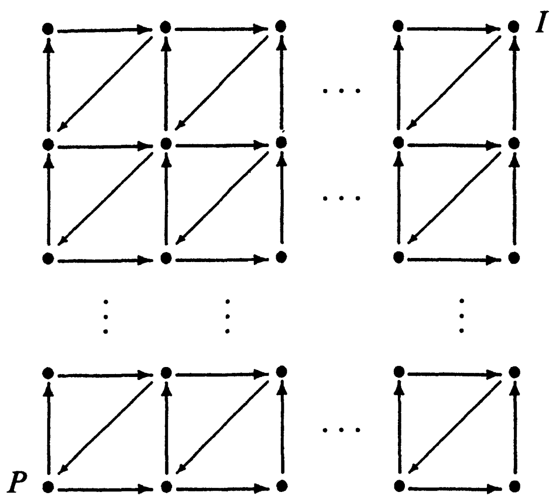
$P = \tau I, X = \tau X$

(2 × 2) 

$P = \tau I$

The occurence of the above minimal rejectable subsets will be discussed in elsewhere. Here we only remark that the last one (2×2) appeared in the sequence of local orders of finite representation type.

1.3.2. For any $n \geq 1$, $m \geq 1$, the following square (with $n \times m$ vertices) is minimal rejectable.

$(n \times m)$ 

where diagonal arrows indicate τ.

There is an infinite sequence of orders of finite representation type $\Lambda_1 \supset \Lambda_2 \supset \cdots$ such that $\mathscr{S}_i = \mathrm{Ind}\,\Lambda_i - \mathrm{Ind}\,\Lambda_{i+1}$ has the above form for any i for one fixed (n, m).

1.4. Our Method. We shall use, so to speak, a kind of *deformation of a complex* of Λ-lattices by means of an almost split sequence. Suppose that the n-th term A_n of a complex $\boldsymbol{A} = (A_l, a_l)$ has the form $A_n = A \oplus T$ with $T \in \mathrm{Ind}\,\Lambda$ and $a_n|_T \in \mathrm{rad}(T, A_{n+1})$. Let $\theta^- T$ denote the target of the source map (= minimal left almost split morphism) from $T, T \to \theta^- T$.

We construct a new complex $\boldsymbol{A}' = (A'_l, a'_l)$ of which n-th term is $A'_n = A \oplus \theta^- T$, together with a chain homomorphism $f : \boldsymbol{A} \to \boldsymbol{A}'$ with some appropriate properties which are described in 3.2. Intuitively said, we can replace T by $\theta^- T$, and we call the process as rejection, rejecting T from A_n. The whole idea is to reach a desirable result by a *successive rejection*, and is extremely simple. However, to keep track of the effect of rejections numerically, we had better work in the $\boldsymbol{Z}$-module $\boldsymbol{Z}(\mathrm{Ind}\,\Lambda)$ rather than in the category of Λ-lattices. Moreover, to cope with apparent obstacles caused by the lack of quotient in the lattice category, we shall prepare some remarks on rationally exactness (2.2). These make up a rather lengthy preliminary § 2.

The method seems to have a variety of applications other than the ones given in this paper.

1.5. Artin algebras. The methods and proofs of this paper apply for an artin algebra Λ. One should of course replace the overorder Γ by the quotient algebra $\Lambda/\mathfrak{a}$. At each step, the proof is the same or simpler in this case. It is simpler because considerations for cofaithfulness and rationally exactness are totally unnecessary in artinian case.

2. Preliminaries.

Let R, K, Λ, $\tilde{\Lambda}$ be as in § 1 and π the prime of R. Recall that we have assumed that $\tilde{\Lambda}$ is semi-simple throughout in this paper. Besides $\mathrm{Ind}\,\Lambda$ in § 1, we fix the following notation:

$$\mathrm{lat}\,\Lambda := \text{the set of isomorphism classes of left } \Lambda\text{-lattices.}$$

$$\mathrm{proj}\,\Lambda := \text{the subset of } \mathrm{Ind}\,\Lambda \text{ consisting of projective lattices.}$$

$$\mathrm{inj}\,\Lambda := \text{the subset of } \mathrm{Ind}\,\Lambda \text{ consisting of injective lattices.}$$

While, by abuse of notation '$X \in \mathrm{lat}\,\Lambda$' is often used to mean 'X is a left Λ-lattice', for example.

We adopt the convention that morphisms will be written on the right side of the object on which they operate.

2.0.1. We want to use the notation in which the dual statement is visible by the original one. The source map from $L \in \mathrm{Ind}\,\Lambda$ is written as $L \to \theta^- L$, the sink map to L is written as $\theta^+ L \to L$. The Auslander (resp. inverse Auslander) translate of L is written as τ^+L (resp. τ^-L). We put τ^-L (resp. $\tau^+L) := 0$ if L is injective (resp. projective). Thus, for example, if $L \notin \operatorname{inj} \Lambda$, the almost split sequence from L is written as $0 \to L \to \theta^- L \to \tau^- L \to 0$.

2.0.2. $\boldsymbol{Z}$(Ind Λ). The set lat Λ is a monoid by the direct sum $\oplus$. We had better work in the group of fractions of the monoid lat Λ. By Krull-Schmidt Theorem we can identify the group of fractions with the free $\boldsymbol{Z}$-module $\boldsymbol{Z}(\operatorname{Ind}\Lambda)$ generated by the base set $\operatorname{Ind}\Lambda$.

On $\boldsymbol{Z}(\operatorname{Ind}\Lambda)$, we introduce the *inner product* $\langle\, , \rangle$ taking $\operatorname{Ind}\Lambda$ as an orthonormal base, then identify lat Λ with the submonoid $\boldsymbol{N}(\operatorname{Ind}\Lambda)$ as

$$\operatorname{lat}\Lambda \ni X = \bigoplus_{L \in \operatorname{Ind}\Lambda} L^{\langle X, L\rangle} = \sum_{L \in \operatorname{Ind}\Lambda} \langle X, L\rangle L \in \boldsymbol{N}(\operatorname{Ind}\Lambda) \subset \boldsymbol{Z}(\operatorname{Ind}\Lambda),$$

where N denotes the submonoid $\{0, 1, 2, \ldots\}$ of $\boldsymbol{Z}$.

Any map ξ from $\operatorname{Ind}\Lambda$ to (a subset of) an abelian group C can be uniquely extended to a $\boldsymbol{Z}$-morphism $\boldsymbol{Z}(\operatorname{Ind}\Lambda) \to C$, which we denote by the same letter ξ. In particular $\theta^- : \operatorname{Ind}\Lambda \to \operatorname{lat}\Lambda \hookrightarrow \boldsymbol{Z}(\operatorname{Ind}\Lambda)$ defines $\theta^- \in \operatorname{End}_{\boldsymbol{Z}}(\boldsymbol{Z}(\operatorname{Ind}\Lambda))$. Using similar reading of τ^-, θ^+, and τ^+, we put

$$\phi^- := 1 - \theta^- + \tau^- \quad (X \mapsto X - \theta^- X + \tau^- X),$$

$$\phi^+ := 1 - \theta^+ + \tau^+ \quad (X \mapsto X - \theta^+ X + \tau^+ X).$$

These endomorphisms of $\boldsymbol{Z}(\operatorname{Ind}\Lambda)$ will play fundamental roles.

Any monoid homomorphism $\lambda : \operatorname{lat}\Lambda \to C$ to an abelian group C uniquely determines a $\boldsymbol{Z}$-morphism $\lambda : \boldsymbol{Z}(\operatorname{Ind}\Lambda) \to C$. Among such λ's, the following will be used in this paper.

$$\widetilde{(\;)} : \operatorname{lat}\Lambda \to \operatorname{mod}\tilde{\Lambda} \quad (X \mapsto \tilde{X} := K \otimes_R X).$$

For an irreducible central idempotent ε (i.e. $\tilde{\Lambda}\varepsilon$ is simple) of $\tilde{\Lambda}$,

$$\varepsilon : \operatorname{lat}\Lambda \to \operatorname{lat}(\Lambda\varepsilon) \quad (X \mapsto \varepsilon X).$$

For an overorder Γ of Λ,

$$\dot{(\;)} : \operatorname{lat}\Lambda \to \operatorname{lat}\Gamma \quad (X \mapsto \dot{X} := \Gamma X),$$

$$\underset{\cdot}{(\;)} : \operatorname{lat}\Lambda \to \operatorname{lat}\Gamma \quad (X \mapsto \underset{\cdot}{X} := \{x \in X \mid \Gamma x \subseteq X\}).$$

Note that $\underset{\cdot}{X} \simeq \operatorname{Hom}_\Lambda(\Gamma, X)$ is the maximum Γ- sublattice of X, while $\dot{X}$ is the minimum Γ-overlattice of X in $\tilde{X}$.

The rational length $l : \boldsymbol{Z}(\operatorname{Ind}\Lambda) \to \boldsymbol{Z}$ is the composite of $\widetilde{(\;)}$ and the length$_{\tilde{\Lambda}}$.

We shall also introduce an *ordering* in $\boldsymbol{Z}(\operatorname{Ind}\Lambda)$ by

$$X \leq Y \Leftrightarrow \langle X, L\rangle \leq \langle Y, L\rangle \text{ for any } L \in \operatorname{Ind}\Lambda.$$

Thus, if $X, Y \in \operatorname{lat}\Lambda$, then $X \leq Y \Leftrightarrow X \mid Y$ i.e. X is (isomorphic to) a summand of Y.

2.0.3. $\boldsymbol{Z}\mathscr{S}$. Let $\mathscr{S}$ be a non-empty subset of $\operatorname{Ind}\Lambda$. The inclusion $\mathscr{S} \subseteq \operatorname{Ind}\Lambda$ induces a $\boldsymbol{Z}$-monomorphism $i_{\mathscr{S}} : \boldsymbol{Z}\mathscr{S} \to \boldsymbol{Z}(\operatorname{Ind}\Lambda)$, by which we often identify as $\boldsymbol{Z}\mathscr{S} \subseteq \boldsymbol{Z}(\operatorname{Ind}\Lambda)$. Then $\boldsymbol{Z}(\operatorname{Ind}\Lambda - \mathscr{S})$ is the orthogonal complement of $\boldsymbol{Z}\mathscr{S}$ with respect to the inner product $\langle\ ,\ \rangle$. Let $p_{\mathscr{S}} : \boldsymbol{Z}(\operatorname{Ind}\Lambda) \to \boldsymbol{Z}\mathscr{S}$ denote the orthogonal projection.

For any $\xi \in \operatorname{End}_{\boldsymbol{Z}}(\boldsymbol{Z}(\operatorname{Ind}\Lambda))$, put $\xi_{\mathscr{S}} := p_{\mathscr{S}} \circ \xi \circ i_{\mathscr{S}} \in \operatorname{End}_{\boldsymbol{Z}}(\boldsymbol{Z}\mathscr{S})$. In particular, the maps $\phi^-_{\mathscr{S}}$, $\theta^-_{\mathscr{S}}$, $\tau^-_{\mathscr{S}}$ will play fundamental roles.

2.0.4. $\mathfrak{A}(\Lambda)$. The Auslander-Reiten quiver $\mathfrak{A}(\Lambda)$ of Λ is, by definition, a valued translation quiver with the vertex set $\operatorname{Ind}\Lambda$; the valued arrow $L \xrightarrow{(a_{LM}, a'_{LM})} M$ with $a_{LM} := \langle L, \theta^+ M\rangle, a'_{LM} := \langle \theta^- L, M\rangle$ (provided $a_{LM} \neq 0$); the translation $\tau := \tau^+$ i.e. $\tau L := \tau^+ L$ (provided $L \notin \operatorname{proj}\Lambda$). Moreover, $\mathfrak{A}(\Lambda)$ has the subadditive function l, the rational length. Whenever we regard $\mathscr{S}$ as a subset of the vertex set of $\mathfrak{A}(\Lambda)$, we consider $\mathscr{S}$ to be a full subquiver of $\mathfrak{A}(\Lambda)$.

Note that the endomorphism $\phi^-_{\mathscr{S}}$ *can be read from the full subquiver structure of* $\mathscr{S}$ in $\mathfrak{A}(\Lambda)$.

2.1. Complex. Let $\boldsymbol{A} = (A_l, a_l)$ denote a complex of Λ-lattices

$$\boldsymbol{A} : \cdots \to A_{l-1} \xrightarrow{a_{l-1}} A_l \xrightarrow{a_l} A_{l+1} \longrightarrow \cdots.$$

Since we write the action of Λ-morphism *from right*, the condition for $\boldsymbol{A}$ to make a complex is given by $a_{l-1}a_l = 0$ for any l. There is associated the group of homology $H^l(\boldsymbol{A}) := \ker a_l / \operatorname{im} a_{l-1}$. We also use an invariant $\chi_l(\boldsymbol{A}) := A_l - A_{l+1} \in \boldsymbol{Z}(\operatorname{Ind}\Lambda)$.

The direct sum $\boldsymbol{A} \oplus \boldsymbol{B}$ of the two complexes $\boldsymbol{A}$ and $\boldsymbol{B} = (B_l, b_l)$ is defined as $\boldsymbol{A} \oplus \boldsymbol{B} := \left(A_l \oplus B_l, \begin{pmatrix} a_l & 0 \\ 0 & b_l \end{pmatrix}\right)$. We obviously have $\chi_l(\boldsymbol{A} \oplus \boldsymbol{B}) = \chi_l(\boldsymbol{A}) + \chi_l(\boldsymbol{B})$.

Although we formulate the result in §3 for infinite complexes since it is a little smoother in this way, main results of this paper will be concerned only for finite complexes.

When we want to look a finite complex, say $0 \to X \to Y \to Z \to 0$ as an infinite complex $\boldsymbol{A}$ placing, say Y at n-th term we write as

$$\boldsymbol{A} : 0 \longrightarrow X \longrightarrow \overset{n}{\check{Y}} \longrightarrow Z \longrightarrow 0,$$

where we understand that all the other term A_l ($l \neq n-1, n, n+1$) is 0.

In particular, for $L \in \operatorname{Ind}\Lambda$, we define as

$$\boldsymbol{I}(\overset{n}{\check{L}}) : 0 \longrightarrow \overset{n}{\check{L}} \xrightarrow{1} L \longrightarrow 0$$

$$\boldsymbol{\Phi}^-(\overset{n}{\check{L}}) : 0 \longrightarrow \overset{n}{\check{L}} \xrightarrow{\nu} \theta^- L \xrightarrow{\mu} \tau^- L \longrightarrow 0$$

$$\boldsymbol{\Phi}^+(\overset{n}{\check{L}}) : 0 \longrightarrow \tau^+ L \longrightarrow \theta^+ L \longrightarrow \overset{n}{\check{L}} \longrightarrow 0.$$

For $T \in \operatorname{lat}\Lambda$, writing as $T = \Sigma L_i$ ($L_i \in \operatorname{Ind}\Lambda$), we define as

$$\boldsymbol{I}(\overset{n}{\check{T}}) := \bigoplus_i \boldsymbol{I}(\overset{n}{\check{L}_i}), \quad \boldsymbol{\Phi}^-(\overset{n}{\check{T}}) := \bigoplus_i \boldsymbol{\Phi}^-(\overset{n}{\check{L}_i}), \quad \boldsymbol{\Phi}^+(\overset{n}{\check{T}}) := \bigoplus_i \boldsymbol{\Phi}^+(\overset{n}{\check{L}_i}).$$

They are uniquely determined by T up to isomorphism of complexes.

As usual, we call that a complex $A = (A_l, a_l)$ is exact at n-th term (or at A_n) if $H^n(A) = 0$, we call that A is exact if it is exact at every term. While we call that A is *rationally exact* at n-th term (or at A_n) if $H^n(A)$ is an R-torsion module.

2.1.1. (1) $\Phi^-(\overset{n}{\check{T}})$ is rationally exact at $(n+1)$-th term $\theta^- T$, exact at every other term. $\Phi^+(\overset{n}{\check{T}})$ is rationally exact at n-th term T, exact at every other term.

(2) If T has no injective (resp. projective) summands, then $\Phi^-(\overset{n}{\check{T}})$ (resp. $\Phi^+(\overset{n}{\check{T}})$) is exact.

2.1.2. For a given complex $A = (A_l, a_l)$ of Λ-lattices, by tensoring $K \otimes_R$, there arises a complex $\tilde{A} = (\tilde{A}_l, \tilde{a}_l)$ of $\tilde{\Lambda}$-modules. Similarly for an overorder Γ, there arises a complex $\dot{A} = (\dot{A}_l, \dot{a}_l)$ with $\dot{A}_l := \Gamma A_l$, and $\dot{a}_l : \dot{A}_l \to \dot{A}_{l+1}$ is the unique extension of a_l. Obviously, we have chain homomorphism $A \to \dot{A} \to \tilde{A}$ induced from natural inclusions, and

(1) If A is rationally exact at A_n, then $\dot{A}$ is rationally exact at $\dot{A}_n$.

(2) If A is rationally exact at A_n, then $\tilde{A}$ is exact at $\tilde{A}_n$. (This is the reason why we used 'rationally'.)

2.2. Rationally exactness. *Let $A : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a complex of Λ-lattices, which is rationally exact at Y and exact at Z.*

(1) *Let $h : Y \to W$ be a Λ-morphism such that $fh = 0$. Then there is a unique Λ-morphism $\bar{h} : Z \to W$ such that $h = g\bar{h}$.*

(2) *Let $i : T \to Y$ be a split monomorphism, and let $\alpha : Y \to Y$ be a Λ-morphism. If $ig \in \operatorname{rad}(T, Z)$ and $f = f\alpha$, then $i\alpha : T \to Y$ is a split monomorphism.*

(3) *Let $B : 0 \to X \xrightarrow{f'} Y' \xrightarrow{g'} Z' \to 0$ be a complex of Λ-lattices, which is rationally exact at Y' and exact at Z'. Let $\rho : Y \to Y'$, $\sigma : Y' \to Y$ be Λ-morphisms such that $f\rho = f'$, $f'\sigma = f$. Then there are Λ-morphisms $\bar{\rho} : Z \to Z'$, $\bar{\sigma} : Z' \to Z$ such that $g\bar{\rho} = \rho g'$, $g'\bar{\sigma} = \sigma g$. If $\sigma\rho \in \operatorname{Aut}(Y')$, then $\bar{\sigma}\bar{\rho} \in \operatorname{Aut}(Z')$ and $\ker \rho \simeq \ker \bar{\rho}$.*

(4) *If f is a split monomorphism, then A is a split exact sequence.*

(5) *If A is exact at X and ε is a central idempotent of $\tilde{\Lambda}$, then $l(X - Y + Z) = l(\varepsilon(X - Y + Z)) = 0$.*

Proof. (1) We shall show that $\ker g \subseteq \ker h$. Pick any $y \in \ker g$. Rationally exactness at Y means that $y = \pi^{-i}(xf)$ by some $x \in X$, so that $yh = \pi^{-i}(xfh) = 0$.

(2) By assumption, $f(\alpha - 1) = 0$. By (1), there is $\gamma : Z \to Y$ such that $\alpha - 1 = g\gamma$. Hence, $i\alpha - i = ig\gamma \in \operatorname{rad}(T, Y)$, so that $i\alpha = i + ig\gamma \in i + \operatorname{rad}(T, Y)$ is a split monomorphism.

(3) The first assertion is obvious from (1).

We assume $\sigma\rho \in \operatorname{Aut}(Y')$. Put $\mu := (\sigma\rho)^{-1}$, then (1) shows that there is $\bar{\mu}$ such that $g'\bar{\mu} = \mu g'$. Since $g'\bar{\mu}\bar{\sigma}\bar{\rho} = \mu g'\bar{\sigma}\bar{\rho} = \mu\sigma\rho g' = g'$ and $g'\bar{\sigma}\bar{\rho}\bar{\mu} = g'$, the second assertion follows.

We shall show that $g|_{\ker \rho}$ induces $\ker \rho \simeq \ker \bar{\rho}$. Obviously $(\ker \rho)g \subseteq \ker \bar{\rho}$. If $y \in \ker(g|_{\ker \rho})$, then we can put $y = \pi^{-i}(xf)$. By $y = \pi^{-i}xf = \pi^{-i}xf'\sigma = \pi^{-i}xf\rho\sigma = y\rho\sigma = 0$, $g|_{\ker \rho}$ is a monomorphism. On the other hand, for all $z \in \ker \bar{\rho}$, exactness of A

at Z shows that we can put $z = yg$. Then we have $(y - y\rho\mu\sigma)\rho = y\rho - y\rho = 0$ and $(y - y\rho\mu\sigma)g = z - yg\bar{\rho}\bar{\mu}\bar{\sigma} = z - z\bar{\rho}\bar{\mu}\bar{\sigma} = z$. Hence $g|_{\ker\rho}$ is an epimorphism.

(4) We shall show that $\ker g \subseteq \operatorname{im} f$. Pick any $y \in \ker g$. There is some $x \in X$ with $y = \pi^{-i}(xf)$. Applying a retraction $\sigma : Y \to X$, $f\sigma = 1$, we get $X \ni y\sigma = \pi^{-i}x$, i.e. $y \in \operatorname{im} f$.

(5) Obviously $\tilde{A} : 0 \to \tilde{X} \to \tilde{Y} \to \tilde{Z} \to 0$ is a split exact sequence of $\tilde{\Lambda}$-modules. Hence $\tilde{X} - \tilde{Y} + \tilde{Z} = 0$ and $\varepsilon(\tilde{X} - \tilde{Y} + \tilde{Z}) = 0$. ■

2.3. Let $\boldsymbol{A} = (A_l, a_l)$ be a complex of Λ-lattices. If A_n has a summand $T \in \operatorname{lat}\Lambda$ $(T \leq A_n)$ such that the restriction $a_n|_T$ is a split monomorphism. Then $\boldsymbol{A}$ decomposes as $\boldsymbol{A} = \boldsymbol{B} \oplus \boldsymbol{I}(\overset{n}{\check{T}})$ up to isomorphism of complexes.

Although it is obvious, we give one explicit split exact sequence which helps to prove our main lemma 3.2.

Write as $A_n =: B_n \oplus T$, $A_{n+1} =: B_{n+1} \oplus T$, $a_n =: \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ ($\delta : T \to T$ is an automorphism) etc.. Putting $B_l := A_l$ $(l \neq n, n+1)$, the following commutative diagram gives the split exact sequence of complexes $0 \to \boldsymbol{I}(\overset{n}{\check{T}}) \to \boldsymbol{A} \to \boldsymbol{B} \to 0$.

$$\begin{tikzcd}
\boldsymbol{I}(\overset{n}{\check{T}}) : \arrow[r] & 0 \arrow[r] \arrow[d] & T \arrow[r, "1"] \arrow[d, "(0\,1)"] & T \arrow[r] \arrow[d, "(\gamma\,\delta)"] & 0 \arrow[r] \arrow[d] & {} \\
\boldsymbol{A} : \arrow[r] & A_{n-1} \arrow[r, "(b_{n-1}\,c_{n-1})"] \arrow[d, "1"] & B_n \oplus T \arrow[r, "\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & B_{n+1} \oplus T \arrow[r, "\left(\begin{smallmatrix} b_{n+1} \\ c_{n+1} \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ -\delta^{-1}\gamma \end{smallmatrix}\right)"] & A_{n+2} \arrow[r] \arrow[d, "1"] & {} \\
\boldsymbol{B} : \arrow[r] & B_{n-1} \arrow[r, "b_{n-1}"] & B_n \arrow[r, "\alpha-\beta\delta^{-1}\gamma"] & B_{n+1} \arrow[r, "b_{n+1}"] & B_{n+2} \arrow[r] & {}
\end{tikzcd}$$

Here, a retraction is given as follows.

$$\begin{tikzcd}
\boldsymbol{I}(\overset{n}{\check{T}}) : \arrow[r] & 0 \arrow[r] & T \arrow[r, "1"] & T \arrow[r] & 0 \arrow[r] & {} \\
\boldsymbol{A} : \arrow[r] & A_{n-1} \arrow[r, "(b_{n-1}\,c_{n-1})"] \arrow[u, "0"] & B_n \oplus T \arrow[r, "\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)"] \arrow[u, "\left(\begin{smallmatrix} \beta\delta^{-1} \\ 1 \end{smallmatrix}\right)"] & B_{n+1} \oplus T \arrow[r, "\left(\begin{smallmatrix} b_{n+1} \\ c_{n+1} \end{smallmatrix}\right)"] \arrow[u, "\left(\begin{smallmatrix} 0 \\ \delta^{-1} \end{smallmatrix}\right)"] & A_{n+2} \arrow[r] \arrow[u, "0"] & {}
\end{tikzcd}$$

2.4. Cofaithfulness. *For a subset $\mathcal{S}$ of $\operatorname{Ind}\Lambda$, the following conditions are mutually equivalent.*

(0) *$\mathcal{S}$ is cofaithful (i.e. $\Lambda(\mathcal{S})$ is an order in $\tilde{\Lambda}$).*

(1) *$\bigoplus_{L \in \operatorname{Ind}\Lambda - \mathcal{S}} L$ is a faithful Λ-module.*

(2) *$\bigoplus_{L \in \operatorname{Ind}\Lambda - \mathcal{S}} \tilde{L}$ is a faithful $\tilde{\Lambda}$-module.*

(3) *For any irreducible central idempotent ε of $\tilde{\Lambda}$ (i.e. $\tilde{\Lambda}\varepsilon$ is simple), there is some $L \in \operatorname{Ind}\Lambda - \mathscr{S}$ such that $\varepsilon L \neq 0$.*

(4) *For any $X, Y \in \operatorname{lat}\Lambda$, there is a Λ-lattice Z without summands in $\mathscr{S}$ such that $\operatorname{Hom}_\Lambda(X, Y) \supseteq \operatorname{Hom}_\Lambda(X, Z)\operatorname{Hom}_\Lambda(Z, Y) \supseteq \pi^a \operatorname{Hom}_\Lambda(X, Y)$ by some $a \geq 0$.*

(5) *There is a faithful Λ-lattice Z without summands in $\mathscr{S}$.*

(6) *There is a Λ-lattice W without summands in $\mathscr{S}$ and there is an exact sequence of Λ-lattices $0 \to \Lambda(\mathscr{S}) \to W \to Y \to 0$.*

Proof. (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4): Obvious since $\tilde{\Lambda}$ is semi-simple.

(4) $\Rightarrow$ (5): Take $X = Y = \Lambda$ in (4), then Z must be Λ-faithful.

(5) $\Rightarrow$ (6): $O_l(Z)$ is an order in $\tilde{\Lambda}$ such that $O_l(Z) \supseteq \Lambda(\mathscr{S}) \supseteq \Lambda$. Hence $O_l(Z)/\Lambda(\mathscr{S})$ is artinian, we can take $X = Z \oplus L_1 \oplus \cdots \oplus L_m$ ($L_i \in \operatorname{Ind}\Lambda - \mathscr{S}$) such that $O_l(X) = \Lambda(\mathscr{S})$.

Put $E := \operatorname{End}_\Lambda X$. Then X is a finitely presented E-module, $E^m \to E^n \to X \to 0$. Taking $\operatorname{Hom}_E(\,, X)$, we get

$$0 \to \operatorname{Hom}_E(X, X) \to \operatorname{Hom}_E(E^n, X) \to \operatorname{Hom}_E(E^m, X).$$

Because X is Λ-faithful, $\operatorname{Hom}_E(X, X) = \Lambda(\mathscr{S})$ and $\operatorname{Hom}_E(E^n, X) = X^n$. We may take as $W := X^n$, $Y := \operatorname{im}(X^n \to X^m)$.

(6) $\Rightarrow$ (0): $\Lambda(\mathscr{S}) \hookrightarrow W$ means that $\Lambda(\mathscr{S})$ is an R-lattice, hence an order.

(0) $\Rightarrow$ (2): Suppose (2) does not hold so that $\bigcap_{L \in \operatorname{Ind}\Lambda - \mathscr{S}} \operatorname{Ann}\tilde{L} \neq 0$, then $\Lambda(\mathscr{S}) \supseteq \bigcap_{L \in \operatorname{Ind}\Lambda - \mathscr{S}} \operatorname{Ann}\tilde{L}$ is not an order. ∎

2.4.1. Let $\mathscr{S}$ be a non-cofaithful subset of $\operatorname{Ind}\Lambda$. Then there is an irreducible central idempotent ε of $\tilde{\Lambda}$ such that

$$\mathscr{S} \supseteq \operatorname{Ind}_\varepsilon \Lambda := \{L \in \operatorname{Ind}\Lambda \mid \varepsilon L \neq 0\}.$$

$\mathscr{S}$ contains at least one projective, at least one injective and at least one irreducible Λ-lattice.

Proof. The first assertion is (3) 2.4. The second assertion is an obvious consequence of the first. ∎

2.4.2. Let $\mathscr{S}$ be a subset of $\operatorname{Ind}\Lambda$ and $\mathscr{S}_p := \mathscr{S} \cap \operatorname{proj}\Lambda$, $\mathscr{S}_i := \mathscr{S} \cap \operatorname{inj}\Lambda$. Suppose $\mathscr{S}_p = \emptyset$ or $\mathscr{S}_i = \emptyset$. Then $\Lambda = \Lambda(\mathscr{S})$, i.e. $\mathscr{S}$ is trivial.

3. Deformation of complexes.

Let $A = (A_l, a_l)$ be a complex of Λ-lattices and T a non-zero Λ-lattice. Assume that, up to the end of 3.2, there is a split monomorphism $i : T \to A_n$ satisfying the following property:

(0) $ia_n \in \operatorname{rad}(T, A_{n+1})$,

where $\operatorname{rad}(T, A_{n+1})$ is defined by the same way as **[R]** 2.5.

3.1. There is a chain homomorphism $u : \Phi^-(\overset{n}{\check{T}}) \to A$ given by the following commutative diagram.

$$\begin{tikzcd}
\Phi^-(\overset{n}{\check{T}}) : \arrow[d, "u"] \arrow[r] & 0 \arrow[r] \arrow[d] & T \arrow[r, "\nu"] \arrow[d, "i"] & \theta^- T \arrow[r, "\mu"] \arrow[d, "\rho"] & \tau^- T \arrow[r] \arrow[d, "\sigma"] & 0 \arrow[r] \arrow[d] & {} \\
A : \arrow[r] & A_{n-1} \arrow[r, "a_{n-1}"] & A_n \arrow[r, "a_n"] & A_{n+1} \arrow[r, "a_{n+1}"] & A_{n+2} \arrow[r, "a_{n+2}"] & A_{n+3} \arrow[r] & {}
\end{tikzcd}$$

PROOF. Since $\nu : T \to \theta^- T$ is a (direct sum of) source map, by the assumption (0), there is ρ. By 2.1.1, we can apply (1) 2.2 and conclude that ρ induces σ. ■

3.2. Main Lemma. *Identifying* $\operatorname{im} i$ *with* T, *we write as* $A_n = A \oplus T$. *Define* $f_l : A_l \to A'_l$, $a'_l : A'_l \to A'_{l+1}$ *as* $A_l = A'_l$ $(l \neq n,\ n+1)$, $a_l = a'_l$ $(l \neq n-1, n, n+1)$, $f_l = 1$ $(l \neq n,\ n+1)$ *and the remaining ones as in the diagram below.*

$$\begin{tikzcd}[column sep=large]
A : \arrow[d, "f"] \arrow[r] & A_{n-1} \arrow[r, "(\alpha\ \beta)"] \arrow[d, "1"] & A \oplus T \arrow[r, "\begin{pmatrix} \gamma \\ \nu\rho \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix}"] & A_{n+1} \arrow[r, "a_{n+1}"] \arrow[d, "(1\ 0)"] & A_{n+2} \arrow[r] \arrow[d, "1"] & {} \\
A' : \arrow[r] & A_{n-1} \arrow[r, "(\alpha\ \beta\nu)"] & A \oplus \theta^- T \arrow[r, "\begin{pmatrix} \gamma & 0 \\ \rho & -\mu \end{pmatrix}"] & A_{n+1} \oplus \tau^- T \arrow[r, "\begin{pmatrix} a_{n+1} \\ \sigma \end{pmatrix}"] & A_{n+2} \arrow[r] & {}
\end{tikzcd}$$

Then we have the following.

(r0) $A' := (A'_l, a'_l)$ *is a complex of* Λ*-lattices and* $f : A \to A$ *is a chain homomorphism.*

(r1) $\chi_n(A') = A'_n - A'_{n+1} = A_n - A_{n+1} - \phi^- T = \chi_n(A) - \phi^- T$.

(r2) *If* $X \in \operatorname{lat} \Lambda$ *and* $\langle X, T \rangle = 0$, *then* $a_{n-1} \operatorname{Hom}_\Lambda(A_n, X) = a'_{n-1} \operatorname{Hom}_\Lambda(A'_n, X)$.

(r3) *There are exact sequences of homology*:

$$0 \to H^n(A) \to H^n(A') \to \ker \mu / \operatorname{im} \nu \to H^{n+1}(A) \to H^{n+1}(A') \to 0$$

$$0 \to H^l(A) \to H^l(A') \to 0 \quad (l \neq n, n+1)$$

DEFINITION. Thus obtained A' (to be precise the chain morphism $f : A \to A'$) will be called the complex obtained from A by *rejecting* T *from* A_n, and will be denoted as

$$A' = A - \overset{n}{\check{T}}.$$

PROOF. (r0) Immediate from the diagram.

(r1) $A'_n - A'_{n+1} = (A + \theta^- T) - (A_{n+1} + \tau^- T) = A_n - A_{n+1} - \phi^- T$.

(r2) Since $a'_{n-1} = a_{n-1} f_n$, $a'_{n-1} \operatorname{Hom}_\Lambda(A'_n, X) \subseteq a_{n-1} \operatorname{Hom}_\Lambda(A_n, X)$. We shall show the opposite inclusion. For a given $h : A_n \to X$, according to the decomposition $A_n = A \oplus T$, write as $h = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$. Then, since $\langle X, T \rangle = 0$, $\eta \in \operatorname{rad}(T, X)$ so that there is $\zeta : \theta^- T \to X$ such that $\eta = \nu\zeta$. The claim follows from the commutative diagram below.

$$\begin{tikzcd}
A'_{n-1} = A_{n-1} \arrow[r, "a_{n-1}"] \arrow[d, "a'_{n-1}"'] & A \oplus T = A_n \arrow[d, "\begin{pmatrix}\xi\\ \eta\end{pmatrix} = h"] \arrow[dl, "\begin{pmatrix}1&0\\0&\nu\end{pmatrix}\ \begin{pmatrix}\xi\\ \zeta\end{pmatrix}"'] \\
A'_n = A \oplus \theta^- T \arrow[r] & X
\end{tikzcd}$$

(r3) We will prove this assertion in the following paragraphs. ■

3.2.1. In general, let $u : \boldsymbol{B} \to \boldsymbol{A}$ be a chain morphism of complexes of Λ-lattices. Then arises a commutative diagram.

$$\begin{tikzcd}
\arrow[r] & A_{l-1} \arrow[r, "a_{l-1}"] \arrow[d, "(1\,0)"] & A_l \arrow[r, "a_l"] \arrow[d, "(1\,0)"] & A_{l+1} \arrow[r] \arrow[d, "(1\,0)"] & {} \\
\arrow[r] & A_{l-1} \oplus B_l \arrow[r, "\begin{pmatrix}a_{l-1} & 0\\ u_l & -b_l\end{pmatrix}"] \arrow[d, "\begin{pmatrix}0\\1\end{pmatrix}"] & A_l \oplus B_{l+1} \arrow[r, "\begin{pmatrix}a_l & 0\\ u_{l+1} & -b_{l+1}\end{pmatrix}"] \arrow[d, "\begin{pmatrix}0\\1\end{pmatrix}"] & A_{l+1} \oplus B_{l+2} \arrow[r] \arrow[d, "\begin{pmatrix}0\\1\end{pmatrix}"] & {} \\
\arrow[r] & B_l \arrow[r, "-b_l"] & B_{l+1} \arrow[r, "-b_{l+1}"] & B_{l+2} \arrow[r] & {}
\end{tikzcd}$$

In other words we have new complexes $\boldsymbol{B}_- := (B_{l+1}, -b_{l+1})$, $\boldsymbol{C} := \left(A_l \oplus B_{l+1}, \begin{pmatrix} a_l & 0 \\ u_{l+1} & -b_{l+1} \end{pmatrix}\right)$ and a short exact sequence of complexes

$$0 \longrightarrow \boldsymbol{A} \xrightarrow{(1\,0)} \boldsymbol{C} \xrightarrow{\begin{pmatrix}0\\1\end{pmatrix}} \boldsymbol{B}_- \longrightarrow 0$$

which induces a long exact sequence of homology,

$$\longrightarrow H^l(\boldsymbol{A}) \longrightarrow H^l(\boldsymbol{C}) \longrightarrow H^l(\boldsymbol{B}_-) = H^{l+1}(\boldsymbol{B}) \longrightarrow H^{l+1}(\boldsymbol{A}) \longrightarrow .$$

3.2.2. We apply 3.2.1 to $u : \Phi^-(\overset{n}{\check{T}}) \to \boldsymbol{A}$, and write down the map $\boldsymbol{A} \xrightarrow{(1\,0)} \boldsymbol{C}$ around $l = n$, according to the decomposition $A_n = A \oplus T$.

$$\begin{tikzcd}[column sep=small]
\boldsymbol{A} : \arrow[r] \arrow[d, "(1\,0)"] & A_{n-2} \arrow[r, "a_{n-2}"] \arrow[d, "1"] & A_{n-1} \arrow[r, "(\alpha\ \beta)"] \arrow[d, "(1\,0)"] & A \oplus T \arrow[r, "\begin{pmatrix}\gamma\\ \nu\rho\end{pmatrix}"] \arrow[d, "\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}"] & A_{n+1} \arrow[r, "a_{n+1}"] \arrow[d, "(1\,0)"] & A_{n+2} \arrow[r] \arrow[d, "1"] & {} \\
\boldsymbol{C} : \arrow[r] & A_{n-2} \arrow[r, "(a_{n-2}\,0)"] & A_{n-1} \oplus T \arrow[r, "\begin{pmatrix}\alpha&\beta&0\\0&1&-\nu\end{pmatrix}"] & A \oplus T \oplus \theta^- T \arrow[r, "\begin{pmatrix}\gamma&0\\ \nu\rho&0\\ \rho&-\mu\end{pmatrix}"] & A_{n+1} \oplus \tau^- T \arrow[r, "\begin{pmatrix}a_{n+1}\\ \sigma\end{pmatrix}"] & A_{n+2} \arrow[r] & {}
\end{tikzcd}$$

Here we have $C_{n-1} = A_{n-1} \oplus T$ and $c_{n-1}|_T = (0 \quad 1 \quad -\nu)$ is a split monomorphism. By 2.3, we have a split exact sequence $0 \to \boldsymbol{I}(\overset{n-1}{\check{T}}) \to \boldsymbol{C} \xrightarrow{p} \boldsymbol{A}' \to 0$.

As is easily seen, the composite $(1\,0)p : \boldsymbol{A} \to \boldsymbol{A}'$ coincides with $f : \boldsymbol{A} \to \boldsymbol{A}'$ of 3.2.

Since $H^l(\boldsymbol{C}) = H^l(\boldsymbol{A}') \oplus H^l(\boldsymbol{I}(\overset{n-1}{\check{T}})) = H^l(\boldsymbol{A}')$, we get (r3) from the long exact sequence in 3.2.1. ■

3.2.3. Remark. It is not hard to see that the chain map $f : \boldsymbol{A} \to \boldsymbol{A}'$ (in 3.2, constructed from $i : T \to A_n$) has the following properties and is characterized uniquely up to isomorphism in the category of complexes of Λ-lattices by the properties

(1) $if_n \in \mathrm{rad}(T, A'_n)$.

(2) If $g : \boldsymbol{A} \to \boldsymbol{B}$ satisfies $ig_n \in \mathrm{rad}(T, B_n)$, then there is $h : \boldsymbol{A}' \to \boldsymbol{B}$ such that $g = fh$.

(3) f is left minimal.

In other words, $f : \boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} \overset{n}{-} \check{T}$ is a sort of substitute for 'source map' in the category of complexes of Λ-lattices.

3.3. Duality. We explain the dual version of the above lemma. Namely, it is obviously valid for right Λ-lattices (i.e. for $\mathrm{lat}\,\Lambda^{op}$) by the same proof. Taking the R-dual $(\ )^* := \mathrm{Hom}_R(\ , R)$, then we get the dual result. However, in this paper, it is more economical to take the dual for final results in § 5, so that we refrain from writing down the dual of 3.2. Instead we give a dictionary of duals here. The map

$$(\ )^* : \boldsymbol{Z}(\mathrm{Ind}\,\Lambda) \to \boldsymbol{Z}(\mathrm{Ind}\,\Lambda^{op}), \quad X \mapsto X^*$$

is a $\boldsymbol{Z}$-isomorphism compatible with the inner product $\langle\ ,\ \rangle$, which induces a bijection $\mathrm{proj}\,\Lambda \to \mathrm{inj}\,\Lambda^{op}$, $\mathrm{inj}\,\Lambda \to \mathrm{proj}\,\Lambda^{op}$. Endomorphism θ^-, τ^-, ϕ^- corresponds to θ^+, τ^+, ϕ^+. For an overorder Γ of Λ, $\dot{(\ )}$ corresponds to $\underset{\cdot}{(\ )}$ in the obvious sense, for examples

$$(\theta^- L)^* = \theta^+(L^*), (\dot{L})^* = (\underset{\cdot}{L^*}) \text{ etc.}$$

3.4. Successive $\mathscr{S}^-$-rejection sequence. Let $\mathscr{S}$ be a subset of $\mathrm{Ind}\,\Lambda$. A (finite or infinite) sequence $(\boldsymbol{A}^{(0)}, \boldsymbol{A}^{(1)}, \boldsymbol{A}^{(2)}, \ldots)$ of complexes of Λ-lattices will be called a *successive $\mathscr{S}^-$-rejection sequence with the initial complex* $\boldsymbol{A}$, if there is a sequence (T_j) with $0 \neq T_j \in N\mathscr{S}$, by which $\boldsymbol{A}^{(j)}$ is defined inductively as $\boldsymbol{A}^{(0)} := \boldsymbol{A}, \boldsymbol{A}^{(j+1)} := \boldsymbol{A}^{(j)} \overset{n}{-} \check{T}_j$.

Where n is an arbitrarily chosen (then fixed) integer. As will be seen in the sequel, the choice of n has not much meaning, so that the reference to n will be omitted.

Of cource, there is implicitly associated a chain homomorphism $f^{(j)} : \boldsymbol{A}^{(j)} \to \boldsymbol{A}^{(j+1)}$ for each j. If that is so, by 3.2, they enjoy the following properties.

(sr0) $A_l^{(j)} = A_l$ $(l \neq n, n+1)$.

(sr1) $\chi_n(\boldsymbol{A}^{(j)}) = A_n^{(j)} - A_{n+1}^{(j)} = \chi_n(\boldsymbol{A}^{(j-1)}) - \phi^- T_{j-1} = \chi_n(\boldsymbol{A}) - \sum_{i=0}^{j-1} \phi^- T_i$.

(sr2) If $X \in \mathrm{lat}\,\Lambda$ and $p_{\mathscr{S}} X = 0$ (i.e. $\langle X, L \rangle = 0$ for any $L \in \mathscr{S}$), then

$$a_{n-1}^{(j)} \mathrm{Hom}_\Lambda(A_n^{(j)}, X) = a_{n-1} \mathrm{Hom}_\Lambda(A_n, X).$$

(sr3) If $\boldsymbol{A}$ is exact at A_{n+1}, then $\boldsymbol{A}^{(j)}$ is exact at $A_{n+1}^{(j)}$.
If $\boldsymbol{A}$ is rationally exact at A_n, then $\boldsymbol{A}^{(j)}$ is rationally exact at $A_n^{(j)}$.
If $\boldsymbol{A}$ is exact at A_n and moreover $\mathscr{S}$ contains no injectives, then $\boldsymbol{A}^{(j)}$ is exact at $A_n^{(j)}$.

3.4.1. Suppose that the initial complex $\boldsymbol{A}$ is three termed finite, $\boldsymbol{A} : 0 \to A_{n-1} \to A_n \to A_{n+1} \to 0$, which is rationally exact at A_n and exact at A_{n+1}. Then for any $j \geq 0$,

$$a_{n-1}^{(j)} \notin a_{n-1}^{(j+1)} \mathrm{Hom}_\Lambda(A_n^{(j+1)}, A_n^{(j)}).$$

Proof. Supposing to be contrary, since we have $\alpha_j : A_n^{(j+1)} \to A_n^{(j)}$ such that $a_{n-1}^{(j)} = a_{n-1}^{(j+1)}\alpha_j$, so $a_{n-1}^{(j)} = a_{n-1}^{(j)} f_n^{(j)} \alpha_j$. For any $j \geq 1$, (sr3) assures that $A^{(j)}$ is rationally exact at n-th term, exact at $(n+1)$-th term. Applying (2) 2.2 to the split monomorphism $i : T_j \to A_n^{(j)}$, we conclude that $if_n^{(j)}\alpha_j$ (hence $if_n^{(j)}$) is a split monomorphism. But this is impossible since, in view of 3.2, $if_n^{(j)}$ should have the form

$$if_n^{(j)} : T_j \xrightarrow{(0\, v)} A \oplus \theta^- T_j$$

with a source map $v : T_j \to \theta^- T_j$. ■

3.4.2. Suppose that the initial complex A is $\boldsymbol{I}(\overset{n-1}{\check{A}})$. Let Γ be an overorder of Λ such that $\mathscr{S} \cap \operatorname{Ind}\Gamma = \emptyset$. Then, the induced Γ-complex $\dot{A}^{(j)}$ is split exact for any $j \geq 1$.

Proof. Let $\iota_l : A_l^{(j)} \to \dot{A}_l^{(j)}$ be natural inclusions. (sr2) assures $a_{n-1}^{(j)} \operatorname{Hom}_\Lambda(A_n^{(j)}, \dot{A}) = \operatorname{Hom}_\Lambda(A, \dot{A}) \ni \iota_{n-1}$, hence ι_{n-1} decomposes as $a_{n-1}^{(j)} f$. We have the following diagram.

$$\begin{tikzcd}
A^{(j)} : 0 \arrow[r] & A \arrow[r, "a_{n-1}^{(j)}"] \arrow[d, "\iota_{n-1}"'] & A_n^{(j)} \arrow[r, "a_n^{(j)}"] \arrow[d, "\iota_n"] \arrow[dl, "f"'] & A_{n+1}^{(j)} \arrow[r] \arrow[d, "\iota_{n+1}"] & 0 \\
\dot{A}^{(j)} : 0 \arrow[r] & \dot{A} \arrow[r, "\dot{a}_{n-1}^{(j)}"] & \dot{A}_n^{(j)} \arrow[r, "\dot{a}_n^{(j)}"] & \dot{A}_{n+1}^{(j)} \arrow[r] & 0
\end{tikzcd}$$

Then f decomposes as $\iota_n \dot{f}$, and $\iota_{n-1}\dot{a}_{n-1}^{(j)}\dot{f} = a_{n-1}^{(j)}\iota_n\dot{f} = \iota_{n-1}$. Since $\tilde{\iota}_{n-1}$ is the identity map of $\tilde{A}$, we get $\dot{a}_{n-1}^{(j)}\dot{f} = 1$, i.e. $\dot{a}_{n-1}^{(j)}$ is a split monomorphism. By (sr3) and (4) 2.2, $\dot{A}^{(j)}$ is a split exact sequence of Γ-lattices. ■

3.5. Let A be a complex of Λ-lattices and $0 \neq T \in \operatorname{lat}\Lambda$. Assume that

$$\sup\{0, \langle \chi_n(A), L\rangle\} \geq \langle T, L\rangle \text{ for any } L \in \operatorname{Ind}\Lambda.$$

Then there is a split monomorphism $i : T \to A_n$ such that $ia_n \in \operatorname{rad}(T, A_{n+1})$.

Proof. Pick an indecomposable summand L of T, and put $t := \langle T, L\rangle > 0$, $r := \langle A_{n+1}, L\rangle \geq 0$. The assumption implies that $\langle A_n, L\rangle = m + r$ with some $m \geq t$. The L-homogeneous part of $a_n : A_n \to A_{n+1}$ can be represented as $(m+r) \times r$ matrix over the local ring $\operatorname{End}_\Lambda L$. Since $m \geq t$, the claim is now obvious. ■

3.6. $\mathscr{S}^-$**-sequence.** Let $\mathscr{S}$ be a subset of $\operatorname{Ind}\Lambda$ and $V \in \boldsymbol{Z}(\operatorname{Ind}\Lambda)$. A (finite or infinite) sequence $(T_j) = (T_0, T_1, \ldots)$ will be called an *$\mathscr{S}^-$-sequence for V* if the following two conditions are satisfied for any j.

(1) $0 \neq T_j \in \boldsymbol{N}\mathscr{S}$.

(2) $\sup\left\{0, \langle V - \sum_{i=0}^{j-1} \phi^- T_i, L\rangle\right\} \geq \langle T_j, L\rangle$ for any $L \in \operatorname{Ind}\Lambda$.

Moreover, put $V_0 := V$, $V_j := V_{j-1} - \phi^- T_{j-1}$ $(j \geq 1)$. Then the sequence $(V_j) = (V_0, V_1, \ldots)$ (in $\boldsymbol{Z}(\operatorname{Ind}\Lambda)$) will be called the *associated $\mathscr{S}^-$-rejected sequence for V*. In terms of (V_j), (2) is written as

(2′) $\sup\{0, \langle V_j, L\rangle\} \geq \langle T_j, L\rangle$ for any $L \in \operatorname{Ind}\Lambda$.

Hence, in view of 3.4 and 3.5, we have the followings.

3.6.1. Let (T_j) be an $\mathscr{S}^-$-sequence for V and $\boldsymbol{A}$ a complex of Λ-lattices with $\chi_n(\boldsymbol{A}) = A_n - A_{n+1} = V$. Then there arises a successive $\mathscr{S}^-$-rejection sequence of complexes $(\boldsymbol{A}^{(0)}, \boldsymbol{A}^{(1)}, \dots)$, $\boldsymbol{A}^{(0)} := \boldsymbol{A}, \boldsymbol{A}^{(j)} := \boldsymbol{A}^{(j-1)} - \overset{n}{\check{T}}_{j-1}$ with $\chi_n(\boldsymbol{A}^{(j)}) = V_j$.

3.6.2. Under (1), the condition (2) is equivalent with
$(2)_{\mathscr{S}}$ $\sup\{0, \langle p_{\mathscr{S}} V - \sum_{i=0}^{j-1} \phi_{\mathscr{S}}^- T_i, L\rangle\} \geq \langle T_j, L\rangle$ for any $L \in \mathscr{S}$.
This means that the condition for (T_j) to be an $\mathscr{S}^-$-sequence for V is read from the full subquiver structure of $\mathscr{S}$ in $\mathfrak{A}(\Lambda)$.
Let (T_j) be an $\mathscr{S}^-$-sequence for V. By $(2)_{\mathscr{S}}$ and definitions, we have
(A) If $p_{\mathscr{S}} V' \geq p_{\mathscr{S}} V$, then (T_j) is an $\mathscr{S}^-$-sequence for V'.
(B) If $\mathscr{S}' \supseteq \mathscr{S}$, then (T_j) is an $(\mathscr{S}')^-$-sequence for V.

3.6.3. $[\mathscr{S}]^- V$ and $\{\mathscr{S}\}^- V$. For a given $V \in \boldsymbol{Z}(\operatorname{Ind}\Lambda)$, let $[\mathscr{S}]^- V$ denote the set consisting of all members of (some) finite $\mathscr{S}^-$-rejected sequence for V. While let $\{\mathscr{S}\}^- V$ denote the set consisting of the last term of some maximal (i.e. not extendable) finite $\mathscr{S}^-$-rejected sequence for V. Namely,

$[\mathscr{S}]^- V \ni U \Leftrightarrow$ There exists an $\mathscr{S}^-$-rejected sequence $(V_0, \dots, V_m)$ with $V_0 = V, V_m = U$.

$\{\mathscr{S}\}^- V \ni U \Leftrightarrow U \in [\mathscr{S}]^- V$ and there exists no non-zero $T \in \boldsymbol{N}\mathscr{S}$ satisfying $\sup\{0, \langle U, L\rangle\} \geq \langle T, L\rangle$ for any $L \in \operatorname{Ind}\Lambda$.

$\Leftrightarrow U \in [\mathscr{S}]^- V$ and $\langle U, L\rangle \leq 0$ for any $L \in \mathscr{S}$.

(1) By definition, $[\mathscr{S}]^- V$ contains V, while $\{\mathscr{S}\}^- V$ might be empty.
(2) If $0 \neq V \in \boldsymbol{N}\mathscr{S}$ and $\phi_{\mathscr{S}}^- V \leq 0$ (i.e. $\langle \phi^- V, L\rangle \leq 0$ for any $L \in \mathscr{S}$), then $(V, V, V, \dots)$ is an infinite $\mathscr{S}^-$-sequence for V.

3.6.4. Assume that $\mathscr{S}$ is not cofaithful, so that by 2.4.1, there is an irreducible central idempotent ε of $\tilde{\Lambda}$ such that $\mathscr{S} \supseteq \operatorname{Ind}_\varepsilon \Lambda$.
(1) If $U \in \boldsymbol{Z}(\operatorname{Ind}\Lambda)$ and $l(\varepsilon U) > 0$, then there is $L \in \operatorname{Ind}_\varepsilon \Lambda$ such that $\langle U, L\rangle > 0$.
(2) If $V \in \boldsymbol{Z}(\operatorname{Ind}\Lambda)$ and $l(\varepsilon V) > 0$ (in particular if $V \in \operatorname{Ind}_\varepsilon \Lambda$), then $\{\mathscr{S}\}^- V = \emptyset$.

PROOF. (1) Put $U = \sum_{L \in \operatorname{Ind}\Lambda} \langle U, L\rangle L$. Then $\varepsilon U = \sum_{L \in \operatorname{Ind}\Lambda} \langle U, L\rangle \varepsilon L$. Hence, $l(\varepsilon U) = \sum_{L \in \operatorname{Ind}_\varepsilon \Lambda} \langle U, L\rangle l(\varepsilon L) > 0$ implies $\langle U, L\rangle > 0$ for some $L \in \operatorname{Ind}_\varepsilon \Lambda$.
(2) By definition, any $U \in [\mathscr{S}]^- V$ has the form $U = V - \phi^- T$, $T \in \boldsymbol{N}\mathscr{S}$. By (5) 2.2, $l(\varepsilon\phi^- T) = 0$, so that $l(\varepsilon U) = l(\varepsilon V) > 0$. By (1), U cannot be an end term. ■

4. Finite finishing of $\mathscr{S}^-$-rejection.

For L, $M \in \operatorname{Ind}\Lambda$, let $|LM|$ denote the distance of L and M in $\mathfrak{A}(\Lambda)$, i.e. $|LM|$ is the length of the shortest path connecting L and M ($|LM| := \infty$ if they are not connected). For $X, Y \in \operatorname{lat}\Lambda$, put

$$|XY| := \inf\{|LM| \mid \langle L, X\rangle > 0, \langle Y, M\rangle > 0, L, M \in \operatorname{Ind}\Lambda\}.$$

In this section, we use the following abbreviation.

$$(X, Y) := \mathrm{Hom}_\Lambda(X, Y)$$

$$(X, Y)_{\mathscr{S}} := \sum_{L \in \mathrm{Ind}\,\Lambda - \mathscr{S}} \mathrm{Hom}_\Lambda(X, L)\mathrm{Hom}_\Lambda(L, Y), \text{ a sub } R\text{-module of} (X, Y).$$

4.1. Let $\mathscr{S}$ be a non-empty bounded subset of $\mathrm{Ind}\,\Lambda$. There is a natural number $m(\mathscr{S}) > 0$ minimal with respect to the following property:

For any $m \geq m(\mathscr{S})$, if $L_i \in \mathscr{S}$ $(0 \leq i \leq m)$, $f_i \in \mathrm{rad}(L_i, L_{i+1})$ $(0 \leq i < m)$, then $\mathrm{im}(f_0 \cdots f_{m-1}) \subseteq \pi L_m$.

Proof. This is a direct consequence of Harada-Sai's Lemma and Maranda's Theorem ([**CR**] 30-19). Note that the separability of $\tilde{\Lambda}$ (assumed in [**CR**] 30-19) is not necessary, since the proof of [**CR**] (30-13) depends only on the existence of maximal orders. ∎

4.2. Let $\mathscr{S}$ be bounded and $m := m(\mathscr{S})$ be the same as defined in 4.1. Then for any $X, Y \in \mathrm{lat}\,\Lambda$, $|XY| \geq m$ implies $(X, Y) = (X, Y)_{\mathscr{S}}$.

Proof. We prove first the assertion for indecomposable lattices.

If $L \notin \mathscr{S}$, then $(L, M)_{\mathscr{S}} \supseteq (L, L)(L, M) = (L, M)$, so that $(L, M)_{\mathscr{S}} = (L, M)$.

Suppose that $|LM| \geq m > 0$, so that $(L, M) = \mathrm{rad}(L, M)$. For any $f \in (L, M)$, there is $g : \theta^- L \to M$ such that $f = vg$ where $v : L \to \theta^- L$ is a source map. In other words, we have $(L, M) = \sum_{M' \mid \theta^- L}(L, M')(M', M)$.

If each $M' \notin \mathscr{S}$, then $(L, M) = (L, M)_{\mathscr{S}}$ and we are through.

For $M' \in \mathscr{S}$, we can further decompose as $(M', M) = \sum_{M'' \mid \theta^- M'}(M', M'')(M'', M)$.

Repeating the procedure, in view of 4.1, we get

$$(L, M) \subseteq (L, M)_{\mathscr{S}} + \pi(L, M).$$

Applying Nakayama's Lemma for R-module (L, M), we get $(L, M) = (L, M)_{\mathscr{S}}$.

In general cases, since $|XY| \geq m \Leftrightarrow |LM| \geq m$ for any $L \mid X$ and $M \mid Y$, the claim is obvious. ∎

4.3. For a commutative diagram of Λ-lattices

$$\begin{tikzcd} X \arrow[r, "g_0"] \arrow[d, "1"] & Y_0 \arrow[d, "u"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

put $\mathscr{S} := \{L \in \mathrm{Ind}\,\Lambda \mid g_0(Y_0, L) \neq g(Y, L)\}$. If $\mathscr{S}$ is bounded, then $\mathscr{S}$ is contained in a finite set $U = U(Y_0, m(\mathscr{S})) := \{M \in \mathrm{Ind}\,\Lambda \mid |Y_0 M| < m(\mathscr{S})\}$.

Proof. Assuming that $M \notin U$, we shall show that $g_0(Y_0, M) \subseteq g(Y, M)$. Indeed, we have

$$\begin{aligned} g(Y, M) &\supseteq \sum_{L \in \mathrm{Ind}\,\Lambda - \mathscr{S}} g(Y, L)(L, M) = \sum_{L \in \mathrm{Ind}\,\Lambda - \mathscr{S}} g_0(Y_0, L)(L, M) \\ &= g_0(Y_0, M)_{\mathscr{S}} = g_0(Y_0, M), \end{aligned}$$

where the last equality is by 4.2. Since $\mathfrak{A}(\Lambda)$ is locally finite, U is a finite set. ∎

4.3.1. PROPOSITION. *A bounded rejectable subset $\mathscr{S}$ of* $\operatorname{Ind}\Lambda$ *is necessarily finite, which is a generalization of Roiter's Theorem* ($=$ *Brauer-Thrall* I).

PROOF. Put $\Gamma := \Lambda(\mathscr{S})$ so that $\mathscr{S} = \operatorname{Ind}\Lambda - \operatorname{Ind}\Gamma$. In 4.3, take as $X = Y_0 = \Lambda$, $Y = \Gamma$, $g_0 = 1$, $g = u = \iota : \Lambda \to \Gamma$. Then by the map $g_0 f \mapsto 1(g_0 f)$, $g_0(Y_0, L) \simeq L$, $g(Y, L) \simeq \{x \in L | \Gamma x \subseteq L\} = L$, so that $\{L \in \operatorname{Ind}\Lambda | g_0(Y_0, L) \neq g(Y, L)\} = \operatorname{Ind}\Lambda - \operatorname{Ind}\Gamma$.

The latter part is obvious, since if $\operatorname{Ind}\Lambda$ is bounded, then $\mathscr{S} := \operatorname{Ind}\Lambda - \operatorname{Ind}\Gamma$ is bounded for any maximal overorder Γ, so that $\mathscr{S}$ is finite. ■

4.4. Consider a family of commutative diagrams of Λ-lattices ($j \geq 0$):

$$\begin{array}{ccc} X & \xrightarrow{g_j} & Y_j \\ \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle u_j} \\ X & \xrightarrow{g_{j+1}} & Y_{j+1}. \end{array}$$

Assume that there exists a bounded cofaithful subset $\mathscr{S}$ of $\operatorname{Ind}\Lambda$ satisfying the following property.

(0) $g_0(Y_0, L) = g_j(Y_j, L)$ for any $L \in N(\operatorname{Ind}\Lambda - \mathscr{S})$ and $j \in N$.

Then, for almost all j, we have

$$g_j(Y_j, L) = g_{j+1}(Y_{j+1}, L) \text{ for any } L \in N \operatorname{Ind}\Lambda.$$

In particular, for each sufficiently large j, there always, exists $\alpha_j : Y_{j+1} \to Y_j$ such that $g_j = g_{j+1}\alpha_j$.

PROOF. Put $\mathscr{S}_j := \{L \in \operatorname{Ind}\Lambda | g_0(Y_0, L) \neq g_j(Y_j, L)\}$ and $\mathscr{S}' := \bigcup_{j \in N} \mathscr{S}_j$.

By (0), each $\mathscr{S}_j$ is contained in $\mathscr{S}$, which is bounded and cofaithful. Applying 4.3, each $\mathscr{S}_j$ is contained in the finite set $U(Y_0, m(\mathscr{S}))$, so that $\mathscr{S}'$ is a finite cofaithful subset of $\operatorname{Ind}\Lambda$. We shall see $\bigoplus_{L \in \operatorname{Ind}\Lambda} g_j(Y_j, L) = \bigoplus_{L \in \operatorname{Ind}\Lambda} g_{j+1}(Y_{j+1}, L)$ for almost all j, or equivalently,

(1) $\bigoplus_{L \in \mathscr{S}'} g_j(Y_j, L) = \bigoplus_{L \in \mathscr{S}'} g_{j+1}(Y_{j+1}, L)$ for almost all j.

Since $\mathscr{S}'$ is finite cofaithful, by (4) 2.4, there is $Z \in N(\operatorname{Ind}\Lambda - \mathscr{S}')$ and $a \geq 0$ such that

(2) $g_j(Y_j, \bigoplus_{L \in \mathscr{S}'} L) \supseteq g_j(Y_j, Z)(Z, \bigoplus_{L \in \mathscr{S}'} L) = g_0(Y_0, Z)(Z, \bigoplus_{L \in \mathscr{S}'} L) \supseteq \pi^a g_0(Y_0, \bigoplus_{L \in \mathscr{S}'} L)$ for all j.

Now (1) is clear from (2). ■

4.5. LEMMA. *Let* $A : 0 \to A_{n-1} \to A_n \to A_{n+1} \to 0$ *be a finite complex which is rationally exact at* A_n *and exact at* A_{n+1}. *Let* $\mathscr{S}$ *be a bounded cofaithful subset of* $\operatorname{Ind}\Lambda$. *Then any successive* $\mathscr{S}^-$*-rejection with the initial complex* A (cf. 3.4) *terminates at a finite number of steps.*

PROOF. Supposing to be contrary, we get a family of commutative diagrams.

$$\begin{array}{ccccc} A_{n-1}^{(j)} & = & A_{n-1} & \xrightarrow{a_{n-1}^{(j)}} & A_n^{(j)} \\ & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle f_n^{(j)}} \\ A_{n-1}^{(j+1)} & = & A_{n-1} & \xrightarrow{a_{n-1}^{(j+1)}} & A_n^{(j+1)} \end{array}$$

The condition (0) of 4.4 is satisfied by (sr2) 3.4. By 4.4, for large enough j, there is $\alpha_j : A_n^{(j+1)} \to A_n^{(j)}$ such that $a_{n-1}^{(j)} = a_{n-1}^{(j+1)}\alpha_j$, which contradicts with 3.4.1. ∎

4.5.1. Remark. Although the above lemma is sufficient for applications in this paper, we can in fact prove: If $\mathscr{S}$ is bounded and cofaithful, then successive $\mathscr{S}^-$-rejection of *any complex* A terminates at a finite number of steps. Indeed, the general case can easily be reduced to the special case treated in the above lemma.

4.5.2. Corollary. *Assume that $\mathscr{S}$ is bounded and cofaithful. Then any $\mathscr{S}^-$-sequence for any $V \in \operatorname{lat} \Lambda$ is finite. In particular, $\{\mathscr{S}\}^- V$ is always non-empty.*

Proof. Suppose to be contrary. Then the complex $A : 0 \to V \xrightarrow{1} \overset{n}{V} \to 0$ has an infinite successive $\mathscr{S}^-$- rejection by 3.6.1. This is not the case by 4.5. ∎

4.6. Criterion for cofaithfulness. *Let $\mathscr{S}$ be a non-empty bounded subset of* $\operatorname{Ind} \Lambda$. *Put $\mathscr{S}_p := \mathscr{S} \cap \operatorname{proj} \Lambda$, $\mathscr{S}_i := \mathscr{S} \cap \operatorname{inj} \Lambda$, $P_{\mathscr{S}} := \bigoplus_{L \in \mathscr{S}_p} L$ and $I_{\mathscr{S}} := \bigoplus_{L \in \mathscr{S}_i} L$. Then the following conditions for $\mathscr{S}$ are equivalent.*

(1) *$\mathscr{S}$ is cofaithful.*

(2) *For any $V \in \operatorname{lat} \Lambda$, every $\mathscr{S}^-$-sequence for V is finite.*

(3) *$\{\mathscr{S}\}^- P_{\mathscr{S}}$ is not empty.*

(4) *$\{\mathscr{S}\}^- I_{\mathscr{S}}$ is not empty.*

Proof. (1) ⇒ (2): By 4.5.2.

(2) ⇒ (3) (resp. (4)): Obvious by definitions.

(3) ⇒ (1): If $\mathscr{S}$ is not cofaithful, then $l(\varepsilon P_{\mathscr{S}}) > 0$ so that $\{\mathscr{S}\}^- P_{\mathscr{S}}$ is empty by (2) 3.6.4 where ε is a central idempotent given in 3.6.4. ∎

Our rejection theory has wide applications for the problem characterizing $\mathfrak{A}(\Lambda)$ with a subquiver of some special type. Among others, we apply our method, in the next 4.7, to give an alternative proof for Wiedemann's Theorem [W].

4.7. Wiedemann's Theorem. *Let Λ be a connected R-order (i.e. Λ has no nontrivial central idempotent) and assume that there is $L_0 \in \operatorname{Ind} \Lambda$ with an irreducible map to itself (i.e. $\mathfrak{A}(\Lambda)$ has a subquiver $\circlearrowleft L_0$). Then $\mathfrak{A}(\Lambda)$ has the following form by some $n \geq 0$; $l(L_i) = 1$ $(0 \leq i \leq n)$, $L_n \in \operatorname{proj} \Lambda \cap \operatorname{inj} \Lambda$, $\tau L_i = L_i$ $(0 \leq i < n)$.*

$$\circlearrowleft L_0 \rightleftarrows L_1 \rightleftarrows \cdots\cdots \rightleftarrows L_{n-1} \rightleftarrows L_n$$

Proof. Let $\mathscr{C}$ be a connected component of $\mathfrak{A}(\Lambda)$ containing L_0. Take a vertex L_n from $\mathscr{C}$ and a path $L_0 - L_1 - \cdots - L_{n-1} - L_n$ connecting L_0 with L_n, and put $\mathscr{S} := \{L_0, \ldots, L_n\}$.

Step 1 Suppose that the following condition (0) is satisfied.

(0) $\tau L_i = L_i (0 \leq i < n)$.

Then it is easy to observe the following (1)–(3).

(1) $l(L_0) \geq l(L_1) \geq \cdots \geq l(L_{n-1}) \geq l(L_n)$

(2) $\mathscr{S}$ contains the following subquiver:

$$\circlearrowleft L_0 \rightleftarrows L_1 \rightleftarrows \cdots\cdots \rightleftarrows L_{n-1} \rightleftarrows L_n$$

(3) If $\mathscr{S} = \operatorname{Ind} \Lambda$ and the diagram in (2) is a full subquiver of $\mathscr{S}$, then $l(L_0) = \cdots = l(L_n)$ and $\tau L_n = 0$.

Step 2 There exist some $L \in \mathscr{C}$ such that $\tau L \neq L$.

If that is not the case, the condition (0) must be satisfied for any $L \in \mathscr{C}$ and any path $L_0 - \cdots - L_n = L$. Hence $l(L) \leq l(L_0)$ by Step 1. Thus $\mathscr{C}$ is bounded, so we get $\mathscr{C} = \mathfrak{A}(\Lambda)$ by Auslander's Theorem [**R**]. This is impossible since any $L = \tau L$ is not projective.

Step 3 From the set $\{L \in \mathscr{C} | \tau L \neq L\}$, pick one L with the shortest distance $|L_0 L| =: n$ from L_0. Then the shortest path $L_0 - \cdots - L_n = L$ satisfies the condition (0). Putting $V := \sum_{i=0}^{n} L_i \in N\mathscr{S}$, by (0) and (2), $\theta_{\mathscr{S}}^{-} V \geq 2 \sum_{i=0}^{n-1} L_i + L_n$, $\tau_{\mathscr{S}}^{-} V = \sum_{i=0}^{n-1} L_i$, so that we have

(4) $\phi_{\mathscr{S}}^{-} V \leq 0$.

(5) $\phi_{\mathscr{S}}^{-} V = 0 \Leftrightarrow$ the diagram in (2) is a full subquiver of $\mathscr{S}$.

By (2) 3.6.3 and 4.5.2, (4) implies that $\mathscr{S}$ is not cofaithful. By 2.4.1, $\operatorname{Ind}_{\varepsilon} \Lambda (\subseteq \mathscr{S})$ contains at least one projective and irreducible lattice. By (1), L_n is irreducible, $l(L_n) = 1$. By (0), L_n is the only projective in $\operatorname{Ind}_{\varepsilon} \Lambda$, so that $\varepsilon M = 0$ for any $M \in \operatorname{proj} \Lambda - \{L_n\}$. Since Λ is connected, $\varepsilon \Lambda = \Lambda$, i.e. $\varepsilon = 1$ and $\mathscr{S} = \operatorname{Ind} \Lambda$. By (5) 2.2, $l(\phi^{-} V) = 0$, so $\phi^{-} V = 0$ by (4). Hence the diagram in (2) is a full subquiver of $\mathscr{S}$ by (5). By (3), we have $l(L_0) = \cdots = l(L_n) = 1$. ∎

5. Main Results.

Let $\mathscr{S}$ be a subset of $\operatorname{Ind} \Lambda$ and $\mathscr{S}_p$, $\mathscr{S}_i$, $P_{\mathscr{S}}$, $I_{\mathscr{S}}$ the same as those of 4.6.

5.1. If the following condition (t4) holds, then $\mathscr{S}$ is trivial (i.e. $\Lambda(\mathscr{S}) = \Lambda$).

(t4) For any $P \in \mathscr{S}_p$, there exists $T \in N(\mathscr{S} - \mathscr{S}_i)$ such that $\phi_{\mathscr{S}}^{-} T \geq P$.

Proof. Suppose that $\mathscr{S}$ is not cofaithful. Then $\mathscr{S} \supseteq \operatorname{Ind}_{\varepsilon} \Lambda$ for a central idempotent ε by 2.4.1. Take $P \in \mathscr{S}_p$ with $\varepsilon P \neq 0$. There exists $T \in N(\mathscr{S} - \mathscr{S}_i)$ such that $\phi_{\mathscr{S}}^{-} T \geq P$ by (t4). By (5) 2.2, we have $l(\varepsilon \phi_{\mathscr{S}}^{-} T) = 0$, so $l(\varepsilon P) \leq l(\varepsilon \phi_{\mathscr{S}}^{-} T) = 0$, a contradiction. Hence $\mathscr{S}$ is cofaithful and $\Gamma := \Lambda(\mathscr{S})$ is an order with $\mathscr{S} \supseteq \operatorname{Ind} \Lambda - \operatorname{Ind} \Gamma$.

For any $P \in \mathscr{S}_p$, take the complex

$$A : 0 \to P \oplus T \to P \overset{n}{\oplus} T \to 0 \to 0.$$

We can reject T from n-th term getting

$$A' := A - \overset{n}{\check{T}} : 0 \to P \oplus T \to A'_n \to A'_{n+1} \to 0.$$

Since T has no injective summand, A' is exact. Since $\chi_n(A') = A'_n - A'_{n+1} = \chi_n(A) - \phi^{-} T = P + T - \phi^{-} T$, we have $\langle P + T - A'_n + A'_{n+1}, L \rangle = \langle \phi^{-} T, L \rangle \geq \langle P, L \rangle \geq 0$ for any $L \in \mathscr{S}$. Applying the next general simple lemma 5.1.1 to the exact sequence A', we have $0 = \langle P + T - A'_n + A'_{n+1}, L \rangle$ for any $L \in \operatorname{Ind} \Lambda - \operatorname{Ind} \Gamma$, so that $\langle P, L \rangle = 0$. Hence $\mathscr{S}_p \subseteq \operatorname{Ind} \Gamma$, i.e. $\Gamma = \Lambda(\mathscr{S}) = \Lambda$. ∎

5.1.1. Let $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ be an exact sequence of Λ-lattices, Γ be an over-order of Λ in $\tilde{\Lambda}$, and $\mathscr{S}' := \operatorname{Ind}\Lambda - \operatorname{Ind}\Gamma$. If $\langle X - Y + Z, L\rangle \geq 0$ for any $L \in \mathscr{S}'$, then $\langle X - Y + Z, L\rangle = 0$ for any $L \in \mathscr{S}'$.

Proof. Denoting as $\dot{X} := \Gamma X$, $\dot{\alpha} :=$ (the unique extension of α) etc., there arises a complex $0 \to \dot{X} \xrightarrow{\dot{\alpha}} \dot{Y} \xrightarrow{\dot{\beta}} \dot{Z} \to 0$ which induces a complex of Λ-modules $0 \to \dot{X}/X \to \dot{Y}/Y \to \dot{Z}/Z \to 0$, which is exact at $\dot{X}/X$ and $\dot{Z}/Z$. Hence we have $0 \geq l_\Lambda(\dot{X}/X) - l_\Lambda(\dot{Y}/Y) + l_\Lambda(\dot{Z}/Z) = \sum_{L \in \operatorname{Ind}\Lambda} \langle X - Y + Z, L\rangle l_\Lambda(\dot{L}/L) = \sum_{L\in\mathscr{S}'}\langle X - Y + Z, L\rangle l_\Lambda(\dot{L}/L)$. This, together with the assumption $\langle X - Y + Z, L\rangle \geq 0$ for any $L \in \mathscr{S}'$, obviously implies that $\langle X - Y + Z, L\rangle = 0$ for any $L \in \mathscr{S}'$. ■

5.2. If $\mathscr{S}$ is trivial, then the following condition (t1) holds.

(t1) $\langle U, I\rangle \leq 0$ for any $U \in [\mathscr{S} - \mathscr{S}_i]^- P_{\mathscr{S}}$ and $I \in \mathscr{S}_i$.

Proof. Suppose that $\Lambda = \Lambda(\mathscr{S})$ and (t1) does not hold. In particular $\mathscr{S}$ is cofaithful and, by (6) 2.4, there is an exact sequence with $p_{\mathscr{S}} W = 0$, which we look as a complex $\boldsymbol{A}$ of Λ-lattices

$$\boldsymbol{A} : 0 \to \overset{n}{\check{P}_{\mathscr{S}}} \to W \to Y \to 0$$

so that $\chi_n(\boldsymbol{A}) = P_{\mathscr{S}} - W$, $p_{\mathscr{S}}\chi_n(\boldsymbol{A}) = P_{\mathscr{S}}$.

Since (t1) does not hold, there exist $I \in \mathscr{S}_i$ and $U \in [\mathscr{S} - \mathscr{S}_i]^- P_{\mathscr{S}}$ such that $\langle U, I\rangle > 0$ and $U = P_{\mathscr{S}} - \sum_{j=0}^{m-1} \phi^- T_j$ by some $(\mathscr{S} - \mathscr{S}_i)^-$-sequence (T_j) for $P_{\mathscr{S}}$. By 3.6.1, there is successive $\mathscr{S}^-$-rejection sequence $\boldsymbol{A}^{(0)} = \boldsymbol{A}, \boldsymbol{A}^{(j)} = \boldsymbol{A}^{(j-1)} - \overset{n}{\check{T}}_{j-1}$. By 3.4, writing $\alpha := a_n^{(m)}$, $\boldsymbol{A}^{(m)}$ has the following form

$$\boldsymbol{A}^{(m)} : 0 \to A_n^{(m)} \xrightarrow{\alpha} A_{n+1}^{(m)} \to Y \to 0$$

with $\chi_n(\boldsymbol{A}^{(m)}) = A_n^{(m)} - A_{n+1}^{(m)} = U - W$.

Since $T_j \in N(\mathscr{S} - \mathscr{S}_i)$ has no injective summands, $\boldsymbol{A}^{(m)}$ is exact by (sr3).

Since $\langle U - W, I\rangle = \langle U, I\rangle > 0$, by 3.5, there is a split monomorphism $i : I \to A_n^{(m)}$ such that $i\alpha \in \operatorname{rad}(I, A_{n+1}^{(m)})$. Since i is split, we get the exact sequence of Λ-lattices

$$0 \to I \xrightarrow{i\alpha} A_{n+1}^{(m)} \to \operatorname{cok} i\alpha \to 0$$

which must be split since I is injective. But it is impossible since $i\alpha \in \operatorname{rad}(I, A_{n+1}^{(m)})$. ■

5.3. Theorem. *Assume that $\mathscr{S}$ is bounded. Then the following five conditions are equivalent.*

(t) *$\mathscr{S}$ is trivial (i.e. $\Lambda(\mathscr{S}) = \Lambda$).*

(t1) *in* 5.2.

(t2) *$\langle U, I\rangle \leq 0$ for any $U \in \{\mathscr{S} - \mathscr{S}_i\}^- P_{\mathscr{S}}$ and $I \in \mathscr{S}_i$.*

(t3) *There exists $U \in \{\mathscr{S} - \mathscr{S}_i\}^- P_{\mathscr{S}}$ such that $\langle U, I\rangle \leq 0$ for any $I \in \mathscr{S}_i$.*

(t4) *in* 5.1.

Proof. (t) $\Rightarrow$ (t1): By 5.2, (t4) $\Rightarrow$ (t): By 5.1, (t1) $\Rightarrow$ (t2): trivial.

(t2) $\Rightarrow$ (t3): By 2.4.1, $\mathscr{S} - \mathscr{S}_i$ is cofaithful, it is bounded by assumption. By 4.5.2, $\{\mathscr{S} - \mathscr{S}_i\}^- P_{\mathscr{S}}$ is not empty. Hence (t2) implies (t3).

(t3) $\Rightarrow$ (t4): By Definition 3.6.3, U (in (t3)) has the form $U = P_{\mathscr{S}} - \phi^{-}T$, $T = \sum_{j=0}^{m-1} T_j$ with some $(\mathscr{S} - \mathscr{S}_i)^{-}$-sequence (T_j). Since $T \in N(\mathscr{S} - \mathscr{S}_i)$, $P_{\mathscr{S}} = p_{\mathscr{S}} P_{\mathscr{S}} = \phi_{\mathscr{S}}^{-}T + p_{\mathscr{S}}U$. $p_{\mathscr{S}}U \leq 0$ shows $P - \phi_{\mathscr{S}}^{-}T \leq P_{\mathscr{S}} - \phi_{\mathscr{S}}^{-}T = p_{\mathscr{S}}U \leq 0$ for any $P \in \mathscr{S}_p$. ■

5.3.1. By duality (cf. 3.3), (t) is equivalent with the condition $(\mathrm{t}l)^{+}$ $(l = 1,2,3,4)$, obtained from $(\mathrm{t}l)$ by interchanging the role of projective with injective, $(\ )^{-}$ with $(\ )^{+}$, for example

$(\mathrm{t4})^{+}$ For any $I \in \mathscr{S}_i$, there exists $T \in N(\mathscr{S} - \mathscr{S}_p)$ such that $\phi_{\mathscr{S}}^{+}T \geq I$.

5.3.2. (1) Any bounded non-trivial subset $\mathscr{S}$ of $\operatorname{Ind} \Lambda$ contains at least one projective and at least one injective.

(2) Let $\mathscr{S}$ be a minimal non-trivial subset of $\operatorname{Ind} \Lambda$. If $\mathscr{S}$ is bounded, then it contains exactly one projective and exactly one injective.

PROOF. (1) By 5.3 and its dual. But the claim is in fact trivial by 2.4.2 without assuming the boundedness.

(2) If $\mathscr{S}$ is non-trivial, it does not satisfy (t1), so that there is some $I \in \mathscr{S}_i$, $U \in [\mathscr{S} - \mathscr{S}_i]^{-}P_{\mathscr{S}}$ with $\langle U, I \rangle > 0$. Again from (t1) and 3.6.2 (B), a subset $(\mathscr{S} - \mathscr{S}_i) \cup \{I\}$ of $\mathscr{S}$ is also non-trivial. By the minimality $\mathscr{S}_i = \{I\}$. By duality, $\mathscr{S}_p = \{P\}$.

However, the claim can in fact be proved without assuming the boundedness. ■

5.4. THEOREM. *Let $\mathscr{S}$ be a finite rejectable subset of* $\operatorname{Ind} \Lambda$ *and put $\Gamma := \Lambda(\mathscr{S})$ i.e.* $\operatorname{Ind} \Lambda - \operatorname{Ind} \Gamma = \mathscr{S}$. *For any $V \in \operatorname{lat} \Lambda$, $\{\mathscr{S}\}^{-}V$ is a singleton set consisting of $\dot{V} = \Gamma V$.*

PROOF. Take any $U \in \{\mathscr{S}\}^{-}V$. By 3.6, $U = V - \sum_{j=0}^{m-1} \phi^{-}T_j$ with an $\mathscr{S}^{-}$-sequence (T_j) for V. We take an initial complex A as below and get successive $\mathscr{S}^{-}$-rejection $A^{(1)}, \dots, A^{(m)}$.

$$A : 0 \longrightarrow V \longrightarrow \overset{n}{\check{V}} \longrightarrow 0 \longrightarrow 0$$

$$A^{(m)} : 0 \longrightarrow V \xrightarrow{a_{n-1}^{(m)}} A_n^{(m)} \xrightarrow{a_n^{(m)}} A_{n+1}^{(m)} \longrightarrow 0$$

By 3.4.2, the assosiated Γ-complex $\dot{A}^{(m)}$ is split exact.

$$\dot{A}^{(m)} : 0 \longrightarrow \dot{V} \xrightarrow{\dot{a}_{n-1}^{(m)}} \dot{A}_n^{(m)} \xrightarrow{\dot{a}_n^{(m)}} \dot{A}_{n+1}^{(m)} \longrightarrow 0 \text{ (split exact)}$$

Let Z be a summand of $A_n^{(m)}$ which is maximal among the summands with the property that $a_n^{(m)}|_z$ is a split monomorphism.

By 2.3, $A^{(m)}$ decomposes as follows.

$$\begin{array}{ccccccccc}
A^{(m)} : 0 & \longrightarrow & V & \longrightarrow & A_n^{(m)} & \longrightarrow & A_{n+1}^{(m)} & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle 1} & & \\
B \oplus I(\overset{n}{\check{Z}}) : 0 & \longrightarrow & V & \xrightarrow{(b_{n-1}\,0)} & B_n \oplus Z & \xrightarrow{\begin{pmatrix} b_n & 0 \\ 0 & 1 \end{pmatrix}} & B_{n+1} \oplus Z & \longrightarrow & 0
\end{array}$$

Since $U = \chi_n(A^{(m)}) = \chi_n(\boldsymbol{B})$ and $\langle U, L\rangle \leq 0$ for any $L \in \mathscr{S}$, we can write as
(1) $B_n = X \oplus E$, $B_{n+1} = X \oplus Y \oplus F$ with $X, Y \in N\mathscr{S}$ and $E, F \in N(\operatorname{Ind}\Lambda - \mathscr{S})$.
The maximality of Z implies that $b_n \in \operatorname{rad}(B_n, B_{n+1})$, so that we have
(2) $\dot{b}_n|_E \in \operatorname{rad}(E, \dot{B}_{n+1})$.
Now we can see the claim as
$\dot{a}_n^{(m)}$: split epimorphism $\Rightarrow \dot{b}_n$: split epimorphism $\overset{(2)}{\Rightarrow} \dot{b}_n|_{\dot{X}}$: split epimorphism $\overset{(1)}{\Rightarrow} (Y = F = 0, E = \dot{V}) \Rightarrow U = \dot{V}$. ∎

5.4.1. By duality, $\{\mathscr{S}\}^+V$ is a singleton set consisting of the maximum Γ-sublattice $\underset{\cdot}{V} \simeq \operatorname{Hom}_\Lambda(\Gamma, V)$.

5.5. Recovering $\mathfrak{A}(\Gamma)$ from $\mathfrak{A}(\Lambda)$. Let $\mathscr{S}$ be a finite rejectable subset of $\operatorname{Ind}\Lambda$ and $\Gamma = \Lambda(\mathscr{S})$. For $M \in \operatorname{Ind}\Gamma = \operatorname{Ind}\Lambda - \mathscr{S}$, we write the Γ-source map from V as

$$0 \to M \xrightarrow{\rho} \dot{\theta}^- M \to \dot{\tau}^- M \to 0.$$

5.5.1. For any $M \in \operatorname{Ind}\Gamma$, we have $\dot{\theta}^- M - \dot{\tau}^- M = (\theta^- M)^{\cdot} - (\tau^- M)^{\cdot}$.

Proof. Since $\nu : M \to \theta^- M$ is in the radical, so is $\dot{\nu} : M \to (\theta^- M)^{\cdot}$, and there exists f such that $\dot{\nu} = \rho f$, getting the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\rho"] \arrow[d, "1"] & \dot{\theta}^- M \arrow[r] \arrow[d, "f"] & \dot{\tau}^- M \arrow[r] \arrow[d, "\bar{f}"] & 0 \\
0 \arrow[r] & M \arrow[r, "\dot{\nu}"] & (\theta^- M)^{\cdot} \arrow[r] & (\tau^- M)^{\cdot} \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, "\nu"] \arrow[u, "1"] & \theta^- M \arrow[r] \arrow[u, "\iota"] & \tau^- M \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where $\bar{f}$ is uniquely induced from f by (1) 2.2.

Since $\rho : M \to \dot{\theta}^- M$ is in the radical, there exists $g : \theta^- M \to \dot{\theta}^- M$ such that $\rho = \nu g$, which decomposes as $g = \iota\dot{g}$, $\dot{g} : (\theta^- M)^{\cdot} \to \dot{\theta}^- M$.

We have $\rho f \dot{g} = \dot{\nu}\dot{g} = \nu\iota\dot{g} = \nu g = \rho$.

Since ρ is left minimal, $f\dot{g}$ is an automorphism of $\dot{\theta}^- M$, so that $\dot{g}$ is a split epimorphism. By (3) 2.2, $\dot{g}$ induces $\bar{\dot{g}} : (\tau^- M)^{\cdot} \to \dot{\tau}^- M$ and $\overline{f\dot{g}}$ is an automorphism of $\dot{\tau}^- M$, $\ker \dot{g} \simeq \ker \bar{\dot{g}}$. These imply that $\dot{\theta}^- M - \dot{\tau}^- M = (\theta^- M)^{\cdot} - (\tau^- M)^{\cdot}$. ∎

5.5.2. Write $(\theta^- M)^{\cdot} - (\tau^- M)^{\cdot}$ as
(1) $(\theta^- M)^{\cdot} - (\tau^- M)^{\cdot} = U - V$, $U, V \in \operatorname{lat}\Gamma$, $\langle U, V\rangle = 0$.
Then we have one of the following three results.
(A) If $V \neq 0$, then $M \notin \operatorname{inj}\Gamma$, $V \in \operatorname{Ind}\Gamma$ and the Γ-almost split sequence from M is given by

$$0 \to M \to U \to V \to 0.$$

(B) If $V = 0$ and $M \in \operatorname{inj}\Gamma$, the complex of the Γ-source map from M is given by

$$0 \to M \to U \to 0 \to 0.$$

(C) If $V=0$ and $M \notin \operatorname{inj}\Gamma$, the Γ-almost split sequence from M is given by

$$0 \to M \to U \oplus M \to M \to 0.$$

Consequently $\mathfrak{A}(\Gamma)$ has a connected component of the form of 4.7.

PROOF. By (1), we can write, with some $X \in \operatorname{lat}\Gamma$, as

(2) $\dot{\theta}^{-}M = U \oplus X$, $\dot{\tau}^{-}M = V \oplus X$

Since $\dot{\tau}^{-}M$ is indecomposable or 0, $V \neq 0$ implies $X=0$, so that $V = \dot{\tau}^{-}M \in \operatorname{Ind}\Gamma$, showing (A).

Assume that $V=0$. If $M \in \operatorname{inj}\Gamma$, then $X=0$ and we meet with the case (B). If $M \notin \operatorname{inj}\Gamma$, $X = \dot{\tau}^{-}M \in \operatorname{Ind}\Gamma$ and Γ-almost split sequence from M has the form $0 \to M \to U \oplus X \to X \to 0$. Then there is an irreducible map from X to itself, by 4.7, $X = M$. ■

ACKNOWLEDGEMENT. The author would like to express his heartfelt thanks to Professor H. Hijikata for valuable suggestions and encouragement.

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