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Regular weights on algebras of unbounded operators

By Atsushi INOUE and Hidekazu OGI

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1. Introduction.

Algebras of unbounded operators called O^*-algebras have been studying from the pure mathematical situations (operator theory, topological $*$-algebras, representations of Lie algebras etc.) and the physical applications (the Wightman quantum field theory, unbounded CCR-algebras, quantum groups etc.). To proceed such studies it is important to study the Tomita-Takesaki theory in O^*-algebras [**11**$\sim$**15**]. Weights on O^*-algebras (that is, linear functionals that take positive, but not necessarily finite valued) are naturally appeared in the studies of the unbounded Tomita-Takesaki theory [**13**$\sim$**15**] and the quantum physics [**4**, **15**]. Thus it is significant to study weights on O^*-algebras for the structure of O^*-algebras and the physical applications. Further, the weights on O^*-algebras occasion some pathological phenomena which don't occur for weights on C^*- and W^*-algebras. From this viewpoint we should study systematically weights on O^*-algebras.

In Section 2 we shall define quasi-weights and weights on O^*-algebras and give the fundamental examples. Let $\mathscr{M}$ be a closed O^*-algebra on a dense subspace $\mathscr{D}$ in a Hilbert space $\mathscr{H}$. We define positive cones $\mathscr{P}(\mathscr{M})$ and $\mathscr{M}_+$ of $\mathscr{M}$ by

$$\mathscr{P}(\mathscr{M}) = \left\{ \sum_{k=1}^{n} X_k^\dagger X_k; X_k \in \mathscr{M}(k = 1, 2, \dots, n), n \in N \right\},$$
$$\mathscr{M}_+ = \{X \in \mathscr{M}; X \geq 0\}.$$

The above positive cones $\mathscr{P}(\mathscr{M})$ and $\mathscr{M}_+$ are different in general [**22**, **25**], and so we need to define the notions of two types of weights as follows: A map φ of $\mathscr{P}(\mathscr{M})$ (resp. $\mathscr{M}_+$) into $\boldsymbol{R}_+ \cup \{+\infty\}$ is said to be a *weight* on $\mathscr{P}(\mathscr{M})$ (resp. $\mathscr{M}_+$) if

$$(\mathrm{W})_1 \qquad \varphi(A + B) = \varphi(A) + \varphi(B),$$
$$(\mathrm{W})_2 \qquad \varphi(\alpha A) = \alpha\varphi(A)$$

for all $A, B \in \mathscr{P}(\mathscr{M})$ (resp. $\mathscr{M}_+$) and $\alpha \geq 0$, where $0 \cdot (+\infty) = 0$. The first phenomenon arises for the GNS-construction of φ which is important for such a study: $\mathfrak{N}_\varphi^0 \equiv \{X \in \mathscr{M}; \varphi(X^\dagger X) < \infty\}$ is a left ideal of $\mathscr{M}$ in the bounded case, but it is not necessarily a left ideal of $\mathscr{M}$. For example, the condition $\varphi(I) < \infty$ doesn't necessarily imply $\varphi(X^\dagger X) < \infty$ for all $X \in \mathscr{M}$. So, using the left ideal $\mathfrak{N}_\varphi$ of $\mathscr{M}$ defined by

$$\mathfrak{N}_\varphi \equiv \{X \in \mathscr{M}; \varphi((AX)^\dagger(AX)) < \infty \text{ for all } A \in \mathscr{M}\},$$

we shall construct the GNS-representation π_φ on the similar method to positive linear functionals, that is, π_φ is a $*$-homomorphism of $\mathscr{M}$ onto the O^*-algebras $\pi_\varphi(\mathscr{M})$ on the dense subspace $\mathscr{D}(\pi_\varphi)$ in the Hilbert space $\mathscr{H}_\varphi$. However, there are non-zero weights φ such that $\mathfrak{N}_\varphi^0$ has many elements but $\mathfrak{N}_\varphi = \{0\}$ (Example 5.1, A) and so the GNS-construction for such a weight is meaningless. We don't treat with such a weight. The second phenomenon arises for the important examples (Example 5.1): For $\xi \in \mathscr{D}^*(\mathscr{M})(\equiv \bigcap_{X \in \mathscr{M}} \mathscr{D}(X^*))\backslash\mathscr{D}$ we put

$$\omega_\xi(X) = (X^{\dagger *}\xi \,|\, \xi), \quad X \in \mathscr{M}.$$

Then ω_ξ is a linear functional on $\mathscr{M}$, but it is not necessarily positive. For $\xi \in \mathscr{H}\backslash\mathscr{D}^*(\mathscr{M})$ even the definition of the above ω_ξ is impossible. Hence, we regard ω_ξ as the map of $\mathscr{P}(\mathfrak{N}_{\omega_\xi})$ into $\boldsymbol{R}_+$ satisfying $(\mathrm{W})_1$ and $(\mathrm{W})_2$ for $\mathscr{P}(\mathfrak{N}_{\omega_\xi})$, where $\mathfrak{N}_{\omega_\xi}$ is a left ideal of $\mathscr{M}$ defined by

$$\mathfrak{N}_{\omega_\xi} = \{X \in \mathscr{M}; \xi \in \mathscr{D}(X^{\dagger *}) \text{ and } X^{\dagger *}\xi \in \mathscr{D}\}.$$

So, we need to study such a map (called *quasi-weight*) which is strictly weaker than the notion of weights. A map φ of the positive cone $\mathscr{P}(\mathfrak{N}_\varphi)$ generated by a left ideal $\mathfrak{N}_\varphi$ of $\mathscr{M}$ into $\boldsymbol{R}_+$ is said to be a *quasi-weight* on $\mathscr{P}(\mathscr{M})$ if it satisfies the above conditions $(\mathrm{W})_1$ and $(\mathrm{W})_2$ for $\mathscr{P}(\mathfrak{N}_\varphi)$. We have felt that the study of quasi-weights is more useful than that of weights in case of O^*-algebras.

We shall give another important (quasi-)weight of a net $\{f_\alpha\}$ of positive linear functionals on $\mathscr{M}$. It is natural to consider whether $\sup_\alpha f_\alpha$ is a (quasi-)weight on $\mathscr{P}(\mathscr{M})$. We show that if $\{f_\alpha\}$ has a certain net property for $\mathscr{P}(\mathscr{M})$ (resp. $\mathscr{P}(\mathfrak{N}_\varphi)$) then $\sup_\alpha f_\alpha$ is a weight (resp. a quasi-weight) on $\mathscr{P}(\mathscr{M})$.

In Section 3 we shall define and study the notions of regularity and singularity for *(quasi-)weights* φ on $\mathscr{P}(\mathscr{M})$, and give the decomposition theorem of φ into the regular part φ_r and the singular part φ_s. A quasi-weight φ on $\mathscr{P}(\mathscr{M})$ is said to be *regular* if $\varphi = \sup_\alpha f_\alpha$ on $\mathscr{P}(\mathfrak{N}_\varphi)$ for some net $\{f_\alpha\}$ of positive linear functionals on $\mathscr{M}$, and φ is said to be singular if there doesn't exist any positive linear functional f on $\mathscr{M}$ such that $f(X^\dagger X) \le \varphi(X^\dagger X)$ for all $X \in \mathfrak{N}_\varphi$ and $f \ne 0$ on $\mathscr{P}(\mathfrak{N}_\varphi)$. Let φ be a quasi-weight on $\mathscr{P}(\mathscr{M})$ such that π_φ is a self-adjoint. Considering the trio-commutant $T(\varphi)'_c$ defined by

$$\begin{aligned} T(\varphi)'_c = \{K = (\pi'(K), \lambda'(K), \lambda'_*(K)) \in \pi_\varphi(\mathscr{M})'_{\mathrm{w}} \times \mathscr{D}(\pi_\varphi) \times \mathscr{D}(\pi_\varphi);\\ \pi'(K)\lambda_\varphi(X) = \pi_\varphi(X)\lambda'(K) \text{ and } \pi'(K)^*\lambda_\varphi(X) = \pi_\varphi(X)\lambda'_*(K), {}^\forall X \in \mathfrak{N}_\varphi\}, \end{aligned}$$

where $\pi_\varphi(\mathscr{M})'_{\mathrm{w}}$ is the weak commutant of the O^*-algebra $\pi_\varphi(\mathscr{M})$, we obtain that the following statements are equivalent:

$(\mathrm{R})_1$ $\qquad \varphi$ is regular.

$(\mathrm{R})_2$ $\qquad \varphi = \sup_\alpha(\omega_{\xi_\alpha} \circ \pi_\varphi) \quad \text{on } \mathscr{P}(\mathfrak{N}_\varphi)$

for some net $\{\xi_\alpha\}$ in $\mathscr{D}(\pi_\varphi)$, where for $\xi \in \mathscr{D}(\pi_\varphi)$ $\omega_\xi \circ \pi_\varphi$ is a positive linear functional on

$\mathcal{M}$ defined by $(\omega_\xi \circ \pi_\varphi)(X) = (\pi_\varphi(X)\xi \mid \xi)$, $X \in \mathcal{M}$.

$(\mathrm{R})_3$ There exists a net $\{K_\alpha\}$ in $T(\varphi)'_c$ such that $0 \leq \pi'(K_\alpha) \leq I$ for each α and $\pi'(K_\alpha) \longrightarrow I$ strongly.

Further, using this result, we show that φ is decomposed into $\varphi = \varphi_r + \varphi_s$, where φ_r is a regular quasi-weight on $\mathscr{P}(\mathcal{M})$ and φ_s is a singular quasi-weight on $\mathscr{P}(\mathcal{M})$.

Let φ be a weight on $\mathscr{P}(\mathcal{M})$. We shall consider when the above $(\mathrm{R})_1$ and $(\mathrm{R})_2$ hold *for all* $A \in \mathscr{P}(\mathcal{M})$, that is, when the following statements $(\mathrm{R})'_1$ and $(\mathrm{R})'_2$ hold:

$(\mathrm{R})'_1$ $$\varphi = \text{is regular} \quad \left(\text{iff } \varphi = \sup_\alpha f_\alpha \text{ on } \mathscr{P}(\mathcal{M})\right).$$

$(\mathrm{R})'_2$ $$\varphi = \sup_\alpha (\omega_{\xi_\alpha} \circ \pi_\varphi) \quad \text{on } \mathscr{P}(\mathcal{M}).$$

For this purpose we define the notions of semifiniteness and normality of φ. Suppose φ is a normal semifinite weight on $\mathscr{P}(\mathcal{M})$ such that π_φ is self-adjoint and normal. Then we obtain the result that the above five statements $(\mathrm{R})_1, (\mathrm{R})_2, (\mathrm{R})_3, (\mathrm{R})'_1$ and $(\mathrm{R})'_2$ are equivalent. Using this result, we show that φ is decomposed into $\varphi = \varphi_r + \varphi_s$, where φ_r is a regular weight on $\mathscr{P}(\mathcal{M})$ and φ_s is a singular weight on $\mathscr{P}(\mathcal{M})$.

In Section 4 we shall define and study an important class in regular (quasi-)weights which is possible to develop the Tomita-Takesaki theory in O^*-algebras. Let φ be a faithful (quasi-)weight on $\mathscr{P}(\mathcal{M})$ such that $\pi_\varphi(\mathcal{M})'_{\mathrm{w}}\mathscr{D}(\pi_\varphi) \subset \mathscr{D}(\pi_\varphi)$. Then, the map $\Lambda_\varphi : \pi_\varphi(X) \rightarrow \lambda_\varphi(X)$, $X \in \mathfrak{N}_\varphi$ is a generalized vector for the O^*-algebra $\pi_\varphi(\mathcal{M})$, that is, it is a linear map of the left ideal $D(\Lambda_\varphi) \equiv \pi_\varphi(\mathfrak{N}_\varphi)$ into $\mathscr{D}(\pi_\varphi)$ satisfying $\Lambda_\varphi(\pi_\varphi(A)\pi_\varphi(X)) = \pi_\varphi(A)\Lambda_\varphi(\pi_\varphi(X))$ for all $A \in \mathcal{M}$ and $X \in \mathfrak{N}_\varphi$. Using (quasi)-standard generalized vectors defined and studied in [**4**, **13**$\sim$**15**], we define the notion of (quasi-) standardness of φ as follows: φ is said to be standard (resp. quasi-standard) if the generalized vector Λ_φ is standard (resp. quasi-standard). And we obtain that if φ is standard, then the modular automorphism group $\{\sigma_t^\varphi\}_{t \in \boldsymbol{R}}$ of $\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^\dagger$ is defined and φ is a $\{\sigma_t^\varphi\}$-KMS (quasi-)weights, and if φ is quasi-standard, then it is extended to a standard quasi-weight $\bar{\varphi}$ on the positive cone $\mathscr{P}(\pi_\varphi(\mathcal{M})''_{\mathrm{wc}})$ of the generalized von Neumann algebra $\pi_\varphi(\mathcal{M})''_{\mathrm{wc}}$.

In Section 5 we shall give some concrete examples of regular (quasi-) weights, singular (quasi-) weights and standard (quasi-) weights. We first investigate the quasi-weights ω_ξ on $\mathscr{P}(\mathcal{M})$ defined by elements ξ in the Hilbert space. When is ω_ξ extended to a weight $\widetilde{\omega_\xi}$ on $\mathscr{P}(\mathcal{M})$ such that $\mathfrak{N}_{\widetilde{\omega_\xi}} = \mathfrak{N}_{\omega_\xi}$? We show that if $\mathcal{M}$ is commutative and integrable then the above question is affirmative. Further, we investigate the regularity, the singularity and the standardness of the quasi-weights ω_ξ. We next apply these results to three physical models, namely the unbounded CCR algebra, a class of interacting boson model in the Fock space and the BCS-Bogolubov model of superconductivity. And we give regular quasi-weights and standard quasi-weights for the relative models.

2. Weights and quasi-weights on O^*-algebras.

We first state some of definitions and the basic properties concerning O^*-algebras [**7**, **18**, **22**, **28**] and define the notions of quasi-weights and weights on O^*-algebras.

Let $\mathscr{D}$ be a dense subspace in a Hilbert space $\mathscr{H}$. We denote by $\mathscr{L}^\dagger(\mathscr{D})$ the set of all linear operators X from $\mathscr{D}$ into $\mathscr{D}$ such that $\mathscr{D}(X^*) \supset \mathscr{D}$ and $X^*\mathscr{D} \subset \mathscr{D}$. Then $\mathscr{L}^\dagger(\mathscr{D})$ is a $*$-algebra with the usual operations and the involution $X \to X^\dagger \equiv X^*\lceil\mathscr{D}$. A $*$-subalgebra of $\mathscr{L}^\dagger(\mathscr{D})$ is called an *O^*-algebra* on $\mathscr{D}$ in $\mathscr{H}$ according to the Schmüdgen book [**28**] though it is also called by an O_p^*-algebra in many papers. Throughout this paper we assume that an O^*-algebra has always an identity operator. Let $\mathscr{M}$ be an O^*-algebra on $\mathscr{D}$. The locally convex topology on $\mathscr{D}$ defined by the family $\{\|\ \|_X; X \in \mathscr{M}\}$ of seminorms: $\|\xi\|_X = \|X\xi\|$ $(\xi \in \mathscr{D})$ is called the *graph topology* on $\mathscr{D}$, which is denoted by $t_\mathscr{M}$. If the locally convex space $\mathscr{D}[t_\mathscr{M}]$ is complete, then $\mathscr{M}$ is said to be *closed*. We put

$$\tilde{\mathscr{D}}(\mathscr{M}) = \bigcap_{X\in\mathscr{M}} \mathscr{D}(\bar{X}) \quad \text{and} \quad \tilde{X} = \bar{X}\lceil\tilde{\mathscr{D}}(\mathscr{M})\ (X \in \mathscr{M}).$$

Then $\tilde{\mathscr{D}}(\mathscr{M})$ equals the completion of $\mathscr{D}[t_\mathscr{M}]$ and $\tilde{\mathscr{M}} \equiv \{\tilde{X}; X \in \mathscr{M}\}$ is a closed O^*-algebra on $\tilde{\mathscr{D}}(\mathscr{M})$ which is the smallest closed extension of $\mathscr{M}$ and it is called the *closure* of $\mathscr{M}$. Hence $\mathscr{M}$ is closed if and only if $\mathscr{D} = \tilde{\mathscr{D}}(\mathscr{M})$. If $\mathscr{D}^*(\mathscr{M}) \equiv \bigcap_{X\in\mathscr{M}} \mathscr{D}(X^*) = \tilde{\mathscr{D}}(\mathscr{M})$, then $\mathscr{M}$ is said to be *essentially self-adjoint*, and if $\mathscr{D}^*(\mathscr{M}) = \mathscr{D}$, then $\mathscr{M}$ is said to be *self-adjoint*. If $X^{\dagger *} = \bar{X}$ for each $X \in \mathscr{M}$, then $\mathscr{M}$ is said to be *integrable* (or *standard*). Clearly, the integrability of $\mathscr{M}$ implies the self-adjointness. We define the *weak commutant* $\mathscr{M}'_{\mathrm{w}}$ of a $\dagger$-invariant subset $\mathscr{M}$ of $\mathscr{L}^\dagger(\mathscr{D})$ as follows:

$$\mathscr{M}'_{\mathrm{w}} = \{C \in \mathscr{B}(\mathscr{H}); (CX\xi \mid \eta) = (C\xi \mid X^\dagger\eta) \text{ for each } \xi, \eta \in \mathscr{D} \text{ and } X \in \mathscr{M}\},$$

where $\mathscr{B}(\mathscr{H})$ is the set of all bounded linear operators on $\mathscr{H}$. Then $\mathscr{M}'_{\mathrm{w}}$ is a $*$-invariant weakly closed subspace of $\mathscr{B}(\mathscr{H})$, but it is not necessarily an algebra. Further, if $\mathscr{M}$ is self-adjoint, then $\mathscr{M}'_{\mathrm{w}}\mathscr{D} \subset \mathscr{D}$, and $\mathscr{M}'_{\mathrm{w}}\mathscr{D} \subset \mathscr{D}$ if and only if $\mathscr{M}'_{\mathrm{w}}$ is a von Neumann algebra and $\bar{X}$ is affiliated with $(\mathscr{M}'_{\mathrm{w}})'$ for each $X \in \mathscr{M}$. Let $\mathscr{M}$ be an O^*-algebra on $\mathscr{D}$ in $\mathscr{H}$. We call the locally convex topology defined by the family $\{P_{\xi,\eta}; \xi, \eta \in \mathscr{D}\}$ (resp. $\{P_\xi; \xi \in \mathscr{D}\}$; $\{P_\xi^*; \xi \in \mathscr{D}\}$) of seminorms;

$$P_{\xi,\eta}(X) = |(X\xi \mid \eta)| \ (\text{resp. } P_\xi(X) = \|X\xi\|;\ P_\xi^* = \|X\xi\| + \|X^\dagger\xi\|), \quad X \in \mathscr{M}$$

the weak topology (resp. strong topology; strong* topology) on $\mathscr{M}$ and denote it by t_{w} (resp. t_s; t_s^*). A closed O^*-algebra $\mathscr{M}$ on $\mathscr{D}$ in $\mathscr{H}$ is said to be a *generalized von Neumann algebra* on $\mathscr{D}$ if $\mathscr{M}'_{\mathrm{w}}\mathscr{D} \subset \mathscr{D}$ and $\mathscr{M} = \mathscr{M}''_{\mathrm{wc}} \equiv \{X \in \mathscr{L}^\dagger(\mathscr{D}); CX \subset XC, {}^\forall C \in \mathscr{M}'_{\mathrm{w}}\}$. It is known that $\mathscr{M}$ is a generalized von Neumann algebra on $\mathscr{D}$ if and only if $\mathscr{M}$ equals the strong*-closure of the O^*-algebra $(\mathscr{M}'_{\mathrm{w}})'\lceil\mathscr{D}$ on $\mathscr{D}$ in $\mathscr{L}^\dagger(\mathscr{D})$ [**14**]. A ($*$-)homomorphism π of a $*$-algebra $\mathscr{A}$ onto an O^*-algebra is said to a *($*$-)representation* of $\mathscr{A}$. A $*$-representation π of $\mathscr{A}$ is said to be *closed* (resp. *self-adjoint*) if the O^*-algebra $\pi(\mathscr{A})$ is closed (resp. self-adjoint). Let π be a $*$-representation of $\mathscr{A}$. We put

$$\mathscr{D}(\tilde{\pi}) = \bigcap_{x \in \mathscr{A}} \mathscr{D}(\overline{\pi(x)}), \quad \tilde{\pi}(x) = \overline{\pi(x)} \lceil \mathscr{D}(\tilde{\pi}),$$
$$\mathscr{D}(\pi^*) = \bigcap_{x \in \mathscr{A}} \mathscr{D}(\pi(x)^*), \quad \pi^*(x) = \pi(x^*)^* \lceil \mathscr{D}(\pi^*), \quad x \in \mathscr{A}.$$

Then $\tilde{\pi}$ is a closed $*$-representation of $\mathscr{A}$ such that $\tilde{\pi}(\mathscr{A}) = \widetilde{\pi(\mathscr{A})}$ and it is called the *closure* of π, and π^* is a closed representation of $\mathscr{A}$ and it is called the *adjoint* of π. A $*$-representation π of an O^*-algebra $\mathscr{M}$ is said to be *weakly continuous* (resp. *strongly continuous*) if it is continuous from the locally convex space $\mathscr{M}[t_w]$ (resp. $\mathscr{M}[t_s]$) onto the locally convex space $\pi(\mathscr{M})[t_w]$ (resp. $\pi(\mathscr{M})[t_s]$).

Throughout the rest of this section let $\mathscr{M}$ be a closed O^*-algebra on $\mathscr{D}$ in $\mathscr{H}$. For a subspace $\mathscr{N}$ of $\mathscr{M}$ we put

$$\mathscr{P}(\mathscr{N}) = \left\{ \sum_{k=1}^{n} X_k^\dagger X_k ; X_k \in \mathscr{N} \ (k = 1, 2, \ldots, n), \ n \in \boldsymbol{N} \right\}$$

and call it the positive cone generated by $\mathscr{N}$.

DEFINITION 2.1. A map φ of $\mathscr{P}(\mathscr{M})$ into $\boldsymbol{R}_+ \cup \{+\infty\}$ is said to be a *weight on* $\mathscr{P}(\mathscr{M})$ if

(i) $$\varphi(A + B) = \varphi(A) + \varphi(B), \quad A, B \in \mathscr{P}(\mathscr{M});$$

(ii) $$\varphi(\alpha A) = \alpha \varphi(A), \quad A \in \mathscr{P}(\mathscr{M}), \quad \alpha \geq 0,$$

where $0 \cdot (+\infty) = 0$. A map φ of the positive cone $\mathscr{P}(\mathfrak{N}_\varphi)$ generated by a left ideal $\mathfrak{N}_\varphi$ of $\mathscr{M}$ into $\boldsymbol{R}_+$ is said to be a quasi-weight on $\mathscr{P}(\mathscr{M})$ if it satisfies the above conditions (i) and (ii) for $\mathscr{P}(\mathfrak{N}_\varphi)$.

Let φ be a quasi-weight on $\mathscr{P}(\mathscr{M})$. We denote by $D(\varphi)$ the subspace of $\mathscr{M}$ generated by $\{X^\dagger X; X \in \mathfrak{N}_\varphi\}$. Since $\mathfrak{N}_\varphi$ is a left ideal of $\mathscr{M}$, we have

$$D(\varphi) = \text{the linear span of } \{Y^\dagger X; X, Y \in \mathfrak{N}_\varphi\},$$

and so each $\sum_k \alpha_k Y_k^\dagger X_k$ ($\alpha_k \in \boldsymbol{C}, X_k, Y_k \in \mathfrak{N}_\varphi$) is represented as $\sum_j \beta_j Z_j^\dagger Z_j$ for some $\beta_j \in \boldsymbol{C}$ and $Z_j \in \mathfrak{N}_\varphi$. Then we can define a linear functional on $D(\varphi)$ by

$$\sum_k \alpha_k Y_k^\dagger X_k \longrightarrow \sum_j \beta_j \varphi(Z_j^\dagger Z_j)$$

and write it by the same φ. It is easily shown that

(2.1) $$|\varphi(Y^\dagger X)|^2 \leq \varphi(Y^\dagger Y)\varphi(X^\dagger X), \quad X, Y \in \mathfrak{N}_\varphi.$$

We put

$$N_\varphi = \{X \in \mathfrak{N}_\varphi; \varphi(X^\dagger X) = 0\}, \quad \lambda_\varphi(X) = X + N_\varphi \in \mathfrak{N}_\varphi / N_\varphi, \quad X \in \mathfrak{N}_\varphi.$$

Then it follows from (2.1) that N_φ is a left ideal of $\mathfrak{N}_\varphi$ and $\lambda_\varphi(\mathfrak{N}_\varphi) \equiv \mathfrak{N}_\varphi / N_\varphi$ is a pre-

Hilbert space with the inner product

$$(\lambda_\varphi(X)|\lambda_\varphi(Y)) = \varphi(Y^\dagger X), \quad X, Y \in \mathfrak{N}_\varphi.$$

We denote by $\mathscr{H}_\varphi$ the Hilbert space obtained by the completion of the pre-Hilbert space $\lambda_\varphi(\mathfrak{N}_\varphi)$. We define a $*$-representation π_φ^0 of $\mathscr{M}$ by

$$\pi_\varphi^0(A)\lambda_\varphi(X) = \lambda_\varphi(AX), \quad A \in \mathscr{M}, X \in \mathfrak{N}_\varphi,$$

and denote by π_φ the closure of π_φ^0. We call the triple $(\pi_\varphi, \lambda_\varphi, \mathscr{H}_\varphi)$ the *GNS-construction* for φ. Let φ be a weight on $\mathscr{P}(\mathscr{M})$ and put

$$\mathfrak{N}_\varphi = \{X \in \mathscr{M}; \varphi((AX)^\dagger(AX)) < \infty \text{ for all } A \in \mathscr{M}\}.$$

Then $\mathfrak{N}_\varphi$ is a left ideal of $\mathscr{M}$ and the restriction $\varphi\lceil\mathscr{P}(\mathfrak{N}_\varphi)$ of φ to the positive cone $\mathscr{P}(\mathfrak{N}_\varphi)$ is a quasi-weight on $\mathscr{P}(\mathscr{M})$ and it is called the quasi-weight on $\mathscr{P}(\mathscr{M})$ generated by φ and is denoted by φ_q. We denote by $(\pi_\varphi, \lambda_\varphi, \mathscr{H}_\varphi)$ the GNS-construction for the quasi-weight φ_q generated by φ. We remark that even if $\varphi \neq 0$ the case of $\varphi_q = 0$ arises (Example 5.1, A), and so the GNS-construction for such a weight is meaningless. We don't treat with such a weight. We next define a weight by another positive cone $\mathscr{M}_+ = \{X \in \mathscr{M}; X \geq 0\}$.

Definition 2.2. A map φ of $\mathscr{M}_+$ into $\boldsymbol{R}_+ \cup \{+\infty\}$ is said to be a *weight on* $\mathscr{M}_+$ if

(i) $$\varphi(X+Y) = \varphi(X) + \varphi(Y), \quad X, Y \in \mathscr{M}_+$$

(ii) $$\varphi(\alpha X) = \alpha\varphi(X), \quad X \in \mathscr{M}_+, \alpha \geq 0.$$

A map φ of a hereditary positive subcone $D(\varphi)_+$ of $\mathscr{M}_+$ into $\boldsymbol{R}_+$ is said to be a quasi-weight on $\mathscr{M}_+$ if it satisfies the above conditions (i) and (ii) for $D(\varphi)_+$. A positive subcone $\mathscr{P}$ of $\mathscr{M}_+$ is said to be hereditary if any element X of $\mathscr{M}_+$ majorized by some element Y of $\mathscr{P}$ (that is, $X \leq Y$) belongs to $\mathscr{P}$.

It is clear that if φ is a weight on $\mathscr{M}_+$ then it is a weight on $\mathscr{P}(\mathscr{M})$. We denote by $\varphi\lceil\mathscr{P}(\mathscr{M})$ the restriction of φ to $\mathscr{P}(\mathscr{M})$. Suppose φ is a weight on $\mathscr{M}_+$. We define the finite part φ_q of φ by

$$D(\varphi_q)_+ = \{X \in \mathscr{M}_+; \varphi(X) < \infty\},$$

$$\varphi_q\left(\sum_k \alpha_k X_k\right) = \sum_k \alpha_k \varphi(X_k), \quad X_k \in D(\varphi_q)_+, \alpha_k \geq 0.$$

Then $D(\varphi_q)_+$ is a hereditary positive subcone of $\mathscr{M}_+$ and φ_q is a quasi-weight on $\mathscr{M}_+$. Suppose φ is a quasi-weight on $\mathscr{M}_+$. We put

$$\mathfrak{N}_\varphi = \{X \in \mathscr{M}; (AX)^\dagger(AX) \in D(\varphi)_+ \text{ for all } A \in \mathscr{M}\}.$$

Then $\mathfrak{N}_\varphi$ is a left ideal of $\mathscr{M}$ and the restriction of φ to $\mathscr{P}(\mathfrak{N}_\varphi)$ is a quasi-weight on

$\mathscr{P}(\mathscr{M})$. In fact, for each $X_1, X_2 \in \mathfrak{N}_\varphi$ and $A \in \mathscr{M}$ we have

$$\begin{aligned}(X_1 + X_2)^\dagger A^\dagger A(X_1 + X_2) + (X_1 - X_2)^\dagger A^\dagger A(X_1 - X_2) \\ = 2(X_1^\dagger A^\dagger A X_1 + X_2^\dagger A^\dagger A X_2) \in D(\varphi)_+,\end{aligned}$$

and since $D(\varphi)_+$ is a hereditary positive subcone of $\mathscr{M}_+$, it follows that $(X_1 + X_2)^\dagger A^\dagger A(X_1 + X_2) \in D(\varphi)_+$, that is, $X_1 + X_2 \in \mathfrak{N}_\varphi$. It is clear that $\alpha X, AX \in \mathfrak{N}_\varphi$ for all $\alpha \in \boldsymbol{C}$, $A \in \mathscr{M}$ and $X \in \mathfrak{N}_\varphi$. Thus, $\mathfrak{N}_\varphi$ is a left ideal of $\mathscr{M}$. Further, since $\mathscr{P}(\mathfrak{N}_\varphi) \subset D(\varphi)_+$, the restriction of φ to $\mathscr{P}(\mathfrak{N}_\varphi)$ is a quasi-weight on $\mathscr{P}(\mathscr{M})$. We denote by $\varphi\lceil\mathscr{P}(\mathscr{M})$ the quasi-weight φ on $\mathscr{M}_+$ regarding it as the quasi-weight on $\mathscr{P}(\mathscr{M})$. The following diagram holds:

$$\begin{tikzcd}
\underset{\text{weight on } \mathscr{M}_+}{\varphi} \arrow[rr] \arrow[d] & & \underset{\text{weight on } \mathscr{P}(\mathscr{M})}{\varphi\lceil\mathscr{P}(\mathscr{M})} \arrow[d] \\
\underset{\text{quasi-weight on } \mathscr{M}_+}{\varphi_q} \arrow[rr] & & \underset{\text{quasi-weight on } \mathscr{P}(\mathscr{M})}{\varphi_q\lceil\mathscr{P}(\mathscr{M}) = (\varphi\lceil\mathscr{P}(\mathscr{M}))_q}
\end{tikzcd}$$

The above equality $\varphi_q\lceil\mathscr{P}(\mathscr{M}) = (\varphi\lceil\mathscr{P}(\mathscr{M}))_q$ follows from

$$\mathfrak{N}_{\varphi_q\lceil\mathscr{P}(\mathscr{M})} = \mathfrak{N}_{\varphi_q} = \mathfrak{N}_\varphi = \mathfrak{N}_{\varphi\lceil\mathscr{P}(\mathscr{M})} = \mathfrak{N}_{(\varphi\lceil\mathscr{P}(\mathscr{M}))_q}.$$

This means that the GNS-constructions of all these (quasi-)weights coincide.

We give two kinds of important examples of weights and quasi-weights on $\mathscr{P}(\mathscr{M})$ or $\mathscr{M}_+$. We first give (quasi-)weights defined by vectors in $\mathscr{H}$. Let $\xi \in \mathscr{H}\backslash\mathscr{D}$. We put

$$\mathfrak{N}_{\omega_\xi} = \{X \in \mathscr{M}; \xi \in \mathscr{D}(X^{\dagger *}) \text{ and } X^{\dagger *}\xi \in \mathscr{D}\},$$

$$\omega_\xi\left(\sum_k X_k^\dagger X_k\right) = \sum_k \|X_k^{\dagger *}\xi\|^2, \quad X_k \in \mathfrak{N}_{\omega_\xi}.$$

Then ω_ξ is a quasi-weight on $\mathscr{P}(\mathscr{M})$. The following question arises: Is ω_ξ extended to a weight on $\mathscr{P}(\mathscr{M})$? In general, this question is inaffirmative, and so this is one of the reasons why we have to consider quasi-weights. In Section 5, we shall investigate such quasi-weights ω_ξ in more details.

We next give some (quasi-)weight defined by a net of positive linear functionals on $\mathscr{M}$. Let $\{f_\alpha\}$ be a net of positive linear functionals on $\mathscr{M}$. We put

$$\sup_\alpha f_\alpha : A \in \mathscr{P}(\mathscr{M}) \longrightarrow \sup_\alpha f_\alpha(A) \in [0, +\infty].$$

Then it is easily shown that

$$\begin{aligned}\max\left(\sup_\alpha f_\alpha(X^\dagger X), \sup_\alpha f_\alpha(Y^\dagger Y)\right) &\leq \sup_\alpha f_\alpha(X^\dagger X + Y^\dagger Y) \\ &\leq \sup_\alpha f_\alpha(X^\dagger X) + \sup_\alpha f_\alpha(Y^\dagger Y)\end{aligned} \tag{2.2}$$

for all $X, Y \in \mathcal{M}$. We define the finite part of $\sup_\alpha f_\alpha$ by

$$\mathfrak{N}^0_{\sup_\alpha f_\alpha} \equiv \left\{ X \in \mathcal{M}; \sup_\alpha f_\alpha(X^\dagger X) < \infty \right\}.$$

Since

$$(X+Y)^\dagger(X+Y) + (X-Y)^\dagger(X-Y) = 2(X^\dagger X + Y^\dagger Y)$$

for each $X, Y \in \mathfrak{N}^0_{\sup_\alpha f_\alpha}$, it follows that $\mathfrak{N}^0_{\sup_\alpha f_\alpha}$ is a subspace of $\mathcal{M}$. But, $(\sup_\alpha f_\alpha)(X^\dagger X + Y^\dagger Y) \neq \sup_\alpha f_\alpha(X^\dagger X) + \sup_\alpha f_\alpha(Y^\dagger Y)$ in general, and we have the following result:

LEMMA 2.3. *Let $\mathcal{N}$ be a subspace of $\mathfrak{N}^0_{\sup_\alpha f_\alpha}$. The following statements are equivalent.*

(1) $(\sup_\alpha f_\alpha)(A+B) = (\sup_\alpha f_\alpha)(A) + (\sup_\alpha f_\alpha)(B)$ *for all $A, B \in \mathcal{P}(\mathcal{N})$.*

(2) *For each finite subset $\{X_1, \dots, X_m\}$ of $\mathcal{N}$ there exists a subsequence $\{\alpha_n\}$ of $\{\alpha\}$ such that*

$$\lim_{n \mapsto \infty} f_{\alpha_n}(X_k^\dagger X_k) = \left(\sup_\alpha f_\alpha\right)(X_k^\dagger X_k), \quad k = 1, 2, \dots, m.$$

PROOF. (1) $\Rightarrow$ (2) Take an arbitrary $\{X_1, \dots, X_m\} \subset \mathcal{N}$. By (2.2), $(\sup_\alpha f_\alpha)\left(\sum_{k=1}^m X_k^\dagger X_k\right) < \infty$, and so there exists a subsequence $\{\alpha'_n\}$ of $\{\alpha\}$ such that

$$\lim_{n \mapsto \infty} f_{\alpha'_n}\left(\sum_{k=1}^m X_k^\dagger X_k\right) = \left(\sup_\alpha f_\alpha\right)\left(\sum_{k=1}^m X_k^\dagger X_k\right).$$

Since $\sup_n f_{\alpha'_n}(X_1^\dagger X_1) \le (\sup_\alpha f_\alpha)(\sum_{k=1}^m X_k^\dagger X_k) < \infty$, there exists a subsequence $\{\alpha''_n\}$ of $\{\alpha'_n\}$ such that

$$\lim_{n \mapsto \infty} f_{\alpha''_n}(X_1^\dagger X_1) = \sup_n f_{\alpha'_n}(X_1^\dagger X_1) \equiv \varphi(X_1^\dagger X_1).$$

Since $\{\alpha''_n\}$ is a subsequence of $\{\alpha'_n\}$, we have

$$\lim_{n \mapsto \infty} f_{\alpha''_n}\left(\sum_{k=1}^m X_k^\dagger X_k\right) = \left(\sup_\alpha f_\alpha\right)\left(\sum_{k=1}^m X_k^\dagger X_k\right), \quad \lim_{n \mapsto \infty} f_{\alpha''_n}(X_1^\dagger X_1) = \varphi(X_1^\dagger X_1).$$

Furthermore, since $\sup_n f_{\alpha''_n}(X_2^\dagger X_2) < \infty$, there exists a subsequence $f\{\alpha'''_n\}$ of $\{\alpha''_n\}$ such that

$$\begin{aligned} &\lim_{n \mapsto \infty} f_{\alpha'''_n}\left(\sum_{k=1}^m X_k^\dagger X_k\right) = \left(\sup_\alpha f_\alpha\right)\left(\sum_{k=1}^m X_k^\dagger X_k\right), \\ &\lim_{n \mapsto \infty} f_{\alpha'''_n}(X_1^\dagger X_1) = \varphi(X_1^\dagger X_1), \\ &\lim_{n \mapsto \infty} f_{\alpha'''_n}(X_2^\dagger X_2) = \left(\sup_n f_{\alpha''_n}\right)(X_2^\dagger X_2) \equiv \varphi(X_2^\dagger X_2). \end{aligned}$$

Repeating this argument, there exists a subsequence $\{\alpha_n\}$ of $\{\alpha\}$ such that

$$(2.3)\qquad \begin{aligned} &\lim_{n\mapsto\infty} f_{\alpha_n}\left(\sum_{k=1}^m X_k^\dagger X_k\right) = \left(\sup_\alpha f_\alpha\right)\left(\sum_{k=1}^m X_k^\dagger X_k\right), \\ &\lim_{n\mapsto\infty} f_{\alpha_n}(X_k^\dagger X_k) = \varphi(X_k^\dagger X_k), \quad k = 1,2,\ldots,m, \end{aligned}$$

which implies by the assumption (1) that

$$(2.4)\qquad \begin{aligned} \sum_{k=1}^m \varphi(X_k^\dagger X_k) &= \lim_{n\mapsto\infty} \sum_{k=1}^m f_{\alpha_n}(X_k^\dagger X_k) = \lim_{n\mapsto\infty} f_{\alpha_n}\left(\sum_{k=1}^m X_k^\dagger X_k\right) \\ &= \left(\sup_\alpha f_\alpha\right)\left(\sum_{k=1}^m (X_k^\dagger X_k)\right) \\ &= \sum_{k=1}^m \left(\sup_\alpha f_\alpha\right)(X_k^\dagger X_k). \end{aligned}$$

Since $0 \le \varphi(X_k^\dagger X_k) \le (\sup_\alpha f_\alpha)(X_k^\dagger X_k)$, $k = 1,2,\ldots,m$, it follows from (2.4) that $\varphi(X_k^\dagger X_k) = (\sup_\alpha f_\alpha)(X_k^\dagger X_k)$, $k = 1,2,\ldots,m$. Therefore, we have by (2.3)

$$\lim_{n\mapsto\infty} f_{\alpha_n}(X_k^\dagger X_k) = \left(\sup_\alpha f_\alpha\right)(X_k^\dagger X_k), \quad k = 1,2,\ldots,m.$$

(2) $\Rightarrow$ (1) Take an arbitrary subset $\{X_1, X_2, \ldots, X_m\}$ of $\mathcal{N}$. By the assumption (2) there exists a subsequence $\{\alpha_n\}$ of $\{\alpha\}$ such that

$$\lim_{n\mapsto\infty} f_{\alpha_n}(X_k^\dagger X_k) = \left(\sup_\alpha f_\alpha\right)(X_k^\dagger X_k), \quad k = 1,2,\ldots,m.$$

The statement (1) follows from

$$\begin{aligned} \left(\sup_\alpha f_\alpha\right)\left(\sum_{k=1}^m X_k^\dagger X_k\right) \le \sum_{k=1}^m \left(\sup_\alpha f_\alpha\right)(X_k^\dagger X_k) &= \lim_{n\mapsto\infty} \sum_{k=1}^m f_{\alpha_n}(X_k^\dagger X_k) \\ &= \lim_{n\mapsto\infty} f_{\alpha_n}\left(\sum_{k=1}^m X_k^\dagger X_k\right) \\ &\le \left(\sup_\alpha f_\alpha\right)\left(\sum_{k=1}^m X_k^\dagger X_k\right). \end{aligned}$$

When $\{f_\alpha\}$ satisfies the condition of Lemma 2.3, (2) we say that $\{f_\alpha\}$ *has the net property for* $\mathscr{P}(\mathcal{N})$ and then denote the restriction of the map $\sup_\alpha f_\alpha$ to $\mathscr{P}(\mathcal{N})$ by $\operatorname{Sup}_\alpha f_\alpha \lceil \mathscr{P}(\mathcal{N})$. In particular, when $\{f_\alpha\}$ has the net property for $\mathscr{P}(\mathfrak{N}^0_{\sup_\alpha f_\alpha})$, we simply say that $\{f_\alpha\}$ *has the net property* and then denote the map $\sup_\alpha f_\alpha$ by $\operatorname{Sup}_\alpha f_\alpha$. By Lemma 2.3 and (2.2) we have the following

PROPOSITION 2.4. *Let $\{f_\alpha\}$ be a net of positive linear functionals on $\mathcal{M}$. Suppose $\{f_\alpha\}$ has the net property for $\mathscr{P}(\mathscr{I})$, where $\mathscr{I}$ is a left ideal of $\mathcal{M}$ which is contained in*

$\mathfrak{N}^0_{\sup_\alpha f_\alpha}$. *Then* $\mathrm{Sup}_\alpha f_\alpha \lceil \mathscr{P}(\mathscr{I})$ *is a quasi-weight on* $\mathscr{P}(\mathscr{M})$. *Suppose* $\{f_\alpha\}$ *has the net property. Then* $\mathrm{Sup}_\alpha f_\alpha$ *is a weight on* $\mathscr{P}(\mathscr{M})$.

Let $\{f_\alpha\}$ be a net of *strongly positive* linear functionals on $\mathscr{M}$. A linear functional f on $\mathscr{M}$ is said to be *strongly positive* if $f(X) \geq 0$ for all $X \in \mathscr{M}_+$. We put

$$\sup_\alpha f_\alpha : X \in \mathscr{M}_+ \longrightarrow \sup_\alpha f_\alpha(X) \in [0, +\infty],$$
$$D\left(\sup_\alpha f_\alpha\right)_+ = \left\{X \in \mathscr{M}_+; \sup_\alpha f_\alpha(X) < \infty\right\}.$$

Then $D(\sup_\alpha f_\alpha)_+$ is a hereditary positive subcone of $\mathscr{M}_+$. Let $\mathscr{P}$ be a positive subcone of $D(\sup_\alpha f_\alpha)_+$. When $\{f_\alpha\}$ satisfies the condition of Lemma 2.3, (2) for $\mathscr{P}$, we say that $\{f_\alpha\}$ *has the net property for* $\mathscr{P}$ and then denote the restriction of the map $\sup_\alpha f_\alpha$ to $\mathscr{P}$ by $\mathrm{Sup}_\alpha f_\alpha \lceil \mathscr{P}$. In particular, when $\{f_\alpha\}$ has the net property for $D(\sup_\alpha f_\alpha)_+$, we simply say that $\{f_\alpha\}$ *has the net property* and then denote the map $\sup_\alpha f_\alpha$ by $\mathrm{Sup}_\alpha f_\alpha$. In similar to the proofs of Lemma 2.3 and Proposition 2.4 we can show the following result:

PROPOSITON 2.5. *Let* $\{f_\alpha\}$ *be a net of strongly positive linear functionals on* $\mathscr{M}$ *and* $\mathscr{P}$ *a hereditary positive subcone of* $D(\sup_\alpha f_\alpha)_+$. *Then* $\{f_\alpha\}$ *has the net property for* $\mathscr{P}$ *if and only if* $\mathrm{Sup}_\alpha f_\alpha \lceil \mathscr{P}$ *is a quasi-weight on* $\mathscr{M}_+$. *Further,* $\{f_\alpha\}$ *has the net property if and only if* $\mathrm{Sup}_\alpha f_\alpha$ *is a weight on* $\mathscr{M}_+$.

Throughout the rest of this paper we treat with only weights and quasi-weights on $\mathscr{P}(\mathscr{M})$.

3. The regularity of quasi-weights and weights.

In this section we define the notions of regularity and singularity of (quasi-)weights and give the decomposition theorem of (quasi-)weights into the regular part and the singular part. Let $\mathscr{M}$ be a closed O^*-algebra on $\mathscr{D}$ in $\mathscr{H}$.

DEFINITION 3.1. A quasi-weight φ on $\mathscr{P}(\mathscr{M})$ is said to be regular if $\varphi = \mathrm{Sup}_\alpha f_\alpha \lceil \mathscr{P}(\mathfrak{N}_\varphi) (= \sup_\alpha f_\alpha$ on $\mathscr{P}(\mathfrak{N}_\varphi)$ by Lemma 2.3) for some net $\{f_\alpha\}$ of positive linear functionals on $\mathscr{M}$, and it is said to be singular if there doesn't exist any positive linear functional f on $\mathscr{M}$ such that $f(X^\dagger X) \leq \varphi(X^\dagger X)$ for each $X \in \mathfrak{N}_\varphi$ and $f \neq 0$ on $\mathscr{P}(\mathfrak{N}_\varphi)$. A weight φ on $\mathscr{P}(\mathscr{M})$ is said to be regular if $\varphi = \mathrm{Sup}_\alpha f_\alpha (= \sup_\alpha f_\alpha$ on $\mathscr{P}(\mathscr{M})$ by Lemma 2.3) for some net $\{f_\alpha\}$ of positive linear functionals on $\mathscr{M}$, and φ is said to be quasi-regular if the quasi-weight φ_q on $\mathscr{P}(\mathscr{M})$ defined by φ is regular. If there doesn't exist any positive linear functional f on $\mathscr{M}$ such that $f(X^\dagger X) \leq \varphi(X^\dagger X)$ for all $X \in \mathscr{M}$ and $f \neq 0$ on $\mathscr{P}(\mathscr{M})$, then φ is said to be singular.

We define trio-commutants $T(\varphi)'_\delta$ and $T(\varphi)'_c$ for a quasi-weight φ which play an important rule for the regularity of φ as follows:

$$T(\varphi)'_{\delta} = \{K = (C, \xi, \eta); C \in \pi_{\varphi}(\mathscr{M})'_{\mathrm{w}}, \xi, \eta \in \mathscr{D}(\pi_{\varphi}^{*})$$
$$\text{s.t. } C\lambda_{\varphi}(X) = \pi_{\varphi}^{*}(X)\xi \text{ and } C^{*}\lambda_{\varphi}(X) = \pi_{\varphi}^{*}(X)\eta \text{ for all } X \in \mathfrak{N}_{\varphi}\},$$
$$T(\varphi)'_{c} = \{K = (C, \xi, \eta) \in T(\varphi)'_{\delta}; \xi, \eta \in \mathscr{D}(\pi_{\varphi})\}.$$

For $K = (C, \xi, \eta) \in T(\varphi)'_{\delta}$ we put

$$\pi'(K) = C, \quad \lambda'(K) = \xi, \quad \lambda'_{*}(K) = \eta.$$

We have the following

LEMMA 3.2. (1) *$T(\varphi)'_{\delta}$ is a $*$-invariant vector space under the following operations and the involution*:

$$K_1 + K_2 = (C_1 + C_2, \xi_1 + \xi_2, \eta_1 + \eta_2), \quad \alpha K = (\alpha C, \alpha\xi, \bar{\alpha}\eta),$$
$$K^{*} = (C^{*}, \eta, \xi)$$

for $K_1 = (C_1, \xi_1, \eta_1)$, $K_2 = (C_2, \xi_2, \eta_2)$ and $K = (C, \xi, \eta)$ in $T(\varphi)'_{\delta}$ and $\alpha \in C$.

(2) *$T(\varphi)'_{c}$ is a $*$-invariant subspace of $T(\varphi)'_{\delta}$. In particular, if $\pi_{\varphi}(\mathscr{M})'_{\mathrm{w}}\mathscr{D}(\pi_{\varphi}) \subset \mathscr{D}(\pi_{\varphi})$, then $T(\varphi)'_{c}$ is a $*$-algebra under the following multiplication*:

$$K_1 K_2 = (C_1 C_2, C_1\xi_2, C_2^{*}\eta_1)$$

for $K_1 = (C_1, \xi_1, \eta_1)$, $K_2 = (C_2, \xi_2, \eta_2) \in T(\varphi)'_{c}$, and π' is a $*$-homomorphism of $T(\varphi)'_{c}$ into the von Neumann algebra $\pi_{\varphi}(\mathscr{M})'_{\mathrm{w}}$ and λ' is a linear map of $T(\varphi)'_{c}$ into $\mathscr{D}(\pi_{\varphi})$ satisfying $\pi'(K_1)\lambda'(K_2) = \lambda'(K_1 K_2)$ for all $K_1, K_2 \in T(\varphi)'_{c}$.

LEMMA 3.3. *Let φ be a quasi-weight on $\mathscr{P}(\mathscr{M})$. Suppose a linear functional f on $\mathfrak{N}_{\varphi}$ satisfies the following conditions* (i) *and* (ii):

(i) *$0 \le f(X^{\dagger}X) \le \varphi(X^{\dagger}X)$ for each $X \in \mathfrak{N}_{\varphi}$.*

(ii) *For any $A \in \mathscr{M}$ there exists $\gamma_A > 0$ such that $|f(A^{\dagger}X)|^2 \le \gamma_A \varphi(X^{\dagger}X)$ for each $X \in \mathfrak{N}_{\varphi}$.*

Then there exists an element $K \in T(\varphi)'_{\delta}$ such that $0 \le \pi'(K) \le I$ and $f(X) = (\lambda_{\varphi}(X) | \lambda'(K))$ for all $X \in \mathfrak{N}_{\varphi}$. Conversely, for each $K \in T(\varphi)'_{\delta}$ with $0 \le \pi'(K) \le I$ we put

$$f(X) = (\lambda_{\varphi}(X) | \lambda'(K)), \quad X \in \mathfrak{N}_{\varphi}.$$

Then f is a linear functional on $\mathfrak{N}_{\varphi}$ satisfying the above (i) *and* (ii).

PROOF. Suppose f is a linear functional on $\mathfrak{N}_{\varphi}$ satisfying the conditions (i) and (ii). In similar to the GNS-construction for quasi-weights, we can define the GNS-construction $(\pi_f, \lambda_f, \mathscr{H}_f)$ for f. By (i) there exists a bounded linear transform C from $\mathscr{H}_{\varphi}$ to $\mathscr{H}_f$ such that $C\lambda_{\varphi}(X) = \lambda_f(X)$ for all $X \in \mathfrak{N}_{\varphi}$. Further, we have

$$C^{*}C \in \pi_{\varphi}(\mathscr{M})'_{\mathrm{w}} \quad \text{and} \quad f(Y^{\dagger}X) = (C^{*}C\lambda_{\varphi}(X) | \lambda_{\varphi}(Y)) \quad {}^{\forall}X, Y \in \mathfrak{N}_{\varphi}. \tag{3.1}$$

It follows from (ii) and the Riesz theorem that there exists an element ξ of $\mathscr{D}(\pi_{\varphi}^{*})$ such that

$$f(X)=(\lambda_\varphi(X)|\,\xi),\quad {}^\forall X\in\mathfrak{N}_\varphi,$$

which implies by (3.1) that

$$(\lambda_\varphi(Y)|\,\pi_\varphi^*(X)\xi)=f(X^\dagger Y)=(\lambda_\varphi(Y)|\,C^*C\lambda_\varphi(X))$$

for all $X,Y\in\mathfrak{N}_\varphi$, and so $C^*C\lambda_\varphi(X)=\pi_\varphi^*(X)\xi$ for all $X\in\mathfrak{N}_\varphi$. Hence, $K=(C^*C,\xi,\xi)\in T(\varphi)'_\delta$, $0\le\pi'(K)\le I$ and $f(X)=(\lambda_\varphi(X)|\,\lambda'(K))$ for all $X\in\mathfrak{N}_\varphi$.

We next show the converse. Take an arbitrary $K\in T(\varphi)'_\delta$ such that $0\le\pi'(K)\le I$. Then it is clear that f is a linear functional on $\mathfrak{N}_\varphi$ and further, since

$$f(X^\dagger X)=(\lambda_\varphi(X)|\,\pi_\varphi^*(X)\lambda'(K))=(\lambda_\varphi(X)|\,\pi'(K)\lambda_\varphi(X)),$$
$$f(A^\dagger X)=(\lambda_\varphi(X)|\,\pi_\varphi^*(A)\lambda'(K))$$

for all $X\in\mathfrak{N}_\varphi$ and $A\in\mathscr{M}$, it follows that f satisfies the conditions (i) and (ii).

REMARK 3.4. For $K\in T(\varphi)'_\delta$ the linear functional $\omega_{\lambda'(K)}\circ\pi_\varphi^*$ on $\mathscr{M}$ defined by

$$(\omega_{\lambda'(K)}\circ\pi_\varphi^*)(X)=(\pi_\varphi^*(X)\lambda'(K)|\,\lambda'(K)),\quad X\in\mathscr{M}$$

is not necessarily positive in case π_φ^* is not a $*$-representation of $\mathscr{M}$. When $K\in T(\varphi)'_c$ and $0\le\pi'(K)\le I$, $\pi_{\lambda'(K)}\circ\pi_\varphi$ is a positive linear functional on $\mathscr{M}$ satisfying

$$(\omega_{\lambda'(K)}\circ\pi_\varphi)(X^\dagger X)\le\varphi(X^\dagger X),\quad {}^\forall X\in\mathfrak{N}_\varphi.$$

But, the above inequality does not hold for all $X\in\mathscr{M}$ because the equality $\pi_\varphi(X)\lambda'(K)=\pi'(K)\lambda_\varphi(X)$ holds for each $X\in\mathfrak{N}_\varphi$ but this doesn't hold for $X\in\mathscr{M}\backslash\mathfrak{N}_\varphi$ in general.

For the regularity and the singularity of quasi-weights we have the following

THEOREM 3.5. *Let φ be a quasi-weight on $\mathscr{P}(\mathscr{M})$.*

I. *Consider the following statements:*

(1) *There exists a net $\{K_\alpha\}$ in $T(\varphi)'_c$ such that $0\le\pi'(K_\alpha)\le I$ for each α and $\pi'(K_\alpha)\to I$ strongly.*

(2) *$\varphi=\mathrm{Sup}_\alpha(\omega_{\xi_\alpha}\circ\pi_\varphi)\lceil\mathscr{P}(\mathfrak{N}_\varphi)$ for some net $\{\xi_\alpha\}$ in $\mathscr{D}(\pi_\varphi)$.*

(3) *φ is regular.*

(4) *There exists a net $\{K_\alpha\}$ in $T(\varphi)'_\delta$ such that $0\le\pi'(K_\alpha)\le I$ for each α and $\pi'(K_\alpha)\to I$ strongly.*

Then the implications $(1)\Rightarrow(2)\Rightarrow(3)\Rightarrow(4)$ hold. In particular, suppose π_φ is self-adjoint, then the statements $(1)\sim(4)$ are equivalent.

II. *Suppose π_φ is self-adjoint. Then φ is singular if and only if there doesn't exist any element K of $T(\varphi)'_c$ such that $\pi'(K)\ge 0$ and $\pi'(K)\ne 0$.*

PROOF. I. $(1)\Rightarrow(2)$ We put $\xi_\alpha=\lambda'(K_\alpha)$. Since

$$(\omega_{\xi_\alpha}\circ\pi_\varphi)(X^\dagger X)=\|\pi'(K_\alpha)\lambda_\varphi(X)\|^2$$

for each $X \in \mathfrak{N}_\varphi$ and α, and $\pi'(K_\alpha) \to I$ strongly, it follows that the net $\{\omega_{\xi_\alpha} \circ \pi_\varphi\}$ of positive linear functionals on $\mathscr{M}$ has the net property for $\mathscr{P}(\mathfrak{N}_\varphi)$ and $\varphi = \mathrm{Sup}_\alpha(\omega_{\xi_\alpha} \circ \pi_\varphi) \lceil \mathscr{P}(\mathfrak{N}_\varphi)$.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (4) This follows Lemma 3.3.

Suppose π_φ is self-adjoint. Then, $T(\varphi)'_\delta = T(\varphi)'_c$, and so the implication (4) $\Rightarrow$ (1) and the statement II follow from Lemma 3.3.

Similarly we have the following result for the regularity of weights:

THEOREM 3.6. *Let φ be a weight on $\mathscr{P}(\mathscr{M})$. Consider the following statements.*

(1) *$\varphi = \mathrm{Sup}_\alpha(\omega_{\xi_\alpha} \circ \pi_\varphi)$ for some net $\{\xi_\alpha\}$ in $\mathscr{D}(\pi_\varphi)$.*

(2) *φ is regular.*

(3) *φ is quasi-regular.*

(4) *There exists a net $\{K_\alpha\}$ in $T(\varphi)'_\delta$ such that $0 \leq \pi'(K_\alpha) \leq I$ for each α and $\pi'(K_\alpha) \longrightarrow I$ strongly.*

Then the following implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold.

Let φ be a weight on $\mathscr{P}(\mathscr{M})$. It follows from the definition of $T(\varphi)'_c$ that the equality

$$\pi_\varphi(X)\lambda'(K) = \pi'(K)\lambda_\varphi(X), \quad (X \in \mathfrak{N}_\varphi, K \in T(\varphi)'_c)$$

holds, but it doesn't hold for *all* $X \in \mathscr{M}$. For this reason, even if π_φ is self-adjoint, the quasi-regularity of φ doesn't necessarily imply the regularity of φ. So, we define the notions of normality and semifiniteness of φ to show the equivalence of the regularity and the quasi-regularity as follows:

DEFINITION 3.7. The symbol $X_\alpha^\dagger X_\alpha \uparrow X^\dagger X$ means that a net $\{X_\alpha\}$ in $\mathscr{M}$ and $X \in \mathscr{M}$ satisfy the following conditions:

(a) $X_\alpha^\dagger X_\alpha \preceq X_\beta^\dagger X_\beta$ (that is, $X_\beta^\dagger X_\beta - X_\alpha^\dagger X_\alpha \in \mathscr{P}(\mathscr{M})$) whenever $\alpha \leq \beta$;

(b) $X_\alpha^\dagger X_\alpha \preceq X^\dagger X, {}^\forall\alpha$;

(c) $\{X_\alpha^\dagger X_\alpha\}$ converges weakly to $X^\dagger X$.

A weight φ on $\mathscr{P}(\mathscr{M})$ is said to be normal if $\varphi(X_\alpha^\dagger X_\alpha) \uparrow \varphi(X^\dagger X)$ whenever $X_\alpha^\dagger X_\alpha \uparrow X^\dagger X$ ($\{X_\alpha\} \subset \mathscr{M}, X \in \mathscr{M}$), and φ is said to be semifinite if for each $X \in \mathscr{M}$ there exists a net $\{X_\alpha\}$ in $\mathfrak{N}_\varphi$ such that $X_\alpha^\dagger X_\alpha \uparrow X^\dagger X$.

A $*$-representation π of $\mathscr{M}$ is said to be normal if $\pi(X_\alpha^\dagger X_\alpha)$ converges weakly to $\pi(X^\dagger X)$ whenever $X_\alpha^\dagger X_\alpha \uparrow X^\dagger X(\{X_\alpha\} \subset \mathscr{M}, X \in \mathscr{M})$.

THEOREM 3.8. *Let φ be a semifinite normal weight on $\mathscr{P}(\mathscr{M})$. Suppose π_φ is self-adjoint and normal. Then the statements (1)$\sim$(4) in Theorem 3.6 are equivalent.*

PROOF. Suppose the statement (4) holds. Since π_φ is self-adjoint, we have $T(\varphi)'_\delta = T(\varphi)'_c$, and so $\lambda'(K_\alpha) \in \mathscr{D}(\pi_\varphi)$ for each α. For each α we put

$$f_\alpha = \omega_{\lambda'(K_\alpha)} \circ \pi_\varphi.$$

In similar to the proof of Theorem 3.5, we can show that f_α is a positive linear functional on $\mathscr{M}$ such that

$$\operatorname*{Sup}_{\alpha} f_\alpha\lceil\mathscr{P}(\mathfrak{N}_\varphi) = \varphi\lceil\mathscr{P}(\mathfrak{N}_\varphi). \tag{3.2}$$

We show that the statement (3.2) holds on $\mathscr{P}(\mathscr{M})$. Take an arbitrary $X \in \mathscr{M}$. By the semifiniteness of φ there exists a net $\{X_\lambda\}$ in $\mathfrak{N}_\varphi$ such that $X_\lambda^\dagger X_\lambda \uparrow X^\dagger X$, and since φ is normal and π_φ is normal, it follows that

$$\begin{aligned} f_\alpha(X^\dagger X) = \|\pi_\varphi(X)\lambda'(K_\alpha)\|^2 &= \lim_{\lambda} \|\pi_\varphi(X_\lambda)\lambda'(K_\alpha)\|^2 \\ &= \lim_{\lambda} \|\pi'(K_\alpha)\lambda_\varphi(X_\lambda)\|^2 \\ &\leq \lim_{\lambda} \varphi(X_\lambda^\dagger X_\lambda) \\ &= \varphi(X^\dagger X). \end{aligned} \tag{3.3}$$

Hence we have $\mathfrak{N}_\varphi^0 \equiv \{X \in \mathscr{M}; \varphi(X^\dagger X) < \infty\} \subset \mathfrak{N}^0_{\sup_\alpha f_\alpha}$. We show the converse inclusion. Suppose $X \notin \mathfrak{N}_\varphi^0$. By the semifiniteness of φ there exists a net $\{X_\lambda\}$ in $\mathfrak{N}_\varphi$ such that $X_\lambda^\dagger X_\lambda \uparrow X^\dagger X$, and it follows from the normality of φ and $\varphi(X^\dagger X) = +\infty$ that for each $\gamma > 0$ there exists an element λ_0 of $\{\lambda\}$ such that $X_{\lambda_0}^\dagger X_{\lambda_0} \preceq X^\dagger X$ and $\varphi(X_{\lambda_0}^\dagger X_{\lambda_0}) > \gamma$. By (3.2) there exists an element α_0 of $\{\alpha\}$ such that

$$\gamma < f_{\alpha_0}(X_{\lambda_0}^\dagger X_{\lambda_0}) \leq f_{\alpha_0}(X^\dagger X),$$

which implies $\sup_\alpha f_\alpha(X^\dagger X) = +\infty$. Hence we have

$$\mathfrak{N}_\varphi^0 = \mathfrak{N}^0_{\sup_\alpha f_\alpha}. \tag{3.4}$$

Take an arbitrary $\{X, Y\} \subset \mathfrak{N}^0_{\sup_\alpha f_\alpha}$. By the normality of φ we have

$$\varphi(X_\lambda^\dagger X_\lambda) \uparrow \varphi(X^\dagger X) \quad \text{and} \quad \varphi(Y_\mu^\dagger Y_\mu) \uparrow \varphi(Y^\dagger Y),$$

where $\{X_\lambda\}$ and $\{Y_\mu\}$ are nets in $\mathfrak{N}_\varphi$ such that $X_\lambda^\dagger X_\lambda \uparrow X^\dagger X$ and $Y_\mu^\dagger Y_\mu \uparrow Y^\dagger Y$. Since $\varphi(X^\dagger X) < \infty$ and $\varphi(Y^\dagger Y) < \infty$ by (3.4), it follows that for each $\varepsilon > 0$ there exist λ_0 and μ_0 such that

$$\varphi(X^\dagger X) - \varepsilon < \varphi(X_{\lambda_0}^\dagger X_{\lambda_0}), \quad \varphi(Y^\dagger Y) - \varepsilon < \varphi(Y_{\mu_0}^\dagger Y_{\mu_0}). \tag{3.5}$$

By (3.2), for X_{λ_0} and Y_{μ_0} there exists a subsequence $\{\alpha_n\}$ of $\{\alpha\}$ such that

$$\lim_{n\to\infty} f_{\alpha_n}(X_{\lambda_0}^\dagger X_{\lambda_0}) = \varphi(X_{\lambda_0}^\dagger X_{\lambda_0}), \quad \lim_{n\to\infty} f_{\alpha_n}(Y_{\mu_0}^\dagger Y_{\mu_0}) = \varphi(Y_{\mu_0}^\dagger Y_{\mu_0}). \tag{3.6}$$

Further, since $X_{\lambda_0}^\dagger X_{\lambda_0} \preceq X^\dagger X$, it follows that

$$f_{\alpha_n}(X_{\lambda_0}^\dagger X_{\lambda_0}) \leq f_{\alpha_n}(X^\dagger X), \quad n \in N,$$

which implies by (3.3), (3.5) and (3.6) that

$$\begin{aligned}\varphi(X^{\dagger}X)-\varepsilon<\lim_{n\to\infty}f_{\alpha_n}(X_{\lambda_0}^{\dagger}X_{\lambda_0})&\le\varliminf_{n\to\infty}f_{\alpha_n}(X^{\dagger}X)\\&\le\varlimsup_{n\to\infty}f_{\alpha_n}(X^{\dagger}X)\\&\le\varphi(X^{\dagger}X).\end{aligned}$$

Hence we have

$$\lim_{n\to\infty}f_{\alpha_n}(X^{\dagger}X)=\varphi(X^{\dagger}X). \tag{3.7}$$

Similarly we have

$$\lim_{n\to\infty}f_{\alpha_n}(Y^{\dagger}Y)=\varphi(Y^{\dagger}Y). \tag{3.8}$$

The same result as (3.7) and (3.8) holds for any finite subset $\{X_1,X_2,\dots,X_m\}$ of $\mathfrak{N}^0_{\sup_\alpha f_\alpha}$. Hence it follows from (3.4) and Lemma 2.3 that $\varphi=\mathrm{Sup}_\alpha f_\alpha=\mathrm{Sup}_\alpha(\omega_{\lambda'(K_\alpha)}\circ\pi_\varphi)$, and so the statement (1) holds. This completes the proof.

As the decomposition theorem of (quasi-)weights we have the following

THEOREM 3.9. (1) *Suppose φ is a quasi-weight on $\mathscr{P}(\mathscr{M})$ such that π_φ is self-adjoint. Then φ is decomposed into*

$$\varphi=\varphi_r+\varphi_s,$$

where φ_r is a regular quasi-weight on $\mathscr{P}(\mathscr{M})$ and φ_s is a singular quasi-weight on $\mathscr{P}(\mathscr{M})$ such that π_{φ_r} and π_{φ_s} are self-adjoint.

(2) *Suppose φ is a normal semifinite weight on $\mathscr{P}(\mathscr{M})$ such that π_φ is self-adjoint and normal. Then φ is decomposed into*

$$\varphi=\varphi_r+\varphi_s,$$

where φ_r is a normal semifinite regular weight on $\mathscr{P}(\mathscr{M})$ and φ_s is a normal semifinite singular weight on $\mathscr{P}(\mathscr{M})$ such that π_{φ_r} and π_{φ_s} are self-adjoint and normal.

PROOF. (1) We denote by P'_φ the projection from $\mathscr{H}_\varphi$ onto the closed subspace of $\mathscr{H}_\varphi$ generated by $\pi'(T(\varphi)'_c)\mathscr{H}_\varphi$. Then, $P'_\varphi\in\pi_\varphi(\mathscr{M})'_{\mathrm{w}}$ and there exists a net $\{K_\alpha\}$ in $T(\varphi)'_c$ such that $0\le\pi'(K_\alpha)\le P'_\varphi$ for each α and $\pi'(K_\alpha)\to P'_\varphi$ strongly. It is clear that the net $\{f_\alpha\equiv\omega_{\lambda'(K_\alpha)}\circ\pi_\varphi\}$ of positive linear functionals on $\mathscr{M}$ has the net property for $\mathscr{P}(\mathfrak{N}_\varphi)$, and so it follows from Lemma 2.3 that $\varphi_r\equiv\mathrm{Sup}_\alpha f_\alpha\lceil\mathscr{P}(\mathfrak{N}_\varphi)$ is a regular quasi-weight on $\mathscr{P}(\mathscr{M})$ such that $\mathfrak{N}_{\varphi_r}=\mathfrak{N}_\varphi$ and

$$\varphi_r(X^{\dagger}X)=\|P'_\varphi\lambda_\varphi(X)\|^2\quad\text{for each } X\in\mathfrak{N}_\varphi. \tag{3.9}$$

We put

$$\varphi_s=\varphi-\varphi_r.$$

Then φ_s is a quasi-weight on $\mathscr{P}(\mathscr{M})$ with $\mathfrak{N}_{\varphi_s}=\mathfrak{N}_\varphi$. It follows from (3.9) that π_{φ_r} (resp. π_{φ_s}) is unitarily equivalent to the induced representation $(\pi_\varphi)_{P'_\varphi}$ (resp. $(\pi_\varphi)_{I-P'_\varphi}$) of π_φ, so that π_{φ_r} and π_{φ_s} are self-adjoint. We show φ_s is singular. Suppose there exists a positive linear functional f on $\mathcal{M}$ such that $f(X^\dagger X) \leq \varphi_s(X^\dagger X)$ for all $X \in \mathfrak{N}_\varphi$ and $f(X_0^\dagger X_0) \neq 0$ for some $X_0 \in \mathfrak{N}_\varphi$. Since $\varphi_s \leq \varphi$, it follows from Lemma 3.3 that there exists an element K of $T(\varphi)'_c$ such that $0 \leq \pi'(K)$, $\pi'(K) \neq 0$ and $f(X) = (\lambda_\varphi(X) | \lambda'(K))$ for all $X \in \mathfrak{N}_\varphi$. Then we have

$$\begin{aligned} |(\pi'(K)\lambda_\varphi(X) | \lambda_\varphi(Y))|^2 &= |f(Y^\dagger X)|^2 \\ &\leq \gamma_Y \|(I - P'_\varphi)\lambda_\varphi(X)\|^2 \end{aligned}$$

for all $X, Y \in \mathfrak{N}_\varphi$, and so

$$\begin{aligned} |(\pi'(K)\lambda_\varphi(X) | \lambda_\varphi(Y))| &= |(P'_\varphi\lambda_\varphi(X) | \pi'(K)\lambda_\varphi(Y))| \\ &= \lim_{n\to\infty} |(\pi'(K)\lambda_\varphi(X_n) | \lambda_\varphi(Y))| \\ &\leq \gamma_Y \lim_{n\to\infty} \|(I - P'_\varphi)\lambda_\varphi(X_n)\|^2 \\ &= 0 \end{aligned}$$

for all $X, Y \in \mathfrak{N}_\varphi$, where $\{X_n\}$ is a sequence in $\mathfrak{N}_\varphi$ such that $\lim_{n\to\infty} \lambda_\varphi(X_n) = P'_\varphi\lambda_\varphi(X)$. Hence, $\pi'(K) = 0$, and so $f(X_0^\dagger X_0) = 0$. This is a contradiction. Hence, φ_s is singular.

(2) By the normality of π_φ, any $f_\alpha \equiv \omega_{\lambda'(K_\alpha)} \circ \pi_\varphi$ is a normal positive linear functional on $\mathcal{M}$. We put

$$\varphi_r(A) = \sup_\alpha f_\alpha(A), \quad A \in \mathscr{P}(\mathcal{M}).$$

By the proof of the above statement (1) we have

$$\varphi_r \lceil \mathscr{P}(\mathfrak{N}_\varphi) = \operatorname*{Sup}_\alpha f_\alpha \lceil \mathscr{P}(\mathfrak{N}_\varphi), \tag{3.10}$$

$$f_\alpha(X^\dagger X) \leq \varphi(X^\dagger X), \quad {}^\forall X \in \mathcal{M}, \tag{3.11}$$

$$\varphi_r(X^\dagger X) = \|P'_\varphi\lambda_\varphi(X)\|^2, \quad {}^\forall X \in \mathfrak{N}_\varphi. \tag{3.12}$$

Further, by the normality of each f_α we have

$$f_\alpha(X^\dagger X) = \lim_\lambda f_\alpha(X_\lambda^\dagger X_\lambda) \leq \lim_\lambda \varphi_r(X_\lambda^\dagger X_\lambda)$$

whenever $X_\lambda^\dagger X_\lambda \uparrow X^\dagger X (\{X_\lambda\} \subset \mathcal{M}, X \in \mathcal{M})$, which implies

$$\varphi_r(X_\lambda^\dagger X_\lambda) \uparrow \varphi_r(X^\dagger X). \tag{3.13}$$

The statements (3.10)$\sim$(3.13) imply by the same proof as in Theorem 3.8 that $\varphi_r = \operatorname{Sup}_\alpha f_\alpha$ and it is a regular normal weight on $\mathscr{P}(\mathcal{M})$. By (3.11) we have

$$\varphi_r(X^\dagger X) \leq \varphi(X^\dagger X), \quad {}^\forall X \in \mathcal{M},$$

and so we put

$$\varphi_s(X^\dagger X) = \begin{cases} \varphi(X^\dagger X) - \varphi_r(X^\dagger X) & \text{if } X \in \mathfrak{N}_\varphi^0, \\ \infty & \text{if otherwise.} \end{cases}$$

Then φ_s is a weight on $\mathscr{P}(\mathscr{M})$ and $\varphi = \varphi_r + \varphi_s$. In similar to the proof of the singularity of φ_s in the statement (1), we can show that φ_s is singular. Since $\varphi_r \leq \varphi$ and $\varphi_s \leq \varphi$, it is easily shown that φ_r and φ_s are normal and semifinite. We finally show that π_{φ_r} and $(\pi_\varphi)_{P'_\varphi}$ are unitarily equivalent, and π_{φ_s} and $(\pi_\varphi)_{1-P'_\varphi}$ are unitarily equivalent. Since $\mathfrak{N}_\varphi \subset \mathfrak{N}_{\varphi_r}$, it follows from (3.12) that $(\pi_\varphi)_{P'_\varphi} \subset \pi_{\varphi_r}$ unitarily, which implies by the self-adjointness of $(\pi_\varphi)_{P'_\varphi}$ that $(\pi_\varphi)_{P'_\varphi} \cong \pi_{\varphi_r}$. Similarly, we have $(\pi_\varphi)_{1-P'_\varphi} \cong \pi_{\varphi_s}$. Hence, π_{φ_r} and π_{φ_s} are self-adjoint and normal. This completes the proof.

4. Standard weights.

In this section we define and study an important class in regular (quasi-)weights which is possible to develop the Tomita-Takesaki theory in O^*-algebras. Let $\mathscr{M}$ be a closed O^*-algebra on $\mathscr{D}$ in $\mathscr{H}$. A quasi-weight φ (resp. weight) on $\mathscr{P}(\mathscr{M})$ is said to be *faithful* if $\varphi(X^\dagger X) = 0, X \in \mathfrak{N}_\varphi$ (resp. $X \in \mathscr{M}$) implies $X = 0$, and φ is said to be *semifinite* if for each $A \in \mathscr{M}$ there exists a net $\{X_\alpha\}$ in $\mathfrak{N}_\varphi$ such that $X_\alpha^\dagger X_\alpha \uparrow A^\dagger A$. It is easily shown that if a quasi-weight φ is faithful, then $\pi_\varphi(X) = 0$, $X \in \mathfrak{N}_\varphi$ implies $X = 0$, and if φ is faithful and semifinite, then π_φ is a $*$-isomorphism of the O^*-algebra $\mathscr{M}$ onto the O^*-algebra $\pi_\varphi(\mathscr{M})$. Let φ be a faithful quasi-weight on $\mathscr{P}(\mathscr{M})$. We put

$$\Lambda_\varphi(\pi_\varphi(X)) = \lambda_\varphi(X), \quad X \in \mathfrak{N}_\varphi.$$

Then Λ_φ is a generalized vector for the O^*-algebra $\pi_\varphi(\mathscr{M})$, that is, it is a linear map of the left ideal $D(\Lambda_\varphi) = \pi_\varphi(\mathfrak{N}_\varphi)$ into $\mathscr{D}(\pi_\varphi)$ and

$$\Lambda_\varphi(\pi_\varphi(A)\pi_\varphi(X)) = \pi_\varphi(A)\Lambda_\varphi(\pi_\varphi(X))$$

for all $A \in \mathscr{M}$ and $X \in \mathfrak{N}_\varphi$. This Λ_φ is called the *generalized vector induced by φ*. Suppose

(S)$_1$ $\quad \pi_\varphi(\mathscr{M})'_{\mathrm{w}}\mathscr{D}(\pi_\varphi) \subset \mathscr{D}(\pi_\varphi)$,

(S)$_2$ $\quad \Lambda_\varphi((D(\Lambda_\varphi) \cap D(\Lambda_\varphi)^\dagger)^2)$ $(= \lambda_\varphi((\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^\dagger)^2))$ is total in $\mathscr{H}_\varphi$.

Then we can define the commutant Λ_φ^c of Λ_φ which is a generalized vector for the von Neumann algebra $\pi_\varphi(\mathscr{M})'_{\mathrm{w}}$ as follows:

$$D(\Lambda_\varphi^c) = \{K \in \pi_\varphi(\mathscr{M})'_{\mathrm{w}};\ {}^\exists\xi_K \in \mathscr{D}(\pi_\varphi) \text{ s.t. } K\Lambda_\varphi(X) = X\xi_K \text{ for all } X \in D(\Lambda_\varphi)\},$$
$$\Lambda_\varphi^c(K) = \xi_K, \quad K \in D(\Lambda_\varphi^c).$$

We have

$$T(\varphi)'_c = \{(K, \Lambda_\varphi^c(K), \Lambda_\varphi^c(K^*));\ K \in D(\Lambda_\varphi^c) \cap D(\Lambda_\varphi^c)^*\}. \tag{4.1}$$

In general, $T(\varphi)'_c$ is always defined, but the commutant Λ_φ^c of the generalized vector Λ_φ

is not necessarily defined without the condition $(S)_2$. Further, suppose

$$(S)_3 \qquad \Lambda_\varphi^c((D(\Lambda_\varphi^c)\cap D(\Lambda_\varphi^c)^*)^2) \text{ is total in } \mathscr{H}_\varphi.$$

We remark that φ is regular by (4.1) and $(S)_3$. We put

$$D(\Lambda_\varphi^{cc}) = \{A \in (\pi_\varphi(\mathscr{M})'_{\mathrm{w}})'; \ {}^\exists\xi_A \in \mathscr{H}_\varphi \text{ s.t. } A\Lambda_\varphi^c(K) = K\xi_A \text{ for all } K \in D(\Lambda_\varphi^c)\},$$
$$\Lambda_\varphi^{cc}(A) = \xi_A, \quad A \in D(\Lambda_\varphi^{cc}).$$

Then Λ_φ^{cc} is a generalized vector for the von Neumann algebra $(\pi_\varphi(\mathscr{M})'_{\mathrm{w}})'$ such that $\Lambda_\varphi^{cc}((D(\Lambda_\varphi^{cc})\cap D(\Lambda_\varphi^{cc})^*)^2)$ is total in $\mathscr{H}_\varphi$, and so the maps $\lambda_\varphi(X) \to \lambda_\varphi(X^\dagger)$, $X \in D(\Lambda_\varphi)\cap D(\Lambda_\varphi)^+ (= \pi_\varphi(\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^\dagger))$ and $\Lambda_\varphi^{cc}(A) \to \Lambda_\varphi^{cc}(A^*)$, $A \in D(\Lambda_\varphi^{cc})\cap D(\Lambda_\varphi^{cc})^*$ are closable in $\mathscr{H}_\varphi$ and their closures are denoted by S_φ and $S_{\Lambda_\varphi^{cc}}$, respectively. Let $S_\varphi = J_\varphi\Delta_\varphi^{1/2}$ and $S_{\Lambda_\varphi^{cc}} = J_{\Lambda_\varphi^{cc}}\Delta_{\Lambda_\varphi^{cc}}^{1/2}$ be the polar decompositions of S_φ and $S_{\Lambda_\varphi^{cc}}$, respectively. Then we see that $S_\varphi \subset S_{\Lambda_\varphi^{cc}}$, and $J_{\Lambda_\varphi^{cc}}(\pi_\varphi(\mathscr{M})'_{\mathrm{w}})'J_{\Lambda_\varphi^{cc}} = \pi_\varphi(\mathscr{M})'_{\mathrm{w}}$ and $\Delta_{\Lambda_\varphi^{cc}}^{it}(\pi_\varphi(\mathscr{M})'_{\mathrm{w}})'\Delta_{\Lambda_\varphi^{cc}}^{-it} = (\pi_\varphi(\mathscr{M})'_{\mathrm{w}})'$ for all $t \in \boldsymbol{R}$ by the Tomita fundamental theorem. But, we don't know how the unitary group $\{\Delta_{\Lambda_\varphi^{cc}}^{it}\}_{t\in\boldsymbol{R}}$ acts on the O^*-algebra $\pi_\varphi(\mathscr{M})$, and so we define a system which has the best condition:

DEFINITION 4.1. A faithful (quasi-)weight φ on $\mathscr{P}(\mathscr{M})$ is said to be quasi-standard if the following conditions (i) and (ii) hold:

(i) The above conditions $(S)_1$, $(S)_2$ and $(S)_3$ hold.

(ii) $\Delta_{\Lambda_\varphi^{cc}}^{it}\mathscr{D}(\pi_\varphi) \subset \mathscr{D}(\pi_\varphi)$ for each $t \in \boldsymbol{R}$.

Further, if

(iii) $\Delta_{\Lambda_\varphi^{cc}}^{it}\pi_\varphi(\mathscr{M})\Delta_{\Lambda_\varphi^{cc}}^{-it} = \pi_\varphi(\mathscr{M})$ for each $t \in \boldsymbol{R}$,

(iv) $\Delta_{\Lambda_\varphi^{cc}}^{it}(D(\Lambda_\varphi)\cap D(\Lambda_\varphi)^\dagger)\Delta_{\Lambda_\varphi^{cc}}^{-it} = D(\Lambda_\varphi)\cap D(\Lambda_\varphi)^\dagger$ for each $t \in \boldsymbol{R}$,

then φ is said to be standard.

THEOREM 4.2. *Let φ be a faithful (quasi-)weight on $\mathscr{P}(\mathscr{M})$. Suppose φ is a standard. Then the following statements hold:*

(1) *$S_\varphi = S_{\Lambda_\varphi^{cc}}$, and so $J_\varphi = J_{\Lambda_\varphi^{cc}}$ and $\Delta_\varphi = \Delta_{\Lambda_\varphi^{cc}}$.*

(2) *These exists a one-parameter group $\{\sigma_t^\varphi\}_{t\in\boldsymbol{R}}$ of $*$-automorphisms of $\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^\dagger$ such that $\pi_\varphi(\sigma_t^\varphi(X)) = \Delta_\varphi^{it}\pi_\varphi(X)\Delta_\varphi^{-it}$ for all $X \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^\dagger$ and $t \in \boldsymbol{R}$.*
Suppose π_φ is a $*$-isomorphism of $\mathscr{M}$ (for example, φ is semifinite). Then $\{\sigma_t^\varphi\}_{t\in\boldsymbol{R}}$ is a one-parameter group of $*$-automorphisms of $\mathscr{M}$.

(3) *φ is a $\{\sigma_t^\varphi\}$-KMS (quasi-)weight, that is, for each $X, Y \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^\dagger$ there exists an element $f_{X,Y}$ of $A(0.1)$ such that*

$$f_{X,Y}(t) = \varphi(Y\sigma_t^\varphi(X)) \equiv (\lambda_\varphi(\sigma_t^\varphi(X))|\lambda_\varphi(Y^\dagger)) \quad \textit{and} \quad f_{X,Y}(t+i) = \varphi(\sigma_t^\varphi(X)Y)$$

for all $t \in \boldsymbol{R}$, where $A(0,1)$ is the set of all complex-valued functions, bounded and continous on $0 \leq I_m z \leq 1$ and analytic in the interior.

PROOF. The standardness of φ implies that of the generalized vector Λ_φ. Hence, this theorem follows from ([**14**] Theorem 2.5).

We next consider quasi-standard (quasi-)weights. We first need a natural extension of a regular quasi-weight φ to the generalized von Neumann algebra $\pi_\varphi(\mathcal{M})''_{\mathrm{wc}}$. Let φ be a faithful regular (quasi-)weight on $\mathscr{P}(\mathcal{M})$ satisfying the conditions $(\mathrm{S})_1$, and $(\mathrm{S})_2$. We put

$$D(\overline{\Lambda_\varphi}) = \{A \in \pi_\varphi(\mathcal{M})''_{\mathrm{wc}};\ ^\exists\xi_A \in \mathscr{D}(\pi_\varphi) \text{ s.t. } A\Lambda_\varphi{}^c(K) = K\xi_A \text{ for all } K \in D(\Lambda_\varphi{}^c)\},$$

$$\overline{\Lambda_\varphi}(A) = \xi_A, \quad A \in D(\overline{\Lambda_\varphi}).$$

Then it is easily shown that $\overline{\Lambda_\varphi}$ is a generalized vector for $\pi_\varphi(\mathcal{M})''_{\mathrm{wc}}$ such that

$$\Lambda_\varphi \subset \overline{\Lambda_\varphi} \quad \text{and} \quad \Lambda_\varphi^c = \overline{\Lambda}_\varphi{}^c. \tag{4.2}$$

We now put

$$\bar{\varphi}\left(\sum_k X_k^\dagger X_k\right) = \sum_k \|\overline{\Lambda_\varphi}(X_k)\|^2, \quad \{X_k\} \subset D(\overline{\Lambda_\varphi}).$$

Then $\bar{\varphi}$ is a faithful regular quasi-weight on $\mathscr{P}(\pi_\varphi(\mathcal{M})''_{\mathrm{wc}})$ such that

$$(\pi_{\bar{\varphi}}(\pi_\varphi(\mathcal{M})''_{\mathrm{wc}}), \lambda_{\bar{\varphi}}) \text{ is unitarily equivalent to } (\pi_\varphi(\mathcal{M})''_{\mathrm{wc}}, \overline{\Lambda_\varphi}), \tag{4.3}$$

that is, there exists a unitary operator U on $\mathscr{H}_\varphi$ onto $\mathscr{H}_{\bar{\varphi}}$ such that $UD(\overline{\Lambda_\varphi}) = \mathfrak{N}_{\bar{\varphi}}$, $U\overline{\Lambda_\varphi}(X) = \lambda_{\bar{\varphi}}(X)$ for each $X \in D(\overline{\Lambda_\varphi})$ and $\pi_{\bar{\varphi}}(A) = UAU^*$ for each $A \in \pi_\varphi(\mathcal{M})''_{\mathrm{wc}}$. The above $\bar{\varphi}$ is said to be the *quasi-weight on* $\mathscr{P}(\pi_\varphi(\mathcal{M})''_{\mathrm{wc}})$ *induced by* φ. By (4.2), (4.3) and Theorem 4.2 we have the following

THEOREM 4.3. *Suppose φ is a faithful quasi-standard (quasi-)weight on $\mathscr{P}(\mathcal{M})$. Then the quasi-weight $\bar{\varphi}$ on $\mathscr{P}(\pi_\varphi(\mathcal{M})''_{\mathrm{wc}})$ induced by φ is standard, and so it is a $\{\sigma_t^{\bar{\varphi}}\}_{t\in\boldsymbol{R}}$-KMS quasi-weight on $\mathscr{P}(\pi_\varphi(\mathcal{M})''_{\mathrm{wc}})$, where $\sigma_t^{\bar{\varphi}}(A) = \Delta_{\Lambda_\varphi^{cc}}^{it} A \Delta_{\Lambda_\varphi^{cc}}^{-it}$, $A \in \pi_\varphi(\mathcal{M})''_{\mathrm{wc}}, t \in \boldsymbol{R}$.*

In [**14**] we have defined and studied standard generalized vectors which are possible to generalize the Tomita-Takesaki theory (in particular, the Connes cocycle theorem and the Pedersen-Takesaki Radon-Nikodym theorem etc.) to generalized von Neumann algebras. As the notion of generalized vectors is spatial, such a generalization is possible to a certain extent, but the notion of (quasi-)weights is purely algebraic and not spatial and the algebraic properties don't reflect to the topological properties in general (for example, $\pi_\varphi(\mathcal{M})$ is not necessarily a generalized von Neumann algebra when $\mathcal{M}$ is a generalized von Neumann algebra), and so the generalizations of the Tomita-Takesaki theory for standard (quasi-)weights have some difficult problems.

5. Examples.

In this section we give some examples of regular (quasi-)weights, singular (quasi-) weights and standard (quasi-)weights. We first investigate the regularity, the singularity and the standardness of the quasi-weights ω_ξ defined by elements ξ of the Hilbert space.

EXAMPLE 5.1. Let $\mathscr{M}$ be a closed O^*-algebra on $\mathscr{D}$ in $\mathscr{H}$ and put

$$\mathscr{D}^*(\mathscr{M}) = \bigcap_{X \in \mathscr{M}} \mathscr{D}(X^*) \quad \text{and} \quad \mathscr{D}^{**}(\mathscr{M}) = \bigcap_{X \in \mathscr{M}} \mathscr{D}((X^* \lceil \mathscr{D}^*(\mathscr{M}))^*).$$

Suppose $\xi \in \mathscr{D}^{**}(\mathscr{M})$ and put

$$\omega_\xi(X) = (X^{\dagger *}\xi \mid \xi), \quad X \in \mathscr{M}.$$

Then ω_ξ is a positive linear functional on $\mathscr{M}$. If $\xi \in \mathscr{D}^*(\mathscr{M}) - \mathscr{D}^{**}(\mathscr{M})$, then ω_ξ is a linear functional on $\mathscr{M}$, but it is not necessarily positive. If $\xi \notin \mathscr{D}^*(\mathscr{M})$, then ω_ξ is not defined, and so we regard ω_ξ as the quasi-weight on $\mathscr{P}(\mathscr{M})$ as follows:

$$\mathfrak{N}_{\omega_\xi} = \{X \in \mathscr{M}; \xi \in \mathscr{D}(X^{\dagger *}) \text{ and } X^{\dagger *}\xi \in \mathscr{D}\},$$
$$\omega_\xi(X^\dagger X) = \|X^{\dagger *}\xi\|^2, \quad X \in \mathfrak{N}_{\omega_\xi}.$$

We here investigate such quasi-weights ω_ξ ($\xi \notin \mathscr{D}^*(\mathscr{M})$) on $\mathscr{P}(\mathscr{M})$ in details.

A. *The extension of ω_ξ to a weight*

Let $\mathscr{M}$ be a commutative integrable O^*-algebra on $\mathscr{D}$ in $\mathscr{H}$ and $\xi \in \mathscr{H} - \mathscr{D}$. We put

$$\widetilde{\omega_\xi}\left(\sum_k X_k^\dagger X_k\right) = \begin{cases} \left(\overline{\sum_k X_k^\dagger X_k}\xi \mid \xi\right) & \text{if } \xi \in \mathscr{D}\left(\overline{\sum_k X_k^\dagger X_k}\right) \\ \infty & \text{if otherwise.} \end{cases}$$

Then $\widetilde{\omega_\xi}$ is a weight on $\mathscr{P}(\mathscr{M})$ such that

$$\mathfrak{N}^0_{\widetilde{\omega_\xi}} \equiv \{X \in \mathscr{M}; \widetilde{\omega_\xi}(X^\dagger X) < \infty\} = \{X \in \mathscr{M}; \xi \in \mathscr{D}(X^*\overline{X})\},$$
$$\mathfrak{N}_{\widetilde{\omega_\xi}} \equiv \{X \in \mathscr{M}; AX \in \mathfrak{N}^0_{\widetilde{\omega_\xi}}, {}^\forall A \in \mathscr{M}\}$$
$$= \mathfrak{N}_{\omega_\xi}$$

and it is an extension of ω_ξ. In fact, since $\mathscr{M}$ is commutative and integrable, it follows that $\xi \in \mathscr{D}\left(\overline{\sum_k X_k^\dagger X_k}\right)$ if and only if there exists a sequence $\{\xi_n\}$ in $\mathscr{D}$ such that $\xi_n \to \xi$ and $\{X_k\xi_n\}$ and $\{X_k^\dagger X_k\xi_n\}$ are Cauchy sequences in $\mathscr{H}$ for each k if and only if $\xi \in \mathscr{D}(\overline{X_k^\dagger X_k}) = \mathscr{D}(X_k^*\overline{X_k})$ for each k, and then $\overline{\sum_k X_k^\dagger X_k}\xi = \sum_k X_k^*\overline{X}_k\xi$, which implies that $\widetilde{\omega_\xi}$ is a weight on $\mathscr{P}(\mathscr{M})$. It is easy to show that $\mathfrak{N}_{\widetilde{\omega_\xi}} = \mathfrak{N}_{\omega_\xi}$ and $\widetilde{\omega_\xi}$ is an extension of ω_ξ. We give a concrete example. Let H be a positive self-adjoint unbounded operator in $\mathscr{H}, \mathscr{D}^\infty(H) \equiv \bigcap_{n \in N} \mathscr{D}(H^n)$ and $H_0 \equiv H \lceil \mathscr{D}^\infty(H)$. Then the polynomial algebra $\mathscr{P}(H_0)$ is a commutative integrable O^*-algebra on $\mathscr{D}^\infty(H)$ in $\mathscr{H}$ and the following statements hold:

(i) If $\xi \notin \mathscr{D}(H^2)$, then $\mathfrak{N}^0_{\widetilde{\omega_\xi}} = \boldsymbol{C}I$ and $\mathfrak{N}_{\widetilde{\omega_\xi}} = \mathfrak{N}_{\omega_\xi} = \{0\}$.

(ii) If $\xi \in \mathscr{D}(H^{2n}) - \mathscr{D}(H^{2n+2})$ ($n \in \boldsymbol{N}$), then

$$\mathfrak{N}^0_{\widetilde{\omega_\xi}} = \{P(H_0); P \text{ is a polynomial with the degree} \leq n\},$$
$$\mathfrak{N}_{\widetilde{\omega_\xi}} = \mathfrak{N}_{\omega_\xi} = \{0\}.$$

B. *The regularity and the singularity of* ω_ξ

(1) Suppose that $\mathscr{M}'_w = CI$, $\mathfrak{N}^\dagger_{\omega_\xi}\mathscr{D}$ is dense in $\mathscr{H}$ and $\mathfrak{N}^{\dagger *}_{\omega_\xi}\xi$ is dense in $\mathscr{D}[t_\mathscr{M}]$. Then ω_ξ is singular. In fact, since $\mathfrak{N}^{\dagger *}_{\omega_\xi}\xi$ is dense in $\mathscr{D}[t_\mathscr{M}]$, $\pi_{\omega_\xi}(\mathscr{M})$ is unitarily equivalent to $\mathscr{M}$, that is, there exists a unitary operator U of $\mathscr{H}_{\omega_\xi}$ onto $\mathscr{H}$ such that $U\lambda_{\omega_\xi}(X) = X^{\dagger *}\xi$ for all $X \in \mathfrak{N}_{\omega_\xi}$ and $U\pi_{\omega_\xi}(A)U^* = A$ for all $A \in \mathscr{M}$. Take an arbitrary $K \in T(\omega_\xi)'_\delta$. Then there is a constant $\alpha \in C$ such that $\alpha X^{\dagger *}\xi = X^{\dagger *}U\lambda'(K)$ for all $X \in \mathfrak{N}_{\omega_\xi}$. Since $\mathfrak{N}^\dagger_{\omega_\xi}\mathscr{D}$ is dense in $\mathscr{H}$, we have $\alpha\xi = U\lambda'(K) \in \mathscr{D}^*(\mathscr{M})$, and so $\alpha = 0$. Hence $K = 0$, which implies by Lemma 3.3 that ω_ξ is singular.

(2) In case $\mathscr{M} = \mathscr{L}^\dagger(\mathscr{D})$, ω_ξ is a singular quasi-weight on $\mathscr{P}(\mathscr{L}^\dagger(\mathscr{D}))$. In fact, this follows since $\mathscr{L}^\dagger(\mathscr{D})$ satisfies all conditions of the above (1).

(3) Suppose $\mathscr{M}$ is self-adjoint and $\mathfrak{N}^{\dagger *}_{\omega_\xi}\xi$ is dense in $\mathscr{D}[t_\mathscr{M}]$. We put

$$\mathscr{C}_\xi = \{C \in \mathscr{M}'_{\mathrm{w}};\, C\xi, C^*\xi \in \mathscr{D}\}, \quad P'_\xi = proj\,\overline{\mathscr{C}_\xi\mathscr{H}}.$$

Then ξ is decomposed into $\xi = \xi_r + \xi_s$, where $\xi_r = P'_\xi\xi$ and $\xi_s = (I - P'_\xi)\xi$. On the other hand, by Theorem 3.9, the quasi-weight ω_ξ on $\mathscr{P}(\mathscr{M})$ is decomposed into $\omega_\xi = \omega^{(r)}_\xi + \omega^{(s)}_\xi$, where $\omega^{(r)}_\xi$ is a regular quasi-weight on $\mathscr{P}(\mathscr{M})$ and $\omega^{(s)}_\xi$ is a singular quasi-weight on $\mathscr{P}(\mathscr{M})$ with $\mathfrak{N}_{\omega^{(r)}_\xi} = \mathfrak{N}_{\omega^{(s)}_\xi} = \mathfrak{N}_{\omega_\xi}$ defined by

$$\omega^{(r)}_\xi(X^\dagger X) = \|P'_\xi X^{\dagger *}\xi\|^2, \quad \omega^{(s)}_\xi(X^\dagger X) = \|(I - P'_\xi)X^{\dagger *}\xi\|^2, \quad X \in \mathfrak{N}_{\omega_\xi}.$$

We have the relation that the quasi-weights ω_{ξ_r} and $\omega^{(r)}_\xi$ are equivalent $(\omega_{\xi_r} \sim \omega^{(r)}_\xi)$, that is, $\pi_{\omega_{\xi_r}}$ and $\pi_{\omega^{(r)}_\xi}$ are unitarily equivalent. In fact, it is clear that $\omega^{(r)}_\xi \subset \omega_{\xi_r}$, that is, $\mathfrak{N}_{\omega^{(r)}_\xi}(= \mathfrak{N}_{\omega_\xi}) \subset \mathfrak{N}_{\omega_{\xi_r}}$ and $\omega^{(r)}_\xi(X^\dagger X) = \omega_{\xi_r}(X^\dagger X)$ for all $X \in \mathfrak{N}_{\omega^{(r)}_\xi}$, and so $\pi_{\omega^{(r)}_\xi} \subset \pi_{\omega_{\xi_r}}$ unitarily and $\pi_{\omega^{(r)}_\xi}$ is self-adjoint. Hence $\pi_{\omega^{(r)}_\xi}$ is unitarily equivalent to $\pi_{\omega_{\xi_r}}$. Similarly, we have $\omega_{\xi^{(s)}} \sim \omega_{\xi_s}$. Thus, ω_{ξ_r} is a regular quasi-weight on $\mathscr{P}(\mathscr{M})$ and ω_{ξ_s} is a singular quasi-weight on $\mathscr{P}(\mathscr{M})$ and $\omega_\xi = \omega_{\xi_r} + \omega_{\xi_s}$ on $\mathscr{P}(\mathfrak{N}_{\omega_\xi})$.

Hence, we call ξ_r and ξ_s the *regular part* and the *singular part* of ξ, respectively. We have the following results:

(a) ω_ξ is singular if and only if $\mathscr{C}_\xi = \{0\}$ if and only if $\xi_r = 0$.

(b) ω_ξ is regular if and only if $\mathscr{C}_\xi$ is a nondegenerate $*$-subalgebra of $\mathscr{M}'_{\mathrm{w}}$ if and only if $\xi_s = 0$.

(c) Suppose $0 \lneqq P'_\xi \lneqq I$. Then ω_ξ is not regular and not singular.

(4) Suppose $\mathscr{M}$ is an O^*-algebra on $\mathscr{D}^\infty(H) = \bigcap_{n=1}^\infty \mathscr{D}(H^n)$ containing

$$\{f(H)\lceil\mathscr{D}^\infty(H);\ f \text{ is a measurable function on } \boldsymbol{R}_+ \text{ such that} \\ |f(t)| \le p(t), t \in \boldsymbol{R}_+ \text{ for some polynomial } p\},$$

where H is a positive self-adjoint operator in $\mathscr{H}$ and $\mathfrak{N}^{\dagger *}_{\omega_\xi}\xi$ is dense in $\mathscr{H}$. Then it is easily shown that $\mathscr{M}$ is self-adjoint and $\mathfrak{N}^{\dagger *}_{\omega_\xi}\xi$ is dense in $\mathscr{D}^\infty(H)[t_\mathscr{M}]$ using the spectral decomposition theorem of H. Hence, the same results as the above (3) hold.

(5) Let $S(\boldsymbol{R})$ be the Schwartz space of infinitely differentiable rapidly decreasing functions and $\{f_n\}_{n=0,1,\ldots} \subset S(\boldsymbol{R})$ an orthonormal basis in the Hilbert space $L^2(\boldsymbol{R})$ of

normalized Hermite functions. We define a number operator N in $L^2(\boldsymbol{R})$ by

$$N = \sum_{n=0}^{\infty}(n+1)f_n \otimes \overline{f_n}.$$

Let $\mathscr{A}$ be the unbounded CCR-algebra for one degree of freedom and π_0 the Schrödinger representation of $\mathscr{A}$. Then $\pi_0(\mathscr{A})$ is a self-adjoint O^*-algebra on $S(\boldsymbol{R})$ satisfying $\pi_0(\mathscr{A}_{\mathrm{w}})' = CI$. Let $\mathscr{M}$ be the O^*-algebra on $S(\boldsymbol{R})$ generated by $\pi_0(\mathscr{A})$ and $\{f(N); f$ is a real-valued continuous function on $\boldsymbol{R}_+$ such that $|f(t)| \leq p(t)(t \in \boldsymbol{R}_+)$ for some polynomial $p\}$. Then it is easily shown that $\mathscr{M}$ is self-adjoint and $\mathfrak{N}^{\dagger *}_{\omega_\xi}\xi$ is dense in $S(\boldsymbol{R})$. Hence it follows from the above (3) and $\xi \notin S(\boldsymbol{R})$ that ω_ξ is singular.

C. *The standardness of* ω_ξ

Suppose

$$(\mathrm{S})_1 \qquad \{YX^{\dagger *}\xi; X, Y \in \mathfrak{N}_{\omega_\xi} \cap \mathfrak{N}^{\dagger}_{\omega_\xi}\} \quad \text{is total in } \mathscr{H},$$

$$(\mathrm{S})_2 \qquad \mathscr{C}_\xi\xi \quad \text{is dense in } \mathscr{H}.$$

Then ω_ξ is a faithful regular quasi-weight on $\mathscr{P}(\mathscr{M})$ and ξ is a cyclic and separating vector for the von Neumann algebra $(\mathscr{M}'_{\mathrm{w}})'$ and denote by Δ''_ξ the modular operator for the left Hilbert algebra $(\mathscr{M}'_{\mathrm{w}})'\xi$. We have the following results:

(1) ω_ξ is quasi-standard if and only if the following condition $(\mathrm{S})_3$ holds:

$$(\mathrm{S})_3 \qquad \Delta''_\xi \mathscr{D} \subset \mathscr{D} \quad \text{for each } t \in \boldsymbol{R}.$$

(2) ω_ξ is standard if and only if the above condition $(\mathrm{S})_3$ and the following condition $(\mathrm{S})_4$ hold:

$$(\mathrm{S})_4 \qquad \Delta''^{it}_\xi \mathscr{M} \Delta''^{-it}_\xi = \mathscr{M} \quad \text{for each } t \in \boldsymbol{R}.$$

We next give some examples of regular quasi-weights, singular quasi-weights and standard quasi-weights defined in the Hilbert space of Hilbert-Schmidt operators, which are important for the quantum physics.

EXAMPLE 5.2. Let $\mathscr{M}$ be a self-adjoint O^*-algebra on $\mathscr{D}$ in $\mathscr{H}$ such that $\mathscr{M}'_{\mathrm{w}} = CI$. We denote by $\mathscr{H} \otimes \overline{\mathscr{H}}$ the set of all Hilbert-Schmidt operators on $\mathscr{H}$, and then it is a Hilbert space with inner product $\langle S \mid T\rangle \equiv trT^*S$. We put

$$\sigma_2(\mathscr{M}) = \{T \in \mathscr{H} \otimes \overline{\mathscr{H}}; T\mathscr{H} \subset \mathscr{D} \text{ and } XT \in \mathscr{H} \otimes \overline{\mathscr{H}}, {}^\forall X \in \mathscr{M}\},$$
$$\pi(X)T = XT, \quad X \in \mathscr{M}, \quad T \in \sigma_2(\mathscr{M}).$$

Then π is a self-adjoint representation of $\mathscr{M}$ on $\sigma_2(\mathscr{M})$ in $\mathscr{H} \otimes \overline{\mathscr{H}}$ such that $\pi(\mathscr{M})'_{\mathrm{w}} = \pi'(\mathscr{B}(\mathscr{H}))$ and $(\pi(\mathscr{M})'_{\mathrm{w}})' = \pi''(\mathscr{B}(\mathscr{H}))$, where $\pi'(A)T = TA$ and $\pi''(A)T = AT$ for $A \in \mathscr{B}(\mathscr{H})$ and $T \in \mathscr{H} \otimes \overline{\mathscr{H}}$ [**4, 12**].

A. Let $\Omega \in \mathscr{H} \otimes \overline{\mathscr{H}} \backslash \sigma_2(\mathscr{M})$ and $\Omega \geq 0$. Then,

$$\mathfrak{N}_{\omega_\Omega} = \{\pi(X); X \in \mathscr{M}, \Omega\mathscr{H} \subset \mathscr{D}(X^{\dagger *}) \text{ and } X^{\dagger *}\Omega \in \sigma_2(\mathscr{M})\}.$$

(1) Suppose $\mathfrak{N}^{\dagger *}_{\omega_\Omega}\Omega$ is dense in $\sigma_2(\mathscr{M})[t_{\pi(\mathscr{M})}]$. We define the quasi-weight φ_Ω on $\mathscr{P}(\mathscr{M})$ by

$$\mathfrak{N}_{\varphi_\Omega} = \{X \in \mathscr{M}; \pi(X) \in \mathfrak{N}_{\varphi_\Omega}\},$$

$$\varphi_\Omega(X^\dagger X) = tr(X^{\dagger *}\Omega)^*(X^{\dagger *}\Omega) = \omega_\Omega(\pi(X)^\dagger \pi(X)), \quad X \in \mathfrak{N}_{\varphi_\Omega}.$$

By Example 5.1, B, (3) we have the following:

(a) φ_Ω is singular if and only if

$$\mathscr{C}_\Omega \equiv \{\pi'(K); K \in \mathscr{B}(\mathscr{H}) \text{ and } \Omega K, \Omega K^* \in \sigma_2(\mathscr{M})\} = \{0\}.$$

(b) φ_Ω is regular if and only if $\mathscr{C}_\Omega$ is a nondegenerate $*$-subalgebra of $\pi'(\mathscr{B}(\mathscr{H}))$.

(c) Ω is decomposed into $\Omega = \Omega_r + \Omega_s$, where Ω_r is the regular part of Ω and Ω_s is the singular part of Ω. Hence, φ_{Ω_r} is a regular quasi-weight on $\mathscr{P}(\mathscr{M})$, φ_{Ω_s} is a singular quasi-weight on $\mathscr{P}(\mathscr{M})$ and $\varphi_\Omega = \varphi_{\Omega_r} + \varphi_{\Omega_s}$ on $\mathscr{P}(\mathfrak{N}_{\varphi_\Omega})$.

(2) Suppose there exists a dense subspace $\mathscr{E}$ in $\mathscr{D}[t_\mathscr{M}]$ such that

(i) $\mathscr{M} \supset \{\xi \otimes \bar{\eta}; \xi, \eta \in \mathscr{E}\}$,

(ii) $\Omega\mathscr{E} \subset \mathscr{D}$ and $\Omega\mathscr{E}$ is dense in $\mathscr{H}$.

Then φ_Ω is regular. In fact, it is easily shown that $\mathfrak{N}_{\varphi_\Omega} \supset \{\pi(\xi \otimes \bar{\eta}); \xi, \eta \in \mathscr{E}\}$, and $\mathfrak{N}^{\dagger *}_{\omega_\Omega}\Omega$ is dense in $\sigma_2(\mathscr{M})[t_{\pi(\mathscr{M})}]$, and further $\mathscr{C}_\Omega \supset \{\pi'(\xi \otimes \bar{\eta}); \xi, \eta \in \mathscr{E}\}$, and so $\mathscr{C}_\Omega \mathscr{H} \otimes \bar{\mathscr{H}}$ is dense in $\mathscr{H} \otimes \bar{\mathscr{H}}$. Hence, φ_Ω is regular by the above (1).

Further, suppose

(iii) Ω^{-1} is densely defined,

(iv) $\Omega^{it}\mathscr{D} \subset \mathscr{D}$ and $\Omega^{it}\mathscr{M}\Omega^{-it} = \mathscr{M}$ for each $t \in \boldsymbol{R}$.

Then φ_Ω is a standard quasi-weight on $\mathscr{P}(\mathscr{M})$ ([**4**] Theorem 3.6).

B. Let Ω be a positive self-adjoint unbounded operator in $\mathscr{H}$. Suppose there exists a subspace $\mathscr{E}$ of $\mathscr{D} \cap \mathscr{D}(\Omega)$ such that

(i) $\mathscr{E}$ is dense in $\mathscr{D}[t_\mathscr{M}]$,

(ii) $\mathscr{M} \supset \{\xi \otimes \bar{\eta}; \xi, \eta \in \mathscr{E}\}$,

(iii) $\Omega\mathscr{E} \subset \mathscr{D}$ and $\Omega\mathscr{E}$ is dense in $\mathscr{H}$.

We put

$$\mathfrak{N}_{\varphi_\Omega} = \{X \in \mathscr{M}; \overline{X^{\dagger *}\Omega} \in \sigma_2(\mathscr{M})\},$$

$$\varphi_\Omega(X^\dagger X) = tr(X^{\dagger *}\Omega)^*(\overline{X^{\dagger *}\Omega}), \quad X \in \mathfrak{N}_{\varphi_\Omega}.$$

Then φ_Ω is a regular quasi-weight on $\mathscr{P}(\mathscr{M})$. In fact, this is shown in similar to the proof of the above A, (2).

Further, suppose

(iv) Ω^{-1} is densely defined and $\mathscr{D} \cap \mathscr{D}(\Omega^{-1})$ is a core for Ω^{-1}.

Then by ([**4**] Theorem 4.2) we have the following results:

(iv)$_1$ Suppose $\Omega^{it}\mathscr{D} \subset \mathscr{D}$ for all $t \in \boldsymbol{R}$. Then φ_Ω is a quasi-standard quasi-weight on $\mathscr{P}(\mathscr{M})$.

(iv)$_2$ Suppose $\Omega^{it}\mathscr{D} \subset \mathscr{D}$ and $\Omega^{it}\mathscr{M}\Omega^{-it} = \mathscr{M}$ for all $t \in \boldsymbol{R}$. Then φ_Ω is a standard quasi-weight on $\mathscr{P}(\mathscr{M})$.

EXAMPLE 5.3. A. We adopt the notations in Example 5.1, B, (5). We put

$$s_+ = \{\{\alpha_n\}_{n=0,1,\dots}; \alpha_n > 0, n = 0, 1, \dots\},$$

$$\Omega_{\{\alpha_n\}} = \sum_{n=0}^{\infty} \alpha_n f_n \otimes \overline{f_n}, \quad \{\alpha_n\}_{n=0,1,\dots} \in s_+,$$

and

$$\mathfrak{N}_{\varphi_{\Omega_{\{\alpha_n\}}}} = \{X \in \mathcal{M}; \overline{X^{\dagger *}\Omega} \in \sigma_2(\mathcal{M})\},$$

$$\varphi_{\Omega_{\{\alpha_n\}}}(X^\dagger X) = tr(X^{\dagger *}\Omega)^*(\overline{X^{\dagger *}\Omega}), \quad X \in \mathfrak{N}_{\varphi_{\Omega_{\{\alpha_n\}}}}.$$

Since the linear span of $\{f_n; n = 0, 1, \dots\}$ satisfies the conditions (i), (ii) and (iii) in Example 5.2, B, it follows that $\varphi_{\Omega_{\{\alpha_n\}}}$ is a regular quasi-weight on $\mathscr{P}(\mathcal{M})$. Further since $\Omega_{\{\alpha_n\}}$ satisfies the conditions (iv) and $(\text{iv})_1$ in Example 5.2, B, it follows that $\varphi_{\Omega_{\{\alpha_n\}}}$ is a quasi-standard quasi-weight on $\mathscr{P}(\mathcal{M})$.

B. We adopt the notations in ([**4**], Example 5.2). The total Hamiltonian of the interacting boson model with a two-body potential is given by a self-adjoint operator H in $\mathscr{F}$

$$H = \bigoplus_{n=0}^{\infty} H_n,$$

where $H_n = d\Gamma_n(h) + V^{(n)}$. We put

$$\Omega = e^{-H/2},$$

and

$$\mathfrak{N}_{\varphi_\Omega} = \{X \in \mathcal{M}; \overline{X^{\dagger *}\Omega} \in \sigma_2(\mathcal{M})\},$$

$$\varphi_\Omega(X^\dagger X) = tr(X^{\dagger *}\Omega)^*(\overline{X^{\dagger *}\Omega}), \quad X \in \mathfrak{N}_{\varphi_\Omega}.$$

Then it is shown in similar to the above A that φ_Ω is a regular quasi-standard quasi-weight on $\mathscr{P}(\mathcal{M})$.

C. We adopt the notations in ([**4**], Example 5.1). The total Hamiltonian of the BCS-model is given by a self-adjoint operator H_B in $\mathscr{H}_{\{N\}}$

$$H_B = \alpha \sum_{p=1}^{\infty} \{\varepsilon_p - (\sigma_p N)\}.$$

We put

$$\Omega = e^{-H_B/2},$$

and

$$\mathfrak{N}_{\varphi_\Omega} = \{X \in \mathcal{M}; \overline{X^{\dagger *}\Omega} \in \sigma_2(\mathcal{M})\},$$

$$\varphi_\Omega(X^\dagger X) = tr(X^{\dagger *}\Omega)^*(\overline{X^{\dagger *}\Omega}), \quad X \in \mathfrak{N}_{\varphi_\Omega}.$$

Then it is shown in similar to the proof of the above A that φ_Ω is a regular quasi-weight on $\mathscr{P}(\mathscr{M})$. Further since Ω satisfies the conditions (iv) and $(\text{iv})_2$ in Example 5.2, B, it follows that φ_Ω is a standard quasi-weight $\mathscr{P}(\mathscr{M})$.

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Atsushi INOUE and Hidekazu OGI

Department of Applied Mathematics
Fukuoka University
Fukuoka
Japan