

Initial-final value problems for ordinary differential equations and applications to equivariant harmonic maps

By Takeyuki NAGASAWA¹ and Keisuke UENO

(Received Nov. 9, 1995)

(Revised July 22, 1996)

1. Introduction.

It is a fundamental problem to show the existence or nonexistence of a harmonic map between complete Riemannian manifolds. In the case of compact manifolds, a remarkable existence result is due to Eells-Sampson [3]. They showed that there exists a harmonic map if the target manifold has nonpositive sectional curvature. However, there is no general theory in the case the target manifold has positive sectional curvature. As for spheres, Smith [5] reduced the harmonic map equation to an ordinary differential equation and solving it, he constructed harmonic maps between spheres (see [2, 12, 13] for details and related topics).

This reduction technique works well even if manifolds are noncompact complete ones. Indeed, Urakawa and the second author [11] proved the existence of harmonic maps between noncompact cohomogeneity-one Riemannian manifolds. Here, a Riemannian manifold M is called a cohomogeneity-one Riemannian manifold if there exists a compact Lie group action G on M such that the quotient space M/G is a one-dimensional manifold. If a harmonic map between cohomogeneity-one Riemannian manifolds is invariant under these Lie group actions, then the harmonic map equation is reduced to an ordinary differential equation. Using this reduction, they showed the existence of harmonic maps between the complex hyperbolic spaces, the real hyperbolic spaces and the standard Euclidean spaces.

Our purpose of this paper is to study the ordinary differential equation (1.4)–(1.5) which appears in [11]. This equation has been already studied in [10], and proved the existence of a global and unbounded solution. In this paper, under different and more relaxed conditions, we present another proof of it and consider the finite time blow-up problem, which asserts the nonexistence of a harmonic map.

Let $f_i(t)$ and $h_i(r)$ ($i = 1, 2$) be given functions defined on $[0, \infty)$ satisfying the following conditions.

¹Partly supported by the Grants-in-Aids for Encouragement of Young Scientists, The Ministry of Education, Science and Culture, Japan.

$$(1.1) \quad \left\{ \begin{array}{l} \cdot f_i(t) > 0 \text{ on } (0, \infty), \text{ and } \dot{f}_i(t) \geq 0 \text{ on } [0, \infty); \\ \cdot \text{ there exist positive constants } a_i \ (i = 1, 2) \text{ such that} \\ \quad f_i(t) = a_i t + O(t^3) \quad (\text{as } t \rightarrow 0); \\ \cdot \text{ for any } t_0 > 0 \text{ there exists a constant } C = C(t_0) \text{ such that} \\ \quad 0 \leq p t \frac{\dot{f}_1(t)}{f_1(t)} + q t \frac{\dot{f}_2(t)}{f_2(t)} - 1 \leq C \text{ on } [0, t_0]; \end{array} \right.$$

$$(1.2) \quad \int_{t_0}^{\infty} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau < \infty.$$

$$(1.3) \quad \left\{ \begin{array}{l} \cdot \{h_i h_i'\}'(r) \geq 0 \text{ on } [0, \infty); \\ \cdot \text{ there exist positive constants } b_i \ (i = 1, 2) \text{ such that} \\ \quad h_i(r) = b_i r + O(r^3) \quad (\text{as } r \rightarrow 0). \end{array} \right.$$

Here \dot{r} (resp. h') means $\frac{dr}{dt}$ (resp. $\frac{dh}{dr}$).

For these given functions f_i and h_i , we consider the following ordinary differential equation between a variable t and unknown function $r = r(t)$.

$$(1.4) \quad \ddot{r}(t) + \left\{ p \frac{\dot{f}_1(t)}{f_1(t)} + q \frac{\dot{f}_2(t)}{f_2(t)} \right\} \dot{r}(t) - \left\{ \mu^2 \frac{h_1(r(t))h_1'(r(t))}{f_1(t)^2} + \nu^2 \frac{h_2(r(t))h_2'(r(t))}{f_2(t)^2} \right\} = 0 \quad \text{on } (0, \infty);$$

$$(1.5) \quad \lim_{t \rightarrow 0} r(t) = 0,$$

where μ and ν are nonnegative numbers. In an application, p and q stand for dimensions of manifolds, but we suppose here that they satisfy the following condition.

$$\left\{ \begin{array}{l} p \geq 1 \text{ and } q \geq 1, \\ \text{or} \\ p + q \geq 1 \text{ if } f_1(t) \equiv c f_2(t) \text{ for some constant } c > 0. \end{array} \right.$$

If $\mu + \nu = 0$, then we can easily verify that $r(t) \equiv 0$ is the unique solution to (1.4)–(1.5). Thus we assume $\mu > 0$.

Let

$$h(r) = \begin{cases} \max\{h_1(r), h_2(r)\} & (\nu > 0), \\ h_1(r) & (\nu = 0). \end{cases}$$

In §2, we prove that if the condition

$$\int^{\infty} \frac{dr}{h(r)} = \infty$$

holds, then every solution to (1.4)–(1.5) is a global and bounded one. Since each solution is increasing in t ([10, Lemma 3.1]), its final value $\lim_{t \rightarrow \infty} r(t)$ always exists. Conversely, for any given $l \in [0, \infty)$, we consider the problem of finding a global solution $r = r(t)$ such that $\lim_{t \rightarrow \infty} r(t) = l$. This problem is called the *final value problem*. In §3, we show this problem is solvable for any final value $l \in [0, \infty)$. This type of problem has been already considered by Ratto-Rigoli [4, §3] in the rotationally symmetric case. However, it seems to the authors to need more discussions. They use the continuous dependence of solutions on its initial value. However, it is generally valid only on finite interval. If we apply their method, we need some information about, for example, the stability of solutions.

In §4, we assume the condition

$$\int^{\infty} \frac{dr}{h(r)} < \infty,$$

which means any solution is not bounded. In this case, we consider the following problem: For any given $T \in (0, \infty]$, find a solution to (1.4)–(1.5) whose life span is $[0, T)$. We call this problem the *prescribed blow-up time problem*. Applying the same argument in §3, for any given $T \in (0, \infty]$, we can construct a solution $r = r(t)$ such that $\lim_{t \rightarrow T} r(t) = \infty$.

In the last section, we apply these results to show the existence of equivariant harmonic maps between real hyperbolic spaces. Our results have more applications, for detail, see [11].

Finally, we note that, under different condition, Tachikawa [6], [7] and Akutagawa-Tachikawa [1] showed the nonexistence results for harmonic maps between noncompact complete manifolds. They assumed only asymptotic conditions at infinity and made use of ordinary differential equations to construct a comparison function.

2. Existence of global solutions.

In this section we prove a sufficient condition for the existence of global solutions. The following lemmas hold under our conditions.

LEMMA 2.1 ([10, Lemma 3.1]). *Let $[0, T)$ be the life span of a solution r to (1.4). Then*

$$\dot{r}(t) > 0 \quad \text{on } (0, T),$$

unless $r(t) \equiv 0$. Furthermore if $T < \infty$, then

$$\lim_{t \uparrow T} r(t) = \infty.$$

LEMMA 2.2 ([10, Lemma 2.6]). *Let $r_i = r_i(t)$ ($i = 1, 2$) be solutions to (1.4)–(1.5) with*

$$r_1(t_0) = r_2(t_0)$$

for some $t_0 > 0$. Then it holds that

$$r_1(t) \equiv r_2(t) \quad \text{on } [0, t_0].$$

LEMMA 2.3 ([10, Lemma 3.3]). Let $r = r(t)$ be a non trivial solution to (1.4)–(1.5). Then it holds that for $t > 0$

$$(2.1) \quad \dot{r}(t)^2 \leq \frac{\mu^2}{f_1(t)^2} h_1(r(t))^2 + \frac{\nu^2}{f_2(t)^2} h_2(r(t))^2.$$

If $r(t) = 0$ for some $t > 0$, then $r(t) \equiv 0$ by Lemma 2.2. Thus it is a global solution. In the remainder of this section, we assume $r(t) > 0$ for all $t > 0$.

From Lemma 2.1 and Lemma 2.3, we have

$$\frac{\dot{r}(t)}{h(r(t))} \leq \gamma \left\{ \frac{1}{f_1(t)} + \frac{1}{f_2(t)} \right\},$$

where $h(r)$ is defined in the previous section and $\gamma = \max\{\mu, \nu\}$. Integrating both sides from t_0 to t , where $t \in [t_0, T]$, we obtain

$$(2.2) \quad \int_{r_0}^{r(t)} \frac{dr}{h(r)} \leq \gamma \int_{t_0}^t \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau.$$

THEOREM 2.4. Let t_0 and r_0 satisfy

$$(2.3) \quad \int_{r_0}^{\infty} \frac{dr}{h(r)} > \gamma \int_{t_0}^{\infty} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau.$$

Then there exists a global solution $r = r(t)$ to (1.4)–(1.5) satisfying $r(t_0) = r_0$ and it is bounded.

PROOF. Let $[0, T)$ be the life span of r , and assume $T < \infty$. Then by virtue of Lemma 2.1 and (2.2), we have

$$\int_{r_0}^{\infty} \frac{dr}{h(r)} = \lim_{t \uparrow T} \int_{r_0}^{r(t)} \frac{dr}{h(r)} \leq \gamma \lim_{t \uparrow T} \int_{t_0}^t \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau \leq \gamma \int_{t_0}^{\infty} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau,$$

which contradicts the assumption. Thus $T = \infty$.

Suppose $r = r(t)$ is not bounded. Then from the same argument we obtain a contradiction:

$$\int_{r_0}^{\infty} \frac{dr}{h(r)} = \lim_{t \rightarrow \infty} \int_{r_0}^{r(t)} \frac{dr}{h(r)} \leq \gamma \int_{t_0}^{\infty} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau. \quad \square$$

COROLLARY 2.5. If

$$(2.4) \quad \int_{r_0}^{\infty} \frac{dr}{h(r)} = \infty,$$

then any solution to (1.4)–(1.5) can be extended globally in t and it is bounded.

PROOF. The assumption of Theorem 2.4 is always valid. \square

REMARK. Since we assume the condition (1.2), the condition (2.3) holds for sufficiently large t_0 . Thus a non-trivial global solution to (1.4)–(1.5) always exists.

3. The final value problem.

Our problem always has bounded global solution. Thus the set

$$L = \left\{ \lim_{t \rightarrow \infty} r(t) \mid r \text{ is a bounded global solution to (1.4)–(1.5)}. \right\}$$

is not empty. In this section, we shall show that $L = [0, \infty)$. Indeed, we can prove the existence of the global solution to (1.4)–(1.5) satisfying $\lim_{t \rightarrow \infty} r(t) = l$ for any given $l \in [0, \infty)$.

PROPOSITION 3.1. L is dense in $[0, \infty)$.

PROOF. Since our problem has trivial solution, $0 \in L$. For any given $l > 0$ and $\varepsilon \in (0, l)$, we shall show

$$(l - \varepsilon, l + \varepsilon) \cap L \neq \emptyset.$$

Let T_0 be a positive number so that

$$\int_{l-\varepsilon}^{\infty} \frac{dr}{h(r)} > \gamma \int_{t_0}^{\infty} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau$$

holds for any $t_0 > T_0$. Then Theorem 2.4 shows the existence of the global solution to (1.4)–(1.5) satisfying

$$r(t_0) = l - \varepsilon > 0.$$

We denote this solution by $r(t; t_0)$. Since it is bounded and increasing in t , the limit $r(\infty; t_0)$ exists and

$$0 < l - \varepsilon < r(\infty; t_0) < \infty.$$

The inequality (2.2) yields

$$\int_{l-\varepsilon}^{r(\infty; t_0)} \frac{dr}{h(r)} \leq \gamma \int_{t_0}^{\infty} \left\{ \frac{1}{f_1(\tau)} + \frac{1}{f_2(\tau)} \right\} d\tau.$$

Since the right hand side tends to 0 as $t_0 \rightarrow \infty$, we have

$$\lim_{t_0 \rightarrow \infty} r(\infty; t_0) = l - \varepsilon.$$

Therefore for sufficiently large t_0 , it holds that

$$r(\infty; t_0) \in (l - \varepsilon, l + \varepsilon). \quad \square$$

THEOREM 3.2. For any $l \in [0, \infty)$, there exists the unique solution to (1.4)–(1.5) satisfying $\lim_{t \rightarrow \infty} r(t) = l$. Therefore $L = [0, \infty)$.

PROOF. If $l = 0$, then $r(t) \equiv 0$ is the desired solution. We assume $l > 0$. Proposition 3.1 asserts that there exist solutions r_i, \bar{r}_j to (1.4)–(1.5) such that

$$l_1 < l_2 < \dots \rightarrow l, \quad \bar{l}_1 > \bar{l}_2 > \dots \rightarrow l,$$

where $l_i = \lim_{t \rightarrow \infty} r_i(t), \bar{l}_j = \lim_{t \rightarrow \infty} \bar{r}_j(t)$. It follows from Lemma 2.2 that

$$r_1(t) < r_2(t) < \dots < \bar{r}_2(t) < \bar{r}_1(t).$$

Therefore, for any fixed $t_0 > 0$, we have two positive numbers $\underline{\alpha}$ and $\bar{\alpha}$ defined by

$$\underline{\alpha} = \lim_{i \rightarrow \infty} r_i(t_0), \quad \bar{\alpha} = \lim_{j \rightarrow \infty} \bar{r}_j(t_0).$$

Let us choose α such that $\underline{\alpha} \leq \alpha \leq \bar{\alpha}$, and let r_α be the solution to (1.4)–(1.5) satisfying $r_\alpha(t_0) = \alpha$. Then, from Lemma 2.2, we have

$$r_i(t) < r_\alpha(t) < \bar{r}_j(t)$$

for $t > 0$. Hence r_α is the global solution. Passing to $t \rightarrow \infty$ firstly, and to $i \rightarrow \infty, j \rightarrow \infty$ secondly, we obtain

$$\lim_{t \rightarrow \infty} r_\alpha(t) = l.$$

Now we show that r_α is the only solution to our problem. Let r be another solution to (1.4)–(1.5) satisfying $\lim_{t \rightarrow \infty} r(t) = l$. We denote $r_\alpha - r$ by ρ . Then it holds that

$$\begin{aligned} \ddot{\rho} + \left(p \frac{\dot{f}_1}{f_1} + q \frac{\dot{f}_2}{f_2} \right) \dot{\rho} \\ = \frac{\mu^2}{f_1^2} (h_1(r_\alpha)h_1'(r_\alpha) - h_1(r)h_1'(r)) + \frac{\nu^2}{f_2^2} (h_2(r_\alpha)h_2'(r_\alpha) - h_2(r)h_2'(r)). \end{aligned}$$

Multiplying both sides by $f_1^p f_2^q \rho$, we have

$$f_1^p f_2^q \rho \times (\text{the right-hand side}) \geq 0.$$

Therefore we obtain

$$\frac{d}{dt} (f_1^p f_2^q \rho \dot{\rho}) \geq f_1^p f_2^q \dot{\rho}^2 \geq 0.$$

Since $(f_1^p f_2^q \rho \dot{\rho})(0) = 0$ ([10, Corollary]) and $f_i(t) > 0$ for $t > 0$, it holds that

$$\rho \dot{\rho} \geq 0.$$

Thus

$$\frac{d}{dt} \{\rho(t)^2\} \geq 0$$

for $t > 0$. We can conclude that ρ is identically zero because

$$\lim_{t \rightarrow \infty} \rho(t)^2 = 0. \quad \square$$

4. The prescribed blow-up time problem.

In this section, we study the prescribed blow-up problem. Namely, for any given T ($0 < T < \infty$), we show that there exists a solution $r = r(t)$ to (1.4)–(1.5) satisfying $\lim_{t \rightarrow T} r(t) = \infty$.

We suppose that $h(r)$ satisfies

$$(4.1) \quad \int_{r_0}^{\infty} \frac{dr}{h(r)} < \infty.$$

REMARK. If the assumption (4.1) does not hold, then every solution is bounded, see Corollary 2.5.

For $t_0 > 0$, $r_0 \geq 0$, let $r = r(t)$ be the solution to (1.4)–(1.5) with $r(t_0) = r_0$. Set

$$\phi(r_0, t_0) = \int_{r_0}^{\infty} \frac{dr}{\sqrt{\gamma_1(t_0)^2 \{h_1(r)^2 - h_1(r_0)^2\} + \gamma_2(t_0)^2 \{h_2(r)^2 - h_2(r_0)^2\} + \beta(r_0, t_0)^2}},$$

where

$$\gamma_1(t_0) = \frac{\mu}{f_1(t_0)}, \quad \gamma_2(t_0) = \frac{\nu}{f_2(t_0)} \quad \text{and} \quad \beta(r_0, t_0) = \dot{r}(t_0).$$

LEMMA 4.1. $\phi(r_0, t_0)$ is well-defined for all $t_0 > 0$, $r_0 > 0$ and

$$\lim_{r_0 \rightarrow \infty} \phi(r_0, t_0) = 0$$

for any fixed $t_0 > 0$.

PROOF. It follows from (1.3) that $(h_i^2)'(r) \geq (h_i^2)'(r - r_0)$ for $r > r_0$. Integrating both sides with respect to r from r_0 to r , we get

$$h_i(r)^2 - h_i(r_0)^2 \geq h_i(r - r_0)^2.$$

Therefore it holds that for $K > 0$

$$\begin{aligned} & \int_{r_0+K}^{\infty} \frac{dr}{\sqrt{\gamma_1(t_0)^2 \{h_1(r)^2 - h_1(r_0)^2\} + \gamma_2(t_0)^2 \{h_2(r)^2 - h_2(r_0)^2\} + \beta(r_0, t_0)^2}} \\ & \leq C(t_0) \int_{r_0+K}^{\infty} \frac{dr}{\sqrt{h_1(r - r_0)^2 + h_2(r - r_0)^2}} \\ & \leq C(t_0) \int_K^{\infty} \frac{dr}{h(r)}. \end{aligned}$$

For any $\varepsilon > 0$, the last side is smaller than ε if K is sufficiently large.

On the other hand, the mean value theorem and (1.3) assert that

$$h_i(r)^2 - h_i(r_0)^2 = 2h_i(\rho_i)h_i'(\rho_i)(r - r_0) > 0,$$

for some $\rho_i \in (r_0, r)$. Hence we obtain

$$\begin{aligned} & \int_{r_0}^{r_0+K} \frac{dr}{\sqrt{\gamma_1(t_0)^2 \{h_1(r)^2 - h_1(r_0)^2\} + \gamma_2(t_0)^2 \{h_2(r)^2 - h_2(r_0)^2\} + \beta(r_0, t_0)^2}} \\ & \leq \frac{1}{\sqrt{2(\gamma_1(t_0)^2 h_1(r_0) h_1'(r_0) + \gamma_2(t_0)^2 h_2(r_0) h_2'(r_0))}} \int_{r_0}^{r_0+K} \frac{dr}{\sqrt{r - r_0}} \\ & \leq C(t_0) \sqrt{K} \min \left\{ \frac{1}{\sqrt{h_1(r_0) h_1'(r_0)}}, \frac{1}{\sqrt{h_2(r_0) h_2'(r_0)}} \right\}. \end{aligned}$$

Now we shall show that the last side converges to zero as $r_0 \rightarrow \infty$. Assume that both $h_1(r)h_1'(r)$ and $h_2(r)h_2'(r)$ are bounded functions. Then, because of (1.3), we get

$$h_i(r)^2 \leq Cr,$$

which yields a contradiction:

$$\int_{r_0}^{\infty} \frac{dr}{h(r)} \geq C \int_{r_0}^{\infty} \frac{dr}{\sqrt{r}} = \infty.$$

Thus

$$\lim_{r_0 \rightarrow \infty} \max \left\{ \sqrt{h_1(r_0) h_1'(r_0)}, \sqrt{h_2(r_0) h_2'(r_0)} \right\} = \infty$$

and we have

$$\lim_{r_0 \rightarrow \infty} \int_{r_0}^{r_0+K} \frac{dr}{\sqrt{\gamma_1(t_0)^2 \{h_1(r)^2 - h_1(r_0)^2\} + \gamma_2(t_0)^2 \{h_2(r)^2 - h_2(r_0)^2\} + \beta(r_0, t_0)^2}} = 0.$$

Summing up these estimates for integrations over $[r_0, r_0 + K]$ and $[r_0 + K, \infty)$, we obtain the assertion. \square

We define a set B as follows.

$$B = \left\{ T \in (0, \infty) \mid \text{There exists a solution } r = r(t) \text{ to (1.4)–(1.5) satisfying } \lim_{t \rightarrow T} r(t) = \infty. \right\}$$

PROPOSITION 4.2. B is dense in $(0, \infty)$.

PROOF. For any fixed $T > 0$ and $\varepsilon \in (0, T)$, we shall show

$$(T - \varepsilon, T + \varepsilon) \cap B \neq \emptyset.$$

Let r_α be the solution to (1.4)–(1.5) satisfying $r_\alpha(T) = \alpha$ and $[0, T_\alpha)$ its life span. Then we have

$$\phi(\alpha, T) \geq f_1(T)^p f_2(T)^q \int_T^{T_\alpha} \frac{d\tau}{f_1(\tau)^p f_2(\tau)^q}$$

([10, Lemma 3.5]). Since the left hand side goes to 0 as α tends to infinity,

$$\lim_{\alpha \rightarrow \infty} T_\alpha = T.$$

Hence $T_\alpha \in (T - \varepsilon, T + \varepsilon)$ for sufficiently large α . \square

THEOREM 4.3. *For any T ($0 < T < \infty$), there exists a solution $r = r(t)$ to (1.4)–(1.5) satisfying $\lim_{t \rightarrow T} r(t) = \infty$.*

PROOF. By virtue of Proposition 4.2 there exist solutions r_i, \bar{r}_j to (1.4)–(1.5) such that

$$0 < \underline{T}_1 < \underline{T}_2 < \cdots \rightarrow T, \quad \bar{T}_1 > \bar{T}_2 > \cdots \rightarrow T,$$

where $[0, \underline{T}_i)$ (resp. $[0, \bar{T}_j)$) is the life span of r_i (resp. \bar{r}_j). It follows from Lemma 2.2 that

$$\bar{r}_1(t) < \bar{r}_2(t) < \cdots < r_2(t) < r_1(t)$$

as long as they exist. Therefore, for any fixed t_0 ($0 < t_0 < \underline{T}_1$), we have two positive numbers $\underline{\alpha}$ and $\bar{\alpha}$ defined by

$$\underline{\alpha} = \lim_{i \rightarrow \infty} r_i(t_0), \quad \bar{\alpha} = \lim_{j \rightarrow \infty} \bar{r}_j(t_0).$$

We choose α such that $\bar{\alpha} \leq \alpha \leq \underline{\alpha}$, and denote by r_α the solution to (1.4)–(1.5) satisfying $r_\alpha(t_0) = \alpha$. Then, for any i and j , it holds that

$$\bar{r}_i(t) < r_\alpha(t) < r_j(t)$$

as long as they exist. Thus

$$\underline{T}_j < T_\alpha < \bar{T}_i,$$

where $[0, T_\alpha)$ is the life span of r_α . Letting i and $j \rightarrow \infty$, we obtain $T_\alpha = T$. \square

Combining Theorem 3.2 and Theorem 4.3, we can prove the existence of a global and bounded solution to (1.4)–(1.5).

COROLLARY 4.4 ([10, Theorem B]). *There exists a global solution $r = r(t)$ satisfying*

$$\lim_{t \rightarrow \infty} r(t) = \infty.$$

PROOF. By virtue of Theorem 3.2 and Theorem 4.3, we can choose sequences of positive numbers $\{l_j\}_{j=1}^\infty$, $\{T_i\}_{i=1}^\infty$ and sequences of solutions $\{r_j\}_{j=1}^\infty$, $\{\bar{r}_i\}_{i=1}^\infty$ to (1.4)–(1.5) so that

$$l_1 < l_2 < \cdots \rightarrow \infty, \quad \lim_{t \rightarrow \infty} r_j(t) = l_j,$$

$$T_1 < T_2 < \cdots \rightarrow \infty, \quad \lim_{t \uparrow T_i} \bar{r}_i(t) = \infty.$$

For any fixed $t_0 < T_1$, put

$$\bar{\alpha} = \lim_{i \rightarrow \infty} \bar{r}_i(t_0), \quad \underline{\alpha} = \lim_{j \rightarrow \infty} r_j(t_0).$$

Note that $\bar{\alpha}$ and $\underline{\alpha}$ are well-defined and $0 < \underline{\alpha} \leq \bar{\alpha} < \infty$. Let α be a number satisfying $\underline{\alpha} \leq \alpha \leq \bar{\alpha}$ and r_α the solution to (1.4)–(1.5) such that $r_\alpha(t_0) = \alpha$. Then, for any i and j , we can easily verify that

$$r_j(t) \leq r_\alpha(t) \leq \bar{r}_i(t)$$

as long as they exist. The second inequality implies that $T_\alpha = \infty$, and the first one asserts

$$\lim_{t \rightarrow \infty} r_\alpha(t) = \infty. \quad \square$$

5. Applications.

As applications, we show the existence of equivariant harmonic maps between real hyperbolic spaces and from real hyperbolic spaces to the real Euclidean spaces. The real hyperbolic space \mathbf{RH}^{m+1} can be considered as the product space $\mathbf{R}_+ \times S^m$ equipped with the metric $dt^2 + \sinh^2 t d\Theta^2$ in the coordinate $(t, \Theta) \in \mathbf{R}_+ \times S^m$, where $\mathbf{R}_+ = \{r \in \mathbf{R} \mid r \geq 0\}$, S^m is the m -dimensional unit sphere and $d\Theta^2$ is the standard metric on S^m . We call a map $\psi : \mathbf{RH}^{m+1} \rightarrow \mathbf{RH}^{n+1}$ is equivariant if it has the following form:

$$\psi : \mathbf{RH}^{m+1} \ni (t, \Theta) \mapsto (r(t), \phi(\Theta)) \in \mathbf{RH}^{n+1},$$

where r (resp. ϕ) is a map from \mathbf{R}_+ (resp. S^m) into \mathbf{R}_+ (resp. S^n). The sufficient and necessary conditions for ψ to be a harmonic map from \mathbf{RH}^{m+1} into \mathbf{RH}^{n+1} are

$$(5.1) \quad \ddot{r}(t) + m \frac{\cosh t}{\sinh t} \dot{r}(t) - 2e(\phi) \frac{\sinh r(t) \cosh r(t)}{\sinh^2 t} = 0, \quad \lim_{t \rightarrow 0} r(t) = 0,$$

and

$$(5.2) \quad \phi : S^m \rightarrow S^n \text{ is harmonic and } e(\phi) \text{ is constant,}$$

where $e(\phi)$ denotes the energy density of ϕ , $\dot{r}(t) = dr/dt$ and $\ddot{r}(t) = d^2r/dt^2$. A map ϕ , which satisfies (5.2), is called an eigenmap and its energy density is given by $2e(\phi) = k(k + m - 1)$ ($k = 1, 2, \dots$). In this case, $f_1(t) = f_2(t) = \sinh t$, $h_1(r) = h_2(r) = \sinh r$ and $p + q = m$, $\mu^2 + \nu^2 = k(k + m - 1)$. Since we can easily verify that these functions satisfy the conditions (1.1)–(1.3) and (4.1), there exist global solutions $r = r(t)$ of (5.1). Hence given an eigenmap from S^m to S^n , then we can construct an equivariant harmonic map from \mathbf{RH}^{m+1} into \mathbf{RH}^{n+1} .

THEOREM 5.1. *Let ϕ be an eigenmap from S^m to S^n . Then there exists an equivariant harmonic map from \mathbf{RH}^{m+1} to \mathbf{RH}^{n+1} . In particular, if ϕ is an onto map, then we can construct an onto harmonic map.*

For the existence and properties of eigenmaps, see [8], [9] and their references.

If the target manifold is the real Euclidean space, then $h(r) = r$ so that conditions (1.1)–(1.3) and (2.4) are fulfilled. Thus, as an application of Corollary 2.5, we have

THEOREM 5.2. *There exist equivariant harmonic maps from \mathbf{RH}^{m+1} into \mathbf{R}^{n+1} and each of them is bounded.*

ACKNOWLEDGEMENT. The authors are grateful to Professor H. Urakawa for his encouragement.

References

- [1] K. Akutagawa and A. Tachikawa, Nonexistence results for harmonic maps between noncompact complete Riemannian manifolds, *Tokyo J. Math.* **16** (1993), 131–145.
- [2] J. Eells and A. Ratto, Harmonic maps and minimal immersions with symmetries, *Ann. of Math. Studies*, **130** (1993), Princeton Univ. Press.
- [3] J. Eells and J. H. Sampson, Harmonic maps of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109–160.
- [4] A. Ratto and M. Rigoli, On the asymptotic behavior of rotationally symmetric harmonic maps, *J. Differential Equations* **101** (1993), 15–27.
- [5] R. T. Smith, Harmonic mappings of spheres, *Amer. J. Math.* **97** (1975), 364–385.
- [6] A. Tachikawa, A non-existence result for harmonic mappings from \mathbf{R}^n into H^n , *Tokyo J. Math.* **11** (1988), 311–316.
- [7] A. Tachikawa, Harmonic mappings from \mathbf{R}^n into an Hadamard manifold, *J. Math. Soc. Japan* **42** (1990), 147–152.
- [8] G. Toth, Harmonic maps and minimal immersions through representation theory, *Perspect. Math.* Academic Press, 1990.
- [9] K. Ueno, Some new examples of eigenmaps from S^m into S^n , *Proc. Japan Acad.* **69** (1993), 205–208.
- [10] K. Ueno, A study of ordinary differential equations arising from equivariant harmonic maps, *Tokyo J. Math.* **17** (1994), 395–416.
- [11] H. Urakawa and K. Ueno, Equivariant harmonic maps associated to large group actions, *Tokyo J. Math.* **17** (1994), 417–437.
- [12] H. Urakawa, Equivariant harmonic maps between compact Riemannian manifolds of cohomogeneity-one, *Michigan Math. J.* **40** (1993), 27–51.
- [13] Y. L. Xin, Riemannian submersion and equivariant harmonic maps, *Proc. Symp. Diff. Geom. in honor of Prof. Su Buchin*, World Sci. Publ. Co., 272–287.

Takeyuki NAGASAWA

Mathematical Institute (Kawauchi)
Faculty of Science
Tôhoku University
Kawauchi, Aoba, Sendai 980-77
Japan

Keisuke UENO

Department of Mathematical Sciences
Faculty of Science
Yamagata University
Yamagata 990
Japan