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On a class of multilinear oscillatory singular integral operators

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1. Introduction.

We will work on $\boldsymbol{R}^n$ $(n \geq 1)$. Let $\Phi(x) \in C^\infty(\boldsymbol{R}^n \setminus \{0\})$ be a real-valued function which satisfies

$$(1) \qquad |D^\alpha \Phi(x)| \leq B_1 |x|^{a-|\alpha|}, \quad |\alpha| \leq 3,$$

and

$$(2) \qquad \sum_{|\alpha|=2} |D^\alpha \Phi(x)| \geq B_2 |x|^{a-2},$$

where a is a fixed real number, B_1 and B_2 are positive constants. Let K_0 be a standard Calderón-Zygmund kernel. Define the oscillatory singular integral operator T by

$$(3) \qquad Tf(x) = \int_{\boldsymbol{R}^n} e^{i\Phi(x-y)} K_0(x-y) f(y)\, dy.$$

For the special case $\Phi(x) = |x|^a$, such operators have been studied by many authors (see [**1**], [**2**], [**7**], [**10**], for example). Recently, Fan and Pan [**6**] considered the operators defined by (3) with smooth phase functions satisfying (1) and (2). They showed that

THEOREM A. *Let $1 < p < \infty$, T be defined as in* (3). *Suppose that Φ satisfies* (1) *and* (2) *for some $a \neq 0$. Then T is bounded on $L^p(\boldsymbol{R}^n)$ with bound $C(n,p)$.*

THEOREM B. *Let T be defined as in* (3). *Suppose that Φ satisfies* (1) *and* (2) *for some $a \neq 0, 1$. Then T is a bounded operator on the Hardy space $H^1(\boldsymbol{R}^n)$.*

The purpose of this paper is to consider a class of multilinear operators related to the operators defined by (3). Let m be a positive integer, K be C^1 away from the origin and satisfy

$$(4) \qquad |K(x)| \leq C|x|^{-n}, \quad |\nabla K(x)| \leq C|x|^{-n-1},$$

and

$$(5) \qquad \int_{a<|x|<b} K(x) x^\alpha\, dx = 0, \quad \text{for any } 0 < a < |x| < b < \infty \text{ and } |\alpha| = m.$$

Let A have derivatives of order m in $\mathrm{BMO}(\boldsymbol{R}^n)$, $R_{m+1}(A; x, y)$ denote the $(m+1)$-th order

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Taylor series remainder of A at x expanded about y, i.e.,

$$R_{m+1}(A;\, x, y) = A(x) - \sum_{|\alpha| \le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha.$$

The operators we consider here are of the form

$$T_A f(x) = \int_{\boldsymbol{R}^n} e^{i\Phi(x-y)} K(x-y) \frac{R_{m+1}(A; x, y)}{|x-y|^m} f(y)\, dy. \tag{6}$$

As well-known, operators of this type related to the standard Calderón-Zygmund singular integral operators were first studied by Cohen [**4**], and then by Cohen and Gosselin [**5**] and Hofmann [**9**]. If the phase functions are replaced by real-valued polynomials on $\boldsymbol{R}^n \times \boldsymbol{R}^n$, the corresponding multilinear operators have been considered by Chen, Hu and Lu [**3**]. Our first result in this paper can be stated as follows.

THEOREM 1. *Let m be a positive integer, $K(x)$ be C^1 away from the origin and satisfy* (4) *and* (5), *A have derivatives of order m in* BMO($\boldsymbol{R}^n$). *Let T_A be defined as in* (6). *Suppose that Φ satisfies* (1) *and* (2) *for some $a \neq 0$. Then for $1 < p < \infty$,*

$$\|T_A f\|_p \le C(n, m, p) \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_p.$$

Let $f^\sharp$ be the sharp function of Fefferman-Stein [**8**], i.e.,

$$f^\sharp(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - m_Q(f)|\, dy,$$

where $m_Q(f)$ is the mean value of f on Q. In this paper, we will establish the sharp function estimate for the operator T_A.

THEOREM 2. *Let m be a positive integer, $K(x)$ be C^1 away from the origin and satisfy* (4) *and* (5), *A have derivatives of order m in* BMO($\boldsymbol{R}^n$). *Let T_A be defined as in* (6). *Suppose that Φ satisfies* (1) *and* (2) *for some $a \neq 0, 1$. Then for any $1 < p < \infty$, there exists a positive constant $C_{m,n,p}$ such that*

$$(T_A f)^\sharp(x) \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} M_p f(x), \quad f \in L_0^\infty(\boldsymbol{R}^n),$$

where M is the Hardy-Littlewood maximal operator, and $M_p f(x) = [M(|f|^p)(x)]^{1/p}$.

As a consequence of Theorem 2, we have the following endpoint estimate for the operator T_A.

COROLLARY. *Under the hypotheses of Theorem* 2, *T_A maps $L^\infty(\boldsymbol{R}^n)$ to* BMO($\boldsymbol{R}^n$) *boundedly, with bound $C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}$.*

2. Proof of Theorem 1.

To begin with, we give some preliminary lemmas.

LEMMA 1. *Let m be a positive integer, $K(x)$ be C^1 away from the origin and satisfy* (4) *and* (5), *A have derivatives of order m in* $\mathrm{BMO}(\boldsymbol{R}^n)$. *Define the operator*

$$\tilde{T}_A f(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} K(x-y)\,\frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy\right|.$$

Then for any $1<p<\infty$,

$$\|\tilde{T}_A f\|_p \le C(n,m,p)\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_p.$$

For the case of $m=1$, this result has been obtained by Cohen [**4**]. For general positive integer m, Lemma 1 can be proved by repeating the argument used in [**4**], together with some computation techniques of Cohen and Gosselin [**5**].

LEMMA 2. (see [**5**]). *Let $b(x)$ be a function on $\boldsymbol{R}^n$ with derivatives of order m in $L^q(\boldsymbol{R}^n)$ for some $n<q\le\infty$. Then*

$$|R_m(b;x,y)| \le C_{m,n}|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}|D^\alpha b(z)|^q\,dz\right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x with diameter $5\sqrt{n}|x-y|$.

LEMMA 3. *Let A have derivatives of order m in* $\mathrm{BMO}(\boldsymbol{R}^n)$. *Then the maximal operator*

$$M_A f(x) = \sup_{r>0} r^{-n-m}\int_{|x-y|<r}|R_{m+1}(A;x,y)f(y)|\,dy,$$

is bounded on $L^p(\boldsymbol{R}^n)$ for $1<p<\infty$ with bound $C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}$.

PROOF. Clearly, it suffices to consider the operator

$$\tilde{M}_A f(x) = \sup_{r>0} r^{-n-m}\int_{r/2<|x-y|\le r}|R_{m+1}(A;x,y)f(y)|\,dy.$$

For fixed $x\in\boldsymbol{R}^n$ and $r>0$, let $Q(x,r)$ be the cube centered at x and having side length r. Set

$$\tilde{A}(y) = A(y) - \sum_{|\alpha|=m}\frac{1}{\alpha!}m_{Q(x,r)}(D^\alpha A)y^\alpha.$$

Note that for each fixed α with $|\alpha|=m$, $D^\beta y^\alpha = 0$ if $|\beta|\ge m+1$. Thus

$$R_{m+1}((\cdot)^\alpha;x,y) = x^\alpha - \sum_{|\beta|\le m}\frac{1}{\beta!}D^\beta(y^\alpha)(x-y)^\beta = 0, \quad |\alpha|=m,$$

which means

$$R_{m+1}(\tilde{A};x,y) = R_{m+1}(A;x,y) - \sum_{|\alpha|=m}\frac{1}{\alpha!}m_{Q(x,r)}(D^\alpha A)R_{m+1}((\cdot)^\alpha;x,y) = R_{m+1}(A;x,y).$$

Note that if $r/2 < |x-y| \le r$, then $\tilde{Q}(x,y)$, the cube centered at x with diameter $5\sqrt{n}|x-y|$, is contained in a fixed multiple of $Q(x,r)$. Thus by Lemma 2, it follows that for some $q > n$,

$$|R_m(\tilde{A};x,y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(|\tilde{Q}(x,y)|^{-1} \int_{\tilde{Q}(x,y)} |D^\alpha A(z) - m_{Q(x,r)}(D^\alpha A)|^q \, dz \right)^{1/q}$$

$$\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} |x-y|^m.$$

Thus for any $1 < t < \infty$,

$$\begin{aligned}\tilde{M}_A f(x) &\le \sup_{r>0} r^{-n-m} \int_{r/2<|x-y|<r} |R_m(\tilde{A};\, x,y)||f(y)|\,dy \\ &\quad + C \sum_{|\alpha|=m} \sup_{r>0} r^{-n} \int_{|x-y|<r} |D^\alpha A(y) - m_{Q(x,r)}(D^\alpha A)||f(y)|\,dy \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} Mf(x) \\ &\quad + C \sum_{|\alpha|=m} \sup_{r>0} r^{-n} \left(\int_{|x-y|<r} |D^\alpha A(y) - m_{Q(x,r)}(D^\alpha A)|^{t'} \, dy \right)^{1/t'} \\ &\quad \times \left(\int_{|x-y|<r} |f(y)|^t \, dy \right)^{1/t} \\ &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} M_t f(x).\end{aligned}$$

For each fixed p, $1 < p < \infty$, we choose t such that $1 < t < p$, then

$$\|\tilde{M}_A f\|_p \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_p.$$

Proof of Theorem 1. Let $\varphi \in C_0^\infty(\boldsymbol{R}^n)$ such that

$$\operatorname{supp} \varphi \subset \{1/2 \le |x| \le 2\} \quad \text{and} \quad \sum_{j=-\infty}^{\infty} \varphi(2^{-j}x) \equiv 1, \quad \text{for } |x| \ne 0.$$

Let $\varphi_j(x) = \varphi(2^{-j}x)$ for integer j. To prove Theorem 1, we consider the following two cases.

Case I $a > 0$. Let $\psi(x) = 1 - \sum_{j=1}^\infty \varphi_j(x)$. It is obvious that $\operatorname{supp}\psi \subset \{|x| \le 4\}$ and $\psi(x) \equiv 1$ if $|x| < 1$. Write

$$\begin{aligned}T_A f(x) &= \int_{\boldsymbol{R}^n} e^{i\Phi(x-y)} K(x-y)\psi(x-y) \frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy \\ &\quad + \sum_{j=1}^{\infty} \int_{\boldsymbol{R}^n} e^{i\Phi(x-y)} K(x-y)\varphi_j(x-y) \frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy \\ &= T_A^0 f(x) + \sum_{j=1}^{\infty} T_A^j f(x).\end{aligned}$$

Let us first consider the term T_A^0. Write

$$\begin{aligned}|T_A^0 f(x)| &\leq \left|\int_{|x-y|\leq 1} K(x-y)\psi(x-y)\,\frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy\right| \\ &\quad + \left|\int_{|x-y|\leq 1} (e^{i\Phi(x-y)}-1)K(x-y)\psi(x-y)\,\frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy\right| \\ &\quad + \left|\int_{|x-y|>1} e^{i\Phi(x-y)}K(x-y)\psi(x-y)\,\frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy\right| \\ &= \mathrm{E}+\mathrm{F}+\mathrm{G}.\end{aligned}$$

Recall that $\psi(x)\equiv 1$ for $|x|\leq 1$. Therefore,

$$\mathrm{E} = \left|\int_{|x-y|\leq 1} K(x-y)\,\frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy\right|.$$

Lemma 1 now tells us that

$$\|\mathrm{E}\|_p \leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_p,\quad 1<p<\infty.$$

On the other hand, by the fact that $a>0$ and (1), trivial computation shows that

$$\mathrm{F}\leq C\int_{|x-y|\leq 1}\frac{|R_{m+1}(A;\ x,y)|}{|x-y|^{n+m-a}}\,|f(y)|\,dy \leq CM_A f(x).$$

This via Lemma 3 leads to that

$$\|\mathrm{F}\|_p \leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_p,\quad 1<p<\infty.$$

Obviously,

$$\mathrm{G}\leq \int_{1\leq|x-y|\leq 4}\frac{|R_{m+1}(A;x,y)|}{|x-y|^{n+m}}\,|f(y)|\,dy \leq CM_A f(x),$$

which in turn implies that

$$\|\mathrm{G}\|_p \leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_p,\quad 1<p<\infty.$$

Combining the estimates for E, F and G yields that

$$\|T_A^0\|_p \leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_p,\quad 1<p<\infty.$$

Now we consider the operator T_A^j for $j\geq 1$. By Lemma 3 we have the following crude estimate

$$\|T_A^j f\|_p \leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_p,\quad 1<p<\infty. \tag{7}$$

Our goal is to obtain a refined L^2 estimate for T_A^j, i.e., we want to show that there exists a positive constant $\varepsilon > 0$ such that

$$\|T_A^j f\|_2 \le C2^{-\varepsilon j} \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_2. \tag{8}$$

If we can do this, an interpolation between the inequalities (7) and (8) then gives

$$\|T_A^j f\|_p \le C2^{-\tilde{\varepsilon} j} \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_p, \quad 1 < p < \infty.$$

Summing over the last inequality for all $j \ge 1$ gives

$$\left\| \sum_{j=1}^{\infty} T_A^j f \right\|_p \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_p, \quad 1 < p < \infty.$$

We turn our attention to the operator

$$\tilde{T}_A^j f(x) = \int_{1/2<|x-y|\le 2} e^{i\Phi(2^j(x-y))} K(x-y)\varphi(x-y) \frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy. \tag{9}$$

By dilation-invariance, we see that the inequality (8) is equivalent to the estimate

$$\|\tilde{T}_A^j f\|_2 \le C2^{-\varepsilon j} \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_2. \tag{10}$$

Write $\boldsymbol{R}^n = \cup_d Q_d$, where each Q_d is a cube with side length 1 and these cubes have disjoint interiors. Set $f_d = f\chi_{Q_d}$. Since the support of $\tilde{T}_A^j f_d$ is contained in a fixed multiple of Q_d, the supports of various terms $\tilde{T}_A^j f_d$ have bounded overlaps. So we have the "almost orthogonality" property

$$\|\tilde{T}_A^j f\|_2^2 \le \sum_d \|\tilde{T}_A^j f_d\|_2^2.$$

Thus we may assume that $\operatorname{supp} f \subset Q$ for some cube with side length 1. Denote by Q^* the cube with the same center as Q but side length $100n$. Let $\phi \in C_0^\infty(\boldsymbol{R}^n)$ such that $0 \le \phi \le 1$, ϕ is identically one on $10nQ$ and vanishes outside of $20nQ$, $\|D^\nu \phi\|_\infty \le C_\nu$ (independent of Q) for all multi-index ν. Let x_0 be a point on the boundary of $40nQ$. Set

$$A^\phi(y) = R_m\Big(A(\cdot) - \sum_{|\beta|=m} \frac{1}{\beta!} m_{Q^*}(D^\beta A)(\cdot)^\beta; y, x_0\Big)\phi(y).$$

The observation of Cohen and Gosselin [5] says that for $y \in Q$ and $x \in 10nQ$,

$$R_{m+1}(A;x,y) = R_{m+1}(A^\phi;x,y).$$

Define the operator

$$\tilde{T}_\alpha^j h(x) = \int_{1/2<|x-y|\le 2} e^{i\Phi(2^j(x-y))} K(x-y)\varphi(x-y) \frac{(x-y)^\alpha}{|x-y|^m} h(y)\,dy.$$

We see that

$$\begin{aligned}\tilde{T}_A^j f(x) &= \tilde{T}_{A^\phi}^j f(x)\\ &= A^\phi(x)\tilde{T}_0^j f(x) - \sum_{|\alpha|<m} \frac{1}{\alpha!}\tilde{T}_\alpha^j((D^\alpha A^\phi)f)(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}\tilde{T}_\alpha^j((D^\alpha A^\phi)f)(x)\\ &= \mathrm{H}+\mathrm{I}+\mathrm{J}.\end{aligned}$$

The estimates for these three terms follows from the following lemma.

LEMMA 4. *Suppose that Φ satisfies* (1) *and* (2). *Then for $j \in \boldsymbol{Z}$ and multi-index α*

$$\|\tilde{T}_\alpha^j h\|_2 \le C2^{-ja/2}\|h\|_2.$$

Lemma 4 can be proved by the same way as in [**6**]. We omit the details for brevity.

We now return to the proof of Theorem 1. Let α be a multi-index such that $|\alpha| \le m$. A straightforward computation (see [**5**, p. 452]) yields that

$$D^\alpha A^\phi(y) = \sum_{\alpha=\mu+\nu} \frac{\alpha!}{\mu!\,\nu!} R_{m-|\mu|}\left(D^\mu\left(A(\cdot) - \sum_{|\beta|=m}\frac{1}{\beta!} m_{Q^*}(D^\beta A)(\cdot)^\beta\right); y, x_0\right) D^\nu\phi(y). \tag{11}$$

Recall that $\operatorname{supp}\phi \subset 20nQ$, Lemma 2 now shows that if $|\alpha| < m$, then

$$\begin{aligned}|D^\alpha A^\phi(y)| &\le C\sum_{|\beta|=m}\left(|\tilde{Q}(y,x_0)|^{-1}\int_{\tilde{Q}(y,x_0)}\left|D^\beta A(z) - m_{Q^*}(D^\beta A)\right|^q dz\right)^{1/q}\\ &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}},\end{aligned}$$

where $n < q < \infty$. So by Lemma 4,

$$\|\mathrm{H}\|_2 \le C\|A^\phi\|_\infty\|\tilde{T}_0^j f\|_2 \le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}2^{-aj/2}\|f\|_2,$$

and

$$\begin{aligned}\|\mathrm{I}\|_2 &\le C\sum_{|\alpha|<m}\left\|\tilde{T}_\alpha^j\left((D^\alpha A^\phi)f\right)\right\|_2 \le C2^{-aj/2}\sum_{|\alpha|<m}\|D^\alpha A^\phi\|_\infty\|f\|_2\\ &\le C2^{-aj/2}\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_2.\end{aligned}$$

To estimate J, observe that

$$\|\tilde{T}_\alpha^j h\|_\infty \le C\|h\|_1,$$

which together with Lemma 4 gives

$$\|\tilde{T}_\alpha^j h\|_q \le C2^{-aj/q}\|h\|_{q'}, \quad 2 < q < \infty, \tag{12}$$

where $q' = q/(q-1)$. If $|\alpha| = m$, by (11) and Lemma 2 we have

$$|D^\alpha A^\phi(y)|$$
$$\le \sum_{\alpha=\mu+\nu,|\mu|<m} C_{\mu,\nu}\left|R_{m-|\mu|}\left(D^\mu\left(A(\cdot) - \sum_{|\beta|=m}\frac{1}{\beta!}m_{Q^*}(D^\beta A)(\cdot)^\beta\right); y, x_0\right)D^\nu\phi(y)\right|$$
$$+ \sum_{|\beta|=m}|(D^\beta A(y) - m_{Q^*}(D^\beta A))\phi(y)|$$
$$\le C\left(\sum_{|\beta|=m}\|D^\beta A\|_{\mathrm{BMO}} + \sum_{|\beta|=m}\left|D^\beta A(y) - m_{Q^*}(D^\beta A)\right|\right)\chi_{Q^*}(y).$$

Thus for any $1 < s < \infty$,

$$\|D^\alpha A^\phi\|_s \le C_s\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}.$$

Choose q_0, q_1 such that $2 < q_0 < \infty$ and $1/q_0' = 1/q_1 + 1/2$. Since supp $\tilde{T}^j_\alpha\left((D^\alpha A)f\right) \subset 20nQ$, it follows from the inequality (12) that

$$\|\mathrm{J}\|_2 \le C\|\mathrm{J}\|_{q_0} \le C\sum_{|\alpha|=m}\left\|\tilde{T}^j_\alpha\left((D^\alpha A^\phi)f\right)\right\|_{q_0}$$
$$\le C2^{-aj/q_0}\sum_{|\alpha|=m}\|(D^\alpha A^\phi)f\|_{q_0'}$$
$$\le C2^{-aj/q_0}\sum_{|\alpha|=m}\|D^\alpha A^\phi\|_{q_1}\|f\|_2$$
$$\le C2^{-aj/q_0}\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_2.$$

This is our desired estimate.

CASE II $a < 0$. Let $\eta(x) = 1 - \sum_{j=-\infty}^{-1}\varphi_j(x)$. Decompose T_A as

$$T_Af(x) = \int_{R^n} e^{i\Phi(x-y)}K(x-y)\eta(x-y)\frac{R_{m+1}(A; x, y)}{|x-y|^m}f(y)\,dy$$
$$+ \sum_{j=-\infty}^{-1}\int_{R^n} e^{i\phi(x-y)}K(x-y)\varphi_j(x-y)\frac{R_{m+1}(A; x, y)}{|x-y|^m}f(y)\,dy$$
$$= \bar{T}^0_Af(x) + \sum_{j=-\infty}^{-1}T^j_Af(x).$$

Noting that $\|\eta\|_\infty \le C$, $\eta(x) \equiv 1$ if $|x| > 1$, and $\eta(x) \equiv 0$ if $|x| < 1/2$, thus as in Case I, we have

$$
\begin{aligned}
|\bar{T}_A^0 f(x)| &\le \left| \int_{|x-y|>1} K(x-y) \frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy \right| \\
&\quad + \int_{|x-y|>1} \frac{|R_{m+1}(A;x,y)|}{|x-y|^{n+m-a}} |f(y)|\,dy \\
&\quad + \int_{1/2 \le |x-y| < 1} \frac{|R_{m+1}(A;x,y)|}{|x-y|^{n+m}} |f(y)|\,dy \\
&\le \left| \int_{|x-y|>1} K(x-y) \frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy \right| + CM_A f(x).
\end{aligned}
$$

So

$$
\|\bar{T}_A^0 f\|_p \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_p, \quad 1 < p < \infty.
$$

Similarly to Case I, it follows that

$$
\|T_A^j f\|_2 \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} 2^{-\varepsilon a j} \|f\|_2, \quad j \le -1,
$$

for some positive constant ε. Hence

$$
\|T_A f\|_p \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \|f\|_p, \quad 1 < p < \infty.
$$

This finishes the proof of Theorem 1 for the case $a < 0$.

3. Proof of Theorem 2.

To prove Theorem 2, we need the following lemma.

LEMMA 5 (see [6]). *Let* $0 < \delta < \infty$, $\xi \in C_0^\infty(\boldsymbol{R}^n)$ *such that* $\xi(x) \equiv 1$ *if* $1 \le |x| \le 2$ *and* $\xi(x) \equiv 0$ *if* $|x| < 3/4$ *or* $|x| > 4$. *Suppose that the real-valued* C^∞ *function* Φ *satisfies* (1) *and* (2). *Then there exists a small positive number* d, *such that for any cube* $Q \subset Q_0 = [-1/2, 1/2]^n$ *with diameter* $\operatorname{diam} Q < d$ *and positive integer* j *with* $2^j \ge 3\sqrt{n}$, *the operator*

$$
S_Q^j f(x) = \xi(2^{-j}x) \int_Q e^{i\Phi(\delta x - \delta y)} f(y)\,dy
$$

is bounded on $L^2(\boldsymbol{R}^n)$ *with bound* $C2^{jn/2}\left[(2^j\delta)^{a-1}\delta\right]^{-1/12}$, *where* C *is independent of* j, d *and* δ.

PROOF OF THEOREM 2. Without loss of generality, we may assume that $\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} = 1$. Let Q_0 be the cube centered at the origin with side length 1, i.e. $Q_0 = [-1/2, 1/2]^n$. For each fixed $\delta > 0$, define

$$
T_{\delta,A} f(x) = \int_{\boldsymbol{R}^n} e^{i\Phi(\delta x - \delta y)} K(x-y) \frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)\,dy.
$$

By Theorem 1 and dilation-invariance, we have

$$\|T_{\delta,A}f\|_p \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}}\|f\|_p, \quad 1<p<\infty.$$

Observe that if $b \in \mathrm{BMO}(\boldsymbol{R}^n)$, then for any $t>0$ and $z \in \boldsymbol{R}^n$, $b_t^z(x) = b(tx+z)$ also belongs to the space $\mathrm{BMO}(\boldsymbol{R}^n)$ and $\|b_t^z\|_{\mathrm{BMO}} = \|b\|_{\mathrm{BMO}}$. Thus it suffices to show that for some $c = c(T_{\delta,A}f)$,

$$\int_{Q_0} |T_{\delta,A}f(y) - c|\,dy \le C \inf_{x\in Q_0} M_pf(x), \quad 1<p<\infty,$$

with C independent of δ. Split f as

$$f = f\chi_{8\sqrt{n}Q_0} + f\chi_{\boldsymbol{R}^n\setminus 8\sqrt{n}Q_0} = f_1 + f_2.$$

Schwarz's inequality then shows that

$$\int_{Q_0} |T_{\delta,A}f_1(y)|\,dy \le \|T_{\delta,A}f_1\|_p \le C_{m,n,p}\|f_1\|_p \le C \inf_{x\in Q_0} M_pf_1(x).$$

Thus our proof can be reduced to proving that for some $c = c(T_{\delta,A}f)$,

$$\int_{Q_0} |T_{\delta,A}f_2(y) - c|\,dy \le C_{m,n,p} \inf_{x\in Q_0} M_pf(x), \quad 1<p<\infty. \tag{13}$$

We consider the following three cases.

Case I $a>1$. Set $r = \max\{8, \delta^{-a/(a-1)}\}$ and

$$f_2 = f\chi_{r\sqrt{n}Q_0\setminus 8\sqrt{n}Q_0} + f\chi_{\boldsymbol{R}^n\setminus r\sqrt{n}Q_0} = f_{21} + f_{22}.$$

We first estimate $T_{\delta,A}f_{21}$. Write

$$T_{\delta,A}h(x) = \int_{\boldsymbol{R}^n} [e^{i\Phi(\delta x-\delta y)} - e^{i\Phi(\delta y)}]K(x-y)\,\frac{R_{m+1}(A;x,y)}{|x-y|^m}\,h(y)\,dy$$
$$+ \int_{\boldsymbol{R}^n} K(x-y)\frac{R_{m+1}(A;x,y)}{|x-y|^m}\,e^{i\Phi(\delta y)}h(y)\,dy.$$

By (1) we see that if $x\in Q_0$, and $y \in \boldsymbol{R}^n\setminus 8\sqrt{n}Q_0$, then

$$|e^{i\Phi(\delta x-\delta y)} - e^{i\Phi(\delta y)}| \le C\delta^a|y|^{a-1}.$$

So we have that for any function h with $\operatorname{supp} h \subset \boldsymbol{R}^n\setminus 8\sqrt{n}Q_0$ and $x\in Q_0$,

$$|T_{\delta,A}h(x) - c| \le C\delta^a \int_{\boldsymbol{R}^n\setminus 8\sqrt{n}Q_0} \frac{|R_{m+1}(A;x,y)|}{|x-y|^{n+m+1-a}}\,|h(y)|\,dy \tag{14}$$
$$+ \left|\int_{\boldsymbol{R}^n} K(x-y)\frac{R_{m+1}(A;x,y)}{|x-y|^m}\,h(y)e^{i\Phi(\delta y)}\,dy - c\right|.$$

Using the techniques of Cohen and Gosselin [5], we can prove that for some c

$$\int_{Q_0}\left|\int_{\boldsymbol{R}^n} K(x-y)\,\frac{R_{m+1}(A;x,y)}{|x-y|^m}\,e^{i\Phi(\delta y)}f_{21}(y)\,dy-c\right|dx$$
$$\leq C_{m,n,p}\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\inf_{x\in Q_0}M_pf(x).$$

For each fixed $j\in\boldsymbol{N}$, set

$$A_j(y)=A(y)-\sum_{|\alpha|=m}\frac{1}{\alpha!}m_{2^j\sqrt{n}Q_0}(D^\alpha A)y^\alpha,$$

then $R_{m+1}(A;x,y)=R_{m+1}(A_j;x,y)$. Let $j_0\in\boldsymbol{N}$ such that $2^{j_0}<r\leq 2^{j_0+1}$. It is easy to find that for $x\in Q_0$,

$$\delta^a\int_{r\sqrt{n}Q_0\backslash 8\sqrt{n}Q_0}\frac{|R_{m+1}(A;x,y)|}{|x-y|^{m+n+1-a}}|f(y)|\,dy$$
$$\leq C\delta^a\sum_{j=1}^{j_0}\int_{2^{j+1}\sqrt{n}Q_0\backslash 2^j\sqrt{n}Q_0}\frac{|R_{m+1}(A_j;x,y)|}{|y|^{m+n+1-a}}|f(y)|\,dy$$
$$\leq C\delta^a\sum_{j=1}^{j_0}\int_{2^{j+1}\sqrt{n}Q_0\backslash 2^j\sqrt{n}Q_0}\frac{|R_m(A_j;x,y)|}{|y|^{n+m+1-a}}|f(y)|\,dy$$
$$+C\delta^a\sum_{|\alpha|=m}\int_{2^{j+1}\sqrt{n}Q_0\backslash 2^j\sqrt{n}Q_0}\frac{\left|D^\alpha A(y)-m_{2^j\sqrt{n}Q_0}(D^\alpha A)\right|}{|y|^{n+1-a}}|f(y)|\,dy$$
$$\leq C\delta^a\sum_{j=2}^{j_0}2^{j(a-1)}\inf_{x\in Q_0}M_pf(x)$$
$$\leq C\delta^a r^{a-1}\inf_{x\in Q_0}M_pf(x),\quad 1<p<\infty.$$

In the second-to-last inequality, we have invoked the fact that

$$|R_m(A_j;x,y)|\leq C|x-y|^m,\quad \text{if } x\in Q_0 \text{ and } y\in 2^{j+1}\sqrt{n}Q_0\backslash 2^j\sqrt{n}Q_0 \quad \text{for } j\geq 2.$$

If $T_{\delta,A}f_{21}\neq 0$, then $r>8$ and $r^{a-1}\delta^a=1$. Thus by (14) we see that for some c,

$$\int_{Q_0}|T_{\delta,A}f_{21}(y)-c|\,dy\leq C\inf_{x\in Q_0}M_pf(x),\quad 1<p<\infty.$$

Now we estimate $T_{\delta,A}f_{22}$. Let $K_A(y,z)=K(y-z)R_{m+1}(A;y,z)|y-z|^{-m}$, $y_0\in 3\sqrt{n}Q_0\backslash 2\sqrt{n}Q_0$. Then

$$T_{\delta,A}f_{22}(y)=\int_{\boldsymbol{R}^n}e^{i\Phi(\delta y-\delta z)}\Big[K_A(y,z)-K_A(y_0,z)\Big]f_{22}(z)\,dz$$
$$+\int_{\boldsymbol{R}^n}e^{i\Phi(\delta y-\delta z)}K_A(y_0,z)f_{22}(z)\,dz$$
$$=Rf_{22}(y)+Sf_{22}(y).$$

Obviously,

$$|Rf_{22}(y)| \le \int_{\mathbf{R}^n\setminus 8\sqrt{n}Q_0} |K_A(y,z) - K_A(y_0,z)|\,|f(z)|\,dz.$$

The standard arguement (see [5]) shows that

$$|Rf_{22}(y)| \le C \inf_{x\in Q_0} M_pf(x), \quad y \in Q_0, \quad 1 < p < \infty.$$

Let d be the small positive constant appeared in Lemma 5. Decompose Q_0 as

$$Q_0 = \bigcup_{k=1}^{N} Q_k,$$

where $N = N(d)$ is a fixed positive integer, each Q_k is a cube with diameter smaller than d, and the cubes $\{Q_k\}$ have disjoint interiors. Let $\xi \in C_0^\infty(\mathbf{R}^n)$ such that $\operatorname{supp}\xi \subset \{3/4 \le |x| \le 4\}$ and $\xi(x) \equiv 1$ if $1 \le |x| \le 2$. Denote by V_j the set $\{2^j < |x| \le 2^{j+1}\}$ for $j \in N$. Define the operator

$$T^j_{Q_k}h(x) = \chi_{Q_k}(x) \int_{\mathbf{R}^n} e^{i\Phi(\delta x - \delta y)}\xi(2^{-j}y)h(y)\,dy$$

and write

$$\begin{aligned} Sf_{22}(y)\chi_{Q_k}(y) &= \sum_{j=0}^{\infty} \chi_{Q_k}(y) \int_{2^j<|z|\le 2^{j+1}} e^{i\Phi(\delta y - \delta z)}\xi(2^{-j}z)K_A(y_0,z)f_{22}(z)\,dz \\ &= \sum_{j=0}^{\infty} T^j_{Q_k}\Big(K_A(y_0,\cdot)\chi_{V_j}f_{22}\Big)(y). \end{aligned}$$

By Lemma 5 and the duality, we see that

$$\|T^j_{Q_k}h\|_2 \le C2^{jn/2}\Big[(2^j\delta)^{a-1}\delta\Big]^{-1/12}\|h\|_2, \quad 2^j \ge 3\sqrt{n}. \tag{15}$$

On the other hand, we have the crude estimate

$$\|T^j_{Q_k}h\|_1 \le \|h\|_1. \tag{16}$$

Interpolation between the inequalities (15) and (16) leads to that

$$\|T^j_{Q_k}h\|_q \le C2^{jn/q'}\Big[(2^j\delta)^{a-1}\delta\Big]^{-1/(6q')}\|h\|_q, \quad 2^j \ge 3\sqrt{n}, \quad 1 < q \le 2. \tag{17}$$

For each fixed p, let $1 < q < \min(2,p)$. Then

$$\begin{aligned} \int_{Q_0} |Sf_{22}(y)|\,dy &\le C \sum_{k=1}^{N} \|(Sf_{22})\chi_{Q_k}\|_q \\ &\le C \sum_{k=1}^{N} \sum_{2^{j+1}\ge r\sqrt{n}} \|T^j_{Q_k}(K_A(y_0,\cdot)\chi_{V_j}f_{22})\|_q \\ &\le C \sum_{k=1}^{N} \sum_{j=j_0-1}^{\infty} 2^{jn/q'}\Big[(2^j\delta)^{a-1}\delta\Big]^{-1/(6q')}\|K_A(y_0,\cdot)\chi_{V_j}f_{22}\|_q. \end{aligned}$$

Set

$$\tilde{A}_j(y) = A(y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} m_{V_j}(D^\alpha A) y^\alpha, \quad j \in \boldsymbol{N}.$$

We have

$$\begin{aligned}\left\|K_A(y_0,\cdot)\chi_{V_j} f_{22}\right\|_q &= \|K_{\tilde{A}_j}(y_0,\cdot)\chi_{V_j} f_{22}\|_q \\ &\le C\left(\int_{V_j}\left(\frac{|R_m(\tilde{A}_j;\, y_0, y)|}{|y-y_0|^{n+m}}\,|f_{22}(y)|\right)^q dy\right)^{1/q} \\ &\quad + C\sum_{|\alpha|=m} 2^{-jn}\left(\int_{V_j}\left(|D^\alpha A(y) - m_{V_j}(D^\alpha)|\,|f_{22}(y)|\right)^q dy\right)^{1/q}.\end{aligned}$$

If $y \in V_j \cap (\boldsymbol{R}^n \backslash r\sqrt{n}Q_0)$, by the familiar arguement involving Lemma 2,

$$|R_m(\tilde{A}_j;\, y_0, y)| \le C|y-y_0|^m.$$

Hölder's inequality now gives

$$\begin{aligned}\left\|K_A(y_0,\cdot)\chi_{V_j} f_{22}\right\|_q &\le C2^{-jn/q'} \inf_{x\in Q_0} M_q f(x) + C2^{-jn/q'} \inf_{x\in Q_0} M_p f(x) \\ &\le C2^{-jn/q'} \inf_{x\in Q_0} M_p f(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{Q_0} |Sf_{22}|\, dy &\le C\sum_{k=1}^{N}\sum_{j=j_0-1}^{\infty}\left[(2^j\delta)^{a-1}\delta\right]^{-1/(6q')} \inf_{x\in Q_0} M_p f(x) \\ &\le C\left(\delta^a r^{a-1}\right)^{-1/(6q')} \inf_{x\in Q_0} M_p f(x) \le C \inf_{x\in Q_0} M_p f(x).\end{aligned}$$

Case II $a<1$, $a \ne 0$ and $\delta^{a/(1-a)} \le 4$. Note that for each $y \in Q_0$ and $1<p<\infty$,

$$\begin{aligned}&\int_{\boldsymbol{R}^n\backslash 8\sqrt{n}Q_0} \frac{|R_{m+1}(A;y,z)|}{|y-z|^{m+n+1-a}}\,|f(z)|\,dz \\ &\le \sum_{j=2}^{\infty}\int_{2^{j+1}\sqrt{n}Q_0\backslash 2^j\sqrt{n}Q_0} \frac{|R_m(A_j;y,z)|}{|y-z|^{m+n+1-a}}\,|f(z)|\,dz \\ &\quad + C\sum_{j=2}^{\infty}\sum_{|\alpha|=m}\int_{2^{j+1}\sqrt{n}Q_0\backslash 2^j\sqrt{n}Q_0} \frac{|D^\alpha A(z) - m_{2^j\sqrt{n}Q_0}(D^\alpha A)|}{|x-y|^{n+1-a}}\,|f(z)|\,dz \\ &\le C\sum_{j=2}^{\infty} 2^{(a-1)j} \inf_{x\in Q_0} M_p f(x) \le C \inf_{x\in Q_0} M_p f(x), \quad 1<p<\infty,\end{aligned}$$

and that for some c,

$$\int_{Q_0}\left|\int_{R^n\setminus 8\sqrt{n}Q_0} K(y-z)\frac{R_{m+1}(A;y,z)}{|y-z|^m}f(z)\,dz - c\right|dy$$
$$\leq C_{m,n,p}\inf_{x\in Q_0} M_p f(x), \quad 1<p<\infty.$$

The inequality (14) then gives our desired estimate (13) in this case.

Case III $a<1$, $a\neq 0$ and $\delta^{a/(1-a)}\geq 4$. Let $r=\delta^{a/(1-a)}$ and $j_0\in N$ such that $2^{j_0}<r\leq 2^{j_0+1}$. For $y\in Q_0$, we have

$$\delta^a\int_{R^n\setminus r\sqrt{n}Q_0}\frac{|R_{m+1}(A;y,z)|}{|y-z|^{m+n+1-a}}|f(z)|\,dz$$
$$=\delta^a\sum_{j=j_0}^{\infty}\int_{2^{j+1}\sqrt{n}Q_0\setminus 2^j\sqrt{n}Q_0}\frac{|R_m(A_j;y,z)|}{|y-z|^{m+n+1-a}}|f(z)|\,dz$$
$$+\delta^a\sum_{j=j_0}^{\infty}\sum_{|\alpha|=m}\int_{2^{j+1}\sqrt{n}Q_0\setminus 2^j\sqrt{n}Q_0}\frac{|D^\alpha A(z)-m_{2^j\sqrt{n}Q_0}(D^\alpha A)|}{|y-z|^{n+1-a}}|f(z)|\,dz$$
$$\leq C\delta^a\sum_{j=j_0}^{\infty}2^{j(a-1)}\inf_{x\in Q_0}M_pf(x)$$
$$\leq C\inf_{x\in Q_0}M_pf(x), \quad 1<p<\infty.$$

Again by (14), we see that for some c,

$$\int_{Q_0}|T_{\delta,A}(f\chi_{R^n\setminus r\sqrt{n}Q_0})(y)-c|\,dy\leq C\inf_{x\in Q_0}M_pf(x), \quad 1<p<\infty.$$

Using inequality (17), as in Case I, we can verify that

$$\int_{Q_0}|T_{\delta,A}(f\chi_{r\sqrt{n}Q_0\setminus 8\sqrt{n}Q_0})(y)|\,dy\leq C\sum_{j=1}^{j_0}\left[(2^j\delta)^{a-1}\delta\right]^{-\gamma}\inf_{x\in Q_0}M_pf(x)$$
$$\leq C\inf_{x\in Q_0}M_pf(x), \quad 1<p<\infty,$$

where γ is a positive constant. This finishes the proof of Theorem 2.

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