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Grassmann geometries on compact symmetric spaces of general type

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Introduction.

This study is a continuation of my papers [12], [13] and [14]. Let M be a compact simply connected riemannian symmetric space of dimension m (≥ 2), and s an integer such that $1 \leq s \leq m$. Let $G^s(T_pM)$ be the set of s-dimensional linear subspaces in a tangent space T_pM and denote by $G^s(TM)$ the Grassmann bundle over M with fibres $G^s(T_pM)$. For an arbitrary subset $\mathscr{V}$ in $G^s(TM)$ an s-dimensional connected submanifold S of M is called a *$\mathscr{V}$-submanifold* if at each point p of S the tangent space T_pS belongs to $\mathscr{V}$. The collection of $\mathscr{V}$-submanifolds, denoted by $\mathscr{S}(M, \mathscr{V})$, constitutes a *$\mathscr{V}$-geometry*. The term "Grassmann geometries" in the title is a collected name for such $\mathscr{V}$-geometries and it has been introduced in R. Harvey-H. B. Lawson [4].

We now consider the following $\mathscr{V}$-geometries. Let G be the isometry group of M. Then it acts transitively on M and at the same time acts on $G^s(TM)$ via the differentials of isometries. If as a subset $\mathscr{V}$ we take a G-orbit on $G^s(TM)$ by this action, the $\mathscr{V}$-geometry gives a class of submanifolds in M with congruent tangent spaces.

We moreover consider G-orbits of the following type. An s-dimensional linear subspace V in T_pM is called *strongly curvature-invariant* if it satisfies that

$$R_p(V, V)V \subset V \quad \text{and} \quad R_p(V^\perp, V^\perp)V^\perp \subset V^\perp,$$

where R denotes the curvature tensor on M and $V^\perp$ denotes the orthogonal complement of V in T_pM. We consider a G-orbit $\mathscr{V}$ through such a subspace V. The $\mathscr{V}$-geometry is also said to be of *strongly curvature-invariant type*. By a result on symmetric space a $\mathscr{V}$-geometry of strongly curvature-invariant type has a unique compact totally geodesic $\mathscr{V}$-submanifold, except the difference by isometries.

In the previous papers we have treated the cases that G is a simple Lie group, and have decided the $\mathscr{V}$-geometries which admit non-totally geodesic $\mathscr{V}$-submanifolds. In the present paper we treat a general case that the Lie group G is not necessarily simple, and we obtain a decomposition theorem for $\mathscr{V}$-submanifolds, thus, for $\mathscr{V}$-geometries of strongly curvature-invariant type. The theorem will clarify the structure of $\mathscr{V}$-geometries of strongly curvature-invariant type, and as a result it will give the classification of symmetric submanifolds in a general compact simply connected riemannian symmetric space.

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§ 1. Preliminaries.

In the following we consider a compact simply connected riemannian symmetric space M and a $\mathscr{V}$-geometry $\mathscr{S}(M,\mathscr{V})$ of strongly curvature-invariant type.

Let $\mathscr{V}_i$'s be the connected components of G-orbit $\mathscr{V}$. They are then realized as orbits by the identity component G^0 of G, where the group G/G^0 acts transitively on the collection $\{\mathscr{V}_i\}$ of G^0-orbits, thus, on the collection of $\mathscr{V}_i$-geometries. Also, since a $\mathscr{V}$-submanifold is always connected, the $\mathscr{V}$-geometry is the union of $\mathscr{V}_i$-geometries. From this observation we hereafter consider only the $\mathscr{V}$-geometries obtained from G^0-orbits.

A $\mathscr{V}$-geometry $\mathscr{S}(M,\mathscr{V})$ of strongly curvature-invariant type is said to be *not substantial* if M is a product of symmetric spaces M_1 and M_2, and it moreover holds that $V \subset TM_1$ for any element V in $\mathscr{V}$. Otherwise it is said to be *substantial.*

Assume that $\mathscr{S}(M,\mathscr{V})$ is not substantial and let G_1^0 be the identity component of isometries on M_1. Then there exists a G_1^0-orbit $\mathscr{V}_1$ over M_1 such that $\mathscr{S}(M,\mathscr{V}) = \{S_1 \times \{q\};\ S_1 \in \mathscr{S}(M_1,\mathscr{V}_1), q \in M_2\}$. We can thus regard the geometry $\mathscr{S}(M,\mathscr{V})$ as a geometry over M_1 of strongly curvature-invariant type. So we consider only substantial ones.

A compact semisimple Lie algebra $\mathfrak{g}$ with involution σ, denoted by $(\mathfrak{g},\sigma)$, is called a *symmetric Lie algebra* if the $(+1)$-eigenspace of σ acts faithfully on the (-1)-eigenspace of σ by the adjoint action. Moreover an ordered pair of symmetric Lie algebras $(\mathfrak{g},\sigma)$ and $(\mathfrak{g},\tau)$ is called a *pairwise symmetric Lie algebra*, abbreviated to PSLA, if the involutions σ and τ are commutative. Hereafter a PSLA is denoted by $(\mathfrak{g},\sigma,\tau)$.

Let $\mathscr{S}(M,\mathscr{V})$ be a substantial geometry. Fix a point p in M and take an element V in $\mathscr{V}$ such that $V \subset T_pM$. Moreover let s_p be the symmetry at p of symmetric space M and t_p be the involutive isometry of M satisfying $t_p(p) = p$ and $(dt_p)_p(v) = -v$, or $(dt_p)_p(v) = v$ according as $v \in V$ or $v \in V^\perp$. The existance of t_p is assured by the strong curvature-invariance of V. We now take the Lie algebra $\mathfrak{g}$ of G and define involutions σ and τ respectively as the differentials of inner automorphisms on G induced from s_p and t_p. Then $(\mathfrak{g},\sigma,\tau)$ is a PSLA. The substantiality of geometry gives a necessary and sufficient condition that $(\mathfrak{g},\tau)$ is a symmetric Lie algebra. This construction, independently of the fixed point p, gives a one-to-one correspondence between the collection of substantial geometries over M and the equivalence classes of PSLA's with the underlying symmetric Lie algebra $(\mathfrak{g},\sigma)$. Here the equivalence is the one by inner automorphisms. (cf. See [**10**] for details.)

Next a substantial geometry $\mathscr{S}(M,\mathscr{V})$ is said to be *not strong* if there exists a product factor M_1 of M, such that $T_pM_1 \subset V$ for any p in M and any V in $\mathscr{V}$ such that $V \subset T_pM$. (Here we admit the case $M_1 = M$.) Otherwise it is said to be *strong*. A substantial geometry $\mathscr{S}(M,\mathscr{V})$ is strong if and only if a geometry $\mathscr{S}(M,\mathscr{V}^\perp)$ is substantial where $\mathscr{V}^\perp$ is the G^0-orbit consisting of orthogonal complements of elements in $\mathscr{V}$. This is moreover equivalent to the fact that the pair $(\mathfrak{g},\sigma\tau)$ is also a symmetric Lie algebra. So, the collection of strongly substantial geometries over M corresponds to the equivalence classes of PSLA's with $(\mathfrak{g},\sigma)$, such that pairs $(\mathfrak{g},\sigma\tau)$ are also symmetric Lie algebras. In the following such a PSLA is called *of strong type*.

REMARK. From a PSLA $(\mathfrak{g},\sigma\tau)$ of strong type we can successively construct the following five PSLA's of strong type: $(\mathfrak{g},\sigma,\sigma\tau)$, $(\mathfrak{g},\tau,\sigma)$, $(\mathfrak{g},\tau,\sigma\tau)$, $(\mathfrak{g},\sigma\tau,\sigma)$ and $(\mathfrak{g},\sigma\tau,\tau)$.

The collection of these six PSLA's is called a *family* and it has much contributed to the classification of PSLA's for the case $\mathfrak{g}$ is simple. (See the previous papers.)

§ 2. The decomposition of substantial geometries.

We first determine the irreducible PSLA's and by using the result we study the decomposition of substantial geometry.

A PSLA is called *irreducible* if it is not decomposed into a sum of more than two PSLA's. A PSLA with simple Lie algebra is irreducible, but an irreducible PSLA does not always have a simple Lie algebra.

LEMMA 2.1. *Let $(\mathfrak{g}, \sigma, \tau)$ be an irreducible PSLA. Then $\mathfrak{g}$ is a compact simple Lie algebra, or a direct sum of two or four copies of a compact simple Lie algebra.*

PROOF. Let $\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i$ be the decomposition into compact simple Lie algebras. Since σ is involutive, we may moreover assume that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s \oplus \mathfrak{g}_{s+1} \oplus \cdots \oplus \mathfrak{g}_{s+t} \oplus \mathfrak{g}_{s+t+1} \oplus \cdots \oplus \mathfrak{g}_{s+2t} \quad (s + 2t = r)$$

where $\sigma(\mathfrak{g}_i) = \mathfrak{g}_i$ $(1 \leq i \leq s)$ and $\sigma(\mathfrak{g}_{s+i}) = \mathfrak{g}_{s+t+i}$ $(1 \leq i \leq t)$.

If $s > 0$, then $\sigma(\mathfrak{g}_1) = \mathfrak{g}_1$. Since τ is an involution commutative to σ, there exists an index j $(1 \leq j \leq s)$ such that $\tau(\mathfrak{g}_1) = \mathfrak{g}_j$. If $j = 1$, the object $(\mathfrak{g}_1, \sigma, \tau)$ is a PSLA, and if $j \neq 1$, the object $(\mathfrak{g}_1 \oplus \mathfrak{g}_j, \sigma, \tau)$ is so. The irreducibility of $(\mathfrak{g}, \sigma, \tau)$ implies that $\mathfrak{g} = \mathfrak{g}_1$ if $j = 1$ and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_j$ if $j \neq 1$. Noting that in the second case $\mathfrak{g}_j$ is isomorphic to $\mathfrak{g}_1$, we can see that in these cases $\mathfrak{g}$ is a compact simple Lie algebra or a direct sum of two copies of a compact simple Lie algebra.

If $s = 0$, then $\sigma(\mathfrak{g}_1) = \mathfrak{g}_{t+1}$. Moreover if τ preserves $\mathfrak{g}_1 \oplus \mathfrak{g}_{t+1}$, similarly as the above argument we can see that $\mathfrak{g}$ is a direct sum of two copies of a compact simple Lie algebra. Otherwise, by the commutativity of σ and τ there exists an index j $(2 \leq j \leq t)$ such that $\tau(\mathfrak{g}_1 \oplus \mathfrak{g}_{t+1}) = \mathfrak{g}_j \oplus \mathfrak{g}_{t+j}$. Again, the irreducibility of PSLA $(\mathfrak{g}, \sigma, \tau)$ implies that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_{t+1} \oplus \mathfrak{g}_j \oplus \mathfrak{g}_{t+j}$. Noting that these Lie algebras are all isomorphic, we can see that $\mathfrak{g}$ is a direct sum of four copies of a compact simple Lie algebra. □

According to the cases of Lemma 2.1 we determine the irreducible PSLA's. In the following let $(\mathfrak{g}, \sigma, \tau)$ be an irreducible PSLA, and let $\mathscr{S}(M, \mathscr{V})$ and N be the corresponding substantial geometry and the compact totally geodesic $\mathscr{V}$-submanifold.

The case (A) that $\mathfrak{g}$ is simple. As described above, such a PSLA, called a *simple* PSLA, is always irreducible. The simplicity of $\mathfrak{g}$ also implies that the simple PSLA is of strong type if and only if $\sigma \neq \tau$. In the case $\sigma = \tau$, the G^0-orbit $\mathscr{V}$ is the tangent bundle TM and thus the substantial geometry $\mathscr{S}(M, \mathscr{V})$ consists of all the connected open subsets in M. Particularly N is M itself. On the other hand the simple PSLA's of strong type have been classified in the previous papers. The associated symmetric spaces M exhaust the irreducible ones of not group type and the compact totally geodesic $\mathscr{V}$-submanifolds N have been also determined in the papers.

The case (B) that $\mathfrak{g}$ is a direct sum of two copies of a compact simple Lie algebra. Let $\mathfrak{l}$ be a compact simple Lie algebra and put $\mathfrak{g} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ where $\mathfrak{l}_1 = \mathfrak{l}_2 = \mathfrak{l}$. In this case the following three subcases are considerable from the proof of Lemma 2.1; the subcase

(B1) that σ preserves $\mathfrak{l}_i$'s and τ exchanges $\mathfrak{l}_i$'s, the subcase (B2) that σ exchanges $\mathfrak{l}_i$'s and τ preserves $\mathfrak{l}_i$'s, and the subcase (B3) that both σ and τ exchange $\mathfrak{l}_i$'s.

The subcase (B1). Suppose that $\tau(X, Y) = (Y, X)$ for $X, Y \in \mathfrak{l}$ and let σ_i $(i = 1, 2)$ be the restrictions of σ on $\mathfrak{l}_i$, which induce involutions on $\mathfrak{l}$. Then by the commutativity of σ and τ, it follows that $\sigma_1 = \sigma_2$ on $\mathfrak{l}$. Hence a PSLA in this case is, by an automorphism of $\mathfrak{g}$, equivalent to a PSLA with the following σ and τ:

$$\sigma(X, Y) = (\hat{\sigma}(X), \hat{\sigma}(Y)), \quad \tau(X, Y) = (Y, X)$$

where $\hat{\sigma}$ is a nontrivial involution of $\mathfrak{l}$. A PSLA of this type is obviously of strong type. Moreover, taking the simply connected symmetric space M^* of not group type which corresponds to $(\mathfrak{l}, \hat{\sigma})$, we can see that $M = M^* \times M^*$ and N is the diagonal subset in $M^* \times M^*$.

The subcase (B2). Suppose that $\sigma(X, Y) = (Y, X)$ for $X, Y \in \mathfrak{l}$. Then by the same way as (B1), a PSLA in this case is, by an automorphism of $\mathfrak{g}$, equivalent to a PSLA with the following σ and τ:

$$\sigma(X, Y) = (Y, X), \quad \tau(X, Y) = (\hat{\tau}(X), \hat{\tau}(Y))$$

where $\hat{\tau}$ is a nontrivial involution of $\mathfrak{l}$. A PSLA of this type is also of strong type. Moreover, taking the simply connected symmetric space M^* of not group type which corresponds to $(\mathfrak{l}, \hat{\tau})$, we can see that M is the universal covering $\hat{L}^*$ of the identity component of isometry group of M^*, and N is the image of Cartan imbedding $M^* \to \hat{L}^*$.

The subcase (B3). Suppose that $\sigma(X, Y) = (Y, X)$ for $X, Y \in \mathfrak{l}$. Then by the same way as (B1), a PSLA in this case is, by an automorphism of $\mathfrak{g}$, equivalent to a PSLA with the following σ and τ:

$$\sigma(X, Y) = (Y, X), \quad \tau(X, Y) = (\hat{\tau}(Y), \hat{\tau}(X))$$

where $\hat{\tau}$ is an involution of $\mathfrak{l}$. (We here admit the case that $\hat{\tau}$ is identical.) A PSLA of this type is of strong type if and only if $\hat{\tau}$ is not identical, that is, $\sigma \neq \tau$. Moreover in the case of strong type, taking the same M^* and $\hat{L}^*$ as in (B2), we can see that M is also $\hat{L}^*$ and N is the compact subgroup $\hat{K}^*$ such that $M^* = \hat{L}^*/\hat{K}^*$.

The case (C) that $\mathfrak{g}$ is a direct sum of four copies of a compact simple Lie algebra. Let $\mathfrak{l}$ be a compact simple Lie algebra and put $\mathfrak{g} = \bigoplus_{i=1}^4 \mathfrak{l}_i$ where $\mathfrak{l}_i = \mathfrak{l}$ $(1 \leq i \leq 4)$. In this case, taking account of the proof of Lemma 2.1, we suppose that

$$\tau(X, Y, Z, W) = (Z, W, X, Y), \quad \sigma(\mathfrak{l}_1) = \mathfrak{l}_2, \quad \sigma(\mathfrak{l}_3) = \mathfrak{l}_4.$$

Then by the commutativity of σ and τ, there exists an automorphism φ of $\mathfrak{l}$ such that

$$\sigma(X, Y, Z, W) = (\varphi(Y), \varphi^{-1}(X), \varphi(W), \varphi^{-1}(Z)).$$

Hence a PSLA in this case is, by an automorphism of $\mathfrak{g}$, equivalent to the PSLA with these σ and τ. A PSLA of this type is obviously of strong type. Moreover we can see that M is, as symmetric space, isomorphic to the product group $L \times L$ of the simply connected compact Lie group L with Lie algebra $\mathfrak{l}$, and N is also the diagonal subgroup in $L \times L$ which is isomorphic to L.

REMARK. Fixing an irreducible symmetric space M^* which appears in the case (B), we can easily see that three PSLA's of subcases (B1), (B2), and (B3) constitute a family. Similarly, fixing a simple Lie algebra $\mathfrak{l}$ in the case (C), we can see that the PSLA in the case constitutes a family by itself.

Now, before we describe the decomposition theorem of substantial geometries, we review four examples of substantial geometries $\mathscr{S}(M, \mathscr{V})$ which admit non-totally geodesic $\mathscr{V}$-submanifolds. See the previous papers for details.

EXAMPLE 1. Let M be the $m(\geq 2)$-dimensional standard sphere S^m and fix an integer s such that $1 \leq s \leq m-1$. Then, if we put $\mathscr{V} = G^s(TM)$, it is only the G^0-orbit and defines a strongly substantial geometry $\mathscr{S}(M, \mathscr{V})$, which consists of all the s-dimensional connected submanifolds of M. The corresponding PSLA of strong type belongs to the case (A) if $m \neq 3$, the case (B2) if $m = 3$ and $s = 2$, and the case (B3) if $m = 3$ and $s = 1$.

EXAMPLE 2. Let M be the n-dimensional complex projective space $\boldsymbol{C}P^n$. (In this case $m = 2n$.) Then the following two geometries are considerable:

Complex type. Fix an integer t such that $1 \leq t \leq n-1$ and let $\mathscr{V}$ be the set of t-dimensional tangential complex subspaces of M. (In this case $s = 2t$.) Then it is a G^0-orbit and defines a strongly substantial geometry which consists of all the t-dimensional connected complex submanifolds of M. The corresponding PSLA of strong type belongs to the case (A).

Totally real type. Let $\mathscr{V}$ be the set of n-dimensional tangential totally real subspaces of M. Then it is a G^0-orbit and defines a strongly substantial geometry which consists of all the n-dimensional connected totally real submanifolds of M. The corresponding PSLA of strong type also belongs to the case (A).

EXAMPLE 3. Let M be the n-dimensional quaternion projective space $\boldsymbol{H}P^n$. (In this case $m = 4n$.) Then the following geometry is considerable: Let $\mathscr{V}$ be the set of $2n$-dimensional tangential totally complex subspaces of M. Then it is a G^0-orbit and defines a strongly substantial geometry which consists of all the $2n$-dimensional connected totally complex submanifolds of M. The corresponding PSLA of strong type also belongs to the case (A).

EXAMPLE 4. The geometries in this case are closely related to the irreducible symmetric R-spaces. They are (not necessarily simply connected) compact symmetric spaces and contain the irreducible compact hermitian symmetric spaces.

First let $\hat{M}$ be a compact irreducible hermitian symmetric space and $\hat{N}$ be a real form of $\hat{M}$, i.e., a half-dimensional complete totally real totally geodesic submanifold. The irreducible symmetric R-spaces of not hermitian type exhaust such real forms. (See M. Takeuchi **[16]**.) Then a tangent space $T_p\hat{N}$ is strongly curvature-invariant and it induces a substantial geometry of strong type. The real form $\hat{N}$ is a unique compact totally geodesic submanifold contained in the geometry. Let $(\mathfrak{g}, \tau, \sigma)$ be the PSLA which corresponds to the geometry, and consider the PSLA $(\mathfrak{g}, \sigma, \tau)$ which belongs to the same family. Then the geometry which corresponds to the new PSLA admits non-totally geodesic submanifolds, and a unique compact totally geodesic submanifold in it is locally isometric to $\hat{N}$. The new geometry obviously belongs to the case (A).

Next consider the case that M^* in the subcase (B1) is an irreducible compact hermitian symmetric space. Then M^* is the diagonal real form in the product space $M^* \times M^*$, provided that the complex structure on the second M^* is the minus multiple of the complex structure on the first M^*. In this case the same arguments as above hold, and the obtained new geometry belongs to the subcase (B2).

A typical example of these geometries is the hypersurface geometry of S^n, which also appeared in Example 1. If $n \neq 3$, it is the first case constructed from the real form S^{n-1} of the $(n-1)$-dimensional complex quadric space CQ^{n-1}, and if $n = 3$, it is the second case constructed from the real form CP^1 $(= S^2)$ of the product space $CP^1 \times CP^1$.

We now give a decomposition theorem of substantial geometries. Let $\mathscr{S}(M, \mathscr{V})$ be a substantial geometry with PSLA $(\mathfrak{g}, \sigma, \tau)$ and decompose the PSLA into irreducible PSLA's $(\mathfrak{g}_i, \sigma_i, \tau_i)$, i.e.,

$$(\mathfrak{g}, \sigma, \tau) = \bigoplus_{i=1}^{r} (\mathfrak{g}_i, \sigma_i, \tau_i).$$

Moreover denote by $\mathscr{S}(M_i, \mathscr{V}_i)$ the substantial geometries which correspond to the PSLA's $(\mathfrak{g}_i, \sigma_i, \tau_i)$. Then we have the following.

THEOREM 2.2. *Let S be a submanifold in $\mathscr{S}(M, \mathscr{V})$. Then,*

(1) *for any point p in S there exist submanifolds S_i in $\mathscr{S}(M_i, \mathscr{V}_i)$ such that near p the imbedding $S \to M$ is the product of the imbeddings $S_i \to M_i$. Moreover, if S is complete, there exist complete submanifolds S_i in $\mathscr{S}(M_i, \mathscr{V}_i)$ such that the imbedding $S \to M$ is the global product of the imbeddings $S_i \to M_i$.*

(2) *Under the above product decomposition, if $\mathscr{S}(M_i, \mathscr{V}_i)$ is none of Examples* 1 *through* 4, *the imbedding $S_i \to M_i$ is always totally geodesic.*

PROOF. We first show the claim (1). Identifying the ambient space M with the product space $M_1 \times \cdots \times M_r$, we have the orbit decomposition $\mathscr{V} = \mathscr{V}_1 \times \cdots \times \mathscr{V}_r$. By this decomposition, each tangent space T_qS is uniquely decomposed into the sum of subspaces S_q^i which respectively belong to $\mathscr{V}_i$, i.e., $T_qS = S_q^1 \oplus \cdots \oplus S_q^r$. This moreover induces the following decomposition of the tangent bundle TS:

$$TS = S^1 \oplus \cdots \oplus S^r \tag{2.1}$$

where $S^i = \sum_{q \in S} S_q^i \subset TM_i$. Similarly, taking the bundles $(S^i)^\perp$ orthogonal to S^i in the bundles TM_i, we have the following decomposition of the normal bundle NS:

$$NS = (S^1)^\perp \oplus \cdots \oplus (S^r)^\perp. \tag{2.2}$$

Denote by α the second fundamental form of the imbedding $S \to M$, and by D and D^S the riemannian connections on M and S, respectively. We now show that

$$D_X^S S^i \subset S^i \quad \text{and} \quad \alpha(TS, S^i) \subset (S^i)^\perp \tag{2.3}$$

for a TS-valued vector field X on S and any index i. In fact, it holds by Gauss formula that

$$D_X Y = D_X^S Y + \alpha(X, Y)$$

for any S^i-valued vector field Y on S. We here note that Y is TM_i-valued. Since the holonomy group of M is the direct product of holonomy groups of M_j's, the parallel translations with respect to D preserve each TM_j. This implies that the vector field $D_X Y$ is also TM_i-valued, and thus it follows by (2.1) and (2.2) that $D^S_X Y$ is S^i-valued and $\alpha(X, Y)$ is $(S^i)^\perp$-valued. These prove (2.3).

As a direct result of (2.3) we obtain that

$$R^S(X, Y)S^i \subset S^i \tag{2.4}$$

for TS-valued vector fields X and Y, where R^S is the curvature tensor on S, and moreover obtain that

$$\alpha(S^i, S^j) = 0 \tag{2.5}$$

for distinct indices i and j. From (2.4) the de Rham decomposition theorem assures that there exist submanifolds S_i in $\mathscr{S}(M_i, \mathscr{V}_i)$ such that near a given point p, $S = S_1 \times \cdots \times S_r$ and $TS_i = S^i$. Along arguments in J. D. Moore [**8**] together with (2.5), we can see that near p, the imbedding $S \to M$ is the product of imbeddings $S_i \to M_i$.

In the case that S is complete, we also obtain our claim along Moore's arguments, where if necessary, we take the universal covering of S.

Next we consider the claim (2). Consider an irreducible substantial geometry $\mathscr{S}(M_i, \mathscr{V}_i)$. If it is not of strong type, it always consists of only totally geodesic submanifolds. We thus assume that it is of strong type and none of geometries in Examples 1 through 4. If it is the case (A), we have proved in the previous papers that it does not admit non-totally geodesic submanifolds. For the cases (B) or (C), we will show the fact in the following sections. □

Remark. In the above theorem, a substantial geometry $\mathscr{S}(M, \mathscr{V})$ is of strong type if and only if each irreducible geometry $\mathscr{S}(M_i, \mathscr{V}_i)$ is also of strong type.

Let M be a compact simply connected symmetric space. Then a connected submanifold S of M is called a *symmetric submanifold* if for any point p in S there exists an extrinsic symmetry t_p of S at p, i.e., it is an isometry on M which satisfies

$$t_p(p) = p, \quad t_p(S) = S, \quad \text{and} \quad (dt_p)(v) = \begin{cases} -v & \text{if } v \in T_pS, \\ v & \text{if } v \in N_pS. \end{cases}$$

As a corollary of Theorem 2.2, we obtain the classification of symmetric submanifolds of M by the following facts:

(1) A symmetric submanifold belongs to some geometry of strongly curvature-invariant type ([**15**]);

(2) A compact totally geodesic submanifold which belongs to a geometry of strongly curvature-invariant type is always a symmetric submanifold ([**15**]);

(3) The symmetric submanifolds which belong to the geometries of Examples 1 through 4 have been classified up to non-totally geodesic ones.

The classifications in (3) have been given by many people: Ferus for the geometries of Example 1, Takagi-Nakagawa for ones of Example 2 (complex type), the author-Takeuchi for ones of Example 2 (totally real type), Tsukada for ones of Example 3, and the author for ones of Example 4. (cf. See [**15**] for Examples 1 through 3, and [**10**, **13**, or **14**] for Example 4.)

§3. The irreducible substantial geometries.

In this section we consider whether a given irreducible substantial geometry of strong type admits non-totally geodesic submanifolds, or not. As described above, the problem is solved for the geometries of case (A). We now study it for the geometries of cases (B) and (C).

We first review the structure of PSLA's of strong type, which has been explained in [**13**] or [**14**]. Let $(\mathfrak{g}, \sigma, \tau)$ be a PSLA of strong type and decompose $\mathfrak{g}$ into the sum of (± 1)-eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ of σ. Moreover by τ decompose $\mathfrak{k}$ and $\mathfrak{p}$ into the sums of (± 1)-eigenspaces $\mathfrak{k}_\pm$ and $\mathfrak{p}_\pm$, respectively, i.e.,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{k} = \mathfrak{k}_+ \oplus \mathfrak{k}_-, \quad \mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-.$$

Then $\mathfrak{t}_+$ is a subalgebra of $\mathfrak{g}$ and vector spaces $\mathfrak{t}_-, \mathfrak{p}_\pm$ are $\mathfrak{t}_+$-modules. Take a maximal abelian subspace $\mathfrak{h}_+$ in $\mathfrak{k}_+$ and fix a fundamental root system $\Pi(\mathfrak{k}_+)$ $(\subset \sqrt{-1}\mathfrak{h}_+)$ of semi-simple part of $(\mathfrak{k}_+)^C$. We then make the Dynkin diagram of $\Pi(\mathfrak{k}_+)$, and next represent the minus nonzero dominant weights of $\mathfrak{k}_+$-modules $\mathfrak{k}_-^C$, $\mathfrak{p}_+^C$ and $\mathfrak{p}_-^C$ by cross circles $\otimes^{\mu_j}$, $\otimes^{\nu_j}$, $\otimes^{v_j}$ with lavels μ, ν and v, while the zero dominant weights are represented by black circles $\bullet^{\mu_j}$, $\bullet^{\nu_j}$, $\bullet^{v_j}$. All dominant weights are counted up to the multiplicity. Here the term *zero dominant weight* is used for the case that a $\mathfrak{k}_+$-module has the nonzero submodule on which $\mathfrak{k}_+$ acts trivially, and the multiplicity implies the dimension of the submodule. We note that the multiplicity of nonzero dominant weight is always one. Next we put lines and arrows between fundamental roots and minus dominant weights according to the rule of Dinkin diagram. An obtained figure is said to be a *P-figure* associated with PSLA. The PSLA's which belong to a family have the same subalgebra $\mathfrak{k}_+$ and the $\mathfrak{k}_+$-modules $\mathfrak{k}_-$, $\mathfrak{p}_\pm$ which coincide except order. So the associated P-figures are the same ones except lavels μ, ν, and v.

We have seen in the previous papers [**13**, **14**] that the simple PSLA's of strong type and their families, appeared in the case (A), are decided only by the P-figures. This is true even for irreducible PSLA's of strong type, thus, for general ones of strong type, which will be seen after the decision of the P-figures associated with irreducible PSLA's in the cases (B) and (C).

REMARK. The irreducible PSLA's of strong type in the cases (B) and (C) are always of *outer* type, i.e., rank $\mathfrak{k}_+ \neq \operatorname{rank} \mathfrak{g}$.

We next review a sufficient condition for a substantial geometry of strong type to be constituted by only totally geodesic submanifolds.

For a PSLA $(\mathfrak{g}, \sigma, \tau)$ of strong type we define a $\mathfrak{k}_+$-homomorphism ρ of $\mathfrak{p}_-^* \otimes \mathfrak{k}_-$ to $\wedge^2(\mathfrak{p}_-^*) \otimes \mathfrak{p}_+$ in the following way:

$$\rho(\lambda)(x, y) = [\lambda(x), y] - [\lambda(y), x]$$

for $\lambda \in \mathfrak{p}_-^* \otimes \mathfrak{k}_-$ and $x, y \in \mathfrak{p}_-$. Here $(\ )^*$ denotes the dual space of vector space. The sufficient condition is given by the following lemma.

LEMMA 3.1 ([12]). *If the homomorphism ρ is injective, the geometry associated with $(\mathfrak{g}, \sigma, \tau)$ is constituted by only totally geodesic submanifolds.*

REMARK. We have seen in the previous papers [**12**, **13**, and **14**] that for a geometry of case (A) this condition is also a necessary condition. After deciding the injectivity of ρ for geometries of cases (B) and (C), we will see from Theorem 2.2 that even for a general geometry of strong type it is a necessary and sufficient condition.

We now study the PSLA's of cases (B) and (C). We first consider the case (C) and next consider the case (B).

Let $(\mathfrak{g}, \sigma, \tau)$ be an irreducible PSLA of case (C) and suppose that σ and τ are the involutions given in §.2. Then the subalgebra $\mathfrak{k}_+$ and the $\mathfrak{k}_+$-modules $\mathfrak{k}_-$, $\mathfrak{p}_\pm$ are given in the following:

$$\begin{aligned}
\mathfrak{k}_+ &= \{(X, \varphi^{-1}(X), X, \varphi^{-1}(X)); X \in \mathfrak{l}\},\\
\mathfrak{k}_- &= \{(Y, \varphi^{-1}(Y), -Y, -\varphi^{-1}(Y)); Y \in \mathfrak{l}\},\\
\mathfrak{p}_+ &= \{(Z, -\varphi^{-1}(Z), Z, -\varphi^{-1}(Z)); Z \in \mathfrak{l}\},\\
\mathfrak{p}_- &= \{(W, -\varphi^{-1}(W), -W, \varphi^{-1}(W)); W \in \mathfrak{l}\}.
\end{aligned}$$

Consider the first projection of these spaces onto $\mathfrak{l}$. Then $\mathfrak{k}_+$ is as Lie algebras identified with $\mathfrak{l}$ and $\mathfrak{k}_-$, $\mathfrak{p}_\pm$ are as vector space identified with it. According to $\mathfrak{k}_-$, $\mathfrak{p}_+$, and $\mathfrak{p}_-$, we here rewrite the identified vector space $\mathfrak{l}$ by $\mathfrak{l}_1$, $\mathfrak{l}_2$, and $\mathfrak{l}_3$. The Lie algebra $\mathfrak{g}$ is then identified with a Lie algebra $\mathfrak{l} \oplus \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \mathfrak{l}_3$ with the bracket product such that $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}$, $[\mathfrak{l}, \mathfrak{l}_i] = \mathfrak{l}_i$, $[\mathfrak{l}_i, \mathfrak{l}_i] = \mathfrak{l}$, and $[\mathfrak{l}_i, \mathfrak{l}_j] = \mathfrak{l}_k$ for distinct indices i, j, and k.

We now see the P-figure associated with this PSLA. Take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{l}$ and fix a fundamental root system $\Pi(\mathfrak{l})$ of $\mathfrak{l}^C$. Since the representations of $\mathfrak{l}$ on $\mathfrak{l}_i^C$ are all adjoint, their dominant weights are given by only the highest root β^0 of $\Pi(\mathfrak{l})$. Hence the P-figure coincides with the extended Dynkin diagram of $\Pi(\mathfrak{l})$, provided that the minus dominant weight $-\beta^0$ has all labels μ, ν, and v.

We next consider the injectivity of $\rho : \mathfrak{l}_3^* \otimes \mathfrak{l}_1 \to (\wedge^2 \mathfrak{l}_3) \otimes \mathfrak{l}_2$. We take the complexification of ρ and argue by the same way as in [**13**] or [**14**].

Let $\mathfrak{r}$ be the root system of $\mathfrak{l}^C$ with respect to $\mathfrak{a}^C$ and take a basis $\{a_{0(1)}, \ldots, a_{0(\ell)}, X_\delta(\delta \in \mathfrak{r})\}$ of $\mathfrak{l}^C$, where $\{a_{0(1)}, \ldots, a_{0(\ell)}\}$ is a basis of $\mathfrak{a}^C$ and X_δ's are root vectors with roots δ. Putting $\Sigma = \{0(1), \ldots, 0(\ell)\} \cup \mathfrak{r}$, we write the basis by $\{T_\alpha; \alpha \in \Sigma\}$. For the $\mathfrak{l}$-module $(\mathfrak{l}_3^C)^* \otimes \mathfrak{l}_1^C$ we represent a maximal weight vector u in the following way:

$$u = \sum_{\alpha : w_\alpha \neq 0} T_\alpha^* \otimes w_\alpha \tag{3.1}$$

where w_α's are weight vectors in $\mathfrak{l}_1^C$ and at least one of them is a maximal vector with the dominant weight β^0. (This follows by a result of representation theory.) Then it follows

$$\rho(u) = \sum_{\alpha' \in \Sigma} \sum_{\alpha : w_\alpha \neq 0} (T_\alpha^* \wedge T_{\alpha'}^*) \otimes [w_\alpha, T_{\alpha'}]. \tag{3.2}$$

We now start from the dominant weight β^0 of $\mathfrak{l}_1^C$ and perform the following procedure. Fix an element α in Σ and put $\lambda(\alpha) = -\alpha + \beta^0$. For a given subset S in Σ we then consider the following condition (C α) on S:

$$(\mathrm{C}\,\alpha) \qquad \gamma + \lambda(\alpha) \in \{0\} \cup \mathfrak{r} \quad \text{for } \gamma \in S.$$

We now put

$$\Omega_0 = \{\gamma \in \Sigma; [T_\gamma, X_{\beta^0}] \neq 0\} \quad \text{and} \quad D_0 = \{\alpha \in \Sigma; (\mathrm{C}\alpha) \text{ holds on } \Omega_0\}.$$

If $D_0 \neq \emptyset$, we next put $B_0 = \{\beta^0\}$ and for each α in D_0 we take the following sets $B_1(\alpha)$, $\Omega_1(\alpha)$ and D_1:

$$B_1(\alpha) = \{\beta \in \mathfrak{r} - B_0; \beta = \gamma + \lambda(\alpha) \text{ for some } \gamma \in \Omega_0\},$$

$$\Omega_1(\alpha) = \{\gamma \in \Sigma; [T_\gamma, X_\beta] \neq 0 \text{ for some } \beta \in B_1(\alpha)\},$$

$$D_1 = \{\alpha \in D_0; (\mathrm{C}\alpha) \text{ holds on } \Omega_1(\alpha)\}.$$

Moreover, for integers k (≥ 2) we inductively define sets $B_k(\alpha)$, $\Omega_k(\alpha)$ and D_k as follows:

$$B_k(\alpha) = \left\{\beta \in \mathfrak{r} - \left(\bigcup_{j=0}^{k-1} B_j(\alpha)\right); \beta = \gamma + \lambda(\alpha) \text{ for some } \gamma \in \Omega_{k-1}(\alpha)\right\},$$

$$\Omega_k(\alpha) = \{\gamma \in \Sigma; [T_\gamma, X_\beta] \neq 0 \text{ for some } \beta \in B_k(\alpha)\},$$

$$D_k = \{\alpha \in D_{k-1}; (\mathrm{C}\alpha) \text{ holds on } \Omega_k(\alpha)\}.$$

From the way of taking vectors T_α, we can easily check the conditions $[T_\gamma, X_\beta] \neq 0$. For a sufficiently large integer k the set $B_k(\alpha)$ is empty and then the set $\Omega_k(\alpha)$ is also regarded as the empty set. Hence it follows that $D_k = D_{k-1}$ for such an integer k. We here put $D = \bigcap_{k \geq 0} D_k$. Then we have the following.

LEMMA 3.2. *$D = \{\beta^0\}$ if $\operatorname{rank} \mathfrak{l} \neq 1$, and $D = \{\beta^0, 0\}$ if $\operatorname{rank} \mathfrak{l} = 1$.*

PROOF. Since $\lambda(\beta^0) = 0$, we can easily see that generally $\beta^0 \in D$. Moreover we note that Ω_0 contains zero and $-\beta^0$.

We first consider the case $\operatorname{rank} \mathfrak{l} \neq 1$. We assume that there exists an element α in D except β^0, and we then deduce a contradiction for each of the following cases; the case $\alpha = 0$, the case that α is a negative root, and the case that α is a positive root.

Suppose that $\alpha = 0$. By the assumption $\operatorname{rank} \mathfrak{l} \neq 1$ the set Ω_0 moreover contains a negative root except $-\beta^0$. Since $\lambda(\alpha) = \beta^0$, the set $B_1(\alpha)$ thus contains a positive root β_1, which is not β^0. Then there exists a positive root γ such that $\gamma + \beta_1$ is also a positive root. This implies that $[T_\gamma, X_{\beta_1}] \neq 0$, and it thus follows that $\gamma \in \Omega_1(\alpha)$. Since $\gamma + \lambda(\alpha)$ is not a root, the condition (C α) does not hold on $\Omega_1(\alpha)$, which is a contradiction.

Suppose that α is a negative root. Then $\lambda(\alpha)$ is not a root. Since $0 \in \Omega_0$, this contradicts that the condition (C α) holds on Ω_0.

Suppose that α is a positive root. We here note that $\lambda(\alpha) > 0$. If $\lambda(\alpha)$ is not a root, we have a contradiction by the same way as the case that α is a negative root. Hence we may assume that $\lambda(\alpha)$ is a positive root, where $\lambda(\alpha) \neq \beta^0$. Since $\beta^0 - \alpha = \lambda(\alpha)$, it follows that in this case $-\alpha \in \Omega_0$ and thus $-\alpha + \lambda(\alpha) = \beta^0 - 2\alpha \in B_1(\alpha)$. From this we also obtain that $\alpha - \lambda(\alpha) \in \Omega_1(\alpha)$ and $\alpha \in B_2(\alpha)$. Again since $\alpha + \lambda(\alpha) = \beta^0$, it follows that $\lambda(\alpha) \in \Omega_2(\alpha)$. Since $2\lambda(\alpha)$ is a not root, this contradicts that the condition (C α) holds on $\Omega_2(\alpha)$.

We next consider the case $\operatorname{rank}\mathfrak{l} = 1$. In the above arguments for the case that α is a negative root, we do not use the assumption $\operatorname{rank}\mathfrak{l} \neq 1$. Moreover we can easily see that in this case $0 \in D$. Hence it follows that $D = \{\beta^0, 0\}$. □

Now we claim that $\rho(u) \neq 0$ for any maximal weight vector u ($\neq 0$) in $(\mathfrak{l}_3^C)^* \otimes \mathfrak{l}_1^C$, which deduces the injectivity of ρ. In fact, if ρ is not injective, the kernel $\operatorname{Ker}\rho$ of ρ is a nontrivial $\mathfrak{l}$-submodule of $(\mathfrak{l}_3^C)^* \otimes \mathfrak{l}_1^C$ since ρ is an $\mathfrak{l}$-homomorphism. Take a maximal weight vector u ($\neq 0$) of $\operatorname{Ker}\rho$. Then $\rho(u) = 0$ and u is also a maximal weight vector of $(\mathfrak{l}_3^C)^* \otimes \mathfrak{l}_1^C$. This is a contradiction.

Let's prove our claim. We assume that there exists a maximal weight vector u ($\neq 0$) in $(\mathfrak{l}_3^C)^* \otimes \mathfrak{l}_1^C$ such that $\rho(u) = 0$, and then induce contradictions. Represent u by $u = \Sigma_{\alpha\in\Sigma} T_\alpha^* \otimes w_\alpha$, and let α_0 be an element in Σ such that w_{α_0} ($\neq 0$) is a maximal weight vector in $\mathfrak{l}_1^C$, where the weight of w_{α_0} is β^0. Then we can inductively see that $\alpha_0 \in D$. We first show $\alpha_0 \in D_0$. Take any γ in Ω_0. Since $[T_\gamma, X_{\beta^0}] \neq 0$, it follows $[T_\gamma, w_{\alpha_0}] \neq 0$. In (3.2), the coefficient of the term $T_{\alpha_0}^* \wedge T_\gamma^*$ equals to $[w_{\alpha_0}, T_\gamma] - [w_\gamma, T_{\alpha_0}]$. Since $\rho(u) = 0$, it holds $[w_\gamma, T_{\alpha_0}] = [w_{\alpha_0}, T_\gamma] \neq 0$ and thus $w_\gamma \neq 0$. This implies that the term $T_\gamma^* \otimes w_\gamma$ appears in the representation (3.1) of u. Since $\lambda(\alpha_0)$ is the weight of u, w_γ is a weight vector in $\mathfrak{l}_1^C$ with weight $\gamma + \lambda(\alpha_0)$. Particularly $\gamma + \lambda(\alpha_0) \in \mathfrak{r} \cup \{0\}$. Hence $\alpha_0 \in D_0$. We next show $\alpha_0 \in D_1$. For any β in $B_1(\alpha_0)$ there exists $\gamma \in \Omega_0$ such that $\beta = \gamma + \lambda(\alpha_0)$. We here note that w_γ is a nonzero weight vector in $\mathfrak{l}_1^C$ with nonzero weight β and the term $T_\gamma^* \otimes w_\gamma$ appears in the representation (3.1) of u. For any δ in $\Omega_1(\alpha_0)$ such that $[T_\delta, X_\beta] \neq 0$, we can see by the same way as above that $\delta + \lambda(\alpha_0) \in \mathfrak{r} \cup \{0\}$. Hence it follows $\alpha_0 \in D_1$. Inductively and by the same way, we can see that $\alpha_0 \in D_k$ for any k. Hence we have $\alpha_0 \in D$.

By Lemma 3.2 it holds $\alpha_0 = \beta^0$ or $\alpha_0 = 0$. The latter case occurs only in the case $\operatorname{rank}\mathfrak{l} = 1$. We induce a contradiction for each case.

We first consider the case $\alpha_0 = \beta^0$. In this case $\lambda(\alpha_0) = 0$. Note that $-\beta^0 \in \Omega_0$ and $0(i) \in \Omega_0$ for some i. Then, since $\rho(u) = 0$, it follows $w_{-\beta^0} \neq 0$ and $w_{0(i)} \neq 0$, i.e., the terms $T_{-\beta^0}^* \otimes w_{-\beta^0}$ and $T_{0(i)}^* \otimes w_{0(i)}$, together with the term $T_{\beta^0}^* \otimes w_{\beta^0}$, appear in the representation (3.1) of u. Since the weights of w_{β^0} and $w_{-\beta^0}$ are respectively β^0 and $-\beta^0$, there exist nonzero numbers a, b such that $w_{\beta^0} = aX_{\beta^0}$ and $w_{-\beta^0} = bX_{-\beta^0}$. In (3.2), the coefficients of terms $T_{\beta^0}^* \wedge T_{-\beta^0}^*$, $T_{\beta^0}^* \wedge T_{0(i)}^*$ and $T_{-\beta^0}^* \wedge T_{0(i)}^*$ are respectively given by $[w_{\beta^0}, T_{-\beta^0}] - [w_{-\beta^0}, T_{\beta^0}]$, $[w_{\beta^0}, T_{0(i)}] - [w_{0(i)}, T_{\beta^0}]$ and $[w_{-\beta^0}, T_{0(i)}] - [w_{0(i)}, T_{-\beta^0}]$. Since $\rho(u) = 0$, it moreover follows

$$a + b = 0, \quad a\beta^0(a_{0(i)}) + \beta^0(w_{0(i)}) = 0, \quad b\beta^0(a_{0(i)}) + \beta^0(w_{0(i)}) = 0,$$

where we note $w_{0(i)} \in \mathfrak{a}^C$. Since $0(i) \in \Omega_0$, it holds $[w_{\beta^0}, T_{0(i)}] \neq 0$ and thus $\beta^0(a_{0(i)}) \neq 0$. Noting this, we obtain that $a = b = 0$, which is a contradiction.

We next consider the case that $\operatorname{rank}\mathfrak{l} = 1$ and $\alpha_0 = 0$. In this case $\lambda(\alpha_0) = \beta^0$. Since $-\beta^0 \in \Omega_0$, $w_{-\beta^0}$ is a nonzero weight vector in $\mathfrak{l}_1^C$ with weight 0. By the assumption $\operatorname{rank}\mathfrak{l} = 1$, it follows $[T_{\beta^0}, w_{-\beta^0}] \neq 0$ and thus w_{β^0} is also a nonzero weight vector in $\mathfrak{l}_1^C$ with weight $2\beta^0$. Particularly $2\beta^0 \in \mathfrak{r}$. This is a contradiction.

Summing up the above arguments, we have the following result.

Proposition 3.3. *A substantial geometry of the case* (C) *admits only totally geodesic submanifolds.*

We next consider a substantial geometry of case (B). We start with the subcase (B1). Since the geometries of subcases (B2) and (B3) constitute families with the ones of subcase (B1), we argue them at the same time as the subcase (B1).

Let $(\mathfrak{l}, \hat{\sigma})$ be a symmetric Lie algebra with compact simple Lie algebra $\mathfrak{l}$. As described in §.2, an irreducible PSLA $(\mathfrak{g}, \sigma, \tau)$ of subcase (B1) is given as follows:

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}, \quad \sigma(X, Y) = (\hat{\sigma}(X), \hat{\sigma}(Y)), \quad \tau(X, Y) = (Y, X)$$

for X and Y in $\mathfrak{l}$. Let $\mathfrak{s}$ and $\mathfrak{m}$ be the (± 1)-eigenspaces of $\hat{\sigma}$, respectively. Then the subalgebra $\mathfrak{k}_+$ and the $\mathfrak{k}_+$-modules $\mathfrak{k}_-$, $\mathfrak{p}_\pm$ are given in the following:

$$\mathfrak{k}_+ = \{(X, X); X \in \mathfrak{s}\}, \quad \mathfrak{k}_- = \{(Y, -Y); Y \in \mathfrak{s}\},$$

$$\mathfrak{p}_+ = \{(Z, Z); Z \in \mathfrak{m}\}, \quad \mathfrak{p}_- = \{(W, -W); W \in \mathfrak{m}\}.$$

Consider the first projection of these spaces onto $\mathfrak{s}$ or $\mathfrak{m}$. Then $\mathfrak{k}_+$ is as Lie algebra identified with $\mathfrak{s}$ and $\mathfrak{k}_-$, $\mathfrak{p}_\pm$ are as vector space identified with $\mathfrak{s}$, $\mathfrak{m}$. According to $\mathfrak{k}_-$, $\mathfrak{p}_\pm$, we here rewrite the identified vector spaces by $\mathfrak{s}_-$, $\mathfrak{m}_\pm$, respectively. The Lie algebra $\mathfrak{g}$ is then identified with a Lie algebra $\mathfrak{s} \oplus \mathfrak{s}_- \oplus \mathfrak{m}_+ \oplus \mathfrak{m}_-$ with the bracket product such that

$$[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}, \qquad [\mathfrak{s}, \mathfrak{s}_-] \subset \mathfrak{s}_-, \quad [\mathfrak{s}, \mathfrak{m}_\pm] = \mathfrak{m}_\pm,$$

$$[\mathfrak{s}_-, \mathfrak{s}_-] \subset \mathfrak{s}, \quad [\mathfrak{m}_\pm, \mathfrak{m}_\pm] \subset \mathfrak{s}, \quad [\mathfrak{s}_-, \mathfrak{m}_\pm] = \mathfrak{m}_\mp,$$

$$[\mathfrak{m}_+, \mathfrak{m}_-] \subset \mathfrak{s}_-.$$

We now see the P-figure associated with this PSLA. Take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{s}$ and fix a fundamental root system $\Pi(\mathfrak{s})$ of the semisimple part of $\mathfrak{s}^C$. Since the representations of $\mathfrak{s}$ on $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are all adjoint, their dominant weights are given as follows: If $\mathfrak{s}$ has trivial center, the dominant weights of $\mathfrak{s}_-^C$ are the highest roots of simple parts of $\Pi(\mathfrak{s})$ and those of $\mathfrak{m}_\pm^C$ are given by the only dominant weight of $\mathfrak{s}$-module $\mathfrak{m}^C$. If $\mathfrak{s}$ has nontrivial center, the $\mathfrak{s}$-module $\mathfrak{s}_-^C$ has the zero dominant weight besides the above ones and the dominant weights of $\mathfrak{m}_\pm^C$ are also given by the just two dominant weights of $\mathfrak{s}$-module $\mathfrak{m}^C$.

From these observations the P-figure is constituted by the fundamental root system $\Pi(\mathfrak{s})$, the minus dominant weights of $\mathfrak{s}$-module $\mathfrak{s}^C$ with label μ, and the minus dominant weights of $\mathfrak{s}$-module $\mathfrak{m}^C$ with label v and υ. Here a figure constituted by $\Pi(\mathfrak{s})$ and the minus dominant weights of $\mathfrak{s}$-module $\mathfrak{m}^C$ is called the *S-figure* associated with $(\mathfrak{l}, \hat{\sigma})$ and it characterizes the symmetric Lie algebra. The S-figures have been already determined (cf. [**5**] or [**9**]) and the P-figures, together with the dominant weights, will be concretely given in §4.

For the PSLA's of (B2) and (B3) which belong to the same family as a PSLA of (B1), the associated P-figures are obtained by permutations of labels μ, v, and υ: The P-figure of (B2) is given by the permutation $(\mu; v; \upsilon \to v; \mu; \upsilon)$ and the P-figure of (B3) is given by the permutation $(\mu; v; \upsilon \to \upsilon; \mu; v)$.

We next consider the injectivity of ρ. We start with the subcase (B1). In this case ρ is a homomorphism of $\mathfrak{m}_-^* \otimes \mathfrak{s}_-$ to $(\wedge^2 \mathfrak{m}_-) \otimes \mathfrak{m}_+$. We again take the complexification of ρ and argue by the same way as in [**13**] or [**14**].

We recall the canonical forms of simple symmetric Lie algebras. (See S. Murakami [**9**] for details.) Let $\mathfrak{l}$ be a compact simple Lie algebra and take a maximal abelian subspace $\mathfrak{h}$ in $\mathfrak{l}$. Fix a fundamental root system $\Pi(\mathfrak{l})$ $(= \{\alpha_1, \dots, \alpha_\ell\} \subset \sqrt{-1}\mathfrak{h})$ and let $\{H_1, \dots, H_\ell\}$ be the dual basis of $\Pi(\mathfrak{l})$ in $\sqrt{-1}\mathfrak{h}$, i.e., $\langle \alpha_i, H_j \rangle = \delta_{ij}$ for all i, j. Next fix a Weyl basis of $\mathfrak{l}^C$ and let t be an involution on $\mathfrak{l}$ which preserves the Dynkin diagram $\Pi(\mathfrak{l})$ and which satisfies the following; $t(X_{\pm\alpha}) = X_{\pm t(\alpha)}$ for α in $\Pi(\mathfrak{l})$ where $X_{\pm\alpha}$, $X_{\pm t(\alpha)}$ are the root vectors with roots $\pm\alpha$, $\pm t(\alpha)$ fixed in the Weyl basis. Represent the minus highest root α_0 for $\Pi(\mathfrak{l})$ as follows:

$$\alpha_0 + m_1\alpha_1 + \cdots + m_\ell\alpha_\ell = 0.$$

Then the symmetric Lie algebras with the following $\hat{\sigma}$, called the *canonical* forms, represent all the equivalence classes of simple symmetric Lie algebras:

(1) $\hat{\sigma} = t$ where t is not identical;

(2) $\hat{\sigma} = t \exp \operatorname{ad}(\pi\sqrt{-1}H_i)$ where $m_i = 1$ or 2, and $t(\alpha_i) = \alpha_i$.

Here the involutions t and $\exp(*)$ are respectively called the Dynkin part and the inner part of $\hat{\sigma}$. If in (2) the Dynkin part is identical, the symmetric Lie algebras with such $\hat{\sigma}$ represent all the simple symmetric Lie algebras of *inner* type.

Fix a canonical form $(\mathfrak{l}, \hat{\sigma})$ in the cases (1) or (2) and take the lexicographic order $<$ on $\sqrt{-1}\mathfrak{h}$ with respect to $\Pi(\mathfrak{l})$. Denote by $\mathfrak{r}$ the root system of $\mathfrak{l}^C$ and define subsets $\mathfrak{r}_0$ and $\mathfrak{r}_1$ in $\mathfrak{r}$ as follows:

$$\mathfrak{r}_0 = \{\alpha \in \mathfrak{r}; t(\alpha) = \alpha\}, \quad \mathfrak{r}_1 = \{\alpha \in \mathfrak{r}; t(\alpha) < \alpha\}.$$

Moreover let $\mathfrak{r}_{0+}$ (resp. $\mathfrak{r}_{0-}$) be the subset in $\mathfrak{r}_0$ which consists of roots α such that $\hat{\sigma}$ is identical (resp. minus identical) on the root subspaces. Then the subsets $\mathfrak{r}_{0\pm}$ are given in the following. (See [**13**].)

LEMMA 3.4. *If* $\mathfrak{l}$ *is of type* A_ℓ (ℓ: *even*) *and* $\hat{\sigma}$ *is the case* (1), *it holds that* $\mathfrak{r}_{0+} = \varnothing$ *and* $\mathfrak{r}_{0-} = \mathfrak{r}_0$, *and if* $\mathfrak{l}$ *is except the above type and* $\hat{\sigma}$ *is the case* (1), *it holds that* $\mathfrak{r}_{0+} = \mathfrak{r}_0$ *and* $\mathfrak{r}_{0-} = \varnothing$.

Next, if $\hat{\sigma}$ *is the case* (2), *it holds that*

$$\mathfrak{r}_{0+} = \{\alpha \in \mathfrak{r}_0; \langle \alpha, H_i \rangle : even\}, \quad \mathfrak{r}_{0-} = \{\alpha \in \mathfrak{r}_0; \langle \alpha, H_i \rangle : odd\}.$$

(*In this case* $\mathfrak{l}$ *is not of type* A_ℓ (ℓ : *even*).)

Let $\mathfrak{a}$ and $\mathfrak{b}$ be the (± 1)-eigenspaces of t in $\mathfrak{h}$. Take a Weyl basis $\Pi(\mathfrak{l}) \cup \{X_\alpha; \alpha \in \mathfrak{r}\}$ of $\mathfrak{l}^C$ and put

$$U_\alpha = X_\alpha + \hat{\sigma}(X_\alpha), \quad V_\alpha = X_\alpha - \hat{\sigma}(X_\alpha)$$

where $\alpha \in \mathfrak{r}$. We then have the following root or weight decompositions:

$$\mathfrak{s}^C = \mathfrak{a}^C \oplus \sum\nolimits_{\alpha \in \mathfrak{r}_{0+} \cup \mathfrak{r}_1} CU_\alpha, \quad \mathfrak{m}^C = \mathfrak{b}^C \oplus \sum\nolimits_{\alpha \in \mathfrak{r}_{0-} \cup \mathfrak{r}_1} CV_\alpha.$$

We here note that $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{s}$. Taking the projection $^-$ of $\sqrt{-1}\mathfrak{h}$ onto $\sqrt{-1}\mathfrak{a}$, we can easily see that the set $\{\bar{\alpha}; \alpha \in \mathfrak{r}_{0+} \cup \mathfrak{r}_1\}$ is a root system of $\mathfrak{s}^C$ with root vectors U_α, and the set $\{\bar{\alpha}; \alpha \in \mathfrak{r}_{0-} \cup \mathfrak{r}_1\}$ is a system of nonzero weights of $\mathfrak{m}^C$ with weight vectors V_α. As an order on $\sqrt{-1}\mathfrak{a}$ we take the restriction of the lexicographic order $<$ on $\sqrt{-1}\mathfrak{h}$ and we fix the fundamental root system $\Pi(\mathfrak{s})$ of $\mathfrak{s}^C$ which consists of simple roots with respect to this order. As the dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ we take the ones with respect to this $\Pi(\mathfrak{s})$.

Returning to the subcase (B1), we now consider the injectivity of ρ. We suppose that $(\mathfrak{l}, \hat{\sigma})$ is a canonical form. Take a basis $\{b_{0(1)}, \dots, b_{0(r)}, V_\delta(\delta \in \mathfrak{r}_{0-} \cup \mathfrak{r}_1)\}$ of $\mathfrak{m}^C$, where $\{b_{0(1)}, \dots, b_{0(r)}\}$ is a basis of $\mathfrak{b}^C$. Putting $\Sigma = \{0(1), \dots, 0(r)\} \cup \mathfrak{r}_{0-} \cup \mathfrak{r}_1$, we write this basis by $\{T_\alpha; \alpha \in \Sigma\}$, and we represent a maximal weight vector u of $(\mathfrak{m}_-^C)^* \otimes \mathfrak{s}_-^C$ in the way of (3.1):

$$u = \sum_{\alpha : w_\alpha \neq 0} T_\alpha^* \otimes w_\alpha$$

where w_α's are weight vectors in $\mathfrak{s}_-^C$, and at least one of them is a maximal vector. Similarly to (3.2) it then follows

$$\rho(u) = \sum_{\alpha' \in \Sigma} \sum_{\alpha : w_\alpha \neq 0} (T_\alpha^* \wedge T_{\alpha'}^*) \otimes [w_\alpha, T_{\alpha'}].$$

We now start from a dominant weight η of $\mathfrak{s}_-^C$ and perform the following procedure. Fix an element α in Σ, and put $\lambda(\alpha) = -\alpha$ if $\eta = 0$ and $\lambda(\alpha) = -\alpha + \beta^0$ if $\eta \neq 0$. In the latter case β^0 is a unique element in Γ such that $\bar{\beta}^0 = \eta$, where $\Gamma = \mathfrak{r}_{0+} \cup \mathfrak{r}_1$. For a given subset S in Σ we then consider the following condition $(\mathrm{C}\,\alpha)$ on S:

$$(\mathrm{C}\,\alpha) \qquad \gamma + \lambda(\alpha) \in \{0\} \cup \Gamma \pmod{\mathfrak{b}^C} \quad \text{for } \gamma \in S.$$

We now put $S_\alpha = U_\alpha$ for α in Γ and moreover put

$$\Omega_0 = \begin{cases} \Sigma & \text{if } \eta = 0, \\ \{\gamma \in \Sigma; [T_\gamma, S_{\beta^0}] \neq 0\} & \text{if } \eta \neq 0, \end{cases}$$

$$D_0(\eta) = \{\alpha \in \Sigma; (\mathrm{C}\alpha) \text{ holds on } \Omega_0\}.$$

If $D_0(\eta) \neq \varnothing$, we next put $B_0 = \varnothing$ or $\{\beta^0\}$ according as $\eta = 0$ or $\eta \neq 0$, and for each α in $D_0(\eta)$ we take the following sets $B_1(\alpha), \Omega_1(\alpha)$ and $D_1(\eta)$:

$$B_1(\alpha) = \{\beta \in \Gamma - B_0; \beta \equiv \gamma + \lambda(\alpha) (\mathrm{mod}\, \mathfrak{b}^C) \text{ for some } \gamma \in \Omega_0\},$$

$$\Omega_1(\alpha) = \{\gamma \in \Sigma; [T_\gamma, S_\beta] \neq 0 \text{ for some } \beta \in B_1(\alpha)\},$$

$$D_1(\eta) = \{\alpha \in D_0(\eta); (\mathrm{C}\,\alpha) \text{ holds on } \Omega_1(\alpha)\}.$$

Moreover for integers k (≥ 2) we inductively define sets $B_k(\alpha)$, $\Omega_k(\alpha)$ and $D_k(\eta)$ as follows:

$$B_k(\alpha) = \left\{\beta \in \Gamma - \left(\bigcup_{j=0}^{k-1} B_j(\alpha)\right); \beta \equiv \gamma + \lambda(\alpha) \text{ for some } \gamma \in \Omega_{k-1}(\alpha)\right\},$$

$$\Omega_k(\alpha) = \{\gamma \in \Sigma; [T_\gamma, S_\beta] \neq 0 \text{ for some } \beta \in B_k(\alpha)\},$$

$$D_k(\eta) = \{\alpha \in D_{k-1}(\eta); (\mathrm{C}\,\alpha) \text{ holds on } \Omega_k(\alpha)\}.$$

Putting $D(\eta) = \bigcap_{k \geq 0} D_k(\eta)$, we then have the following lemma by a similar argument to the case (C). But, in the case $\eta = 0$, we must note the following: A maximal weight vector H ($\neq 0$) in $\mathfrak{s}^C$ with zero dominant weight belongs to the center of $\mathfrak{s}^C$, and $\mathrm{ad}(H)$ is nondegenerate on $\mathfrak{m}^C$. Therefore, it holds $[T_\alpha, H] \neq 0$ for all α in Σ.

LEMMA 3.5. *The homomorphism ρ is injective if $D(\eta) = \emptyset$ for all dominant weights η of $\mathfrak{s}^C$.*

These arguments are also useful for the subcases (B2) or (B3). If it is the subcase (B2), ρ is a homomorphism of $\mathfrak{m}_-^* \otimes \mathfrak{m}_+$ to $(\wedge^2 \mathfrak{m}_-) \otimes \mathfrak{s}_-$. In this case we may do the following replacements in the above procedure for (B1): η is a dominant weight of $\mathfrak{m}_+^C$; Σ and $T_\alpha (\alpha \in \Sigma)$ are the same ones as (B1); $\Gamma = \mathfrak{r}_{0-} \cup \mathfrak{r}_1$; $S_\alpha = V_\alpha$ for α in Γ.

If it is the subcase (B3), ρ is a homomorphism of $\mathfrak{s}_-^* \otimes \mathfrak{m}_+$ to $(\wedge^2 \mathfrak{s}_-) \otimes \mathfrak{m}_-$. In this case we take a basis $\{a_{0(1)}, \ldots, a_{0(s)}\}$ of $\mathfrak{a}^C$ and moreover fix a basis $\{a_{0(1)}, \ldots, a_{0(s)}, U_\delta (\delta \in \mathfrak{r}_{0+} \cup \mathfrak{r}_1)\}$ of $\mathfrak{s}^C$. We here put $\Sigma = \{0(1), \ldots, 0(s)\} \cup \mathfrak{r}_{0+} \cup \mathfrak{r}_1$ and write the fixed basis by $\{T_\alpha; \alpha \in \Sigma\}$. Moreover, for η, Γ and S_α $(\alpha \in \Gamma)$ we may do the same replacements as (B2) in the procedure for (B1). In the subcases (B2) and (B3) we note that $\eta \neq 0$.

After these replacements, Lemma 3.5 also holds for the subcases (B2) and (B3). To use this lemma, we must determine the set $D(\eta)$ for each dominant weight η. We there need to see the conditions $[T_\gamma, S_\beta] \neq 0$ for the determination of sets $\Omega_k(\alpha)$. But the conditions can be ascertained by the following facts:

(1) $[V_\gamma, V_\beta] \neq 0$ for γ and β in $\mathfrak{r}_{0-} \cup \mathfrak{r}_1$ if and only if (i) $\gamma + \beta$ is zero or a root, or (ii) $\gamma + t(\beta)$ is zero or a root which is not contained in $\mathfrak{r}_{0-}$;

(2) $[V_\gamma, U_\beta] \neq 0$ for γ in $\mathfrak{r}_{0-} \cup \mathfrak{r}_1$ and β in $\mathfrak{r}_{0+} \cup \mathfrak{r}_1$ if and only if (i) $\gamma + \beta$ is a root, or (ii) $\gamma + t(\beta)$ is zero or a root which is not contained in $\mathfrak{r}_{0+}$.

These facts are obtained by a similar way to [**13**].

After the above consideration, we explicitly determine the sets $D(\eta)$ case by case. (See §4.) From the results we can see the following. If a geometry is a case of (B1), the sets $D(\eta)$ are all empty except the cases of type GDIV and type GBII ($i = \ell$). Hence if it is not one of the exceptional cases, Lemma 3.5 induces the injectivity of ρ. Though we can not use the lemma for the exceptional cases, we also have the injectivity of ρ by using a more detailed argument. (See Type GDIV in §4.)

Next if a geometry is a case of (B3), the sets $D(\eta)$ are all empty except a case of type GAI. Hence, except the case, we again by Lemma 3.5 have the injectivity of ρ. For the exceptional case ρ is really not injective. This case corresponds to the geometry of curves in S^3.

Let's consider the subcase (B2). Generally, by the irreducibility of $\mathfrak{s}$-module $\mathfrak{m}$, the $\mathfrak{s}$-module $\mathfrak{m}^C$, thus $\mathfrak{m}_+^C$, has at most two dominant weights. The case with just two dominant weights occurs if and only if the Lie algebra $\mathfrak{s}$ has 1-dimensional center, i.e., the symmetric Lie algebra $(\mathfrak{l}, \hat{\sigma})$ is of hermitian type. In this case the corresponding geometry of (B2) is a case in Example 4 and particularly ρ is not injective. We now consider a case of (B2) that $\mathfrak{m}_+^C$ has just one dominant weight η. In this case the set $D(\eta)$, by the case by case determination, consists of only one element. The element is the unique element β^0 in $\mathfrak{r}_{0-} \cup \mathfrak{r}_1$ $(\subset \Sigma)$ such that $\eta = \bar{\beta}^0$. Noting that in this case $\lambda(\beta^0) = 0$, we can easily see that $\beta^0 \in D(\eta)$. Under these consideration, we have the following.

LEMMA 3.6. *If the $\mathfrak{s}$-module $\mathfrak{m}_+^C$ has just one dominant weight, the homomorphism ρ for the subcase* (B2) *is injective.*

PROOF. By virture of the arguments in the case (C), we may show that $\rho(u) \neq 0$ for a maximal weight vector u in $\mathfrak{m}_-^{*C} \otimes \mathfrak{m}_+^C$. We now assume that there exists a maximal weight vector u such that $\rho(u) = 0$, and deduce a contradiction.

Represent u by $u = \Sigma_{\alpha \in \Sigma} T_\alpha^* \otimes w_\alpha$ and let α_0 be an element in Σ such that w_{α_0} ($\neq 0$) is a maximal weight vector in $\mathfrak{m}_+^C$. Since $\mathfrak{m}_+^C$ has just one dominant weight η, the weight of w_{α_0} is η. Also, similarly to the arguments in the case (C), we have that $\alpha_0 \in D(\eta)$. Since $D(\eta)$ consists of one element β^0, it holds $\alpha_0 = \beta^0$ and thus the weight of u equals to zero. Rewrite u by

$$u = \sum_{\alpha \in \boldsymbol{B}} c_\alpha T_\alpha^* \otimes T_\alpha \quad (\text{i.e., } w_\alpha = c_\alpha T_\alpha)$$

where $\boldsymbol{B} = \{\alpha \in \Sigma; c_\alpha \neq 0\}$, which contains β^0. Since $\rho(u) = 0$, it follows by the representation of $\rho(u)$ described before

$$c_\alpha + c_{\alpha'} = 0 \tag{3.1}$$

for α and α' in $\boldsymbol{B}$ such that $[T_\alpha, T_{\alpha'}] \neq 0$.

We first note the following. Let α and α' be roots in $\mathfrak{r}_{0-} \cup \mathfrak{r}_1$ ($\subset \Sigma$) such that $[U_\gamma, \boldsymbol{R}T_\alpha] = \boldsymbol{R}T_{\alpha'}$ for some γ in $\mathfrak{r}_{0+} \cup \mathfrak{r}_1$. Then it follows that

$$[T_\alpha, T_{-t(\alpha')}] \neq 0. \tag{3.2}$$

In fact, since $[U_\gamma, T_\alpha] = [U_\gamma, V_\alpha] \neq 0$, it holds that (i) $\gamma + \alpha$ is a root, or (ii) $\gamma + t(\alpha)$ is zero or a root which is not contained in $\mathfrak{r}_{0+}$. We here note that $\bar{\gamma} + \bar{\alpha} = \bar{\gamma} + \overline{t(\alpha)} = \bar{\alpha}'$, and moreover note that $\bar{\delta} = \bar{\delta}'$ for δ and δ' in $\mathfrak{r}$ if and only if $\delta' = \delta$ or $\delta' = t(\delta)$. If the case (i) occurs, it follows that $\gamma + \alpha = \alpha'$ or $\gamma + \alpha = t(\alpha')$, thus, $\alpha + t(-t(\alpha')) = -\gamma \notin \mathfrak{r}_{0-}$ or $\alpha + (-t(\alpha')) = -\gamma$. These induce (3.2). Similarly, noting that $\gamma + t(\alpha) \neq 0$, we can show (3.2) for the case (ii).

We next note that

$$[T_{\alpha'}, T_{-t(\alpha')}] (= [V_{\alpha'}, V_{-t(\alpha')}]) \neq 0 \tag{3.3}$$

for α' in $\mathfrak{r}_{0-} \cup \mathfrak{r}_1$. This follows since $\alpha' + t(-t(\alpha')) = 0$.

Let $\alpha \in \boldsymbol{B} \cap (\mathfrak{r}_{0-} \cup \mathfrak{r}_1)$ and $\alpha' \in \mathfrak{r}_{0-} \cup \mathfrak{r}_1$. If there exists γ in $\mathfrak{r}_{0+} \cup \mathfrak{r}_1$ such that $[U_\gamma, \boldsymbol{R}T_\alpha] = \boldsymbol{R}T_{\alpha'}$, it follows by (3.1) and (3.2) that $-t(\alpha') \in \boldsymbol{B}$ and $c_\alpha + c_{-t(\alpha')} = 0$, and moreover follows by (3.1) and (3.3) that $\alpha' \in \boldsymbol{B}$ and $c_{\alpha'} + c_{-t(\alpha')} = 0$. Consequently, it holds that $\alpha' \in \boldsymbol{B}$ and $c_{\alpha'} = c_\alpha$.

We now start from the root β^0, which belongs to $\boldsymbol{B} \cap (\mathfrak{r}_{0-} \cup \mathfrak{r}_1)$. Let δ be any root in $\mathfrak{r}_{0-} \cup \mathfrak{r}_1$ which is not β^0. By a result of representation theory there then exist γ_i ($1 \leq i \leq s$) in $\mathfrak{r}_{0+} \cup \mathfrak{r}_1$ such that $[U_r, [\cdots, [U_1, \boldsymbol{R}T_{\beta^0}] \cdots] = \boldsymbol{R}T_\delta$. Hence we inductively obtain that $\delta \in \boldsymbol{B}$ and $c_\delta = c_{\beta^0}$. Again noting that $c_\delta + c_{-t(\delta)} = 0$ for any δ in $\mathfrak{r}_{0-} \cup \mathfrak{r}_1$, we then have a contradiction. $\square$

Summarizing these arguments, we have the following.

PROPOSITION 3.7. *A substantial geometry of the case* (B) *admits non-totally geodesic submanifolds if and only if it is one of the following; the geometries of* (B2) *with hermitian symmetric spaces M^* and the geometry of curves in S^3* ($=\mathrm{SU}(2)$), *which is a special geometry of* (B3).

§.4 Case by case arguments.

In this section we case by case study the strongly substantial geometries of case (B). The items to be clarified are the P-figure associated with each geometry and the injectivity of ρ. Concerning the second item, we determine the sets $D(\eta)$ for all η. To do this for each η, we should generally see the sets $D_k(\eta)$ for all k, but practically it may be sufficient to see sets $D_0(\eta)$ and $D_1(\eta)$ except cases of type GDIV, (B1) and type GBII, (B1). For the exceptional cases we must also see the set $D_2(\eta)$. A detailed argument will be done only for the type GDIV, and for the type GBII it will be omitted since it is a similar way to the type GDIV.

Now, before starting case by case arguments, we recall the following, described in §3. The dominant weights of $\mathfrak{s}$-modules $\mathfrak{s}_-^C$, $\mathfrak{m}_+^C$ and $\mathfrak{m}_-^C$ are respectively represented with labels μ, ν and v. Moreover the dominant weights of $\mathfrak{s}_-^C$ are the ones of $\mathfrak{s}^C$ and the dominant weights of $\mathfrak{m}_\pm^C$ are the ones of $\mathfrak{m}^C$. We start from a simple symmetric Lie algebra $(\mathfrak{l}, \hat{\sigma})$ and take a fundamental root system $\Pi(\mathfrak{l})$ of $\mathfrak{l}^C$ with respect to $\mathfrak{h}^C$. Moreover we take the lexicographic order $<$ on $\sqrt{-1}\mathfrak{h}$ with respect to $\Pi(\mathfrak{l})$ and restrict it on $\sqrt{-1}\mathfrak{a}$. As the fundamental root system $\Pi(\mathfrak{s})$ of $\mathfrak{s}^C$ we then take the system of simple roots with respect to this order. We now start from $\mathfrak{l}$ of classical type. We refer to [**2**] for the root systems.

Let $\mathfrak{l}$ be the compact simple Lie algebra $\mathfrak{su}(\ell+1)$ of type A_ℓ ($\ell \geq 1$). Then the Dynkin diagram of $\Pi(\mathfrak{l})$ is given in the following, where α_0 denotes the minus highest root of $\Pi(\mathfrak{l})$.

$$\underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_{\ell-1}}{\circ} - \underset{\alpha_\ell}{\circ} \qquad \alpha_0 + \alpha_1 + \cdots + \alpha_\ell = 0$$

In the following we identify a vector $a_1\alpha_1 + \cdots + a_\ell\alpha_\ell$ with an ℓ-tuple $(a_1, \ldots, a_\ell)$.

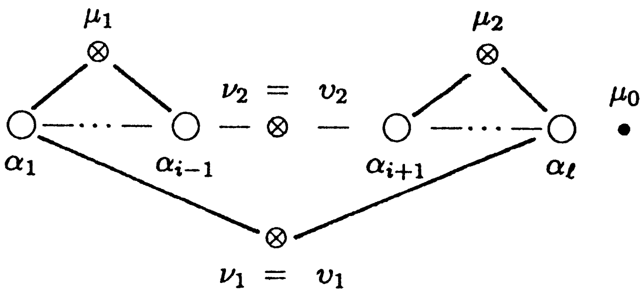
Type GAI. Let $\hat{\sigma} = \exp \mathrm{ad}(\pi\sqrt{-1}H_i)$ $(1 \leq i \leq \ell)$. Then $\mathfrak{s} = \mathfrak{su}(i) \oplus \mathfrak{su}(\ell - i + 1) \oplus \boldsymbol{T}$ and $\Pi(\mathfrak{s}) = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_\ell\}$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_0 = 0$$

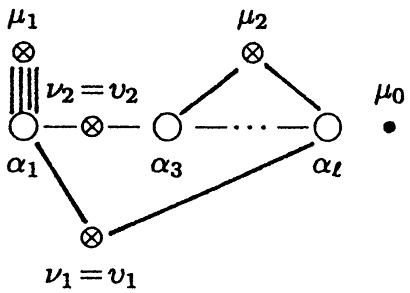
$$\mu_1 = -(1 \cdots 1 \overset{i}{0} \cdots 0) \quad \mu_2 = -(0 \cdots \overset{i}{0} 1 \cdots 1)$$

$$\nu_1 = v_1 = \alpha_0 \quad \nu_2 = v_2 = \alpha_i$$

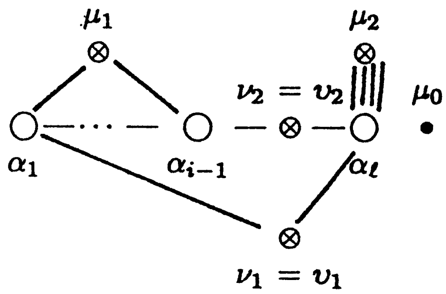
[**1**] The P-figure associated with geometry of subcase (B1).



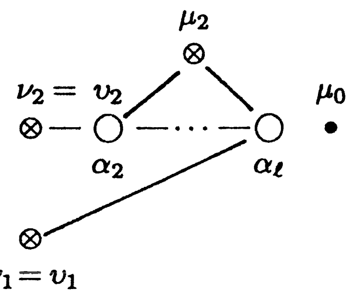
$(\ell \geq 5\ ,\ 3 \leq i \leq \ell-2)$



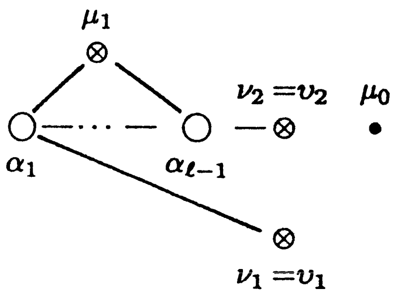
$(\ell \geq 4\ ,\ i = 2)$



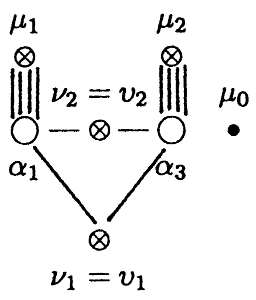
$(\ell \geq 4\ ,\ i = \ell-1)$



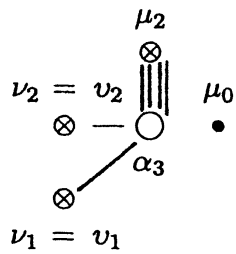
$(\ell \geq 3\ ,\ i = 1)$



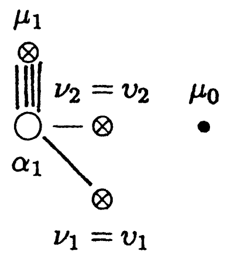
$(\ell \geq 3\ ,\ i = \ell)$



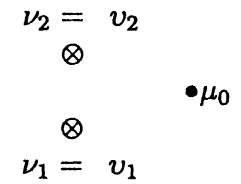
$(\ell = 3,\ i = 2)$



$(\ell = 2,\ i = 1)$



$(\ell = 2,\ i = 2)$



$(\ell = i = 1)$

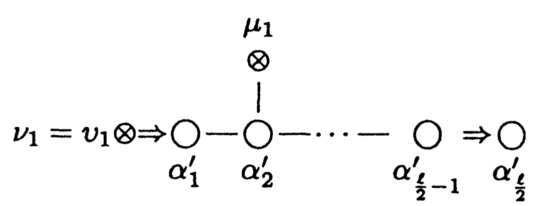
[2] The injectivity of ρ. In the subcase (B1) the sets $D(\eta)$ are all empty, where $\eta = -\mu_i$ $(i = 0, 1, 2)$, and in the subcase (B3) they are empty except the case $\ell = i = 1$, where $\eta = -\nu_i$ $(i = 1, 2)$. The exceptional case corresponds to the geometry of curves in S^3. Moreover the geometry of subcase (B2) is a case in Example 4 since $(\mathfrak{l}, \hat{\sigma})$ is hermitian.

Type GAII. We assume that ℓ is even. Let $\hat{\sigma} = t$ where the Dynkin part t is determined by the following involution on $\Pi(\mathfrak{l}) : t(\alpha_i) = \alpha_{\ell - i + 1}$ for all i. Then $\mathfrak{s} = \mathfrak{so}(\ell + 1)$ and $\Pi(\mathfrak{s}) = \{\alpha'_1, \ldots, \alpha'_{\ell/2}\}$ where $\alpha'_i = (1/2)(\alpha_i + t(\alpha_i))$ $(1 \le i \le \ell/2)$. The minus dominant weights of $\mathfrak{s}^C_-$ and $\mathfrak{m}^C_\pm$ are given in the following.

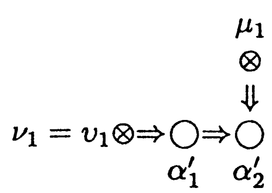
$$\mu_1 = -\frac{1}{2}(12\cdots21) = -(\alpha'_1 + 2\alpha'_2 + \cdots + 2\alpha'_{\ell/2})$$

$$\nu_1 = v_1 = -\frac{1}{2}(2\cdots2) = \alpha_0$$

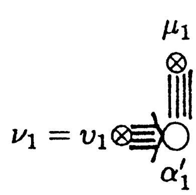
[1] The P-figure associated with geometry of subcase (B1).



$(\ell \ge 6)$



$(\ell = 4)$



$(\ell = 2)$

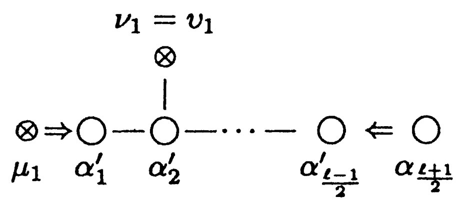
[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = -\alpha_0$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Type GAIII. We assume that ℓ is odd and more than 3. Let $\hat{\sigma} = t$ where the Dynkin part t is the same as that of Type GAII. Then $\mathfrak{s} = \mathfrak{sp}((\ell+1)/2)$ and $\Pi(\mathfrak{s}) = \{\alpha_1', \dots, \alpha_{(\ell-1)/2}', \alpha_{(\ell+1)/2}\}$ where $\alpha_i' = (1/2)(\alpha_i + t(\alpha_i))$ $(1 \le i \le (\ell-1)/2)$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

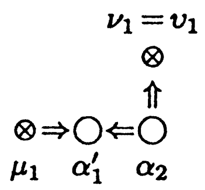
$$\mu_1 = -\frac{1}{2}(2\cdots 2) = \alpha_0 = -(2\alpha_1' + 2\alpha_2' + \cdots + 2\alpha_{(\ell-1)/2}' + \alpha_{(\ell+1)/2})$$

$$\nu_1 = v_1 = -\frac{1}{2}(12\cdots 21) = -(\alpha_1' + 2\alpha_2' + \cdots + 2\alpha_{(\ell-1)/2}' + 2\alpha_{(\ell+1)/2})$$

[1] The P-figure associated with geometry of subcase (B1).



$(\ell \ge 5)$



$(\ell = 3)$

[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = (1\cdots 10)$, which belongs to $\mathfrak{r}_1$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

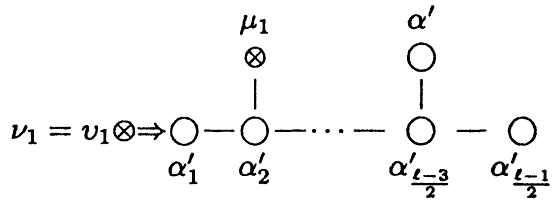
Type GAIV. We assume that ℓ is odd and more than 3. Let $\hat{\sigma} = t \exp \operatorname{ad}(\pi\sqrt{-1}H_{(\ell+1)/2})$ where the Dynkin part t is the same as that of Type GAIII. Then $\mathfrak{s} = \mathfrak{so}(\ell+1)$ and $\Pi(\mathfrak{s}) = \{\alpha_1', \dots, \alpha_{(\ell-1)/2}', \alpha'\}$ where α_i' $(1 \le i \le (\ell-1)/2)$ are the same as those of Type GAIII and $\alpha' = (1/2)(0\cdots 01210\cdots 0)$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_1 = -\frac{1}{2}(12\cdots 21) = -(\alpha' + \alpha_1' + 2\alpha_2' + \cdots + 2\alpha_{(\ell-3)/2}' + \alpha_{(\ell-1)/2}')$$

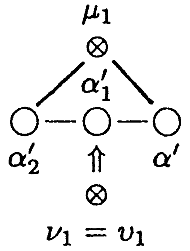
$$\mu_2 = -\frac{1}{2}(101) \quad (\ell = 3)$$

$$\nu_1 = v_1 = -\frac{1}{2}(2\cdots 2) = \alpha_0$$

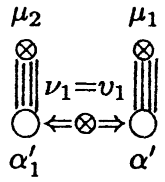
[1] The P-figure associated with geometry of subcase (B1).



$(\ell \ge 7)$



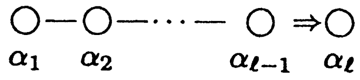
$(\ell = 5)$



$(\ell = 3)$

[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = (1 \cdots 1)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Let $\mathfrak{l}$ be the compact simple Lie algebra $\mathfrak{so}(2\ell + 1)$ of type B_ℓ $(\ell \geq 3)$. Then the Dynkin diagram of $\Pi(\mathfrak{l})$ is given in the following, where α_0 denotes the minus highest root of $\Pi(\mathfrak{l})$.



$$\alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + 2\alpha_\ell = 0$$

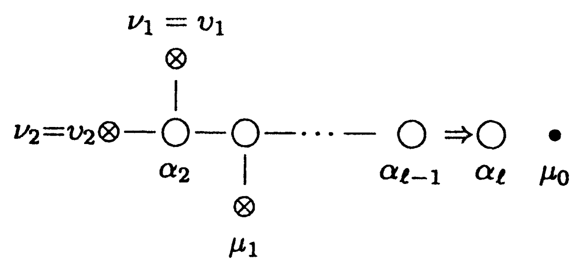
In the following we identify a vector $a_1\alpha_1 + \cdots + a_\ell\alpha_\ell$ with an ℓ-tuple $(a_1, \ldots, a_\ell)$.

Type GBI. Let $\hat{\sigma} = \exp \operatorname{ad}(\pi\sqrt{-1}H_1)$. Then $\mathfrak{s} = \mathfrak{so}(2\ell - 1) \oplus \boldsymbol{T}$ and $\Pi(\mathfrak{s}) = \{\alpha_2, \ldots, \alpha_\ell\}$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

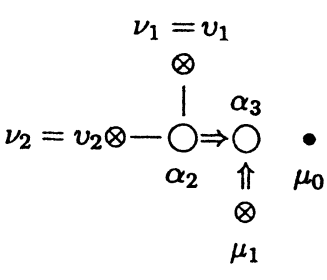
$$\mu_0 = 0 \qquad \mu_1 = -(012\cdots 2)$$

$$\nu_1 = v_1 = \alpha_0 \qquad \nu_2 = v_2 = \alpha_1$$

[1] The P-figure associated with geometry of subcase (B1).



$(\ell \geq 4)$



$(\ell = 3)$

[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and the geometry of subcase (B2) is a case in Example 4 since $(\mathfrak{l}, \hat{\sigma})$ is hermitian.

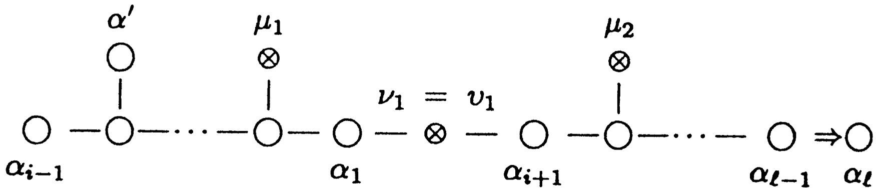
Type GBII. Let $\hat{\sigma} = \exp \operatorname{ad}(\pi\sqrt{-1}H_i)$ $(2 \leq i \leq \ell)$. Then $\mathfrak{s} = \mathfrak{so}(2i) \oplus \mathfrak{so}(2\ell - 2i + 1)$ and $\Pi(\mathfrak{s}) = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_\ell, \alpha'\}$ where $\alpha' = (0 \cdots 0\overset{i}{1}2 \cdots 2)$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_1 = -(12\cdots 2) = \alpha_0 \qquad \mu_2 = -(0\cdots \overset{i}{0}12\cdots 2) \quad (i \le \ell - 1)$$

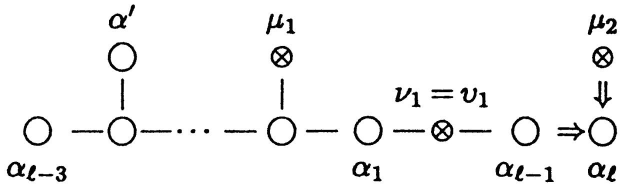
$$\mu_{-1} = -(10\cdots 0) \quad (i = 2)$$

$$\nu_1 = v_1 = -(1\cdots \overset{i}{1}2\cdots 2)$$

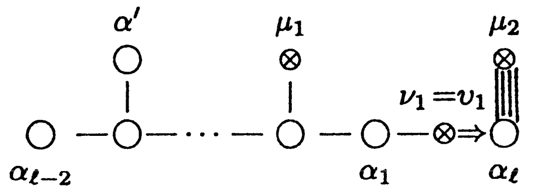
[1] The P-figure associated with geometry of subcase (B1).



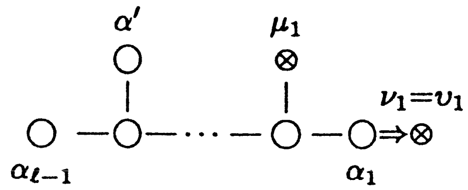
$(\ell \ge 7\ ,\ 4 \le i \le \ell - 3)$



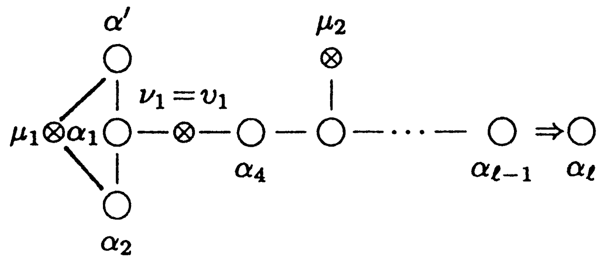
$(\ell \ge 6\ ,\ 4 \le i = \ell - 2)$



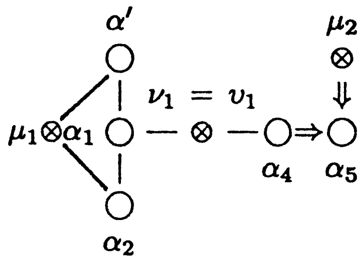
$(\ell \ge 5\ ,\ 4 \le i = \ell - 1)$



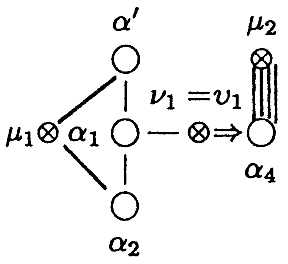
$(\ell \ge 4\ ,\ i = \ell)$

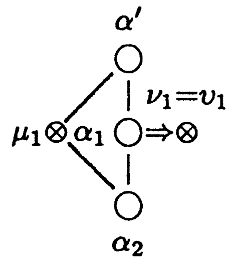


$(\ell \ge 6\ ,\ i = 3)$

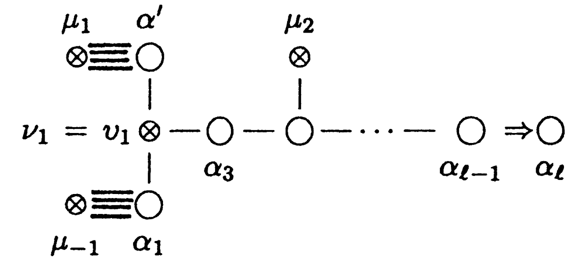


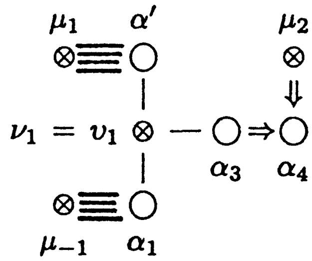
$(\ell = 5\ ,\ i = 3)$



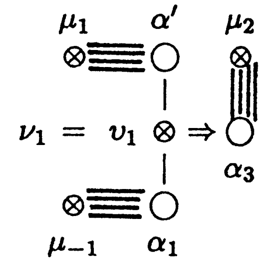


($\ell = 4$, $i = 3$) ($\ell = i = 3$)





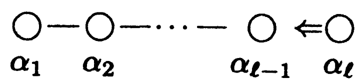
($\ell \geq 5$, $i = 2$) ($\ell = 4$, $i = 2$)



($\ell = 3$, $i = 2$)

[2] The injectivity of ρ. In the subcase (B3) the set $D(\eta)$ is empty. In the subcase (B1) ($i = \ell$), the set $D(\eta)$ is not empty. But, by a similar way to the type GDIV, (B1), we can see that in this case ρ is injective. (The proof is omitted.) For the other cases of (B1) the sets $D(\eta)$ are all empty. In the subcase (B2) the set $D(\eta)$ $(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = (1 \cdots \overset{i}{1} 2 \cdots 2)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Let $\mathfrak{l}$ be the compact simple Lie algebra $\mathfrak{sp}(\ell)$ of type C_ℓ ($\ell \geq 2$). Then the Dynkin diagram of $\Pi(\mathfrak{l})$ is given in the following, where α_0 denotes the minus highest root of $\Pi(\mathfrak{l})$.



$$\alpha_0 + 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell = 0$$

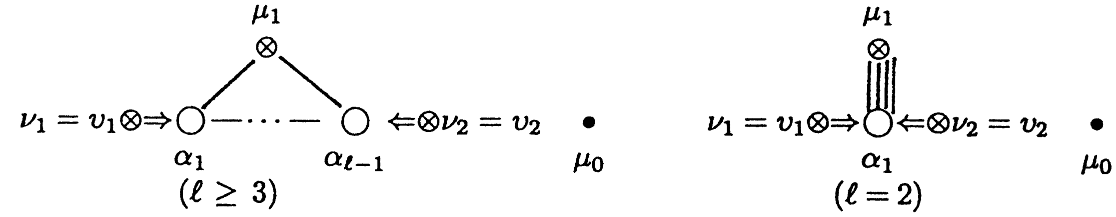
In the following we identify a vector $a_1\alpha_1 + \cdots + a_\ell\alpha_\ell$ with an ℓ-tuple $(a_1, \ldots, a_\ell)$.

Type GCI. Let $\hat{\sigma} = \exp \operatorname{ad}(\pi\sqrt{-1}H_\ell)$. Then $\mathfrak{s} = \mathfrak{su}(\ell) \oplus \boldsymbol{T}$ and $\Pi(\mathfrak{s}) = \{\alpha_1, \ldots, \alpha_{\ell-1}\}$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_0 = 0 \qquad \mu_1 = -(1\cdots 10)$$

$$\nu_1 = v_1 = \alpha_0 \qquad \nu_2 = v_2 = \alpha_\ell$$

[1] The P-figure associated with geometry of subcase (B1).



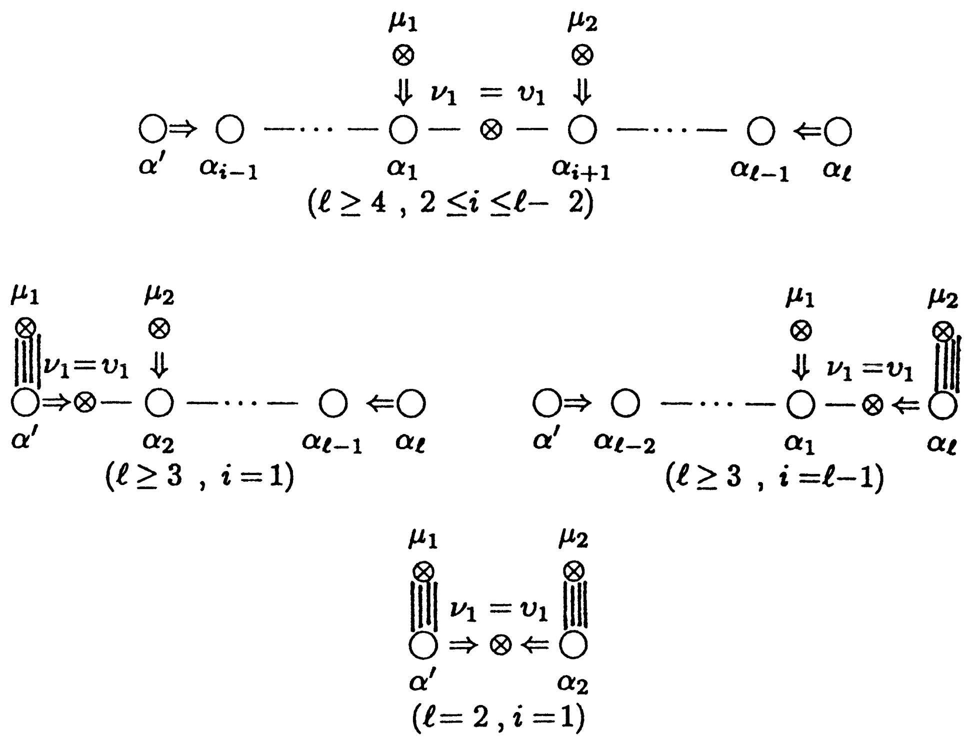
[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and the geometry of subcase (B2) is a case in Example 4 since $(\mathfrak{l}, \hat{\sigma})$ is hermitian.

Type GCII. Let $\hat{\sigma} = \exp \operatorname{ad}(\pi\sqrt{-1}H_i)$ $(1 \leq i \leq \ell - 1)$. Then $\mathfrak{s} = \mathfrak{sp}(i) \oplus \mathfrak{sp}(\ell - i)$ and $\Pi(\mathfrak{s}) = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_\ell, \alpha'\}$ where $\alpha' = (0\cdots 0\overset{i}{2}\cdots 21)$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_1 = -(2\cdots 21) = \alpha_0 \qquad \mu_2 = -(0\cdots 0\overset{i}{2}\cdots 21)$$

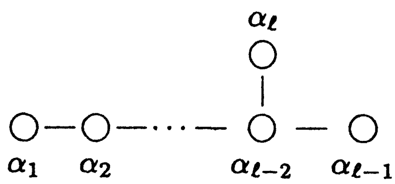
$$\nu_1 = v_1 = -(1\cdots \overset{i}{1}2\cdots 21)$$

[1] The P-figure associated with geometry of subcase (B1).



[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = (1\cdots \overset{i}{1}2\cdots 21)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Let $\mathfrak{l}$ be the compact simple Lie algebra $\mathfrak{so}(2\ell)$ of type D_ℓ ($\ell \geq 4$). Then the Dynkin diagram of $\Pi(\mathfrak{l})$ is given in the following, where α_0 denotes the minus highest root of $\Pi(\mathfrak{l})$.



$$\alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell = 0$$

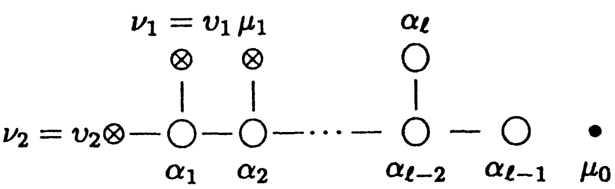
In the following we identify a vector $a_1\alpha_1 + \cdots + a_\ell\alpha_\ell$ with an ℓ-tuple $(a_1, \ldots, a_{\ell-2}|a_{\ell-1}, \alpha_\ell)$.

Type GDI. Let $\hat{\sigma} = \exp \operatorname{ad}(\pi\sqrt{-1}H_1)$. Then $\mathfrak{s} = \mathfrak{so}(2\ell - 2) \oplus \boldsymbol{T}$ and $\Pi(\mathfrak{s}) = \{\alpha_2, \ldots, \alpha_\ell\}$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_0 = 0 \qquad \mu_1 = -(012\cdots2|11)$$

$$\nu_1 = v_1 = \alpha_0 \qquad \nu_2 = v_2 = \alpha_1$$

[1] The P-figure associated with geometry of subcase (B1).



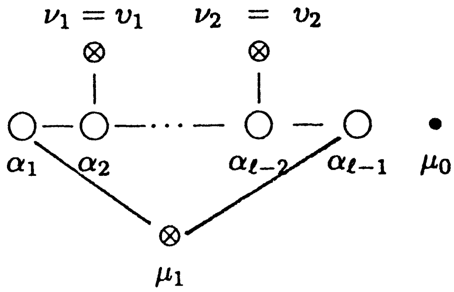
[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and the geometry of subcase (B2) is a case in Example 4 since $(\mathfrak{l}, \hat{\sigma})$ is hermitian.

Type GDII. Let $\hat{\sigma} = \exp \operatorname{ad}(\pi\sqrt{-1}H_\ell)$. Then $\mathfrak{s} = \mathfrak{su}(\ell) \oplus \boldsymbol{T}$ and $\Pi(\mathfrak{s}) = \{\alpha_1, \ldots, \alpha_{\ell-1}\}$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_0 = 0 \qquad \mu_1 = -(1\cdots1|10)$$

$$\nu_1 = v_1 = \alpha_0 \qquad \nu_2 = v_2 = \alpha_\ell$$

[1] The P-figure associated with geometry of subcase (B1).



[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and the geometry of subcase (B2) is a case in Example 4 since $(\mathfrak{l}, \hat{\sigma})$ is hermitian.

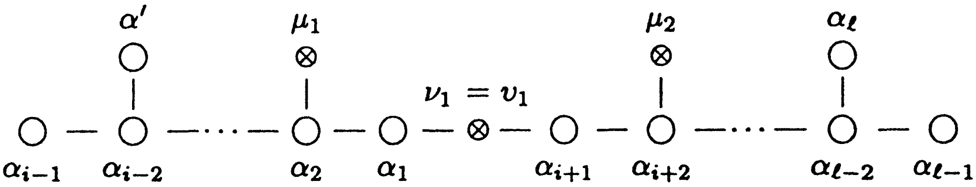
Type GDIII. Let $\hat{\sigma} = \exp \mathrm{ad}(\pi\sqrt{-1}H_i)$ $(2 \le i \le \ell - 2)$. Then $\mathfrak{s} = \mathfrak{so}(2i) \oplus \mathfrak{so}(2\ell - 2i)$ and $\Pi(\mathfrak{s}) = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_\ell, \alpha'\}$ where $\alpha' = (0\cdots0\overset{i}{1}2\cdots2|11)$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_1 = -(12\cdots2|11) = \alpha_0 \qquad \mu_2 = -(0\cdots0\overset{i}{1}2\cdots2|11) \quad (i \le \ell - 3)$$

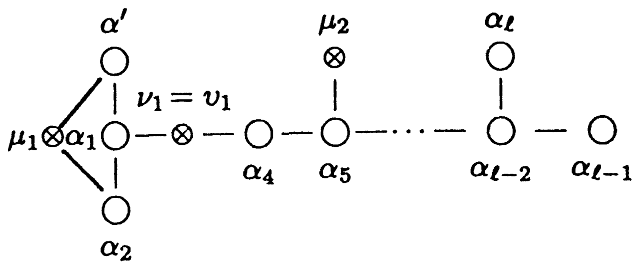
$$\mu_{-1} = -(10\cdots0|00) \quad (i = 2) \qquad \mu_3 = -(0\cdots0|01) \quad (i = \ell - 2)$$

$$\nu_1 = v_1 = -(1\cdots\overset{i}{1}2\cdots2|11) \qquad \mu_{-3} = -(0\cdots0|10) \quad (i = \ell - 2)$$

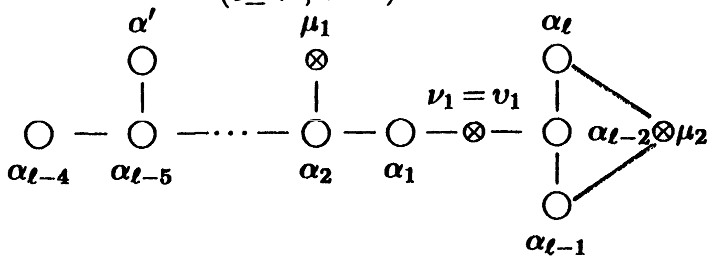
[1] The P-figure associated with geometry of subcase (B1).



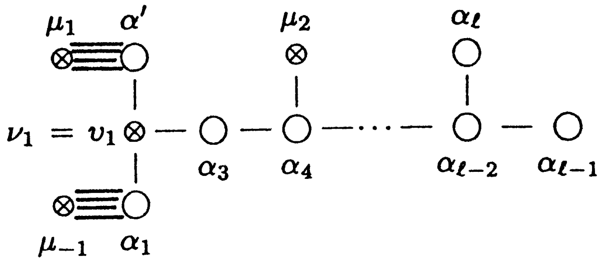
$(\ell \ge 8\ ,\ 4 \le i \le \ell - 4)$



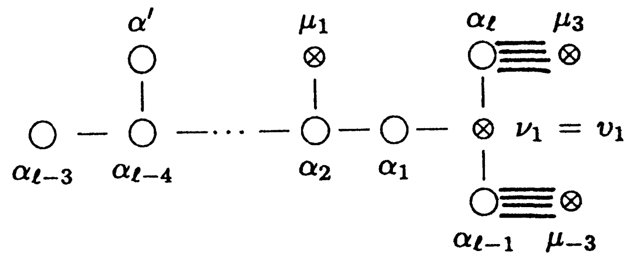
$(\ell \ge 7\ ,\ i = 3)$



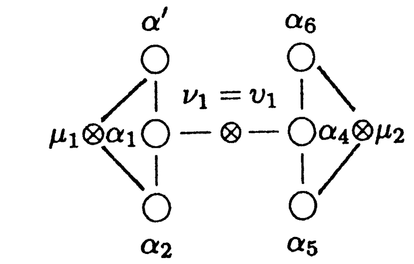
$(\ell \ge 7\ ,\ i = \ell - 3)$



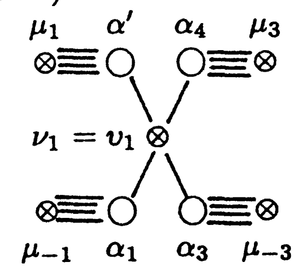
$(\ell \ge 6\ ,\ i = 2)$



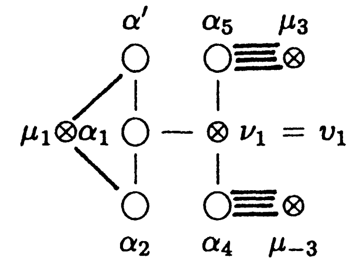
$(\ell \geq 6\ ,\ i = \ell - 2)$



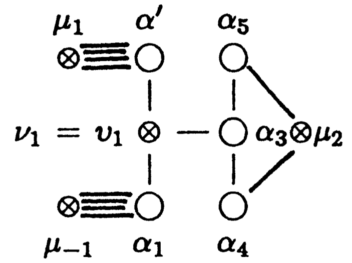
$(\ell = 6\ , i = 3)$



$(\ell = 4\ , i = 2)$



$(\ell = 5\ ,\ i = 3)$



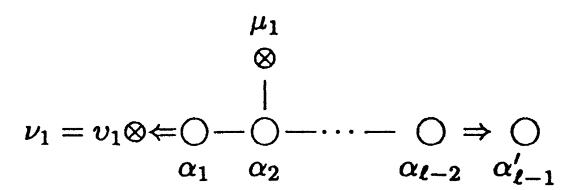
$(\ell = 5\ ,\ i = 2)$

[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = (1 \cdots \overset{}{\underset{i}{1}}2 \cdots 2|11)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Type GDIV. Let $\hat{\sigma} = t$ where the Dynkin part t is determined by the following involution on $\Pi(\mathfrak{l}) : t(\alpha_i) = \alpha_i$ $(1 \leq i \leq \ell - 2)$, $t(\alpha_{\ell-1}) = \alpha_\ell$, and $t(\alpha_\ell) = \alpha_{\ell-1}$. Then $\mathfrak{s} = \mathfrak{so}(2\ell - 1)$ and $\Pi(\mathfrak{s}) = \{\alpha_1, \ldots, \alpha_{\ell-2}, \alpha'_{\ell-1}\}$ where $\alpha'_{\ell-1} = (1/2)(\alpha_{\ell-1} + t(\alpha_\ell))$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_1 = \alpha_0$$
$$\nu_1 = v_1 = -\frac{1}{2}(2 \cdots 2|11)$$

[1] The P-figure associated with geometry of subcase (B1).



[2] The injectivity of ρ. In the subcase (B3) the set $D(\eta)$ is empty, and in the subcase (B2) the set $D(\eta)(=D(-\nu_1))$ consists of one element β^0 where $\beta^0=(1\cdots1|10)$, which belongs to $\mathfrak{r}_1$. Taking account of Lemma 3.6, we can see that in these cases ρ is injective.

We consider the subcase (B1). Then it follows that

$$\Omega_0=\{-(1\cdots1|01),-(01\cdots1|01)\}\quad\text{and}$$

$$D(\eta)=\{-(00\cdots\overset{k}{0}1\cdots1|01);\ 2\le k\le \ell-2\}.$$

In this case we need a detailed argument. Fix an integer k and put $\alpha^0=-(00\cdots\overset{k}{0}1\cdots1|01)$. Then $\lambda(\alpha^0)=(12\cdots\overset{k}{2}3\cdots3|12)$ and the set $B_0\cup(\bigcup_{i\ge1}B_i(\alpha^0))$ consists of the following elements;

$$\beta^0=(12\cdots2|11),\quad \beta^1=(01\cdots\overset{k}{1}2\cdots2|11),\quad \beta^2=(11\cdots\overset{k}{1}2\cdots2|11).$$

The corresponding elements α^i in Σ, which satisfy $\beta^i\equiv\alpha^i+\lambda(\alpha^0)\ (\mathrm{mod}\,\mathfrak{b}^C)$, are given in the following.

$$\alpha^0=-(00\cdots\overset{k}{0}1\cdots1|01),\quad \alpha^1=-(1\cdots1|01),\quad \alpha^2=-(01\cdots1|01).$$

We suppose that a maximal weight vector u of $(\mathfrak{m}_-^C)^*\otimes\mathfrak{s}_-^C$ has weight $\bar{\lambda}(\alpha^0)$ and the elements α^i $(i=0,1,2)$ appear in the summation: $u=\sum_{\alpha:w_\alpha\neq0}T_\alpha^*\otimes w_\alpha$. Note that $T_{\alpha^i}=V_{\alpha^i}$, and $w_{\alpha^i}=c_iU_{\beta^i}$ for some nonzero constants c_i. If $\rho(u)=0$, it then follows

$$c_i[V_{\alpha^j},U_{\beta^i}]=c_j[V_{\alpha^i},U_{\beta^j}]$$

for distinct indices i and j. Since in this case $U_{\beta^i}=2X_{\beta^i}$ and $V_{\alpha^i}=X_{\alpha^i}-\hat{\sigma}(X_{\alpha^i})$, this moreover induces the following: $c_i[X_{\alpha^j},X_{\beta^i}]=c_j[X_{\alpha^i},X_{\beta^j}]$ and thus

$$c_iN_{\alpha^j,\beta^i}=c_jN_{\alpha^i,\beta^j}. \tag{4.1}$$

We now put $\gamma=\alpha^2-\alpha^1=\beta^2-\beta^1$, which is a positive root of $\mathfrak{s}^C$ and belongs to $\mathfrak{r}_{0+}$. Since u is a maximal vector, it holds $X_\gamma\cdot u=0$ and it consequently follows

$$c_2N_{\gamma,\alpha^1}=c_1N_{\gamma,\beta^1}. \tag{4.2}$$

Next, noting that $\alpha^1+\beta^1$ is not a root, we have the following.

$$\begin{aligned}N_{\alpha^2,\beta^1}X_{\alpha^2+\beta^1}&=(1/N_{\alpha^1,\gamma})[[X_{\alpha^1},X_\gamma],X_{\beta^1}]\\&=-(1/N_{\alpha^1,\gamma})([[X_\gamma,X_{\beta^1}],X_{\alpha^1}]+[[X_{\beta^1},X_{\alpha^1}],X_\gamma])\\&=-(1/N_{\alpha^1,\gamma})N_{\gamma,\beta^1}N_{\beta^2,\alpha^1}X_{\beta^2+\alpha^1}\end{aligned}$$

and so $N_{\alpha^2,\beta^1}N_{\alpha^1,\gamma}=-N_{\gamma,\beta^1}N_{\beta^2,\alpha^1}$. This, together with (4.2), implies $c_1N_{\alpha^2,\beta^1}=-c_2N_{\alpha^1,\beta^2}$, which contradicts (4.1). (The above argument is similar to that of Lemma 2.4 in [**12**].)

By these arguments we obtain that in the subcase (B1) the homomorphism ρ is injective.

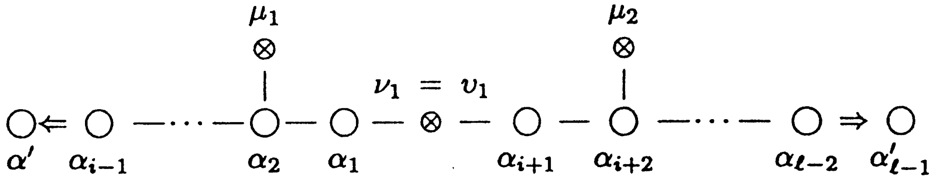
Type GDV. Let $\hat{\sigma}=t\exp\mathrm{ad}(\pi\sqrt{-1}H_i)$ where the Dynkin part t is the same as that of Type GDIV and $2\le i\le\ell-2$. Then $\mathfrak{s}=\mathfrak{so}(2i+1)\oplus\mathfrak{so}(2\ell-2i-1)$ and $\Pi(\mathfrak{s})=$

$\{\alpha_1, \ldots, \underset{i}{,} \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{\ell-2}, \alpha'_{\ell-1}, \alpha'\}$ where $\alpha'_{\ell-1} = (1/2)(\alpha_{\ell-1} + t(\alpha_{\ell-1}))$ and $\alpha' = (1/2)$ $(0 \cdots 0\underset{i}{2} \cdots 2|11)$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

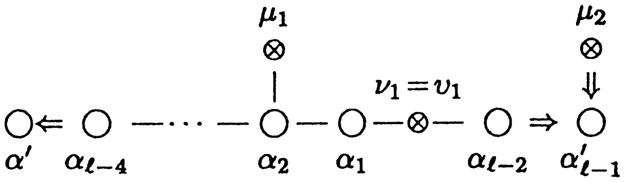
$$\mu_1 = -(12\cdots 2|11) = \alpha_0 \qquad \mu_2 = -(0\cdots 0\overset{i}{1}2\cdots 2|11)$$

$$\nu_1 = v_1 = -(1\cdots \overset{i}{1}2\cdots 2|11)$$

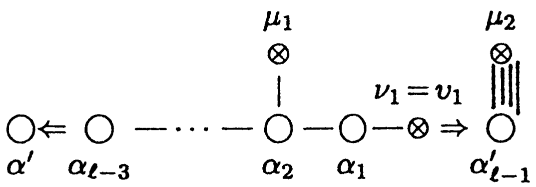
[1] The P-figure associated with geometry of subcase (B1).



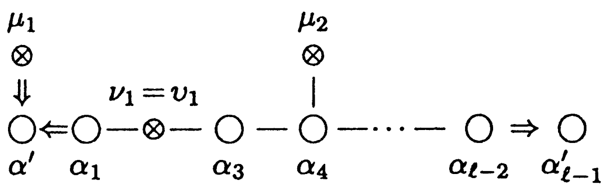
$(\ell \geq 7\ ,\ 3 \leq i \leq \ell - 4)$



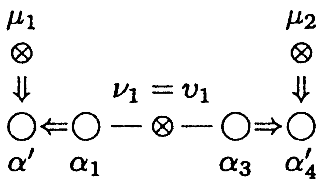
$(\ell \geq 6\ ,\ i = \ell - 3)$



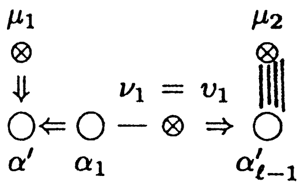
$(\ell \geq 5\ ,\ i = \ell - 2)$



$(\ell \geq 6\ ,\ i = 2)$



$(\ell = 5\ , i = 2)$

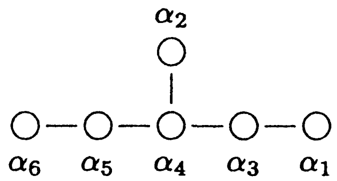


$(\ell = 4\ ,\ i = 2)$

[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(=D(-\nu_1))$ consists of one element β^0 where $\beta^0 = (1\cdots\underset{i}{1}2\cdots2|11)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

We next treat $\mathfrak{l}$ of exceptional type. We refer to [3] for the root systems.

Let $\mathfrak{l}$ be the compact simple Lie algebra of type E_6. Then the Dynkin diagram of $\Pi(\mathfrak{l})$ is given in the following, where α_0 denotes the minus highest root of $\Pi(\mathfrak{l})$.



$$\alpha_0+\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6=0$$

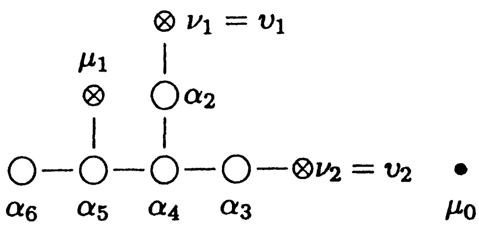
In the following we identify a vector $a_1\alpha_1+a_2\alpha_2+a_3\alpha_3+a_4\alpha_4+a_5\alpha_5+a_6\alpha_6$ with a 6-tuple $\left(\begin{smallmatrix} & & a_2 & & \\ a_6 & a_5 & a_4 & a_3 & a_1\end{smallmatrix}\right)$.

Type GE$_6$I. Let $\hat{\sigma}=\exp \mathrm{ad}(\pi\sqrt{-1}H_1)$. Then $\mathfrak{s}=\mathfrak{so}(10)\oplus T$ and $\Pi(\mathfrak{s})=\{\alpha_2,\dots,\alpha_6\}$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_0=0 \qquad \mu_1=-\left(\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 2 & 1 & 0\end{smallmatrix}\right)$$

$$\nu_1=v_1=\alpha_0 \qquad \mu_2=v_2=\alpha_1$$

[1] The P-figure associated with geometry of subcase (B1).



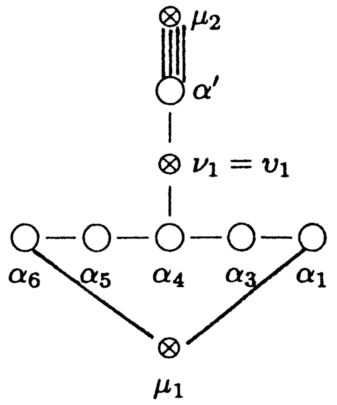
[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and the geometry of subcase (B2) is a case in Example 4 since $(\mathfrak{l},\hat{\sigma})$ is hermitian.

Type GE$_6$II. Let $\hat{\sigma}=\exp \mathrm{ad}(\pi\sqrt{-1}H_2)$. Then $\mathfrak{s}=\mathfrak{su}(6)\oplus\mathfrak{su}(2)$ and $\Pi(\mathfrak{s})=\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha'\}$ where $\alpha'=-\alpha_0$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_1=-\left(\begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 1 & 1\end{smallmatrix}\right) \qquad \mu_2=\alpha_0$$

$$\nu_1=v_1=-\left(\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 3 & 2 & 1\end{smallmatrix}\right)$$

[1] The P-figure associated with geometry of subcase (B1).



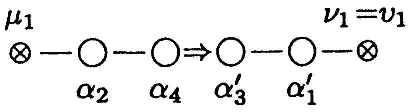
[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = \left(\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}\right)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Type GE$_6$III. Let $\hat{\sigma} = t$ where the Dynkin part t is determined by the following involution on $\Pi(\mathrm{I}) : t(\alpha_i) = \alpha_i$ $(i = 0, 2, 4)$, $t(\alpha_1) = \alpha_6$, $t(\alpha_6) = \alpha_1$, $t(\alpha_3) = \alpha_5$, and $t(\alpha_5) = \alpha_3$. Then $\mathfrak{s} = \mathfrak{f}_4$ and $\Pi(\mathfrak{s}) = \{\alpha_1', \alpha_2, \alpha_3', \alpha_4\}$ where $\alpha_1' = (1/2)(\alpha_1 + t(\alpha_1))$ and $\alpha_3' = (1/2)(\alpha_3 + t(\alpha_3))$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_1 = \alpha_0$$

$$\nu_1 = v_1 = -\frac{1}{2}\left(\begin{smallmatrix} & & 2 & & \\ 2 & 3 & 4 & 3 & 2 \end{smallmatrix}\right)$$

[1] The P-figure associated with geometry of subcase (B1).

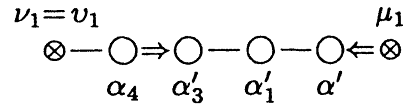


[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are empty, and in the subcase (B2) the set $D(\eta)$ $(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = \left(\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 2 & 2 & 1 \end{smallmatrix}\right)$, which belongs to $\mathfrak{r}_1$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Type GE$_6$IV. Let $\hat{\sigma} = t \exp \mathrm{ad}(\pi\sqrt{-1}H_2)$ where the Dynkin part t is the same as that of Type GE$_6$III. Then $\mathfrak{s} = \mathfrak{sp}(4)$ and $\Pi(\mathfrak{s}) = \{\alpha_1', \alpha_3', \alpha_4, \alpha'\}$ where $\alpha' = (1/2)\left(\begin{smallmatrix} & & 2 & & \\ 0 & 1 & 2 & 1 & 0 \end{smallmatrix}\right)$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

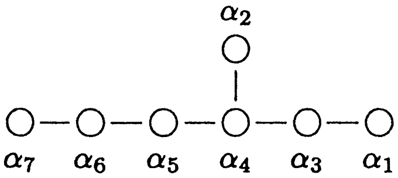
$$\mu_1 = \alpha_0 \quad \nu_1 = v_1 = -\left(\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}\right)$$

[1] The P-figure associated with geometry of subcase (B1).



[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are empty, and in the subcase (B2) the set $D(\eta)$ $(=D(-\nu_1))$ consists of one element β^0 where $\beta^0=\left(1\,2\,\overset{1}{3}\,2\,1\right)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Let $\mathfrak{l}$ be the compact simple Lie algebra of type E_7. Then the Dynkin diagram of $\Pi(\mathfrak{l})$ is given in the following, where α_0 denotes the minus highest root of $\Pi(\mathfrak{l})$.



$$\alpha_0+2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7=0$$

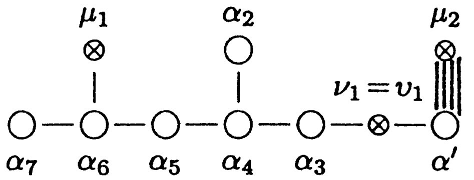
In the following we identify a vector $a_1\alpha_1+a_2\alpha_2+a_3\alpha_3+a_4\alpha_4+a_5\alpha_5+a_6\alpha_6+a_7\alpha_7$ with a 7-tuple $\left(a_7\,a_6\,a_5\,\overset{a_2}{a_4}\,a_3\,a_1\right)$.

Type GE$_7$I. Let $\hat{\sigma}=\exp\operatorname{ad}(\pi\sqrt{-1}H_1)$. Then $\mathfrak{s}=\mathfrak{so}(12)\oplus\mathfrak{su}(2)$ and $\Pi(\mathfrak{s})=\{\alpha',\alpha_2,\dots,\alpha_7\}$ where $\alpha'=-\alpha_0$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_1=-\left(1\,2\,2\,\overset{1}{2}\,1\,0\right)\qquad \mu_2=-\left(1\,2\,3\,\overset{2}{4}\,3\,2\right)=\alpha_0$$

$$\nu_1=v_1=-\left(1\,2\,3\,\overset{2}{4}\,3\,1\right)$$

[1] The P-figure associated with geometry of subcase (B1).

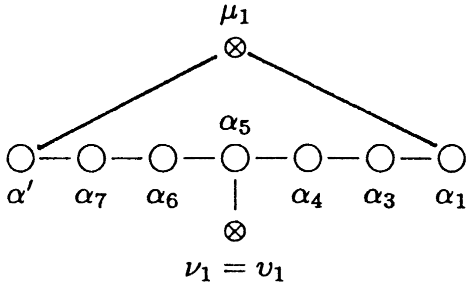


[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(=D(-\nu_1))$ consists of one element β^0 where $\beta^0=\left(1\,2\,3\,\overset{2}{4}\,3\,1\right)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Type GE$_7$II. Let $\hat{\sigma}=\exp\operatorname{ad}(\pi\sqrt{-1}H_2)$. Then $\mathfrak{s}=\mathfrak{su}(8)$ and $\Pi(\mathfrak{s})=\{\alpha_1,\alpha_3,\dots,\alpha_7,\alpha'\}$ where $\alpha'=\left(0\,1\,2\,\overset{2}{3}\,2\,1\right)$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_1=-\left(1\,2\,3\,\overset{2}{4}\,3\,2\right)\quad \nu_1=v_1=-\left(1\,2\,3\,\overset{1}{3}\,2\,1\right)$$

[1] The P-figure associated with geometry of subcase (B1).



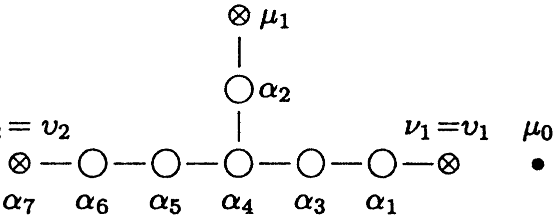
[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(=-D(\nu_1))$ consists of one element β^0 where $\beta^0 = \left(\begin{smallmatrix} &&&1&&\\ 1&2&3&3&2&1\end{smallmatrix}\right)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Type GE$_7$III. Let $\hat{\sigma} = \exp \mathrm{ad}(\pi\sqrt{-1}H_7)$. Then $\mathfrak{s} = \mathfrak{e}_6 \oplus \boldsymbol{T}$ and $\Pi(\mathfrak{s}) = \{\alpha_1, \ldots, \alpha_6\}$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_0 = 0 \qquad \mu_1 = -\left(\begin{smallmatrix} &&&2&&\\ 0&1&2&3&2&1\end{smallmatrix}\right)$$

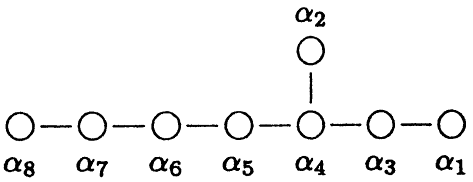
$$\nu_1 = \upsilon_1 = \alpha_0 \qquad \nu_2 = \upsilon_2 = \alpha_7$$

[1] The P-figure associated with geometry of subcase (B1).



[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and the geometry of subcase (B2) is a case in Example 4 since $(\mathfrak{l}, \hat{\sigma})$ is hermitian.

Let $\mathfrak{l}$ be the compact simple Lie algebra of type E$_8$. Then the Dynkin diagram of $\Pi(\mathfrak{l})$ is given in the following, where α_0 denotes the minus highest root of $\Pi(\mathfrak{l})$.



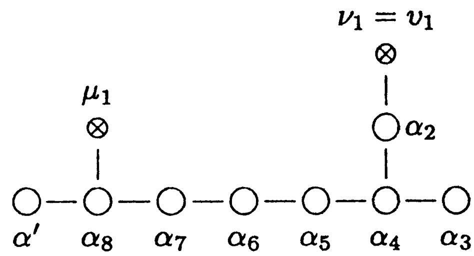
$$\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 = 0$$

In the following we identify a vector $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 + a_5\alpha_5 + a_6\alpha_6 + a_7\alpha_7 + a_8\alpha_8$ with an 8-tuple $\left(\begin{smallmatrix} &&&&a_2&&\\ a_8&a_7&a_6&a_5&a_4&a_3&a_1\end{smallmatrix}\right)$.

Type GE$_8$I. Let $\hat{\sigma} = \exp \mathrm{ad}(\pi\sqrt{-1}H_1)$. Then $\mathfrak{s} = \mathfrak{so}(16)$ and $\Pi(\mathfrak{s}) = \{\alpha', \alpha_2, \ldots, \alpha_8\}$ where $\alpha' = \left(\begin{smallmatrix} &&&&2&&\\ 0&1&2&3&4&3&2\end{smallmatrix}\right)$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_1 = -\left(\begin{smallmatrix} &&&&3&&\\ 2&3&4&5&6&4&2\end{smallmatrix}\right) \quad \nu_1 = \upsilon_1 = -\left(\begin{smallmatrix} &&&&3&&\\ 1&2&3&4&5&3&1\end{smallmatrix}\right)$$

[1] The P-figure associated with geometry of subcase (B1).



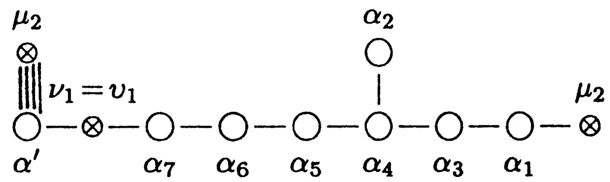
[**2**] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = \left(\begin{smallmatrix} &&&&3&& \\ 1&2&3&4&5&3&1 \end{smallmatrix}\right)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Type GE$_8$II. Let $\hat{\sigma} = \exp \mathrm{ad}(\pi\sqrt{-1})H_8)$. Then $\mathfrak{s} = \mathfrak{e}_7 \oplus \mathfrak{su}(2)$ and $\Pi(\mathfrak{s}) = \{\alpha', \alpha_1, \dots, \alpha_7\}$ where $\alpha' = -\alpha_0$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_1 = -\left(\begin{smallmatrix} &&&&2&& \\ 0&1&2&3&4&3&2 \end{smallmatrix}\right) \qquad \mu_2 = \alpha_0$$

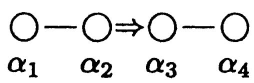
$$\nu_1 = \upsilon_1 = -\left(\begin{smallmatrix} &&&&3&& \\ 1&3&4&5&6&4&2 \end{smallmatrix}\right)$$

[1] The P-figure associated with geometry of subcase (B1).



[**2**] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = \left(\begin{smallmatrix} &&&&3&& \\ 1&3&4&5&6&4&2 \end{smallmatrix}\right)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Let $\mathfrak{l}$ be the compact simple Lie algebra of type F_4. Then the Dynkin diagram of $\Pi(\mathfrak{l})$ is given in the following, where α_0 denotes the highest root of $\Pi(\mathfrak{l})$.



$$\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = 0$$

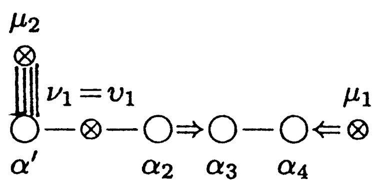
In the following we identify a vector $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4$ with a 4-tuple (a_1, a_2, a_3, a_4).

Type GF$_4$I. Let $\hat{\sigma} = \exp \mathrm{ad}(\pi\sqrt{-1}H_1)$. Then $\mathfrak{s} = \mathfrak{su}(2) \oplus \mathfrak{sp}(3)$ and $\Pi(\mathfrak{s}) = \{\alpha', \alpha_2, \alpha_3, \alpha_4\}$ where $\alpha' = -\alpha_0$. The minus dominant weights of $\mathfrak{s}_-^C$ and $\mathfrak{m}_\pm^C$ are given in the following.

$$\mu_1 = -(0122) \qquad \mu_2 = -(2342) = \alpha_0$$

$$\nu_1 = \upsilon_1 = -(1342)$$

[1] The P-figure associated with geometry of subcase (B1).

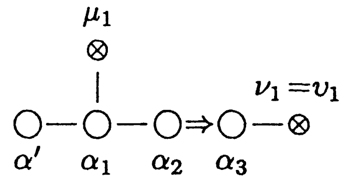


[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = (1342)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Type GF$_4$II. Let $\hat{\sigma} = \exp \operatorname{ad}(\pi\sqrt{-1}H_4)$. Then $\mathfrak{s} = \mathfrak{so}(9)$ and $\Pi(\mathfrak{s}) = \{\alpha', \alpha_1, \alpha_2, \alpha_3\}$ where $\alpha' = (0122)$. The minus dominant weights of $\mathfrak{s}_-^{\boldsymbol{C}}$ and $\mathfrak{m}_\pm^{\boldsymbol{C}}$ are given in the following.

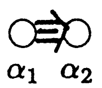
$$\mu_1 = -(2342) \qquad \nu_1 = \upsilon_1 = -(1231)$$

[1] The P-figure associated with geometry of subcase (B1).



[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = (1231)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Let $\mathfrak{l}$ be the compact simple Lie algebra of type G$_2$. Then the Dynkin diagram of $\Pi(\mathfrak{l})$ is given in the following, where α_0 denotes the minus highest root of $\Pi(\mathfrak{l})$.



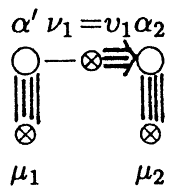
$$\alpha_0 + 2\alpha_1 + 3\alpha_2 = 0$$

In the following we identify a vector $a_1\alpha_1 + a_2\alpha_2$ with a 2-tuple (a_1, a_2).

Type GG$_2$. Let $\hat{\sigma} = \exp \operatorname{ad}(\pi\sqrt{-1}H_1)$. Then $\mathfrak{s} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\Pi(\mathfrak{s}) = \{\alpha', \alpha_2\}$ where $\alpha' = -\alpha_0$. The minus dominant weights of $\mathfrak{s}_-^{\boldsymbol{C}}$ and $\mathfrak{m}_\pm^{\boldsymbol{C}}$ are given in the following.

$$\mu_1 = -(01) \qquad \mu_2 = -(23) = \alpha_0$$
$$\nu_1 = \upsilon_1 = -(13)$$

[1] The P-figure associated with geometry of subcase (B1).



[2] The injectivity of ρ. In the subcases (B1) and (B3) the sets $D(\eta)$ are all empty, and in the subcase (B2) the set $D(\eta)$ $(= D(-\nu_1))$ consists of one element β^0 where $\beta^0 = (13)$, which by Lemma 3.4 belongs to $\mathfrak{r}_{0-}$. Taking account of Lemma 3.6, we can see that ρ is injective for all cases.

Comparing the P-figures associated with strongly substantial geometries of cases (A), (B) and (C), we last have the following.

THEOREM 4.1. *Generally, the PSLA's of strong type, thus, the strongly substantial geometries are decided by their P-figures.*

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