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Equivariant algebraic vector bundles over cones with smooth one dimensional quotient

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Introduction.

This paper is concerned with aspects of the general problem of constructing and distinguishing equivariant algebraic vector bundles over a base space which is an affine variety with an algebraic action of a complex reductive group G i.e. an affine G-variety.

In previous papers (See references.) we have been interested in general aspects of equivariant stably trivial vector bundles over a base which is an arbitrary affine G-variety. In those papers special emphasis was placed on the case where the base is a representation. In this paper we introduce a new class of equivariant varieties as base space. For these we give a complete description of a naturally defined subset of stably trivial equivariant bundles and give applications.

There are two important classes of base spaces for equivariant vector bundles. These are homogeneous spaces and representations. In the case of homogeneous spaces all equivariant vector bundles are known and easy to describe. For representations, far less is known but an interesting picture is developing. What we do in this paper is to introduce a class of affine G-varieties called *weighted G-cones* which from the point of view of the G-action are somewhat more complex than homogenous spaces but far simpler than representations. We are able to describe some of the equivariant vector bundles over weighted G-cones in Theorem A. As an application we apply the results to describe families of equivariant vector bundles over representations. See Theorems B and C.

Here is some of the history which inspired this work. It illustrates the state of the subject. The history begins with two important problems—the Equivariant Serre Problem and the Linearity Problem.

LINEARITY PROBLEM. *Is every algebraic action of G on C^n conjugate to a linear action?*

Kambayashi in [**Ka**] conjectured an affirmative answer and treated this in a paper with Russell in [**KR**]. This stimulated a lot of research. Somewhat later Bass-Haboush [**BH1**] tied this in with the Equivariant Serre Problem and showed how to produce a negative answer to the Linearity Problem from a negative answer to the Equivariant Serre Problem.

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EQUIVARIANT SERRE PROBLEM. *Is an algebraic G-vector bundle over a representation B of G trivial, i.e., isomorphic to $B \times F$ for some representation F of G?*

The answer to both questions is unknown for all groups and in the case of the Equivariant Serre Problem is rather subtle. It is true for some groups and false for others. Schwarz [**S**] gave the first negative solutions to both problems for several non abelian infinite groups. There are also negative solutions for finite groups. See [**MP1**, **MP2**]. The authors in [**MMP3**] show that the Equivariant Serre Problem does have an affirmative answer for abelian groups.

Another interesting feature of this problem is that in the cases where there is a non trivial G-vector bundle over a representation, there are continuous families of G-vector bundles over this representation. This means that we can expect the problem of distinguishing equivariant vector bundles in the algebraic category to be more difficult than in the topological category. In fact the problem of classification of equivariant vector bundles is probably hopeless and one should concentrate on giving a general construction of equivariant vector bundles and methods for distinguishing them. In [**MP1**] and [**MP2**], we give a construction E_Φ (See section 3.) of equivariant vector bundles over an arbitrary affine G-variety and give an invariant ρ (See below.) of these bundles. This construction and invariant were used to establish the existence of families of negative solutions to the Equivariant Serre Problem for certain families of finite groups mentioned above ([**MP1**], [**MP2**]) and for non abelian connected reductive groups ([**MMP1**], [**MMP2**]).

Prior to the papers cited above Bass and Haboush made several important contributions to the Equivariant Serre Problem [**BH1**] and [**BH2**]. In particular they showed that equivariant vector bundles over representations are stably trivial. Their work together with our experience with stably trivial vector bundles in topology led us to consider a subset $VEC(B, F; S)$ of stably trivial vector bundles over the affine G-variety B depending on representations F and S of G. This is the set of isomorphism classes of G-vector bundles over B whose Whitney sum with $\mathbf{S}$ is $\mathbf{F} \oplus \mathbf{S}$, where $\mathbf{F}$ resp. $\mathbf{S}$ denotes the trivial G-vector bundle over B whose total space is $B \times F$ resp. $B \times S$. There are two important features of the sets $VEC(B, F; S)$: Varying F and S gives all equivariant vector bundles over B when B is a representation. The set $VEC(B, F; S)$ is naturally isomorphic via Theorem 1.1 to the orbit space of an action of the automorphism group of the bundle $\mathbf{F} \oplus \mathbf{S}$ on the space of surjective G-vector bundle morphisms of $\mathbf{F} \oplus \mathbf{S}$ to $\mathbf{S}$.

In [**MP1**] we produce a function

$$\rho : VEC(B, F; S) \to (R/I)^*/\Gamma$$

where R is the ring of G-vector bundle endomorphisms of $\mathbf{S}$, I is an ideal in R, $(R/I)^*$ denotes the units of this quotient ring and Γ is a subgroup of units. (See section 1). This is essential to all that we do here and is responsible for our contributions to the existence of the negative solutions to the Equivariant Serre Problem mentioned above.

As mentioned the problem of classifying equivariant vector bundles over an arbitrary affine G-variety B seems to be out of the question. A more tractable problem is to identify and distinguish a natural class of equivariant vector bundles. The construction E_Φ and invariant ρ are our main tools for this and the set $VEC(B, F; S)$ is a convenient subset of G-vector bundles for organizing this problem.

We now define the weighted G-cones mentioned above. More specifically these G-varieties are called *weighted G-cones with smooth one dimensional quotients*, and will be abbreviated as weighted G-cones in this paper. Here is a brief picture of the equivariant algebraic geometry of these varieties. A variety B is a weighted G-cone if it is an affine $G \times \boldsymbol{C}^*$-variety whose quotient by $\boldsymbol{C}^*$ is a point and whose quotient by G is the complex line $\boldsymbol{C}$. Each weighted G-cone B contains *a special G-cone B^s* (section 2) which may be viewed as a parametrized family of homogeneous spaces in the sense that there is a G-invariant map $\Delta : B^s \to \boldsymbol{C}$ with $\Delta^{-1}(t) \cong G/H$ for $t \neq 0$. Here H is a fixed reductive subgroup of G. The special G-cone B^s captures all the equivariant bundle information of B because $VEC(B, F; S) \to VEC(B^s, F; S)$ is bijective. See 1.3 and 2.2.

The use of cones to treat equivariant vector bundles over representations began informally several years ago. It started with our realization that the invariant ρ which we introduced in [**MP1**] to distinguish equivariant vector bundles over representations could also be fruitfully employed as an invariant for the restrictions of these bundles to the closure of an orbit of a linear subspace. These closures are $G \times \boldsymbol{C}^*$-varieties and cones. The $\boldsymbol{C}^*$-action comes from scalar multiplication in the representation. The point is that the determination of the invariant for the restriction of the bundle to the cone is often more tractable than for the bundle itself. From this informal beginning developed the wish to find a simple useful setting where the invariant led to an isomorphism. We found this setting in the notion of weighted and special G-cones. (The weighted G-cones here are associated with the closure of the orbit of a line in a representation.) An announcement of a preliminary version of this work appears in [**MJ**].

One simple but important class of weighted G-cones is the set of representations with one dimensional quotient. Kraft and Schwarz treated vector bundles over such representations from a different viewpoint in [**KS**].

We now discuss the main results of the paper. For this B denotes a weighted G-cone. An essential feature of a weighted G-cone is that its ring of G-invariant functions is a polynomial ring in one variable $\boldsymbol{C}[\Delta]$. The ideal in this ring generated by Δ^m is denoted by (Δ^m). In our case here $S = \boldsymbol{C}$ and B is a weighted G-cone. In this case it is easy to see that $R = \boldsymbol{C}[\Delta]$.

THEOREM A. (1) *$\rho : VEC(B, F; \boldsymbol{C}) \to (R/I)^*/\Gamma$ is bijective.*

(2) *The multiplicative group $(R/I)^*/\Gamma$ is isomorphic to a vector space $(\Delta)/(\Delta^{d+1})$ $(\cong \boldsymbol{C}^d)$ via a logarithmic map* log *which converts multiplication to addition. Here $d = d(B, F)$ is a non-negative integer depending on B and F.*

Hence $\log\rho : VEC(B, F; \boldsymbol{C}) \to (\Delta)/(\Delta^{d+1}) \cong \boldsymbol{C}^d$ *is bijective.*

DEFINITION. We say that a representation V of G is *multiplicity free with respect to* H if each irreducible representation of H occurs in V (viewed as a representation of H) with multiplicity at most 1.

For applications of Theorem A, one must determine $d(B,F)$. For that purpose we define an explicit integer $e(B,F)$ which gives good upper bound.

THEOREM B. *There is an explicit integer* $e(B,F)$ *(defined in section* 6*) such that* $d(B,F) \leq e(B,F)$ *with equality holding if* F *is a multiple of a representation of* G *which is multiplicity free with respect to a principal isotropy group of* B.

This result allows us to give explicit examples of non-trivial equivariant vector bundles.

Here is a brief picture of applications of the theory on cones to equivariant vector bundles over representations. Let V be a representation of G such that its ring of invariants $\mathcal{O}(V)^G$ is not $\boldsymbol{C}$, in other words, there is a non-zero $x \in V$ whose orbit is closed. From the closure of the $G \times \boldsymbol{C}^*$-orbit of x, we produce (section 7) a special G-cone $B = B(V;x)$ (or $B_1(V;x)$) together with a G-morphism $\imath : B \to V$. This induces a map

$$\imath^* : \mathcal{O}(V)_1^G \to (\Delta)/(\Delta^{d(B,F)+1}) \cong \boldsymbol{C}^d$$

where $\mathcal{O}(V)_1^G$ is the ideal of invariant functions on V with zero constant term. The image of $\imath^*$ is a complex vector space whose dimension we denote by $f(V,F;x)$. This leads to a lower bound for equivariant vector bundles over V. Let

$$\log \rho_x : VEC(V,F;\boldsymbol{C}) \to (\Delta)/(\Delta^{d(B,F)+1}) \cong \boldsymbol{C}^d$$

be the composition of the pull-back $VEC(V,F;\boldsymbol{C}) \to VEC(B,F;\boldsymbol{C})$ by $\imath$ and $\log \rho$.

THEOREM C. *Suppose that there is an equivariant morphism* $\Phi : V \to F^* = Hom(F,\boldsymbol{C})$ *such that* $\Phi^{-1}(0) = 0$. *(We call such* Φ *a weak duality.) Then the image of* $\log \rho_x$ *agrees with* $\imath^*\mathcal{O}(V)_1^G$, *in particular* $\log \rho_x$ *maps* $VEC(V,F;\boldsymbol{C})$ *onto* $\boldsymbol{C}^{f(V,F;x)}$.

Theorems A–C are our main general theorems. In sections 7–11, we give several applications to specific families of groups where we give a description of $e(B,F)$ and $f(V,F;x)$ for various weighted G-cones and representations, and the calculations of $d(B,F)$ use Theorem B. E.g. in section 8 we do this for dihedral groups and in section 9, we give a lower bound for $e(B,F)$ when $B = B(V;x)$ and V and F are multiples of irreducible representations of a connected reductive group (See 9.3). The lower bound is explicit in the dominant weights of V and F. The integer $e(B,F)$ depends on the equivariant algebraic geometry of B and the representation theory of F. In the case of a connected reductive group, it can be determined from information involving the maximal unipotent subgroup. This connection is made in section 10 and applied in section 11 to treat equivariant vector bundles over weighted G-cones and representations associated with spherical subgroups of rank 1. It is clear to us that the Equivariant

Serre Problem for many families of non-abelian finite groups and representations may be treated by using Theorem C as we do for the dihedral groups in section 8. As evidence of this we cite the related treatment of this issue for several families of non-abelian finite groups in [**MP2**].

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§1. Review and general remarks.

In this section we review some notation and some results from [**MP2**] which we will use, and give two general remarks (Propositions 1.3 and 1.4) on closed G-subvarieties which contain all closed orbits in a given affine G-variety.

Let G be a reductive group, F and S be representations of G, and B be an affine G-variety (reduced, but not necessarily irreducible). Denote by $sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$ the set of surjective G-vector bundle morphisms from $\mathbf{F} \oplus \mathbf{S}$ to $\mathbf{S}$, and by $aut(\mathbf{F} \oplus \mathbf{S})$ the group of G-vector bundle automorphisms of $\mathbf{F} \oplus \mathbf{S}$. We note that if $L \in sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$, then its kernel $E(L)$ defines an element in $VEC(B, F; S)$ (see [**MP2**, I.1.1]).

THEOREM 1.1 ([**MP2**, I.1.2]). *The map which sends $L \in sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$ to $E(L)$ defines a bijection*

$$sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})/aut(\mathbf{F} \oplus \mathbf{S}) \cong VEC(B, F; S).$$

This theorem tells us that the study of $VEC(B, F; S)$ can be divided into two steps: one is the study of $sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$ and the other is that of $aut(\mathbf{F} \oplus \mathbf{S})$. The former corresponds to construction of G-vector bundles and the latter to distinguishing them. As for their construction, the following easy lemma describes a criterion to find elements of $sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$. Denote by $mor(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$ the set of G-vector bundle morphisms from $\mathbf{F} \oplus \mathbf{S}$ to $\mathbf{S}$. There is a simple characterization for an element of $mor(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$ to be in $sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$. Note that an element of $mor(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$ is of the form (Φ, T) where $\Phi \in mor(\mathbf{F}, \mathbf{S})$ and $T \in end(\mathbf{S}) = mor(\mathbf{S}, \mathbf{S})$.

LEMMA 1.2. *$(\Phi, T) \in sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$ if and only if there are $\Psi \in mor(\mathbf{S}, \mathbf{F})$ and $Y \in end(\mathbf{S})$ such that $\Phi\Psi + TY = 1$ where 1 denotes the identity element in $end(\mathbf{S})$.*

PROOF. It is known in [**BH**] that any surjective G-vector bundle morphism has a splitting. If $(\Phi, T) \in sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$, then we take $(\Psi, Y) \in mor(\mathbf{S}, \mathbf{F} \oplus \mathbf{S})$ to be the splitting of (Φ, T). This proves the "only if" part. The converse is evident. □

We will review the definition of the invariant ρ. To define ρ, we take a closer look at $sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$ and $aut(\mathbf{F} \oplus \mathbf{S})$. Let U and V be representations of G. To emphasize the source and target of a G-vector bundle morphism D from $\mathbf{U}$ and $\mathbf{V}$, we write such a morphism as $D(U, V)$. For example, any $A \in aut(\mathbf{F} \oplus \mathbf{S})$ has components $A(U, V)$ for

$U = F, S$ and $V = F, S$ and similarly any $L \in sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$ has components $L(F, S)$ and $L(S, S)$.

Denote by R the ring $end(\mathbf{S})$. Let I be the two-sided ideal in R generated by the elements which factor through $\mathbf{F}$. That is, the generators of I are given by all the elements of the form $\Phi\Psi \in end(\mathbf{S})$, where $\Phi \in mor(\mathbf{F}, \mathbf{S})$ and $\Psi \in mor(\mathbf{S}, \mathbf{F})$. It is shown in [**MP2**] that for any $A \in aut(\mathbf{F} \oplus \mathbf{S})$ the class of $A(S, S)$ in R/I is a unit. As a consequence, we can consider the following subgroup of the group of units $(R/I)^*$ of R/I:

$$\Gamma = \{A(S, S) \in (R/I)^* \mid A \in aut(\mathbf{F} \oplus \mathbf{S})\}.$$

Using Theorem 1.1 one shows that the class of $L(S, S)$ is a unit in R/I for any $L \in sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$ and that there is a well-defined map

$$\rho : VEC(B, F; S) \to (R/I)^*/\Gamma$$

given by $\rho(E(L)) = [L(S, S)] \in (R/I)^*/\Gamma$ for $L \in sur(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})$, where $[L(S, S)]$ means the class of $L(S, S)$ in $(R/I)^*/\Gamma$. In other words, *$\rho(E)$ is an invariant for the isomorphism class of the G-vector bundle E.* This invariant is used to distinguish the G-vector bundles.

We close this section with two useful results to compare the equivariant vector bundles over B and over a closed G-subvariety which contains all the closed orbits in B.

PROPOSITION 1.3. *Suppose that B' is a closed G-subvariety of B which contains all closed orbits of B. Then the restriction map $VEC(B, F; S) \to VEC(B', F; S)$ is bijective.*

PROOF. Each stably trivial G-vector bundle is the kernel of a surjective G-vector bundle morphism between trivial bundles by Theorem 1.1. Any such surjective G-vector bundle morphism over B' extends to a surjective G-vector bundle morphism over B by the Equivariant Nakayama Lemma ([**BH**, §6]) since B' contains all closed orbits in B. This proves the surjectivity.

If two G-vector bundles over B are isomorphic when restricted to B', then the isomorphism extends to an isomorphism over B again by the Equivariant Nakayama Lemma. This proves the injectivity. □

PROPOSITION 1.4. *Let B' be the same as in Proposition* 1.3. *Then the restriction map $\mathcal{O}(B)^G \to \mathcal{O}(B')^G$ is an isomorphism.*

PROOF. The restriction map $\mathcal{O}(B) \to \mathcal{O}(B')$ is surjective, since B' is a closed subvariety. Thus the restriction to the G-invariants is also surjective, since G is reductive (see e.g. [**BH**]). This proves the surjectivity.

If f is in the kernel, it vanishes on every closed orbit of B. However, since it is invariant, it is constant on the closure of any orbit. Thus it is zero everywhere, and the map is injective. □

REMARK. As a consequence, if $S = \boldsymbol{C}$ the trivial one-dimensional representation, then the ring $R = \mathcal{O}(B)^G$, the ideal I and the subgroup Γ used to calculate the invariants are the same for a G-vector bundles over B and their restrictions over a closed G-subvariety of B containing all closed orbits of B.

§2. Weighted G-cones and special G-cones.

In this section, we will describe the type of affine G-variety that will be used as the base of the G-vector bundles. They will be called weighted G-cones (with smooth one-dimensional quotients).

Consider the following two conditions on an affine $G \times \boldsymbol{C}^*$-variety B:

(C0) $\mathcal{O}(B)^{\boldsymbol{C}^*} = \boldsymbol{C}$ and $\mathcal{O}(B)$ is positively graded with respect to the $\boldsymbol{C}^*$-action,

(C1) $\mathcal{O}(B)^G = \boldsymbol{C}[\Delta]$ for some non-constant homogeneous $\Delta \in \mathcal{O}(B)^G$.

(We say that $f \in \mathcal{O}(B)$ is homogeneous of degree d if $f(cx) = c^d f(x)$ for any $c \in \boldsymbol{C}^*$ and $x \in B$.)

Definition. We call B *a weighted G-cone* if (C0) is satisfied and *a weighted G-cone with smooth one dimensional quotient* if in addition (C1) is satisfied.

Remarks. (1) In this paper we consider only weighted G-cones with smooth one dimensional quotient and refer to them simply as weighted G-cones. We emphasize that this means both conditions are satisfied.

(2) We also remark that in the presence of (C1), the condition that $\mathcal{O}(B)$ is positively (or negatively) graded is a consequence of the other two conditions. If $\mathcal{O}(B)$ is negatively graded, then we change the $\boldsymbol{C}^*$-action via the automorphism of $\boldsymbol{C}^*$ which takes each element to its inverse. With this new $\boldsymbol{C}^*$-action, $\mathcal{O}(B)$ is positively graded. This reconciles the definition in the introduction with that here.

(3) For any affine G-variety B, the (algebraic) quotient is defined to be the affine variety whose coordinate ring is $\mathcal{O}(B)^G$. A representation whose quotient is one-dimensional automatically satisfies condition (C1). However, in general, a one-dimensional quotient of a G-variety is not necessarily smooth, and therefore does not suffice for condition (C1) (see section 8).

Once again B will denote a weighted G-cone unless otherwise stated. The following lemma is a consequence of condition (C0).

Lemma 2.1. (1) *Any closed $\boldsymbol{C}^*$-subvariety X of B satisfies condition (C0) and $\mathcal{O}(X)^* = \boldsymbol{C}^*$, where $\mathcal{O}(X)^*$ denotes the group of units in $\mathcal{O}(X)$.*

(2) *The only closed $\boldsymbol{C}^*$-orbit in B is in fact a $\boldsymbol{C}^*$-fixed point. It is also fixed by the action of G.*

Proof. (1) Since X is closed and $\boldsymbol{C}^*$-invariant, the restriction map $\mathcal{O}(B) \to \mathcal{O}(X)$ is surjective and $\boldsymbol{C}^*$-equivariant. Thus the restriction to the $\boldsymbol{C}^*$-invariants is also surjective since $\boldsymbol{C}^*$ is reductive. This implies that X inherits condition (C0) from B. Thus the former statement has been proved.

Now we prove the latter statement. Since $\boldsymbol{C}^*$ is connected, X is a union of irreducible closed $\boldsymbol{C}^*$-subvarieties of X. Any $\boldsymbol{C}^*$-variety contains at least one closed

C^*-orbit and X has only one closed C^*-orbit by (C0), so the intersection of the irreducible components is not empty. Therefore it suffices to show that $\mathcal{O}(X)^* = C^*$ when X is irreducible.

Suppose f is a unit in $\mathcal{O}(X)$. Decompose f and $h = f^{-1}$ into its homogeneous parts, and denote by $f_{(n)}$ resp. $h_{(m)}$ the highest non-zero homogeneous part of f resp. h. Since X is irreducible, $f_{(n)}h_{(m)} \neq 0$, and thus $n = m = 0$. In other words, f is a non-zero constant.

(2) The only closed C^*-orbit in B is isomorphic to C^* or a C^*-fixed point. By (1) the former case does not occur. Since the C^*-fixed point is unique and G-invariant, it must be fixed by the action of G. $\square$

The G-invariant function $\Delta : B \to C$ becomes C^*-equivariant when we give the target space C the usual C^*-action with weight $\deg \Delta$. As a matter of fact, the quotient $B//G$ is isomorphic to C by condition (C1) and then the map Δ is nothing but the quotient map. Denote the fiber $\Delta^{-1}(t)$ $(t \in C)$ by B_t. By the C^*-equivariance of Δ, the fibers B_t $(t \neq 0)$ are isomorphic to each other. They are called the *generic fibers* while the fiber B_0 is called the *null-fiber*. The closed orbits in the generic fibers B_t $(t \neq 0)$ are isomorphic to G/H where H is a principal isotropy group of B (as a G-variety).

DEFINITION. We say that B is *special* or *a special G-cone* if $B = \overline{(G \times C^*)x}$ for some $x \in B - B_0$ whose G-orbit is closed. (Overline denotes closure.)

Throughout this paper H will denote a principal isotropy group of B (as a G-variety) unless otherwise stated. If B is special, $B = \overline{G(B - B_0)^H}$.

LEMMA 2.2. (1) *The subvariety $\overline{G(B - B_0)^H}$ is a special G-cone and contains all closed G-orbits in B.*

(2) *If B is special, then the generic fibers B_t $(t \neq 0)$ are isomorphic to G/H.*

(3) *If B is special and irreducible, then the field of G-invariant rational functions on B is the field $C(\Delta)$.*

PROOF. (1) Set $B^s = \overline{G(B - B_0)^H}$. Clearly B^s is a closed $G \times C^*$-subvariety which contains all closed G-orbits in B. It follows from Lemma 2.1(1) and Proposition 1.4 that B^s is a weighted G-cone. Then it is obvious from the construction that B^s is special.

(2) For $t \neq 0$, B_t contains a closed orbit isomorphic to G/H. If the dimension of the fiber is the dimension of G/H, then in fact the fiber consists only of this one orbit. This is easily seen to be the case, since

$$\dim G/H \le \dim B_t < \dim B = \dim(G \times C^*) - \dim H = \dim G/H + 1.$$

(3) This is an easy consequence of the fact from (2). In fact, suppose that f is a G-invariant rational function on B. Then set of poles is contained in a finite number of fibers, since it is G-stable. Let h be a G-invariant regular non-zero function which

vanishes on those fibers. By multiplying f with a large enough multiple of h, one obtains a regular G-invariant function. Thus f is in the quotient field of $\boldsymbol{C}[\Delta]$. $\square$

REMARK. By virtue of Proposition 1.3 and Lemma 2.2(1) we may assume that B is special whenever we study $VEC(B, F; S)$.

Given any representation W of G, denote by $Mor(B, W)^G$ the set of equivariant morphisms from B to W. Note that there is a natural isomorphism

$$Mor(B, Hom(U, V))^G \cong mor(\mathbf{U}, \mathbf{V})$$

where U and V are representations of G. A general theory tells us that $Mor(B, W)^G$ is a finitely generated $\mathcal{O}(B)^G$-module ([**Kr**, II.3.2]). In our case $\mathcal{O}(B)^G = \boldsymbol{C}[\Delta]$ and we can say more about $Mor(B, W)^G$ if B is special.

LEMMA 2.3. *Suppose that B is special. Then*

(1) *The vector space $Mor(B, W)^G \otimes_{\boldsymbol{C}[\Delta]} \boldsymbol{C}[\Delta]/(\Delta - t)$ is isomorphic to W^H if $t \neq 0$, where the isomorphism is given by evaluating elements of $Mor(B, W)^G$ at a point of B_t whose isotropy group is H.*

(2) *The $\boldsymbol{C}[\Delta]$-module $Mor(B, W)^G$ is free of rank* $\dim W^H$ *for any representation W of G.*

PROOF. (1) For $t \neq 0$, $\Delta - t$ generates the ideal of regular functions which vanish on B_t since $\Delta : B \to \boldsymbol{C}$ is smooth off B_0 ([**L**]). It follows that the module $Mor(B, W)^G$ tensored with $\boldsymbol{C}[\Delta]/(\Delta - t)$ over $\boldsymbol{C}[\Delta]$ agrees with $Mor(B_t, W)^G$. The result then follows because $B_t \cong G/H$ and $Mor(G, W)^G \cong W^H$.

(2) First we show that $Mor(B, W)^G$ is a free $\boldsymbol{C}[\Delta]$-module. Since $Mor(B, W)^G \cong (\mathcal{O}(B) \otimes W)^G$ as $\boldsymbol{C}[\Delta]$-modules, it suffices to show that $\mathcal{O}(B)$ is free as a $\boldsymbol{C}[\Delta]$-module. Choose $f \in \mathcal{O}(B)$ and $h \neq 0 \in \boldsymbol{C}[\Delta]$ such that $hf = 0$. Then, for some $x \in B - B_0$, h is not identically 0 on $\boldsymbol{C}^*x$, thus $f = 0$ on $\boldsymbol{C}^*x$. Similarly, $f = 0$ on $\boldsymbol{C}^*gx$ for any $g \in G$ since h is G-invariant. Since B is special, $f = 0$ on B by continuity of the function. This proves that $\mathcal{O}(B)$ is torsion-free as a $\boldsymbol{C}[\Delta]$-module. Thus it is free since $\boldsymbol{C}[\Delta]$ is a principal ideal domain.

The rank of $Mor(B, W)^G$ is calculated by part (1). $\square$

We prepare a lemma which will be used in Section 6. Denote the normalizer of H in G by $N_G(H)$.

LEMMA 2.4. *The restriction map $j^* : \boldsymbol{C}[\Delta] = \mathcal{O}(B)^G \to \mathcal{O}(B^H)^{N_G(H)}$ is injective and $\mathcal{O}(B^H)^{N_G(H)}$ viewed as a $\boldsymbol{C}[\Delta]$-module via j^* has rank one free part.*

PROOF. Let $B' = \overline{GB^H}$. Then $B'^H = B^H$ and the map j^* splits into two restriction maps

$$\mathcal{O}(B)^G \to \mathcal{O}(B')^G \to \mathcal{O}(B'^H)^{N_G(H)} = \mathcal{O}(B^H)^{N_G(H)}.$$

The former restriction map is an isomorphism by Proposition 1.4 since B' contains all closed G-orbits in B, and the latter one is injective since the G-orbit of B'^H is dense in B'. This proves the injectivity of j^*.

The above discussion also says that we may assume $B = \overline{GB^H}$. Then $B - B_0$ is isomorphic to the balanced product $G \times_{N_G(H)} (B - B_0)^H$. Therefore the restriction map

$$\mathcal{O}(B - B_0)^G \to \mathcal{O}((B - B_0)^H)^{N_G(H)}$$

is an isomorphism. Here since $B_0 = \Delta^{-1}(0)$, $\mathcal{O}((B - B_0)^H)^{N_G(H)}$ is obtained from $\mathcal{O}(B^H)^{N_G(H)}$ by tensoring with $\mathcal{O}(B - B_0)^G \cong \boldsymbol{C}[\Delta, \Delta^{-1}]$ over $\boldsymbol{C}[\Delta]$. This implies the latter statement of the lemma since this operation does not change the rank of the free part. $\square$

§3. Bijectivity of the invariant ρ.

In this section we prove Theorem A(1), i.e. bijectivity of the invariant ρ when B is a weighted G-cone and $S = \boldsymbol{C}$.

For $\Phi \in mor(\mathbf{F}, \mathbf{C})$ we define

$$R(\Phi)_* = \{T \in 1 + (\Delta) | (\Phi, T) \in sur(\mathbf{F} \oplus \mathbf{C}, \mathbf{C})\} \subset R = \boldsymbol{C}[\Delta].$$

Since B has a G-fixed point by Lemma 2.1(2), any element of $VEC(B, F; \boldsymbol{C})$ is represented by the kernel of $(\Phi, T) \in sur(\mathbf{F} \oplus \mathbf{C}, \mathbf{C})$ for some $\Phi \in mor(\mathbf{F}, \mathbf{C})$ and $T \in R(\Phi)_*$ ([**MP2**, I.3.2]). Note that $VEC(B, F; \boldsymbol{C}) = \{*\}$ if $mor(\mathbf{F}, \mathbf{C}) = \{0\}$. Denote by $E_\Phi(T) \in VEC(B, F; \boldsymbol{C})$ the isomorphism class of the kernel of (Φ, T). The key result which we prove is that $E_\Phi(T)$ does not depend on Φ (Theorem 3.5).

We begin with some elementary preliminaries.

PROPOSITION 3.1. *Let $\Phi \in mor(\mathbf{F}, \mathbf{C})$ and $\Psi \in mor(\mathbf{C}, \mathbf{F})$. Suppose that $\Psi\Phi \in end(\mathbf{F})$ and $\Phi\Psi \in \boldsymbol{C}[\Delta]$ are divisible by Δ^l $(l \geq 0)$ in $end(\mathbf{F})$ and $\boldsymbol{C}[\Delta]$ respectively. Then, for polynomials P, P', Q and $Q' \in \boldsymbol{C}[\Delta]$, the endomorphism of $\mathbf{F} \oplus \mathbf{C}$ defined by*

$$\begin{pmatrix} 1 + P\Psi\Phi/\Delta^l & Q\Psi \\ Q'\Phi & 1 + P'\Phi\Psi/\Delta^l \end{pmatrix}$$

is an automorphism if P, P', Q and Q' satisfy the equation

$$P + P' + PP'\Phi\Psi/\Delta^l - \Delta^l QQ' = 0.$$

PROOF. In fact, an elementary check shows that

$$\begin{pmatrix} 1 + P'\Psi\Phi/\Delta^l & -Q\Psi \\ -Q'\Phi & 1 + P\Phi\Psi/\Delta^l \end{pmatrix}$$

is the inverse of the given endomorphism if the given condition is satisfied. $\square$

PROPOSITION 3.2. *Let $\Phi, \Phi' \in mor(\mathbf{F}, \mathbf{C})$, $\xi \in \boldsymbol{C}[\Delta]$ and $T \in R(\xi\Phi + \Phi')_*$. Then $E_{\xi\Phi+\Phi'}(T) = E_{\Phi+\Phi'}(T)$ if there is $\Psi \in mor(\mathbf{C}, \mathbf{F})$ such that $(\xi\Phi\Psi, T) = (1)$ and $\Phi'\Psi = 0$.*

PROOF. Since $(\xi\Phi\Psi, T) = (1)$, there are polynomials $\alpha, \beta \in C[\Delta]$ such that

$$\alpha\xi\Phi\Psi + \beta T = 1 - \xi.$$

One sees that

$$A = \begin{pmatrix} 1 + \alpha\Psi\Phi & \alpha T\Psi \\ \beta\Phi & 1 - \alpha\xi\Phi\Psi \end{pmatrix}$$

is an automorphism by Proposition 3.1 and that $(\xi\Phi + \Phi', T)A = (\Phi + \Phi', T)$ where we use the condition $\Phi'\Psi = 0$. This shows that $E_{\xi\Phi+\Phi'}(T) = E_{\Phi+\Phi'}(T)$. $\square$

COROLLARY 3.3. *Let $\Phi \in mor(\mathbf{F}, \mathbf{C})$, $\xi \in C[\Delta]$ and $T \in R(\xi\Phi)_*$. Then $E_{\xi\Phi}(T) = E_\Phi(T)$.*

PROOF. By Proposition 3.2 it suffices to prove the existence of $\Psi \in mor(\mathbf{C}, \mathbf{F})$ such that $(\xi\Phi\Psi, T) = (1)$. Since $T \in R(\xi\Phi)_*$, $(\xi\Phi, T) \in sur(\mathbf{F} \oplus \mathbf{C}, \mathbf{C})$ by definition. It follows from Lemma 1.2 that there are $\Psi \in mor(\mathbf{C}, \mathbf{F})$ and $Y \in C[\Delta]$ such that $\xi\Phi\Psi + TY = 1$. This shows that the Ψ is the desired element. $\square$

PROPOSITION 3.4. *If B is special, then there are a non-negative integer c and elements $\phi_i \in mor(\mathbf{F}, \mathbf{C})$ and $\psi_i \in mor(\mathbf{C}, \mathbf{F})$ which form a basis over $C[\Delta, \Delta^{-1}]$ (when the modules are localized by Δ) such that $\phi_i\psi_j = \delta_{ij}\Delta^c$ where δ_{ij} denotes Kronecker delta.*

PROOF. This follows from the fact that the natural pairing $mor(\mathbf{C}, \mathbf{F}) \times mor(\mathbf{F}, \mathbf{C}) \to end(\mathbf{C}) \cong C[\Delta]$ is non degenerate over $C[\Delta, \Delta^{-1}]$ when the modules are localized by Δ. This in turn follows from the fact that these $C[\Delta]$-modules are free by Lemma 2.3 and the fact that the induced pairing over $C[\Delta]/(\Delta - t)(\cong C)$ for $t \neq 0$ is non degenerate. Note: Over $C[\Delta]/(\Delta - t)$ this is the non degenerate natural pairing $F^H \times F^{*H} \to C$ by Lemma 2.3(1). $\square$

THEOREM 3.5. *Let $\Phi, \Phi' \in mor(\mathbf{F}, \mathbf{C})$. Then $E_\Phi(T) = E_{\Phi'}(T)$ for any $T \in R(\Phi)_* \cap R(\Phi')_*$. Hence, if Φ is nowhere zero on $B - B_0$, then $R(\Phi)_* = 1 + (\Delta)$ and*

$$E_\Phi : R(\Phi)_* = 1 + (\Delta) \to VEC(B, F; C)$$

is surjective.

PROOF. Since each element in $VEC(B, F; C)$ is $E_\Phi(T)$ for some Φ and $T \in R(\Phi)_*(\subset 1 + (\Delta))$, the second assertion is a consequence of the first.

By the remark after Lemma 2.2 we may assume B to be special. If $F^H = 0$, then $mor(\mathbf{F}, \mathbf{C}) = \{0\}$ by Lemma 2.3 since $mor(\mathbf{F}, \mathbf{C})$ is a free module of dimension $\dim F^H$. Therefore the theorem is trivial in this case.

Now we assume $\dim F^H > 0$. Let ϕ_i and ψ_i for $i = 1, \dots, n = \dim F^H$ be the elements of $mor(\mathbf{F}, \mathbf{C})$ and $mor(\mathbf{C}, \mathbf{F})$ in Proposition 3.4. We prove that $E_\Phi(T) = E_{\phi_i}(T)$ for any i. The same argument shows that $E_{\Phi'}(T) = E_{\phi_i}(T)$. Then the theorem follows. From now on we fix i.

Note that $T \in R(\Delta^N\Phi)_*$ for any non-negative integer N. By Corollary 3.3

$E_{\Delta^N\Phi}(T) = E_\Phi(T)$. Therefore we can raise the degree of Φ as we want, so that we may assume $\Phi = \sum \xi_j\phi_j$ with $\xi_j \in C[\Delta]$. Here $(\xi_1, \dots, \xi_n, T) = (1)$ since $(\Phi, T) \in sur(\mathbf{F} \oplus \mathbf{C}, \mathbf{C})$. We may also assume that Δ^c divides each ξ_j by taking N sufficiently large.

Choose $q_j \in C[\Delta]$ $(j \neq i)$ and define $A \in end(\mathbf{F} \oplus \mathbf{C})$ by

$$\begin{pmatrix} 1 + \sum_{j \neq i} q_j\psi_j\phi_i & 0 \\ 0 & 1 \end{pmatrix}.$$

One can easily see that A is an automorphism since $\phi_i\psi_j = 0$ unless $i = j$, and that

$$(\Phi, T)A = \left(\left(\xi_i + \Delta^c \sum_{j \neq i} q_j\xi_j\right)\phi_i + \sum_{j \neq i} \xi_j\phi_j, T\right).$$

CLAIM. *There exist polynomials q_j such that $(\xi_i + \Delta^c \sum_{j \neq i} q_j\xi_j, T) = (1)$.*

PROOF. Since $C[\Delta]$ is a principal ideal domain, there are $\xi, r \in C[\Delta]$ such that

$$(\Delta^c)(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n) = (\xi), \quad (\xi_i, T) = (r).$$

Since $(\xi_1, \dots, \xi_n, T) = (1)$ and $T(0) = 1$, we have $(\xi, r) = (1)$. Let s be the product of all factors of T/r which are relatively prime to r. Note that $(\xi_i, s) = (1)$. One can see that $(\xi_i + s\xi, T) = (1)$, which proves the claim.

By the claim above we may assume $(\xi_i, T) = (1)$ and hence $(\Delta^N\xi_i, T) = (1)$ for any $N \geq 0$ since $T(0) = 1$. Since $\phi_j\psi_i = \delta_{ij}\Delta^c$, Proposition 3.2 can be applied with $\Phi = \phi_i$, $\Phi' = \sum_{j \neq i} \xi_j\phi_j$, $\xi = \xi_i$ and $\Psi = \psi_i$. Hence we can replace ξ_i by 1.

Remember that Δ^c divides each ξ_j. We define $A' \in aut(\mathbf{F} \oplus \mathbf{C})$ by

$$\begin{pmatrix} 1 - \sum_{j \neq i} \xi_j/\Delta^c\psi_i\phi_j & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $(\Phi, T)A' = (\phi_i, T)$. This proves that $E_\Phi(T) = E_{\phi_i}(T)$. □

COROLLARY 3.6. *The map : $VEC(B, F; C) \to VEC(B, F \oplus F'; C)$ defined by $E \to E \oplus \mathbf{F}'$ is surjective for any representations F and F' of G with $\dim F^H \neq 0$.*

PROOF. We may assume B to be special as before. Since $\dim F^H \neq 0$, there is a non-zero homogeneous $\Phi \in mor(\mathbf{F}, \mathbf{C})$. Then Φ is nowhere zero on $B - B_0$ and hence

$$E_\Phi : R(\Phi)_* \to VEC(B, F; C)$$

is surjective by Theorem 3.5. Let $\bar{\Phi} \in mor(\mathbf{F} \oplus \mathbf{F}', \mathbf{C})$ be the element induced from Φ via the projection of $F \oplus F'$ on F. Then again it follows from Theorem 3.5 that

$$E_{\bar{\Phi}} : R(\bar{\Phi})_* \to VEC(B, F \oplus F'; C)$$

is surjective. Since $E_{\bar{\Phi}}(T) = E_\Phi(T) \oplus \mathbf{F}'$, this proves the corollary. □

PROOF OF THEOREM A(1). We may assume B to be special by the remarks after Proposition 1.4 and Lemma 2.2. If $\dim F^H = 0$, then $mor(\mathbf{F}, \mathbf{C}) = \{0\}$; so

$VEC(B, F; C) = \{*\}$ and $I = (0)$. Therefore $(R/I)^* = C^*$ and since $\Gamma \supset C^*$, ρ is trivially bijective in this case.

Suppose $\dim F^H > 0$. Then $I = (\Delta^m)$ for some $m \geq 0$ since I is a homogeneous ideal in $R = C[\Delta]$, and there are non-zero homogeneous elements $\Phi \in mor(\mathbf{F}, \mathbf{C})$ and $\Psi \in mor(\mathbf{C}, \mathbf{F})$ such that $\Phi\Psi = \Delta^m$. Then $R(\Phi)_* = 1 + (\Delta)$. Since $\Gamma \supset C^*$ and $\rho(E_\Phi(T)) = [T]$ by definition, this means that ρ is surjective.

Now we prove injectivity of ρ. Suppose $[T] = [T']$ in $(R/I)^*/\Gamma$. By Theorem 3.5 it suffices to prove that $E_\Phi(T) = E_\Phi(T')$. It follows from the definition of Γ that there is $A \in aut(\mathbf{F} \oplus \mathbf{C})$ such that $T \equiv T'A(C, C) \bmod I$. Set $(\Phi'', T'') = (\Phi, T')A$. Then $E_\Phi(T') = E_{\Phi''}(T'') = E_\Phi(T'')$ where the latter identity follows from Theorem 3.5. Thus it suffices to prove that $E_\Phi(T) = E_\Phi(T'')$. Since $T'' = \Phi A(C, F) + T'A(C, C)$, we have $T'' \equiv T'A(C, C) \bmod I$ and hence $T \equiv T'' \bmod I$. This means that $T = T'' + p\Delta^m$ with some $p \in C[\Delta]$. Define $A' \in aut(\mathbf{F} \oplus \mathbf{C})$ by

$$\begin{pmatrix} 1 & p\Psi \\ 0 & 1 \end{pmatrix}.$$

Then $(\Phi, T'')A' = (\Phi, T)$ since $\Phi\Psi = \Delta^m$. This proves that $E_\Phi(T'') = E_\Phi(T)$ and completes the proof. □

§4. The group structure of $VEC(B,F;C)$.

We give the set $VEC(B, F; C)$ the group structure of $(R/I)^*/\Gamma$ via ρ. It turns out that the group structure is related to Whitney sum. This is similar to the abelian group structure described in [S] and [**KS**, §4 in Chapter VII] when B is a representation of G with a one-dimensional quotient (and the bundles are not necessarily trivialized by C). Our result of this section was inspired by this description. We denote the group operation on $VEC(B, F; C)$ by $+$.

THEOREM 4.1. *Let $E, E' \in VEC(B, F; C)$. Then $E \oplus E' \in VEC(B, F \oplus F; C)$ and $(E + E') \oplus \mathbf{F} = E \oplus E'$ in $VEC(B, F \oplus F; C)$.*

PROOF. Since E and E' are trivialized by $\mathbf{C}$, $E \oplus E' \oplus \mathbf{C} = E \oplus \mathbf{F} \oplus \mathbf{C} = \mathbf{F} \oplus \mathbf{F} \oplus \mathbf{C}$. This shows that $E \oplus E' \in VEC(B, F \oplus F; C)$.

We prove the latter statement in the theorem. We may assume B to be special as usual. If $I = (0)$ (in other words, $\dim F^H = 0$), then $VEC(B, F; C) = \{*\}$ and the statement is trivial. Therefore we may assume $I = (\Delta^m)$ for some $m \geq 0$ and there are non-zero homogeneous elements $\Phi \in mor(\mathbf{F}, \mathbf{C})$ and $\Psi \in mor(\mathbf{C}, \mathbf{F})$ such that $\Phi\Psi = \Delta^m$. By Theorem 3.5 E and E' are represented by $E_\Phi(T)$ and $E_\Phi(T')$ respectively for some $T, T' \in R(\Phi)_*$. Then $E + E'$ is represented by $E_\Phi(TT')$. Thus it suffices to prove that $E_\Phi(TT') \oplus \mathbf{F} = E_\Phi(T) \oplus E_\Phi(T')$. We view the elements at both sides in $VEC(B, F \oplus F'; C \oplus C)$. Noting that $VEC(B, F \oplus F; C)$ is naturally a subset of

$VEC(B, F \oplus F; C \oplus C)$, it suffices to prove that there is $A \in aut(\mathbf{F} \oplus \mathbf{F} \oplus \mathbf{C} \oplus \mathbf{C})$ such that

$$\begin{pmatrix} \Phi & 0 & TT' & 0 \\ 0 & \Phi & 0 & 1 \end{pmatrix} A = \begin{pmatrix} \Phi & 0 & T & 0 \\ 0 & \Phi & 0 & T' \end{pmatrix}.$$

LEMMA 4.2. *Let* $p_i, q_i \in C[\Delta]$ $(1 \leq i \leq 4)$. *If* $p_i \equiv q_i \bmod \Delta^m$, *then there is* $A' \in aut(\mathbf{F} \oplus \mathbf{F} \oplus \mathbf{C} \oplus \mathbf{C})$ *such that*

$$\begin{pmatrix} \Phi & 0 & p_1 & p_2 \\ 0 & \Phi & p_3 & p_4 \end{pmatrix} A' = \begin{pmatrix} \Phi & 0 & q_1 & q_2 \\ 0 & \Phi & q_3 & q_4 \end{pmatrix}.$$

PROOF. Set $r_i = (q_i - p_i)/\Delta^m$, which is in $C[\Delta]$ by assumption. We define $A' \in aut(\mathbf{F} \oplus \mathbf{F} \oplus \mathbf{C} \oplus \mathbf{C})$ by

$$A' = \begin{pmatrix} 1 & 0 & r_1\Psi & r_2\Psi \\ 0 & 1 & r_3\Psi & r_4\Psi \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly A' is an automorphism and it is easy to check that A' is the desired one.

Now we return to the proof of Theorem 4.1. Since $T \in R(\Phi)_*$, there is $Y \in C[\Delta]$ such that $TY \equiv 1 \bmod \Delta^m$. Set $P = 1 - YT$ and $Q = Y(Y-1)$. Then $P \equiv 0 \bmod \Delta^m$ and $TQ \equiv Y - 1 \bmod \Delta^m$. Consider $A_1 \in end(\mathbf{F} \oplus \mathbf{F} \oplus \mathbf{C} \oplus \mathbf{C})$ defined by

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & Y - PQ & P \\ 0 & 0 & Y - 1 - (T+P)Q & T + P \end{pmatrix}.$$

One sees that A_1 is an automorphism by applying the elementary operation to A_1. Observe that

$$\begin{pmatrix} \Phi & 0 & TT' & 0 \\ 0 & \Phi & 0 & 1 \end{pmatrix} A_1 = \begin{pmatrix} \Phi & 0 & T' + s_1 & s_2 \\ 0 & \Phi & s_3 & T + s_4 \end{pmatrix}$$

where $s_i \in C[\Delta]$. By Lemma 4.2 there is $A_2 \in aut(\mathbf{F} \oplus \mathbf{F} \oplus \mathbf{C} \oplus \mathbf{C})$ such that

$$\begin{pmatrix} \Phi & 0 & T' + s_1 & s_2 \\ 0 & \Phi & s_3 & T + s_4 \end{pmatrix} A_2 = \begin{pmatrix} \Phi & 0 & T' & 0 \\ 0 & \Phi & 0 & T \end{pmatrix}.$$

Therefore A_1A_2 is the desired element A. $\square$

§5. The structure of $(R/I)^*/\Gamma$.

In this section we prove Theorem A(2), i.e. that the multiplicative group $(R/I)^*/\Gamma$ is isomorphic to a vector group $(\Delta)/(\Delta^{d+1})$ $(\cong C^d)$ by a logarithmic map. The group

$(R/I)^*$ is well understood since $R = C[\varDelta]$ and I is a homogeneous ideal in R; so we analyze the subgroup $\varGamma$.

Our C^*-action on B defines positive gradings on $end(\mathbf{F} \oplus \mathbf{C})$ etc. Hence an element A in $end(\mathbf{F} \oplus \mathbf{C})$ is expressed as $A = \sum_{i \geq 0} A_{(i)}$ where $A_{(i)}$ is homogeneous of degree i. Note that $end(\mathbf{F} \oplus \mathbf{C})$ is a $C[\varDelta]$-module. Associated to $A \in end(\mathbf{F} \oplus \mathbf{C})$ and $z \in C[\varDelta]$, we define

$$A_z = \sum_{i \geq 0} z^i A_{(i)} \in end(\mathbf{F} \oplus \mathbf{C}).$$

LEMMA 5.1. *If* $A \in aut(\mathbf{F} \oplus \mathbf{C})$, *then* $A_z \in aut(\mathbf{F} \oplus \mathbf{C})$ *for any* $z \in C[\varDelta]$; *hence* $[A_z(C, C)] \in \varGamma \subset (R/I)^*$ *where* $[\ ]$ *denotes the class in* $(R/I)^*$.

PROOF. Since the C^*-action on B commutes with the G-action, we may think of an element of C^* as a G-variety automorphism of B. Then the induced element c^*A is in $aut(\mathbf{F} \oplus \mathbf{C})$ for any $c \in C^*$. By the definition of the grading we have

$$c^*A = \sum_{i \geq 0} c^i A_{(i)} = A_c.$$

Since $(c^*A)(c^*A^{-1}) = c^*(AA^{-1}) = 1$ for any $c \in C^*$, this shows that $A_z(A^{-1})_z = 1$ whenever $z \in C^* \subset C[\varDelta]$. Therefore the identity $A_z(A^{-1})_z = 1$ holds even when z is viewed as an indeterminate. In turn this implies that it holds for any $z \in C[\varDelta]$, i.e. A_z has an inverse in $end(\mathbf{F} \oplus \mathbf{C})$. □

Since the ideal I is homogeneous, the quotient R/I inherits the grading from R. Since $(R/I)^*_{(0)} = C^*$ and $\varGamma \supset C^*$, we have

$$(R/I)^*/\varGamma = (1 + (R/I)_1)^*/\varGamma_*$$

where $(R/I)_1$ denotes the ideal in R/I with zero constant term and

$$\varGamma_* = \{\gamma \in \varGamma \mid \gamma_{(0)} = 1\}.$$

If $I = (0)$ or (1), then $(R/I)^*/\varGamma$ is trivial; so we assume $I = (\varDelta^m)$ for some $m \geq 1$. Then $(1 + (R/I)_1)^* = 1 + (\varDelta)/(\varDelta^m)$ and we have a logarithmic map

$$\log : 1 + (\varDelta)/(\varDelta^m) \to (\varDelta)/(\varDelta^m)$$

defined using the formal power series of the logarithmic function. The map log is an isomorphism between the multiplicative group and the additive group. In fact, the inverse is given by an exponential map.

Since the map log is C^*-equivariant and $\varGamma_*$ is a multiplicative group, $\log \varGamma_*$ is a C^*-invariant additive subgroup of $(\varDelta)/(\varDelta^m)$. We will show that it is in fact a homogeneous ideal in $C[\varDelta]/(\varDelta^m)$. The homogeneity comes from the C^*-action, and the fact that it is an ideal will come from Lemma 5.1 and the following elementary proposition.

PROPOSITION 5.2. *For any positive integer k, every polynomial in* $C[\Delta]$ *can be written as a finite sum of kth powers.*

PROOF. Consider the subset $Z \subset C[\Delta]$ of polynomials which can be written as a finite sum of kth powers. It is a subvector space of $C[\Delta]$. It is stable by C^*, since if $z = \sum x_i^k$, then $c * z = \sum (c * x_i)^k$, where $c \in C^*$, and the star denotes the C^*-action. Thus it is generated by homogeneous elements. For each i, $(\Delta^i + 1)^k \in Z$, so each homogeneous part is in Z. In particular, Z contains Δ^i. Thus Z is the entire ring $C[\Delta]$. □

PROPOSITION 5.3. $\log \Gamma_*$ *is a homogeneous ideal in* $C[\Delta]/(\Delta^m)$. *That is, there exists a non-negative integer* $d = d(B, F) \leq m - 1$ *such that* $\log \Gamma_* = (\Delta^{d+1})/(\Delta^m)$.

PROOF. Consider the subvector space $V = C \log \Gamma_* \subset C[\Delta]/(\Delta^m)$. It is a C^*-module, and therefore it is generated by homogeneous elements. If $f \in V$ is homogeneous of degree k, then by Lemma 5.1, $z^k f$ is also in V for any polynomial z. By Proposition 5.2, V is an ideal. Finally, $V = \log \Gamma_*$, since each homogeneous element of V has non-zero degree, and by the C^*-stability, it is in $\log \Gamma_*$. □

PROOF OF THEOREM A(2). If $I = (0)$ or (1), then $(R/I)^*/\Gamma$ is trivial; so $d = 0$ in this case. Suppose $I = (\Delta^m)$ for some $m \geq 1$. Then $(R/I)^* = (1 + (R/I)_1)^* = 1 + (\Delta)/(\Delta^m)$ and it is isomorphic to $(\Delta)/(\Delta^m)$ via the logarithmic map. Thus Theorem A(2) follows from Proposition 5.3. □

§6. Estimate of $d(B, F) = \dim(R/I)^*/\Gamma$.

In this section we prove Theorem B and give two cases where $VEC(B, F; C) = \{*\}$. As before H will denote a principal isotropy group of B (as a G-variety).

DEFINITION. Suppose B is special. Then we define $e(B, F) = 0$ when $F^H = 0$, and

$$e(B, F) = \min_{\Phi, \Psi, l} \{\deg(\Psi\Phi/\Delta^l)/\deg \Delta - 1 \,|\, \Psi\Phi/\Delta^l \in end(\mathbf{F})_1 \text{ and } \Phi\Psi \neq 0 \in C[\Delta]\}$$

when $F^H \neq 0$, where Φ and Ψ range over non-zero homogeneous elements in $mor(\mathbf{F}, \mathbf{C})$ and $mor(\mathbf{C}, \mathbf{F})$ respectively and $end(\mathbf{F})_1$ denotes the subset of $end(\mathbf{F})$ with zero constant term. If B is not special, we define $e(B, F)$ to be $e(B^s, F)$ where B^s is the special G-cone in B.

REMARK. If $\dim F^H = 1$, then a non-zero $\Phi \in mor(\mathbf{F}, \mathbf{C})$ or $\Psi \in mor(\mathbf{C}, \mathbf{F})$ restricted to B^H is an isomorphism from F^H to C or from C to F^H on a dense subset of B^H. This shows that the condition $\Phi\Psi \neq 0$ in the above definition is automatically satisfied when $\dim F^H = 1$.

Here are some properties of $e(B, F)$.

LEMMA 6.1. (1) *$e(B, F)$ is a non-negative integer.*

(2) *If $I = (0)$ or (1), then $e(B, F) = 0$, and if $I = (\Delta^m)$ for some $m \geq 1$, then $e(B, F) \leq m - 1$.*

(3) *If* $F^H \neq 0$, *then* $e(B,F)$ *is equal to the minimum among* $e(B,F_i)$ *where* F_i *are irreducible G-subrepresentations of* F *such that* $F_i^H \neq 0$.

PROOF. (1) We may assume $F^H \neq 0$, otherwise the statement is trivial by definition. Let Φ and Ψ be the elements in the definition of $e(B,F)$. Since $\Phi\Psi \in \boldsymbol{C}[\Delta]$ is non-zero, $\deg \Psi\Phi = \deg \Phi\Psi$ is divisible by $\deg \Delta$ and hence $\deg(\Psi\Phi/\Delta^l)/\deg \Delta$ in the definition is a positive integer because $\Psi\Phi/\Delta^l \in end(\mathbf{F})_1$. This proves (1).

(2) We note that $I = (0)$ (resp. $I = (1)$) if and only if $F^H = 0$ (resp. $F^G \neq 0$). Therefore the statement is trivial by definition when $I = (0)$. When $I = (1)$, there is a one-dimensional subspace W in F^G. We take homogeneous elements Φ and Ψ in $mor(\mathbf{W},\mathbf{C}) \subset mor(\mathbf{F},\mathbf{C})$ and $mor(\mathbf{C},\mathbf{W}) \subset mor(\mathbf{C},\mathbf{F})$ respectively such that $\Phi\Psi \neq 0$. Then, since $\dim W = 1$ and $W^G = W$, $\Psi\Phi$ can be thought of as a non-zero element of $\boldsymbol{C}[\Delta]$. Therefore we can take $l = \deg \Psi\Phi/\deg \Delta - 1$ if $\deg \Psi\Phi > 0$ and this proves $e(B,F) = 0$ when $I = (1)$.

Now suppose $m \geq 1$. Then there are homogeneous elements $\Phi \in mor(\mathbf{F},\mathbf{C})$ and $\Psi \in mor(\mathbf{C},\mathbf{F})$ such that $\Phi\Psi = \Delta^m$. Since these specific Φ and Ψ are members in the definition of $e(B,F)$, we have

$$e(B,F) \leq \deg(\Psi\Phi/\Delta^l)/\deg \Delta - 1 = m - l - 1 \leq m - 1.$$

(3) Since $F = \oplus_i F_i$, Φ and Ψ in the definition of $e(B,F)$ are written as $\Phi = \oplus_i \Phi_i$ and $\Psi = \oplus_i \Psi_i$ with homogeneous elements $\Phi_i \in mor(\mathbf{F}_i,\mathbf{C})$ and $\Psi_i \in mor(\mathbf{C},\mathbf{F}_i)$. Since $\Psi\Phi = \oplus_{i,j} \Psi_i \Phi_j$ and $\Psi_i \Phi_j$ is in $mor(\mathbf{F}_j,\mathbf{F}_i)$, Δ^l divides $\Psi\Phi$ if and only if it divides all $\Psi_i\Phi_j$. Moreover it is clear that $\Phi_j\Psi_i = 0$ unless $i = j$. Therefore, in the definition of $e(B,F)$, it suffices to take elements in $mor(\mathbf{F}_i,\mathbf{C})$ and $mor(\mathbf{C},\mathbf{F}_i)$ for all F_i such that $F_i^H \neq 0$. This proves the statement (3). □

PROPOSITION 6.2. $d(B,F) \leq e(B,F)$.

PROOF. If $I = (0)$ or (1), then $VEC(B,F;\boldsymbol{C}) = \{*\}$ as noted in the proof of Theorem A(2), and $e(B,F) = 0$ by Lemma 6.1(2); so the proposition is trivial in this case. Suppose $I = (\Delta^m)$ for some $m \geq 1$ (hence $F^H \neq 0$). Let Φ, Ψ be the homogeneous elements which attain the minimum in the definition of $e(B,F)$, so $\Phi\Psi/\Delta^l = \Delta^{e(B,F)+1}$ up to a scalar non-zero constant. Then the equation in Proposition 3.1 has a solution for any polynomial P' since $e(B,F) + 1 \geq 1$. It follows that the $\boldsymbol{C}[\Delta]$-module $\log \Gamma_*$ contains $(\Delta^{e(B,F)+1})/(\Delta^m)$, which means $d(B,F) \leq e(B,F)$ by Proposition 5.3. □

Here are two results derived from Proposition 6.2.

COROLLARY 6.3. *If* $\dim F = 1$, *then* $e(B,F) = 0$ *and hence* $VEC(B,F;\boldsymbol{C}) = \{*\}$.

PROOF. We may assume $F^H \neq 0$ (so $\dim F^H = 1$) since otherwise $e(B,F) = 0$ by definition. Then there are non-zero homogeneous elements $\Phi \in mor(\mathbf{F},\mathbf{C})$ and $\Psi \in mor(\mathbf{C},\mathbf{F})$. Since F is one-dimensional, the action of G on $End(F)$ is trivial. Therefore

$end(\mathbf{F}) \cong \boldsymbol{C}[\Delta]$ and $\Psi\Phi$ can be thought of as a monomial of Δ; so we can take $l = \deg \Psi\Phi / \deg \Delta - 1$ in the definition of $e(B, F)$. Thus $e(B, F) = 0$. $\square$

THEOREM 6.4. *If G is abelian, then any stably trivial G-vector bundle over B is in fact trivial.*

PROOF. The theorem is equivalent to the statement that $VEC(B, F; S) = \{*\}$ for any representations F and S of G. Since G is abelian, any representation of G decomposes into a sum of one-dimensional representations of G. Thus it suffices to prove $VEC(B, F; S) = \{*\}$ when S is one-dimensional. Moreover by tensoring with the dual representation of S it suffices to prove $VEC(B, F; \boldsymbol{C}) = \{*\}$ for any F. As usual we may assume $F^H \neq 0$. Then, since G is abelian, F contains a one dimensional representation of G on which H acts trivially. It follows from Corollaries 3.6 and 6.3 that $VEC(B, F; \boldsymbol{C}) = \{*\}$. $\square$

We prepare a proposition before the proof of Theorem B. Let W be a representation of G. In the following the subscript 1 will indicate the set of elements with zero constant term as is used in $end(\mathbf{F})_1$ in the definition of $e(B, F)$. Suppose that B is special and consider the restriction map

$$r : end(\mathbf{W})_1 = Mor(B, End(W))_1^G \to Mor(B^H, End(W)^H)_1^{N_G(H)}.$$

This map is a $\boldsymbol{C}[\Delta]$-module homomorphism when the right hand side is viewed as a $\boldsymbol{C}[\Delta]$-module through the restriction map : $\boldsymbol{C}[\Delta] = \mathcal{O}(B)^G \to \mathcal{O}(B^H)^{N_G(H)}$, and is injective since the G-orbit of B^H is dense in B. Set

$$M = Mor(B^H, End(W^H))_1^{N_G(H)} \subset Mor(B^H, End(W)^H)_1^{N_G(H)}.$$

PROPOSITION 6.5. *Suppose* $\dim W^H = 1$ *and let* $\tilde{A} \in end(\mathbf{W})$. *Then*

$$\det \tilde{A} \equiv a\ constant \mod \Delta^{e(B,W)+1}$$

if $\tilde{A} - \tilde{A}_{(0)} \in end(\mathbf{W})_1$ is in $r^{-1}(M)$, where $\tilde{A}_{(0)}$ denotes the constant term of $\tilde{A}$.

PROOF. If $W^G \neq 0$, then $e(B, W) = 0$ as is shown in the proof of Lemma 6.1(2). Therefore the proposition is trivial in this case. We assume $W^G = 0$ in the following.

Since $\dim W^H = 1$, $End(W^H)$ is one-dimensional and the action of $N_G(H)$ on it is trivial. Therefore M is isomorphic to $\mathcal{O}(B^H)_1^{N_G(H)}$. By Lemma 2.3 $end(\mathbf{W})$ is free as a $\boldsymbol{C}[\Delta]$-module, so the submodule $r^{-1}(M)$ is also free since $\boldsymbol{C}[\Delta]$ is a principal ideal domain. Since the map r is injective and the rank of the free part of $M \cong \mathcal{O}(B^H)_1^{N_G(H)}$ as a $\boldsymbol{C}[\Delta]$-module is one by Lemma 2.4, the free submodule $r^{-1}(M)$ must be of rank one.

Again since $\dim W^H = 1$, $mor(\mathbf{W}, \mathbf{C})$ and $mor(\mathbf{C}, \mathbf{W})$ are free of rank one as $\boldsymbol{C}[\Delta]$-modules by Lemma 2.3. Let Φ' and Ψ' be their homogeneous generators respectively which are of positive degrees since $W^G = 0$. Let l be the maximum integer such that Δ^l divides $\Psi'\Phi'$ in $end(\mathbf{W})_1$. It follows from Schur's Lemma that $\Psi'\Phi'/\Delta^l$ restricted to

B^H is an endomorphism of W^H, so $\Psi'\Phi'/\Delta^l$ is in $r^{-1}(M)$. Since $\deg(\Psi'\Phi'/\Delta^l)/\deg\Delta = e(B, W) + 1$ by definition, it suffices to prove

CLAIM. *$\Psi'\Phi'/\Delta^l$ generates the submodule $r^{-1}(M)$ over $\boldsymbol{C}[\Delta]$.*

PROOF. Let $\mu \in r^{-1}(M)$ be a generator which is homogeneous. Then $\Psi'\Phi'/\Delta^l = p(\Delta)\mu$ with a non-zero monomial p. If $\deg p \geq 1$, then Δ divides $\Psi'\Phi'/\Delta^l$ in $r^{-1}(M) \subset end(\mathbf{W})_1$, which contradicts the definition of l. Thus p is a non-zero constant, and hence $\Psi'\Phi'/\Delta^l$ is a generator. This proves the claim and completes the proof of the proposition. □

PROOF OF THEOREM B. By Proposition 6.2 it remains to prove $d(B,F) = e(B,F)$ when F is a multiple of a representation of G which is multiplicity free with respect to H. We may assume $I = (\Delta^m)$ for some $m \geq 1$ as noted in the proof of Proposition 6.2 and B to be special as usual.

Let $A \in aut(\mathbf{F} \oplus \mathbf{C})$. Remember that A is of the form

$$A = \begin{pmatrix} A(F,F) & A(\boldsymbol{C},F) \\ A(F,\boldsymbol{C}) & A(\boldsymbol{C},\boldsymbol{C}) \end{pmatrix}.$$

Looking at the degrees, one sees that $\det A \equiv \det A(F,F) \det A(\boldsymbol{C},\boldsymbol{C}) \bmod \Delta^m$. Here $\det A$ is a non-zero constant since A is an automorphism, and $\det A(\boldsymbol{C},\boldsymbol{C}) = A(\boldsymbol{C},\boldsymbol{C})$ since $A(\boldsymbol{C},\boldsymbol{C})$ is 1×1 matrix. Therefore the congruence above reduces to

$$a \det A(F,F) \equiv A(\boldsymbol{C},\boldsymbol{C}) \mod \Delta^m \tag{6.6}$$

with some non-zero constant a.

By assumption $F = kW$ for some positive integer k and a representation W of G which is multiplicity free with respect to H. Then $\dim W^H = 0$ or 1 and it suffices to treat the latter case because if $\dim W^H = 0$, then $\dim F^H = 0$ and hence $d(B,F) = e(B,F) = 0$.

We define a map

$$\mathrm{Det} : end(\mathbf{F}) = end(k\mathbf{W}) \to end(\mathbf{W})$$
$$P \mapsto \sum_{\sigma} (-1)^{\operatorname{sgn}\sigma} \prod_{i=1}^{k} P_{i\sigma(i)}$$

where σ runs over all permutations on k letters and P_{ij} is the (i,j)th entry of P which sends the jth factor W of kW to the ith factor W of kW. Set $\tilde{A} = \mathrm{Det}(A(F,F))$. The restrictions of $A(F,F)$ and $\tilde{A}$ to B^H decompose into sum of factors $A(F,F)_\chi$ and $\tilde{A}_\chi$ respectively according to irreducible H-representations χ which occur in W viewed as a representation of H. Since $F = kW$, $A(F,F)$ consists of k^2 blocks where each block is a square matrix of size $\dim W$ and its restriction to B^H is a diagonal matrix by Schur's Lemma and the multiplicity free condition. Therefore one sees that

$$\det A(F,F)_\chi = \det \tilde{A}_\chi \quad \text{for each } \chi. \tag{6.7}$$

Taking the product at the both sides of (6.7) over χ, one obtains that $\det A(F,F) = \det \tilde{A}$ on B^H and hence on the whole space B because $\det A(F,F)$ and $\det \tilde{A}$ are both invariant functions on B and the G-orbit of B^H is dense in B. Moreover $\tilde{A} - \tilde{A}_{(0)} \in r^{-1}(M)$. To see this, note that since W is multiplicity free with respect to H, $\tilde{A}_\chi$ is multiplication by some $a_\chi \in \mathcal{O}(B^H)$. On the other hand since A is an automorphism, $\det A(F,F)_\chi \in \mathcal{O}(B^H)$ is a unit and hence $\det \tilde{A}_\chi$ is a constant by Lemma 2.1(1) and (6.7) unless χ is the trivial representation. Hence a_χ must be a constant for non-trivial χ, which means that $\tilde{A} - \tilde{A}_{(0)} \in r^{-1}(M)$.

Thus it follows from Proposition 6.5 that

$$\det A(F,F) = \det \tilde{A} \equiv \text{a constant} \mod \Delta^{e(B,W)+1}.$$

Here $e(B,W) = e(B,kW) = e(B,F)$ by Lemma 6.1(3) and $e(B,F) + 1 \leq m$ by Lemma 6.1(2). Hence it follows from (6.6) that

$$A(\boldsymbol{C},\boldsymbol{C}) \equiv \text{a constant} \mod \Delta^{e(B,F)+1}.$$

This shows that $\log \Gamma_*$ is contained in $(\Delta^{e(B,F)+1})/(\Delta^m)$, which means $d(B,F) \geq e(B,F)$ by Proposition 5.3. The opposite inequality is established in Proposition 6.2, so $d(B,F) = e(B,F)$. □

§7. Pulling back G-vector bundles over representations to cones.

Suppose V is a representation of G such that $\mathcal{O}(V)^G \neq \boldsymbol{C}$, in other words, there exists a non-zero point $x \in V$ whose orbit is closed. We define $B' = B'(V;x) \subset V$ to be $\overline{(G \times \boldsymbol{C}^*)x}$ where the $\boldsymbol{C}^*$-action is scalar multiplication. The $G \times \boldsymbol{C}^*$-variety B' satisfies condition (C0) and has a one-dimensional irreducible quotient $B'//G$; however $B'//G$ is not necessarily isomorphic to $\boldsymbol{C}$, i.e. B' does not necessarily satisfy condition (C1) (see e.g. the next section). There are two ways to get around this trouble. One is to take the fiber product of B' and the normalization of $B'//G$ over $B'//G$, and the other is to take the normalization of B'.

Let L be the line $\boldsymbol{C}x$ in B' and N the stabilizer of L in G. Observe that $L//N$ is isomorphic to $\boldsymbol{C}$ and the natural map $\tau : L//N \to B'//G$ is bijective since L intersects all the closed orbits of B'.

PROPOSITION 7.1. *The reduced fiber product $B(V;x)$ of B' and the normalization of $B'//G$ over $B'//G$ is a special G-cone.*

REMARK. We specify that the fiber product should be reduced, because in algebraic geometry, the usual definition of fiber product (see e.g. [H]), does not necessarily require this. Here we mean the fiber product which satisfies the necessary universal property for all reduced varieties. It is the same as the reduced variety associated to the (unreduced) fiber product of [H]. It is not hard to see that the fiber product of $\alpha : X \to Z$ and $\beta : Y \to Z$ is the subvariety of $X \times Y$ of elements (x,y) such that $\alpha(x) = \beta(y)$.

PROOF. Let $B = B(V; x)$. The observation made just above the proposition shows that $L//N$ is the normalization of $B'//G$. By definition,

$$B = \{(x', c) \in B' \times L//N \mid \pi'(x') = \tau(c)\}$$

where $\pi' : B' \to B'//G$ is the quotient map. The projection of B on $L//N$ induces a morphism $B//G \to L//N$. The universal property of fiber products shows that this is an isomorphism. Thus $B//G \cong L//N \cong \boldsymbol{C}$ and B satisfies condition (C1). It is easy to check that B satisfies condition (C0).

Let $\kappa : B \to B'$ be the projection. Note that the map κ is bijective and finite, and B and B' have unique open $G \times \boldsymbol{C}^*$-orbits respectively on which κ is an isomorphism. Let C be the closure of the open $G \times \boldsymbol{C}^*$-orbit in B. Then its image $\kappa(C)$ is closed by the finiteness of κ. Since $\kappa(C)$ contains the open $G \times \boldsymbol{C}^*$-orbit in B', it follows from the construction of B' that $\kappa(C) = B'$. Now it follows from the bijectivity of κ that $C = B$ which means that B is special. □

Now we consider the normalization of B'. In this case we need to assume G to be connected so that B' is irreducible and then the normalization of B' makes sense. Since B' is an affine $G \times \boldsymbol{C}^*$-variety, so is its normalization (see [**Kr**, p. 261]).

PROPOSITION 7.2. *Suppose G is connected. Then the normalization $B_1(V; x)$ of B' is a special G-cone.*

PROOF. Let $B = B_1(V; x)$. First note that the quotient $B//G$ is isomorphic to the normalization of $B'//G$. This is because the natural map $B//G \to B'//G$ is a finite and birational map between rational curves. Thus in particular, the map $B//G \to B'//G$ is one-to-one. Also, this proves condition (C1).

To prove that B satisfies condition (C0), it is enough to know that the inverse image of the origin under the normalization map is a unique point. Then the closure of every $\boldsymbol{C}^*$-orbit would pass through this point. This is clearly the case, since if it contained several points, each would be fixed by G, and one could separate them with an invariant function, which is impossible since $B//G \to B'//G$ is one-to-one. The ring $\mathcal{O}(B)$ inherits the positive grading from that of $\mathcal{O}(B')$.

Since B' has a unique open $G \times \boldsymbol{C}^*$-orbit, so does B. Since B is irreducible, the closure of this open orbit is B, and thus it is special. □

PROOF OF THEOREM C. Since Φ is a weak duality, $E_\Phi(1 + P) \in VEC(V, F; \boldsymbol{C})$ if $P \in \mathcal{O}(V)_1^G$. The ρ invariant of the pull-back bundle of $E_\Phi(1 + P)$ by $\imath$ is $1 + \imath^* P$. Since $\log(1 + \imath^*(\mathcal{O}(V)_1^G)) = \imath^*(\mathcal{O}(V)_1^G)$, this proves the theorem. □

§8. Applications to dihedral groups.

In this section we illustrate the previous material by treating the group $O(2)$ and its dihedral subgroups. We begin with $O(2)$, a base representation V_1 which is a weighted G-cone and $F = V_m$ a representation which is multiplicity free with respect to a principal isotropy group of V_1. We describe $VEC(V_1, V_m; \boldsymbol{C})$ in this case. We then restrict it to dihedral subgroups. The representation V_1 itself no longer satisfies condition (C1) for these subgroups. Thus to treat vector bundles over V_1 when the group is dihedral, we first describe $VEC(B, F; \boldsymbol{C})$ for a special G-cone B associated to V_1, as described in the previous section. We then use Theorem C to get a lower bound for equivariant vector bundles over V_1.

Remember that $O(2) = \boldsymbol{C}^* \rtimes \boldsymbol{Z}/2\boldsymbol{Z}$. Denote by g an element of $\boldsymbol{C}^*$, and by h the non-trivial element of $\boldsymbol{Z}/2\boldsymbol{Z}$. For a positive integer k we define the two-dimensional representation V_k of $O(2)$ by

$$g(a,b) = (g^k a, g^{-k} b) \qquad h(a,b) = (b,a)$$

where $(a,b) \in V_k = \boldsymbol{C}^2$.

Let $G = O(2)$, $B = V_1$ and $F = V_m$. We find that $\mathcal{O}(B)^G = \boldsymbol{C}[\Delta]$ where $\Delta = ab$ and that B is a special G-cone whose principal isotropy group H is $\boldsymbol{Z}/2\boldsymbol{Z}$. Since B is a representation, B has the $\boldsymbol{C}^*$-action defined by scalar multiplication. Note that F is multiplicity free with respect to H. It is easy to check that $mor(\mathbf{F}, \mathbf{C})$ is free and of rank one as a $\boldsymbol{C}[\Delta]$-module and the generator Φ is given by

$$\Phi(a,b)(x,y) = b^m x + a^m y$$

where $(a,b) \in B = V_1$ and $(x,y) \in F = V_m$. Observe that Φ is a weak duality. Similarly $mor(\mathbf{C}, \mathbf{F})$ is also free and of rank one as a $\boldsymbol{C}[\Delta]$-module and the generator Ψ is given by

$$\Psi(a,b)(z) = (a^m z, b^m z)$$

where $z \in \boldsymbol{C}$. Thus we have

$$\Phi\Psi(a,b)(z) = \Delta^m z$$

$$\Psi\Phi(a,b)(x,y) = \begin{pmatrix} \Delta^m & a^{2m} \\ b^{2m} & \Delta^m \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The former identity implies that $I = (\Delta^m)$ and the latter implies that $e(B,F) = m-1$ since the 2×2 matrix above is not divisible by Δ. Thus it follows from Theorems A and B that

THEOREM 8.1. *Suppose $G = O(2)$, $B = V_1$ and $F = V_m$. Then*
(1) $VEC(B, F; \boldsymbol{C}) \cong \boldsymbol{C}^{m-1}$. ([**S**].)
(2) *More precisely any element in $VEC(B, F; \boldsymbol{C})$ is represented by $E_\Phi(T)$ for some $T \in R(\Phi)_* = 1 + (\Delta)$, and $E_\Phi(T) = E_\Phi(T')$ in $VEC(B, F; \boldsymbol{C})$ for $T, T' \in R(\Phi)_*$ if and only if $T \equiv T' \bmod \Delta^m$.* ([**MP2**], [**MP3**].)

Now we consider the dihedral groups D_n where $D_n = \boldsymbol{Z}/n\boldsymbol{Z} \rtimes \boldsymbol{Z}/2\boldsymbol{Z}$. Since D_n is a subgroup of $O(2)$, the $O(2)$-vector bundles above can also be thought of as D_n-vector bundles. We will show that for n large enough, they stay non-isomorphic as D_n-vector bundles. The basic difference between this and the $O(2)$ case is that the base representation no longer has a one-dimensional quotient, and thus we use the special G-cone associated to V_1.

In the following we set $G = D_n$ and regard V_m as a representation of G via restriction. Then $V_m = V_{m+n}$ and if $m \leq n$, then $V_m = V_{n-m}$; thus we may assume that $2m \leq n$. It is not hard to check that $\mathcal{O}(V_1)^G = \boldsymbol{C}[ab, a^n + b^n]$.

Let $x = (1,1) \in V_1$. The isotropy group of x is $H = \boldsymbol{Z}/2\boldsymbol{Z}$. Let $B' = B'(V_1; x)$ from section 7. Then

$$B' = \{(c\zeta, c\zeta^{-1}) \mid c \in \boldsymbol{C} \text{ and } \zeta \text{ is an } n\text{th root of unity}\}.$$

It is a union of lines through the origin. One finds that

$$\mathcal{O}(B') = \begin{cases} \boldsymbol{C}[a,b]/(a^{n/2} - b^{n/2}) & \text{if } n \text{ is even} \\ \boldsymbol{C}[a,b]/(a^n - b^n) & \text{if } n \text{ is odd} \end{cases}$$

and

$$\mathcal{O}(B')^G = \begin{cases} \boldsymbol{C}[ab] & \text{if } n \text{ is even} \\ \boldsymbol{C}[ab, a^n + b^n]/((a^n - b^n)^2) (\cong \boldsymbol{C}[c^2, c^n]) & \text{if } n \text{ is odd.} \end{cases}$$

Therefore if n is even, B' is a special G-cone; so we take $B = B'$. If n is odd, $B'//G$ is a cusp; so B' is not even a weighted G-cone, but $B = B(V_1; x)$, defined in Proposition 7.1, is a special G-cone. In either case the principal isotropy group of B is H. It is easy to check that

$$B = \{(a,b,c) \in \boldsymbol{C}^3 \mid ab = c^2, a^n = b^n = c^n\} \tag{8.2}$$

when n is odd. Thus $\mathcal{O}(B)^G = \boldsymbol{C}[\varDelta]$ in any case where

$$\varDelta = \begin{cases} ab & \text{if } n \text{ is even} \\ c & \text{if } n \text{ is odd.} \end{cases}$$

We have a natural map $\imath : B \to V_1$. Denote by $\imath^*\Phi$ and $\imath^*\Psi$ the pull-back of Φ and Ψ by $\imath$ where Φ and Ψ are given above for $O(2)$.

LEMMA 8.3. *$\imath^*\Phi$ and $\imath^*\Psi$ generate $mor(\mathbf{F}, \mathbf{C})$ and $mor(\mathbf{C}, \mathbf{F})$ as $\boldsymbol{C}[\varDelta]$-modules respectively.*

PROOF. We prove the claim only for $\imath^*\Phi$ since the proof for $\imath^*\Psi$ is the same. Since B is special and $\dim F^H = 1$, the rank of $mor(\mathbf{F}, \mathbf{C})$ as a $\boldsymbol{C}[\varDelta]$-module is one. Since $\deg \imath^*\Phi = m$, it suffices to prove that any homogeneous element of $mor(\mathbf{F}, \mathbf{C}) = Mor(B, F^*)^G$ has degree at least m.

Suppose n is even. Then B is a closed G-subvariety of V_1 and hence the restriction map $Mor(V_1, F^*)^G \to Mor(B, F^*)^G$ is surjective. Thus it suffices to prove that any homogeneous element Q of $Mor(V_1, F^*)^G = Mor(V_1, V_m)^G$ (note that $F = V_m$ is self-dual) has degree at least m. It follows from the H-equivariance that Q is of the form

$$Q(a,b) = (q(a,b), q(b,a))$$

with a polynomial q. If $q(a,b) = \sum c_{ij}a^i b^j$ with $c_{ij} \in \boldsymbol{C}$, then $i - j \equiv m \bmod n$ by the $\boldsymbol{Z}/n\boldsymbol{Z}(\subset D_n)$-equivariance. Since $2m \le n$, this congruence implies $i + j \ge m$; so $\deg Q \ge m$. If n is odd, we can apply the same reasoning after noting that B is a closed G-subvariety in $V_1 \oplus \boldsymbol{C}$. Therefore the lemma has been proven. □

THEOREM 8.4. *Suppose $0 < 2m \le n$. For $G = D_n$, B constructed above and $F = V_m$,*

$$d(B,F) = \begin{cases} \min\{m-1, n/2 - m - 1\} & \text{if } n \text{ is even } \neq 2m \\ 0 & \text{if } n = 2m \\ \min\{2m-1, n-2m-1\} & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. Since $F = V_m$ is multiplicity free with respect to $H = \boldsymbol{Z}/2\boldsymbol{Z}$, Theorem B can be applied; so we compute the number $e(B,F)$ in the following. By virtue of Lemma 8.3 the computation of the number $e(B,F)$ reduces to finding the maximum integer l such that Δ^l divides $\iota^*\Psi\iota^*\Phi$ in $end(\mathbf{F})_1$. Note that $e(B,F) = m - l - 1$ by definition.

Calculation of l. We note that

$$\iota^*\Psi\iota^*\Phi = \begin{pmatrix} \Delta^m & a^{2m} \\ b^{2m} & \Delta^m \end{pmatrix}$$

at $(a,b) \in B$ or $(a,b,c) \in B$ according as n is even or odd. This shows that Δ^l divides $\iota^*\Psi\iota^*\Phi$ if and only if it divides both a^{2m} and b^{2m}. By the H-equivariance Δ^l divides a^{2m} if and only if it divides b^{2m}. Thus Δ^l divides $\iota^*\Psi\iota^*\Phi$ if and only if it divides a^{2m}.

CASE 1. The case where n is even $\neq 2m$.
Remember that $\Delta = ab$ in this case. If Δ^l divided a^{2m}, then one could solve

$$a^{2m} - f\Delta^l \in (a^{n/2} - b^{n/2}) \subset \boldsymbol{C}[a,b] \tag{8.5}$$

for some polynomial f of a and b since $\mathcal{O}(B) = \boldsymbol{C}[a,b]/(a^{n/2} - b^{n/2})$.

If $n \ge 4m$, then one checks the term of degree $2m$ to see that there is no polynomial f which satisfies (8.5). This means that $l = 0$ if $n \ge 4m$. Suppose $2m < n < 4m$. Then $\Delta^{2m-n/2}$ does divide a^{2m}, in fact, if we take $f = b^{n-2m}$, then

$$a^{2m} - f\Delta^{2m-n/2} = a^{2m-n/2}(a^{n/2} - b^{n/2}).$$

Since $n > 2m$, $\deg f > 0$ which means that $\iota^*\Psi\iota^*\Phi/\Delta^{2m-n/2}$ is in $end(\mathbf{F})_1$; so $l \ge 2m - n/2$. We will show equality. Now $a^{2m} = b^{n-2m}\Delta^{2m-n/2}$ in $\mathcal{O}(B)$. We must show that Δ does not divide b^{n-2m}. If it did, one could solve

$$b^{n-2m} - f'\Delta \in (a^{n/2} - b^{n/2}) \subset \boldsymbol{C}[a,b]$$

for some polynomial f'. But this is not possible since $n - 2m < n/2$ (remember that we assume $n < 4m$).

CASE 2. The case where $n = 2m$.

The argument developed just above almost works when $n = 2m$. The only thing which we have to be careful about is that the degree of the f is zero as $n = 2m$. Therefore we must take $f = b^{n-2m+2}$ in this case. Then one sees that $l = 2m - n/2 - 1 = m - 1$.

CASE 3. The case where n is odd.

Remember that $\Delta = c$ in this case. If Δ^l divided a^{2m}, then we could write

$$a^{2m} = c^l f(a, b, c) \in \mathcal{O}(B) \tag{8.6}$$

for some homogeneous polynomial f of degree $2m - l$. Solving the defining equations (8.2) of B, we have

$$a = \zeta c, \qquad b = \zeta^{-1} c$$

with an nth root ζ of unity. Replace a and b in (8.6) by ζc and $\zeta^{-1} c$ respectively and look at the exponents of ζ at the both sides. Then one sees that l cannot be positive if $n \geq 4m$. Thus $l = 0$ in this case. Now suppose $n < 4m$. Then since $a = \zeta c$ and $b = \zeta^{-1} c$, we have $a^{2m} = c^{4m-n} b^{n-2m}$ in $\mathcal{O}(B)$; so $l \geq 4m - n$. Just like the above case where $n \geq 4m$, one checks that c does not divide b^{n-2m}. Therefore $l = 4m - n$ when $n < 4m$.

Noting that $e(B, F) = m - l - 1$, one sees that this proves the theorem. $\square$

THEOREM 8.7 (cf. [**MP2**, II.1.3]). *Suppose that $G = D_n$, $0 < 2m \leq n$ and $x = (1, 1) \in V_1$. Then $\log \rho_x : VEC(V_1, V_m; \boldsymbol{C}) \to \boldsymbol{C}^{f(V_1, V_m; x)}$ is surjective, where*

$$f(V_1, V_m; x) = \begin{cases} \min\{m - 1, [(n-1)/2] - m\} & \text{if } n \neq 2m \\ 0 & \text{if } n = 2m. \end{cases}$$

PROOF. Theorem C applies since $\Phi^{-1}(0) = 0$. It is not difficult to see that the image under ι^* of $\mathcal{O}(V_1)^G$ in $\mathcal{O}(B)^G = \boldsymbol{C}[\Delta]$ is $\boldsymbol{C}[\Delta]$ or $\boldsymbol{C}[\Delta^2]$ according as n is even or odd. Hence the theorem follows from Theorems C and 8.4. $\square$

The lower bound for $VEC(V_1, V_m; \boldsymbol{C})$ in the theorem above is obtained by pulling back bundles over V_1 to B. Actually in some cases a better bound is obtained by using the equivariant cone B'. We illustrate this in the next two results.

PROPOSITION 8.8. *Suppose n is odd and $2m < n < 4m$. Suppose also that T and T' are polynomials of $\Delta = ab$ with 1 as the constant term. If the G-vector bundles $E_\Phi(T)$ and $E_\Phi(T')$ over V_1 are isomorphic over B', then $T \equiv T' \bmod \Delta^m$.*

REMARK. This is not implied by Theorem 8.7, since some of the vector bundles there have been shown to pull back to equivalent vector bundles on B.

PROOF. The natural dominant morphism from B to B' induces an inclusion $\mathcal{O}(B') \subset \mathcal{O}(B)$. This means that the set of automorphisms of the G-vector bundle $B' \times (F \oplus \boldsymbol{C})$ are included in those of $B \times (F \oplus \boldsymbol{C})$.

Suppose $E_\Phi(T) = E_\Phi(T')$ in $VEC(B', F; \boldsymbol{C})$. Then there is an automorphism A of $B' \times (F \oplus \boldsymbol{C})$ which takes one to the other. Since A can be thought of as an automorphism of $B \times (F \oplus \boldsymbol{C})$ as remarked above, we know from the proof of Proposition 6.5 and the proof of Theorem 8.4 (see Case 3) that

$$(A - A_{(0)})(F, F) = p(c)\imath^* \Psi \imath^* \Phi / c^{4m-n}$$

with some polynomial p, in other words,

$$(A - A_{(0)})(F, F) = p(c)\begin{pmatrix} c^{n-2m} & a^{n-2m} \\ b^{n-2m} & c^{n-2m} \end{pmatrix}.$$

We will show that in order for all the terms to be in $\mathcal{O}(B')$, $p(c)$ must be divisible by c^{4m-n}. Then we are done, since then $A(\boldsymbol{C}, \boldsymbol{C})$ would have to be a constant modulo c^{2m}. (See the proof of Theorem B.)

CLAIM. (1) *If $p(c) \in \mathcal{O}(B')$, then $p(c)$ has no odd terms of degree $< n$.*
(2) *If $p(c)a^k \in \mathcal{O}(B')$, then $p(c)$ has no odd terms of degree $< n - 2k$.*

PROOF. We note that $p(c)$ is G-invariant as so is c. On the other hand we know that $\mathcal{O}(B')^G = \boldsymbol{C}[c^2, c^n]$, so the first case follows. Multiplying $p(c)a^k$ by b^k, the second case reduces to the first one.

By the claim above, $p(c)a^{n-2m} \in \mathcal{O}(B')$ implies that $p(c)$ has no odd terms of degree $< 4m - n$. Also, $p(c)c^{n-2m} \in \mathcal{O}(B')$ implies that $p(c)$ has no even terms of degree $< 2m$ (since n is odd). Thus $p(c)$ is divisible by c^{4m-n} because $4m - n < 2m$ by assumption. □

The above proposition gives a better estimate for $VEC(V_1, V_m; \boldsymbol{C})$ than Theorem 8.7 when n is odd. The following corollary follows from Theorem 8.7 and Proposition 8.8.

COROLLARY 8.9. *Suppose that $G = D_n$, $0 < 2m \leq n$ and n is odd. Then $VEC(V_1, V_m; \boldsymbol{C})$ contains a subspace isomorphic to $\boldsymbol{C}^{m-1}$.*

§9. Applications to connected reductive groups.

Suppose G is a connected reductive group. Given a dominant weight μ of G, we denote by $V(\mu)$ the irreducible representation of G with highest weight μ, and by μ^* the highest weight of $V(\mu)^*$. In **[MMP2]** we investigated $VEC(V, F; S)$ where

$$V = jV(\beta), \quad F = kV(\phi), \quad S = lV(\sigma)$$

for some dominant weights β, ϕ, σ and positive integers j, k, l. In fact we gave a lower bound for $VEC(V, F; S)$ in terms of the weights when there is a reductive subgroup H such that $V(\beta)^H \neq 0$ and $V(\phi)$ is multiplicity free with respect to H. Some of the non-trivial equivariant vector bundles obtained there can also be distinguished using the method developed here when $S = \boldsymbol{C}$ (i.e. $l = 1$ and $\sigma = 0$). We form $B = B(V; x)$ from Proposition 7.1 where x is a non-zero point of V^H. In this section we give a lower bound for $e(B, F)$ in terms of the weights β and ϕ (Proposition 9.3) and use this via Theorems B and C to obtain a lower bound for $VEC(V, F; \boldsymbol{C})$ (Theorem 9.7).

We consider the partial ordering on the additive group of weights of G tensored with $\boldsymbol{Q}$: $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a sum of positive roots with non-negative *rational* coefficients. This ordering behaves well with respect to addition and scalar multiplication.

LEMMA 9.1. *Any irreducible representation in $\mathcal{O}(B)_{(i)}$ has the highest weight at most $i\beta^*$.*

PROOF. By construction B is a closed $G \times \boldsymbol{C}^*$-subvariety of $jV(\beta) \times \boldsymbol{C}$, so the restriction map : $\mathcal{O}(jV(\beta) \times \boldsymbol{C}) \to \mathcal{O}(B)$ is surjective, degree preserving and G-equivariant. This implies the lemma since $\mathcal{O}(jV(\beta) \times \boldsymbol{C})_{(i)} = \bigoplus_{s+t=i} \mathcal{O}(jV(\beta))_{(s)} \otimes \mathcal{O}(\boldsymbol{C})_{(t)}$ and $\mathcal{O}(jV(\beta))_{(s)}$ is the sth symmetric product of $jV(\beta)^* = jV(\beta^*)$. □

LEMMA 9.2. *If $F^H \neq 0$, then there is a non-negative integer i such that $\phi + \phi^* \leq i\beta^*$.*

PROOF. Since $F^H \neq 0$, there is a non-zero homogeneous element $\Phi \in mor(\mathbf{F}, \mathbf{C})$ by Lemma 2.3. The map Φ induces a non-trivial equivariant linear map : $F \to \mathcal{O}(B)_{(\deg \Phi)}$. This means that $\mathcal{O}(B)_{(\deg \Phi)}$ contains $V(\phi)$ since $F = kV(\phi)$. Therefore we obtain an inequality $\phi \leq (\deg \Phi)\beta^*$ by Lemma 9.1. The above argument applied to F^* instead of F shows that $\phi^* \leq (\deg \Phi)\beta^*$. Thus $\phi + \phi^* \leq 2(\deg \Phi)\beta^*$ and this proves the lemma. □

DEFINITION. For fixed weights β and ϕ, consider the inequality $\phi + \phi^* \leq r\beta^*$ for a non-negative integer r. Lemma 9.2 ensures that this inequality has a solution if $F^H \neq 0$. We define an integer $r(\beta, \phi)$ to be 0 when $F^H = 0$, and to be the smallest (non-negative) integer such that the above inequality holds when $F^H \neq 0$.

The definition of $r(\beta, \phi)$ allows us to estimate the integer $e(B, F)$ as follows.

PROPOSITION 9.3. $e(B, F) \geq r(\beta, \phi)/\deg \Delta - 1$.

PROOF. The following proof is essentially the same as in [**MMP2**, Proposition 3.2]. If $F^H = 0$, then $e(B, F) = r(\beta, \phi) = 0$ by definition; so the proposition is trivial in this case. We assume $F^H \neq 0$ in the following. Let Φ and Ψ be non-zero homogeneous elements in $mor(\mathbf{F}, \mathbf{C})$ and $mor(\mathbf{C}, \mathbf{F})$ respectively. Suppose Δ^l divides $\Psi\Phi$ in $end(\mathbf{F})_1$. The element $\Psi\Phi/\Delta^l$ induces the equivariant linear map $\Omega : F^* \otimes F \to \mathcal{O}(B)_{(i)}$ defined by

$$\Omega(v_* \otimes v)(u) = \langle v_*, (\Psi\Phi/\Delta^l)(u)(v) \rangle$$

for $u \in B$, $v \in F$ and $v_* \in F^*$, where $i = \deg(\Psi\Phi/\Delta^l)$ and $\langle\ ,\ \rangle$ denotes the natural pairing between F^* and F. Let v^+ and v_*^+ be the highest weight (or maximal) vectors of F and F^* respectively.

CLAIM. $\Omega(v_*^+ \otimes v^+) \neq 0$.

We admit this claim for a moment and complete the proof of the proposition. Since the weights of v_*^+ and v^+ are respectively ϕ^* and ϕ, we obtain an inequality $\phi^* + \phi \leq i\beta^*$ from Lemma 9.1 and the claim above. Thus $\deg(\Psi\Phi/\Delta^l) = i \geq r(\beta, \phi)$ by the definition of $r(\beta, \phi)$ and hence the proposition follows from the definition of $e(B, F)$.

Now we prove the claim above. We regard Φ and Ψ as elements of $Mor(B, F^*)^G$ and $Mor(B, F)^G$ respectively. Since $B - B_0$ is dense in B, there is a point $w \in B - B_0$ on which Φ and Ψ are both non-zero. It is not difficult to see that there is $g \in G$ such that

$$\langle gv_*^+, \Psi(w)\rangle \neq 0 \quad \text{and} \quad \langle \Phi(w), gv^+\rangle \neq 0$$

(see e.g. [**MMP2**, Lemma 3.3]). Then

$$\begin{aligned}\Omega(v_*^+ \otimes v^+)(g^{-1}w) &= \langle v_*^+, (\Psi\Phi/\Delta^l)(g^{-1}w)(v^+)\rangle \\ &= \langle gv_*^+, (\Psi\Phi/\Delta^l)(w)(gv^+)\rangle \\ &= \langle \Phi(w), gv^+\rangle\langle gv_*^+, \Psi(w)\rangle/\Delta^l(w) \neq 0.\end{aligned}$$

This proves the claim. $\square$

The case where $\phi = m\beta^*$ is an important example in [**MMP1**] and [**MMP2**]. In this case the integer $r(\beta, \phi)$ can be estimated in terms of β.

DEFINITION. We define $q(\beta)$ to be the smallest *rational* number such that $\beta \leq q(\beta)\beta^*$ if it exists.

The number $q(\beta)$ exists if and only if $\beta \neq 0$ and β or β^* restricts to zero on the identity component of the center of G, and it is easily computed. One sees that $q(\beta^*) = q(\beta)$ and $q(\beta) \geq 1$ for any β with equality holding if and only if $\beta^* = \beta$. The following lemma follows directly from the definition of $r(\beta, \phi)$ and $q(\beta)$.

LEMMA 9.4. *If $\phi = m\beta^*$ or $m\beta$ and the number $q(\beta)$ exists, then $r(\beta, \phi) \geq (q(\beta) + 1)m$. If $\beta^* = \beta$ moreover, then $r(\beta, \phi) \geq 2m$.*

Combining Proposition 9.3 with Theorem B and Lemma 9.4, we obtain

COROLLARY 9.5. *Let the situation be as above. If $V(\phi)$ is multiplicity free with respect to H, then*

$$d(B, F) \geq r(\beta, \phi)/\deg \Delta - 1.$$

If $\phi = m\beta^*$ or $m\beta$ and $q(\beta)$ exists moreover, then

$$d(B, F) \geq (q(\beta) + 1)m/\deg \Delta - 1.$$

Now we apply this via Theorem C using the natural map $\iota : B \to V = jV(\beta)$. As an example we take $\phi = m\beta^*$ and $j \leq k$, since in this case we know that there is a weak duality in $Mor(V, F^*)^G$ (see [**MMP2**, §4]); so it suffices to find the image $\iota^*\mathcal{O}(V)_1^G$.

LEMMA 9.6. *Suppose that $\beta^* = \beta$ and* $\dim V(\beta)^H = 1$. *Then there is a degree* 2 *homogeneous function $\Delta_V \in \mathcal{O}(V)^G$ such that $\iota^*\Delta_V \in \mathcal{O}(B)^G = \mathbf{C}[\Delta]$ is non-zero. Therefore* $\deg \Delta$ *must be* 1 *or* 2, *and accordingly $\iota^*\Delta_V = \Delta^2$ or Δ (up to a scalar non-zero constant).*

Proof. The assumption $\beta^* = \beta$ means that $V(\beta)$ is self-dual, so there is a non-degenerate invariant bilinear form $\langle\, ,\rangle$ on $V(\beta)$. It is still non-trivial on $V(\beta)^H$ since $\dim V(\beta)^H = 1$. Take a projection $\pi : V = jV(\beta) \to V(\beta)$ such that $\pi(x) \neq 0$ where x is the chosen point to construct B. We define $\Delta_V \in \mathcal{O}(V)^G$ by $\Delta_V(v) = \langle \pi(v), \pi(v)\rangle$. Since $\pi(x) \neq 0$, $\Delta_V(x) \neq 0$. This shows that $\iota^*\Delta_V \in \mathcal{O}(B)^G$ is non-zero. □

Theorem 9.7 (cf. [**MMP2**, Theorem 4.6]). *Suppose $\beta^* = \beta$ $(\neq 0)$ and there is a reductive subgroup H such that*

(1) $\dim V(\beta)^H = 1$

(2) *$V(m\beta)$ is multiplicity free with respect to H.*

Then $VEC(V, F; \mathbf{C})$ surjects onto $\mathbf{C}^{m-1}$ where $V = jV(\beta)$ and $F = kV(m\beta)$.

Proof. When $j \leq k$, there is a weak duality in $Mor(V, F^*)^G$ as remarked above; so the theorem follows from Theorem C, Corollary 9.5 and Lemma 9.6. When $j > k$, we consider the projection from $V = jV(\beta)$ on $kV(\beta)$. It induces an injection of $VEC(kV(\beta), F; \mathbf{C})$ to $VEC(V, F; \mathbf{C})$. Since the above result can be applied to $kV(\beta)$, this shows that the statement holds even when $j > k$. □

§10. U-invariants.

Suppose G is a connected reductive group. Let U be a maximal unipotent subgroup of G and $\mathcal{O}(B)^U$ the ring of U-invariants. In this section we will see that the integer $e(B, F)$ introduced in Section 6 can be described in a different way, only depending on $\mathcal{O}(B)^U$, under certain conditions. This is useful when $\mathcal{O}(B)^U$ is easy to describe. We will see an example of this in the next section.

An element $\Phi \in mor(\mathbf{F}, \mathbf{C})$ defines an equivariant linear map $\tilde{\Phi} : F \to \mathcal{O}(B)$ and vice versa. In fact, the correspondence is given by $\tilde{\Phi}(v)(u) = \Phi(u)(v)$ for $u \in B$ and $v \in F$. Let F_i be an irreducible subrepresentation of F and let $\tilde{\Phi}_i \in \mathcal{O}(B)^U$ be the image of the highest weight vector of F_i by $\tilde{\Phi}$. The element $\tilde{\Phi}_i$ is defined up to a non-zero constant. We can define $\tilde{\Psi}_i$ similarly for $\Psi \in mor(\mathbf{C}, \mathbf{F})$ with F_i replaced by F_i^*.

Lemma 10.1. *If B is normal, then Δ^l divides $\Psi\Phi$ in $end(\mathbf{F})$ if and only if it divides $\tilde{\Psi}_i\tilde{\Phi}_j$ in $\mathcal{O}(B)^U$ for all i, j.*

Proof. First we prove that Δ^l divides $\Psi\Phi$ in $end(\mathbf{F})$ if and only if it divides $(g\tilde{\Psi}_i)(g'\tilde{\Phi}_j)$ in $\mathcal{O}(B)$ for any $g, g' \in G$. Note that $\Psi\Phi$ corresponds to an equivariant linear map $\Xi : F^* \otimes F \to \mathcal{O}(B)$ and that Δ^l divides $\Psi\Phi$ if and only if it divides each element in the image of Ξ. Since the image of Ξ is the linear span of elements of the form $(g\tilde{\Psi}_i)(g'\tilde{\Phi}_j)$ $(g, g' \in G)$, the assertion follows.

Next we prove that Δ^l divides $(g\tilde{\Psi}_i)(g'\tilde{\Phi}_j)$ in $\mathcal{O}(B)$ for any $g, g' \in G$ if and only if it divides $\tilde{\Psi}_i\tilde{\Phi}_j$ in $\mathcal{O}(B)$. Since B is normal, we know that Δ^l divides $\tilde{\Psi}_i\tilde{\Phi}_j$ in $\mathcal{O}(B)$ if and only if it divides $\tilde{\Psi}_i\tilde{\Phi}_j$ in every local ring $\mathcal{O}_{B,X}$ for codimension one irreducible subvarieties X of B. For any subvariety not in the null-fiber B_0, Δ is invertible. Thus we need only consider irreducible components X of B_0. Since G is connected, X is stable by the action of G and the valuation v on $\mathcal{O}_{B,X}$ is thus stable under the action of G. Thus

$$v(\tilde{\Psi}_i\tilde{\Phi}_j) = v(\tilde{\Psi}_i) + v(\tilde{\Phi}_j) = v(g\tilde{\Psi}_i) + v(g'\tilde{\Phi}_j) = v((g\tilde{\Psi}_i)(g'\tilde{\Phi}_j))$$

for any $g, g' \in G$. This shows that Δ^l divides $\tilde{\Psi}_i\tilde{\Phi}_j$ in $\mathcal{O}_{B,X}$ if and only if it divides $(g\tilde{\Psi}_i)(g'\tilde{\Phi}_j)$ for any $g, g' \in G$ in the same ring. Thus the assertion follows.

Finally since B is irreducible and Δ^l is G-invariant (in particular, U-invariant), the divisibility of $\tilde{\Psi}_i\tilde{\Phi}_j$ by Δ^l in $\mathcal{O}(B)$ is the same as in $\mathcal{O}(B)^U$. $\square$

Observe that $\tilde{\Phi}_i$ are homogeneous if Φ is, and that the weight of $\tilde{\Phi}_i$ agrees with the highest weight of F_i. The same observation applies to $\tilde{\Psi}_i$. Motivated by Lemma 10.1 together with this observation we introduce a non-negative integer analogous to $e(B, F)$.

DEFINITION. Suppose that B is special. Then we define $e^U(B, F) = 0$ when $F^H = 0$, and

$$e^U(B, F) = \min_{f^*, f, l}\{\deg(f^*f/\Delta^l)/\deg\Delta - 1 \mid f^*f/\Delta^l \in \mathcal{O}(B)_1^U\}$$

when $F^H \neq 0$, where f^* and f run over non-zero homogeneous elements in $\mathcal{O}(B)^U$ whose weights agree with the highest weights of F_i^* and F_i respectively for some i. (Note that such f^* and f exist if and only if $F_i^H \neq 0$.) Unless B is special, we define $e^U(B, F)$ to be $e^U(B^s, F)$ where B^s is the special G-cone in B as before.

PROPOSITION 10.2. *Suppose B is normal. Then $e(B, F) \geq e^U(B, F)$ for any F and the equality holds if $\dim F_i^H \leq 1$ for every irreducible subrepresentation F_i of F (e.g. if F is a multiple of a representation which is multiplicity free with respect to H).*

PROOF. We assume $F^H \neq 0$ since otherwise the statement is trivial. It is clear from the definition that $e^U(B, F)$ is the minimum of $e^U(B, F_i)$ over F_i with $F_i^H \neq 0$. This together with Lemma 6.1(3) says that it suffices to prove the statement for such F_i. By Lemma 10.1 the divisibility conditions in the definitions of $e(B, F_i)$ and $e^U(B, F_i)$ are equivalent. Since there is an extra condition in the definition of $e(B, F_i)$, we have the inequality $e(B, F_i) \geq e^U(B, F_i)$. But if $\dim F_i^H = 1$, then the extra condition is automatically satisfied as remarked after the definition of $e(B, F)$ in Section 6; so the equality holds. $\square$

Remember that if the ideal I is (0) or (1), then $VEC(B, F; C) = \{*\}$; so we assume $I = (\Delta^m)$ for some $m \geq 1$ in the following. Since the integer m depends on B and F, we specify them as $m(B, F)$. The integer $m(B, F)$ is easier to compute than $e(B, F)$, and $e(B, F)$ is closely related to the dimension of $VEC(B, F; C)$ as observed in Section 6. Therefore it would be meaningful to compare these two integers. By Lemma 6.1(2) $e(B, F) \leq m(B, F) - 1$ and the equality does not necessarily hold as is seen in the dihedral case. But we have

PROPOSITION 10.3. *Suppose B is normal and* $\dim F_i^H \le 1$ *for all irreducible subrepresentations F_i of F. Then*

(1) $e(B,F) = m(B,F) - 1$ *or* $m(B,F) - 2$ *if the null-fiber B_0 is irreducible,*

(2) $e(B,F) = m(B,F) - 1$ *if B is UFD and G is semisimple.*

This proposition follows immediately from Proposition 10.2 and this lemma.

LEMMA 10.4. *Suppose B is normal. Let f^*, f be non-zero homogeneous elements of $\mathcal{O}(B)^U$ which are not divisible by Δ in $\mathcal{O}(B)^U$. Suppose Δ^l divides f^*f in $\mathcal{O}(B)^U$ for some $l \ge 0$. Then*

(1) $l \le 1$ *if the null-fiber B_0 is irreducible,*

(2) $l = 0$ *if B is UFD and G is semisimple.*

PROOF. (1) Since B is normal and B_0 is irreducible, the local ring $\mathcal{O}_{B,B_0}$ is a valuation ring for a valuation v. If f^*f/Δ^2 is in this ring, then $2v(\Delta) \le v(f^*) + v(f)$. But then we would have that $v(\Delta) \le v(f^*)$ or $v(f)$, and that would mean that Δ would divide f^* or f in $\mathcal{O}_{B,B_0}$. But Δ is invertible in the other local rings of irreducible subvarieties of codimension 1. Thus Δ would divide f^* or f in $\mathcal{O}(B)$, hence in $\mathcal{O}(B)^U$ as remarked at the end of the proof of Lemma 10.1. This is a contradiction.

(2) Since $\mathcal{O}(B)$ is UFD and G is semisimple, one easily sees that the generator Δ of $\mathcal{O}(B)^G$ is irreducible. Hence, if Δ divides f^*f, then it must divide f^* or f, which is a contradiction. □

THEOREM 10.5. *Let B be a representation of a semisimple group G with one dimensional quotient. If the generic fiber of $B \to B//G$ is isomorphic to G/H for some reductive subgroup H and F is a multiple of a representation which is multiplicity free with respect to H, then* $VEC(B,F;\boldsymbol{C}) \cong \boldsymbol{C}^{m(B,F)-1}$.

PROOF. The theorem follows from Theorem B and Proposition 10.3(2). □

EXAMPLES. Theorem 10.5 can be applied to the cases listed in the following table (cf. [**KS**, TABLE III in chapter VII]). For the verification of the multiplicity free condition, see section 5 of [**MMP2**].

Table 1

G	B	F	H
$SL(n)$ $n \ge 2$	φ_1^2	$k\varphi_1^{2m}$ or $k(\varphi_1^{2m})^*$	$SO(n)$
$SL(n)$ $n \ge 3$	$\varphi_1 + \varphi_1^*$	$k\varphi_1^m$ or $k(\varphi_1^m)^*$	$SL(n-1)$
$SO(n)$ $n \ge 3$	φ_1	$k\varphi_1^m = k(\varphi_1^m)^*$	$SO(n-1)$
G_2	φ_1	$k\varphi_1^m = k(\varphi_1^m)^*$	$SL(3)$
E_6	φ_1	$k\varphi_1^m$ or $k(\varphi_1^m)^*$	F_4

In this table φ_1 denotes the standard representations for $SL(n)$ and $SO(n)$, and the non-trivial irreducible representations of dimension 7 and 26 for G_2 and E_6 respectively. The m and k are positive integers and φ_1^p denotes the pth Cartan power of φ_1 (i.e. the irreducible representation whose highest weight is p times the highest weight of φ_1). For all the examples above $m(B,F)=m$ and hence $VEC(B,F;\boldsymbol{C}) \cong \boldsymbol{C}^{m-1}$.

§11. Spherical subgroups of rank one.

In this section we compute the integer $e(B,F)$ when B is a special G-cone associated to a spherical subgroup of G. We continue to assume G to be connected reductive and denote by $V(\mu)$ the irreducible representation of G with highest weight μ as before. We suppose that there is a reductive spherical subgroup H of rank one. This means that a Borel subgroup of G has a dense orbit in G/H and the minimal codimension of a U-orbit in it is one. In the following we fix the Borel subgroup, the unipotent subgroup U and the maximal torus T in the Borel subgroup. The spherical rank one property is equivalent to saying that $\mathcal{O}(G/H)^U = \boldsymbol{C}[w]$ where w is an eigenvector for the maximal torus T, or that $\mathcal{O}(G/H) \cong \bigoplus_{k\geq 0} V(k\lambda)$ where λ is the character of w, (see [**Br**]). We call λ the *spherical character* of H in G.

LEMMA 11.1. (1) $\dim V(\mu)^H = 1$ *if* $\mu = k\lambda$ *for some* $k \geq 0$ *and* 0 *otherwise. Moreover* $V(k\lambda)$ *is self-dual.*

(2) *The isotropy subgroup* H_k *at a non-zero element of* $V(k\lambda)^H$ *has index* 1 *or* 2 *in its normalizer* $N_G(H_k)$.

(3) $H_1 = H$.

REMARK. The statement (3) does not necessarily hold for H_k unless $k=1$. For example, when $G = SL(2)$ and $H = \boldsymbol{C}^*$, $H_k = H$ if k is odd, but $H_k = N_G(H)$ if k is even.

PROOF. (1) Since $\mathcal{O}(G) \cong \bigoplus_\mu V(\mu) \otimes V(\mu)^*$ as is well known ([**Kr**, II.3.1, Satz 3]), where μ runs over all the dominant weights of G, we have

$$\mathcal{O}(G/H) \cong \mathcal{O}(G)^H \cong \bigoplus_\mu V(\mu) \otimes (V(\mu)^*)^H.$$

On the other hand since H is spherical of rank one, we have $\mathcal{O}(G/H) \cong \bigoplus_{k\geq 0} V(k\lambda)$. Since $\dim(V(\mu)^*)^H = \dim V(\mu)^H$, this implies (1).

(2) By (1) the action of $N_G(H_k)/H_k$ on $V(k\lambda)^H$ induces an injective homomorphism $\nu_k : N_G(H_k)/H_k \to \boldsymbol{C}^*$ defined by $gv = \nu_k(g)v$ for $v \in V(k\lambda)^H$ and $g \in N_G(H_k)/H_k$, and there is an invariant non-degenerate bilinear form $\langle\, , \rangle$ on $V(k\lambda)$. It follows that

$$\langle v, v\rangle = \langle gv, gv\rangle = \langle \nu_k(g)v, \nu_k(g)v\rangle = \nu_k(g)^2 \langle v, v\rangle$$

for any $v \in V(k\lambda)^H$ and $g \in N_G(H_k)/H_k$. Here $\langle v, v\rangle \neq 0$ if $v \neq 0$, because the bilinear form is non-degenerate and $\dim V(k\lambda)^H = 1$. Hence $\nu_k(g)^2 = 1$ for any $g \in N_G(H_k)/H_k$. Since ν_k is injective, this proves (2).

(3) Since $V(\lambda)^{H_1} \neq 0$, $V(k\lambda)^{H_1} \neq 0$ for any k. This together with (1) implies that $V(k\lambda)^{H_1} = V(k\lambda)^H$, equivalently $(V(k\lambda)^*)^{H_1} = (V(k\lambda)^*)^H$ for any k. Moreover, unless μ is a multiple of λ, $(V(\mu)^*)^{H_1} = 0$ because $H_1 \supset H$ and $(V(\mu)^*)^H = 0$ by (1). Thus

$$\begin{aligned} \mathcal{O}(G/H_1) = \mathcal{O}(G)^{H_1} &= \bigoplus_\mu V(\mu) \otimes (V(\mu)^*)^{H_1} \\ &= \bigoplus_\mu V(\mu) \otimes (V(\mu)^*)^H = \mathcal{O}(G)^H = \mathcal{O}(G/H), \end{aligned}$$

which implies $H_1 = H$. $\square$

Let $B^{(k)}$ be the normalization of $\overline{GV(k\lambda)^H}$, i.e. $B^{(k)} = B_1(V(k\lambda); x)$ from section 7 where x is a non-zero point in $V(k\lambda)^H$. This is an especially nice situation, since as we shall see in the next proposition, the ring of U-invariants is simply a polynomial ring.

PROPOSITION 11.2. *$\mathcal{O}(B^{(k)})^U = \bigoplus_{i=0}^\infty \boldsymbol{C}[\alpha, \Delta]_{(ki)}$, where α has degree 1 and weight λ, and Δ has degree 1 or 2 and weight 0.*

PROOF. First we consider the case where $k = 1$. Let $B = B^{(1)}$, $B' = \overline{GV(\lambda)^H} \subset V(\lambda)$ and let $\kappa : B \to B'$ be the normalization map. Remember that B and B' have unique open $G \times \boldsymbol{C}^*$-orbits respectively on which κ is an isomorphism. Denote by B'_0 the complement of the open $G \times \boldsymbol{C}^*$-orbit in B'. It is not zero since $\dim B'_0 = \dim G/H \geq 1$ ([**Kr**, Satz 1 in p. 37]). The complement of the open $G \times \boldsymbol{C}^*$-orbit in B is nothing but the null-fiber B_0. Since κ is surjective, we know $\kappa(B_0) = B'_0$. Now let α' be the function of $B' \subset V(\lambda)$ defined by the highest weight vector of $V(\lambda)^*$ and let $\alpha = \alpha'\kappa$. Note that $\alpha \in \mathcal{O}(B)^U$. Since B'_0 is G-invariant and $V(\lambda)$ is irreducible, the linear span of B'_0 agrees with $V(\lambda)$. Hence α' is not identically 0 on B'_0. This shows that α is also not identically 0 on B_0 since $\kappa(B_0) = B'_0$.

We know that $\mathcal{O}(B)^U$ is generated by homogeneous eigenvectors of the maximal torus T. Suppose β is a homogeneous eigenvector of character η. We show that η is a multiple of λ. Restrict β to a generic fiber of $B \to B//G$ on which β is not identically 0. The generic fiber is isomorphic to G/H by Lemma 11.1(3) and β is an eigenvector of the character η. Therefore, by the rank one spherical property, we know that $\eta = s\lambda$ for some $s \geq 0$.

Consider the rational function β/α^s. This is a rational function on B which is invariant under the Borel subgroup TU. Since the generic fiber is isomorphic to G/H, the Borel subgroup has an open orbit in it by the spherical property; so we know that β/α^s is G-invariant. Since the function is homogeneous, it is an integral power of Δ (up to a non-zero constant) by Lemma 2.2(3), where Δ is the generator of $\mathcal{O}(B)^G$. If that power were negative, then α would be identically 0 on B_0, which is not the case as observed above. That is, $\beta = \alpha^s\Delta^t$ (up to a non-zero constant) for some $t \geq 0$.

Now we show that there are no relations between α and Δ in $\mathcal{O}(B)^U$. If there were, then $B//U$ would be 1-dimensional where $B//U$ is the affine variety whose coordinate ring is $\mathcal{O}(B)^U$. But $B//U$ contains the quotient of a generic fiber which is isomorphic to $(G/H)//U$. It is 1-dimensional (since $\mathcal{O}(G/H)^U = \boldsymbol{C}[w]$), contained in $B//U$ properly and $B//U$ is irreducible ([**Kr**, III.3.3]), so $\dim B//U \geq 2$. Thus the assertion follows.

We know that $\kappa : B \to B' = \overline{GV(\lambda)^H}$ is isomorphic on $B - B_0$. This shows that $B - B_0$ is equivariantly isomorphic to the balanced product $G \times_{N_G(H)} (V(\lambda)^H - 0)$. Therefore we have an isomorphism

$$\mathcal{O}(B - B_0)^G \cong \mathcal{O}(V(\lambda)^H - 0)^{N_G(H)}.$$

Since $\dim V(\lambda)^H = 1$, the right hand side is a Laurent polynomial ring $\boldsymbol{C}[\delta, \delta^{-1}]$ with a homogeneous generator δ of degree $|N_G(H)/H|$ which is 1 or 2 by Lemma 11.1(2)(3). On the other hand $\mathcal{O}(B - B_0)^G = \boldsymbol{C}[\Delta, \Delta^{-1}]$. This shows that $\deg \Delta = \deg \delta$ and it is 1 or 2.

Finally note that $B^{(k)}$ is the quotient of $B^{(1)} = B$ by the cyclic group of order k. Thus $\mathcal{O}(B^{(k)})^U$ is just the subring of $\mathcal{O}(B)^U$ which consists of homogeneous elements of degree divisible by k. This completes the proof of the proposition. □

THEOREM 11.3. *Suppose H is a reductive spherical subgroup of rank one. Let λ be the spherical character of H in G and j a non-negative integer. Then*

$$e(B^{(k)}, V(j\lambda)) = \begin{cases} \{2j/k\} - 1 & \text{if } \begin{array}{l} \deg \Delta = 1 \text{ and } j > 0, \text{ or} \\ \deg \Delta = 2 \text{ and } k, j \text{ even} > 0 \end{array} \\ \{j/k\} - 1 & \text{if } \deg \Delta = 2,\ k \text{ odd and } j > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\{r\}$ denotes the least integer greater than or equal to r.

PROOF. By Proposition 10.2 and Lemma 11.1(1) we may compute $e^U(B^{(k)}, V(j\lambda))$. The proof follows from a case-by-case study using Proposition 11.2. For example, suppose $\deg \Delta = 1$ and $j > 0$. Since $V(j\lambda)$ is irreducible and self-dual, the elements f^* and f in the definition of $e^U(B^{(k)}, V(j\lambda))$, which have weight $j\lambda$, can be written as

$$f^* = \alpha^j \Delta^{rk-j}, \qquad f = \alpha^j \Delta^{sk-j}$$

(up to a non-zero constant) with some positive integers r and s by Proposition 11.2. Therefore $f^* f = \alpha^{2j} \Delta^{(r+s)k-2j}$. On the other hand $\mathcal{O}(B^{(k)})^G = \boldsymbol{C}[\Delta^k]$ again by Proposition 11.2. This shows that the generator Δ_k of $\mathcal{O}(B^{(k)})^G$ is given by Δ^k. Therefore $f^* f/\Delta_k^l$ is in $\mathcal{O}(B^{(k)})_1^U$ if and only if $l \leq [r + s - 2j/k]$ (here we use the assumption that $j > 0$). Since

$$\deg(f^* f/\Delta_k^l)/\deg \Delta_k = r + s - l$$

and $[r + s - 2j/k] = r + s - \{2j/k\}$, we obtain the desired formula.

The other cases will be treated in a similar way. □

Using Theorem 11.3 together with Theorems A and B, we obtain the description of $VEC(B^{(k)}, F; \boldsymbol{C})$ if F is a multiple of $V(j\lambda)$ and $V(j\lambda)$ is multiplicity free with respect to H. There are some examples from [**Br**] which do satisfy the multiplicity free condition.

EXAMPLES. For the cases listed in the following table, H is a spherical subgroup of rank one of G and $V(j\lambda)$ is multiplicity free with respect to H for any j.

Table 2

G	H	λ	$\deg \Delta$
$SL(2)$	$\boldsymbol{C}^*$	$\varepsilon = 2\lambda_1$	2
$SL(n)\ n \geq 3$	$GL(n-1)$	ε	1
$Spin(n)\ n \geq 3$	$Spin\ (n-1)$	λ_1	2
G_2	$SL(3)$	λ_1	2
F_4	$Spin(9)$	λ_1	1

In this table, ε denotes the highest weight of the adjoint representation and λ_1 denotes that of the non-trivial irreducible G-representation of the smallest dimension. For the verification of the multiplicity free condition, see section 5 of [**MMP2**]. The degree of Δ is calculated using the fact that it is the order of $N_G(H)/H$.

Remark. Using Theorems B and C we obtain a lower bound for $VEC(V, F; \boldsymbol{C})$ when $V = V(k\lambda)$ and $F = V(mk\lambda)^*$. In this case there is a weak duality $\Phi : V \to F^*$ defined by the m-th Cartan power $\Phi(u) = u^m$ for $u \in V$. See [**MMP2**].

References

[BH] H. Bass and W. Haboush, Linearizing certain reductive group actions, Trans. AMS. **292** (1985), 463–482.

[Br] M. Brion, On spherical varieties of rank one, Proceedings of the Conference on Group Actions and Invariant Theory, Montreal 1988, CMS Conference Proceedings, vol. 10, 1988, p. 31–41.

[H] R. Hartshorne, Algebraic Geometry, GTM 52, Springer Verlag, New York, Heidelberg, Berlin, 1977.

[Ka] T. Kambayashi, Automorphism group of a polynomial ring and algebraic group actions of affine space, J. Algebra **60** (1979), 439–451.

[KR] T. Kambayashi and P. Russel, On linearizing algebraic torus actions, J. Pure Appl. Algebra **23** (1982), 243–250.

[Kr] H. Kraft, Geometrische Methoden in der Invariantentheorie, Aspekte der Mathematik vol. D1 Vieweg, Braunschweig, 1984.

[KS] H. Kraft and G. Schwarz, Reductive group actions with one dimensional quotient, Publ. Math. IHES **76** (1992), 1–97.

[L] D. Luna, Slices étales, Bull. Soc. Math. France, Mémoire **33** (1973), 81–105.

[MP1] M. Masuda and T. Petrie, Equivariant algebraic vector bundles over representations of reductive groups: Theory, Proc. Natl. Acad. Sci. USA **88** (1991), 9061–9064.

[MP2] M. Masuda and T. Petrie, Stably trivial equivariant algebraic vector bundles, J. Amer. Math. Soc. **8** (1995), 687–714.

[MP3] M. Masuda and T. Petrie, Algebraic families of $O(2)$-actions on affine space $\boldsymbol{C}^4$, Symp. in Pure Math. **56 Part 1** (1994), 347–354.

[MMP1] M. Masuda, L. Moser-Jauslin and T. Petrie, Equivariant algebraic vector bundles over representations of reductive groups: Applications, Proc. Natl. Acad. Sci. USA **88** (1991), 9065–9066.

[MMP2] M. Masuda, L. Moser-Jauslin and T. Petrie, Invariants of equivariant algebraic vector bundles and inequalities for dominant weights, Topology **37** (1998), 161–177.

[MMP3] M. Masuda, L. Moser-Jauslin and T. Petrie, The equivariant Serre problem for abelian groups, Topology, **35** (1996), 329–334.

[MJ] L. Moser-Jauslin, Algebraic equivariant vector bundles and the linearity problem, Symp. in Pure Math. **56 Part 1** (1994), 355–364.

[S] G. Schwarz, Exotic algebraic group actions, C.R. Acad. Sci. Paris **309** (1989), 89–94.

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