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The Cauchy problem for Schrödinger type equations with variable coefficients

Dedicated to Professor Toshinobu Muramatsu on his 60th birthday

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§1. Introduction

In this article we consider the following Cauchy problem in $(0, T) \times \boldsymbol{R}^n$,

$$(1.1)\qquad \begin{aligned} &L[u(t,x)] = f(t,x), \quad (t,x) \in (0,T) \times \boldsymbol{R}^n \\ &u(0,x) = u_0(x), \quad x \in \boldsymbol{R}^n, \end{aligned}$$

where $L[u] = \partial_t u - \sqrt{-1} \sum_{j,k} \partial_j \{a_{jk}(x) \partial_k u\} - \sum_j b_j(t,x) \partial_j u - c(t,x)u$ and $\partial_t = \partial/\partial t$ and $\partial_j = \partial/\partial x_j$. We assume that $a_{jk}(x)$ belong to B^∞ and $b_j(t,x), c(t,x)$ are in $C^0([0,T]; B^\infty)$, where B^∞ stands for the set of complex valued functions defined in $\boldsymbol{R}^n$ whose all derivatives are bounded in $\boldsymbol{R}^n$. For a topological space X, a non negative integer k and an interval I in R^1 we denote by $C^k(I; X)$ the set of functions k times continuously differentiable with respect to $t \in I$ in the topology of X. Moreover we assume that $a_{jk}(x) = a_{kj}(x)$ are real valued and there is $c_0 > 0$ such that

$$(1.2)\qquad \sum_{j,k} a_{jk}(x) \xi_j \xi_k \geq c_0 |\xi|^2, \quad x, \xi \in R^n.$$

Let $T > 0$ and X a topological space. We say that the Cauchy problem (1.1) is X-well posed in $(0, T)$, if for any u_0 in X and any f in $C^0([0,T]; X)$ there exists a unique solution u in $C^0([0,T]; X)$ of (1,1).

We shall prove that the Cauchy problem (1.1) is X-well posed in $(0, T)$ under some assumptions, if we take $X = L^2(\boldsymbol{R}^n)$ the set of square integrable functions in $\boldsymbol{R}^n$ or $X = H^\infty$ the sobolev space in $\boldsymbol{R}^n$.

We know a necessary condition in order that the Cauchy problem is L^2 (resp. H^∞)-well posed in $(0, T)$. To state this we need the classical orbit associated to L. Put

$$(1.3)\qquad a_2(x,\xi) = \sum_{j,k} a_{jk}(x) \xi_j \xi_k$$

and let $(X(t,y,\eta), \Xi(t,y,\eta))$ be the solution of the following ordinary differential equations

$$(1.4)\qquad \begin{aligned} &(d/dt)X_j(t) = (\partial/\xi_j) a_2(X(t), \Xi(t)), \quad X_j(0) = y_j \\ &(d/dt)\Xi_j(t) = -(\partial/\partial x_j) a_2(X(t), \Xi(t)), \quad \Xi_j(0) = \eta_j, \end{aligned}$$

where $j = 1, \dots, n$. Then it follows from **[7]** and **[5]** that if the Cauchy problem (1.1) is L^2 (resp. H^∞)-well posed in $(0, T)$, the coefficients $b_j = b_j(t, x)$ satisfy

$$
\begin{aligned}
(1.5)\qquad & \sup_{y,\eta \in R^n, |\eta|=1} \left| \int_0^\rho \sum_j \operatorname{Im} b_j(0, X(t, y, \eta)) \Xi_j(t, y, \eta)\, dt \right| \\
& \leq C(\text{resp. } C\log(1 + \rho))
\end{aligned}
$$

for $\rho > 0$.

Now we assume that there are $C > 0$ and $\sigma \geq 1$ such that the coefficients $b_j(t, x)$ satisfy

$$
(1.6)\qquad |\operatorname{Im} b_j(t, x)| \leq C\langle x\rangle^{-\sigma}
$$

for $(t, x) \in [0, T] \times \boldsymbol{R}^n$ and $j = 1, \dots, n$, where $\langle x\rangle = (1 + |x|^2)^{1/2}$. Then we can see in Lemma 2.3 later on that (1.6) implies (1.5). Furthermore we suppose there are $c_0 > 0$ and $\delta > 0$ such that $a_2(x, \xi)$ satisfies

$$
(1.7)\qquad \sum_j \{(\partial_{\xi_j} a_2)(x, \xi)\xi_j - (\partial_{x_j} a_2)(x, \xi)x_i\} \geq c_0|\xi|^2
$$

for $(x, \xi) \in [0, T] \times \boldsymbol{R}^n$ and the coefficients $a_{jk}(x)$ of $a_2(x, \xi)$ satisfy

$$
(1.8)\qquad |(\partial/\partial x)^\alpha a_{jk}(x)| \leq C_\alpha \langle x\rangle^{-|\alpha|-\delta}
$$

for $x \in \boldsymbol{R}^n, \alpha \in \boldsymbol{N}^n (\alpha \neq 0)$ and $j, k = 1, \dots, n$.

Now we can state our main result.

THEOREM 1.1. *Assume that the conditions* (1.2), (1.6), (1.7) *and* (1.8) *are fulfilled and the coefficients* $b_j(t, x)$, $c(t, x)$ *belong to* $C^0([0, T]; B^\infty)$. *Then the Cauchy problem* (1.1) *is* L^2 (resp. H^∞)*-well posed in* $(0, T)$ *if* $\sigma > 1$ (resp. $\sigma = 1$).

It should be remarked that when $a_2(x, \xi) = |\xi|^2/2$, this problem is treated in **[2, 4, 6, 9, 10, 11]** and that when $a_2(x, \xi)$ has variable coefficients, the case where $\{\operatorname{Im} b_j\}$ is integrable is considered in **[8]**.

§2. Properties of classical orbits

Put $a(x, \xi) = \{2a_2(x, \xi)\}^{1/2}$. We consider the classical orbits associated to the Hamiltonian function $a(x, \xi)$ instead of $a_2(x, \xi)$. Let $\{x(t, y, \eta), \xi(t, y, \eta)\}$ be the solution of the following Hamiltonian system

$$
(2.1)\qquad
\begin{aligned}
&(d/dt)x_j(t) = (\partial a/\partial \xi_j)(x(t), \xi(t)), x_j(0) = y_j, \\
&(d/dt)\xi_j(t) = -(\partial a/\partial x_j)(x(t), \xi(t)), \xi_j(0) = \eta_j
\end{aligned}
$$

for $j = 1, \dots, n$. Since $a(x, \xi)$ is homogenuous in ξ of degree one, there exists globally in t the solution of (2.1) and $x(t, y, \eta)$ and $\xi(t, y, \eta)$ are homogenuous in η of degree zero and one respectively.

To investigate the growth order of the solution of (2.1) tending t to infinite we need a following preliminary lemma.

LEMMA 2.1. *Let $\rho_i(t), f_i(t), i = 1, 2$ be positive functions such that $\rho_1(t)$, $f_2(t)$ are in $L^1((0,\infty))$ and $f_1(t)$ in $L^\infty((0,\infty))$. Assume that $\rho_2(t)$ is differentiable and satisfies that $\rho_2'(t) \le 0$ and $\rho_2(t) \le M/2$, where $M \ge 1$ and we write $\rho_2'(t) = (d/dt)\rho_2(t)$. If positive and differentiable functions $u_i(t), i = 1, 2$ satisfy*

$$\begin{aligned} u_1'(t) &\le \rho_1(t)u_1(t) + Mu_2(t) + f_1(t) \\ u_2'(t) &\le \rho_2(t)^2 u_1(t) + \rho_1(t)u_2(t) + f_2(t) \end{aligned} \tag{2.2}$$

for $t \ge 0$, then there is a positive contant C independent of ρ_2 such that

$$\begin{aligned} u_1(t) \le C\Big[&u_1(0) + u_2(0) + \exp\Big\{\int_0^t M\rho_2(s)\,ds\Big\}\Big\{t\rho_2(0)u_1(0) + tu_2(0) \\ &+t\int_0^t (\rho_2(s)f_1(s) + f_2(s))\,ds\Big\} + \int_0^t (f_1(s) + f_2(s))\,ds\Big] \\ u_2(t) \le C\exp&\Big\{\int_0^t M\rho_2(s)\,ds\Big\}\Big[\rho_2(0)u_1(0) + u_2(0) + \int_0^t \{\rho_2(s)f_1(s) + f_2(s)\}\,ds\Big] \end{aligned} \tag{2.3}$$

for $t \ge 0$.

PROOF. Set

$$v_i(t) = u_i(t)\exp\Big\{-\int_0^t \rho_1(s)\,ds\Big\},$$

$$g_i(t) = f_i(t)\exp\Big\{-\int_0^t \rho_1(s)\,ds\Big\}.$$

$(i = 1, 2)$. Then v_i satisfies

$$\begin{aligned} v_1'(t) &\le Mv_2(t) + g_1(t) \\ v_2'(t) &\le \rho_2(t)^2 v_1(t) + g_2(t) \end{aligned} \tag{2.4}$$

for $t \ge 0$. Put $w_1(t) = Mv_1(t) + v_2(t)$ and $w_2(t) = \rho_2(t)v_1(t) + v_2(t)$. Then we have from (2.4)

$$\begin{aligned} w_1'(t) &\le (M^2 + \rho_2(t))w_2(t) + Mg_1(t) + g_2(t) \\ w_2'(t) &\le M\rho_2(t)w_2(t) + \rho_2(t)g_1(t) + g_2(t) \end{aligned}$$

for $t \ge 0$ and consequently

$$\begin{aligned} w_2(t) &\le \exp\Big\{M\int_0^t \rho_2(s)\,ds\Big\}\Big\{w_2(0) + \int_0^t (M\rho_2(s)g_1(s) + g_2(s))\,ds\Big\} \\ w_1(t) &\le w_1(0) + (M^2 + 3M/2)\exp\Big\{\int_0^t \rho_2(s)\,ds\Big\}\Big\{tw_2(0) + t\int_0^t (\rho_2(s)g_1(s) + g_2(s))\,ds\Big\} \\ &\quad + \int_0^t (Mg_1(s) + g_2(s))\,ds \end{aligned}$$

for $t \geq 0$. Therefore taking account of the following relation

$$v_1(t) = (M - \rho_2(t))^{-1}(w_1(t) - w_2(t)) \leq 2M^{-1}w_1(t)$$
$$v_2(t) = (M - \rho_2(t))^{-1}(Mw_2(t) - \rho_2(t)w_1(t)) \leq 2w_2(t),$$

we obtain (2.3). Q.E.D.

If $\rho_2(t)$ belongs to $L^1((0,\infty))$, then (2.3) gives

$$\begin{aligned} u_1(t) &\leq C\left\{(1 + t\rho_2(0))u_1(0) + (1 + t)u_2(0) + t\left(\sup_{0\leq s\leq t} f_1(s) + \int_0^t f_2(s)\,ds\right)\right\} \\ u_2(t) &\leq C\left\{\rho_2(0)u_1(0) + u_2(0) + \sup_{0\leq s\leq t} f_1(s) + \int_0^t f_2(s)\,ds\right\} \end{aligned} \tag{2.5}$$

for $t \geq 0$.

We denote

$$\Gamma = \left\{(t, y, \eta) \in \boldsymbol{R} \times \boldsymbol{R}^n \times \boldsymbol{R}^n; y\eta = \sum_{j=1}^n y_j\eta_j = 0, \eta \neq 0\right\}.$$

LEMMA 2.2. *Let $\{x(t,y,\eta), \xi(t,y,\eta)\}$ be the solution of* (2.1). *Then there is $C > 0$ such that*

$$C^{-1}|x(t,y,\eta)| \leq |t| + |y| \leq C|x(t,y,\eta)| \tag{2.6}$$

$$C^{-1}|\eta| \leq |\xi(t,y,\eta))| \leq C|\eta| \tag{2.7}$$

for $(t,y,\eta) \in \Gamma$ and moreover

$$C^{-1}|\eta| \leq t^{-1}\sum_{j=1}^n x_j(t,y,\eta)\xi_j(t,y,\eta) \leq C|\eta| \tag{2.8}$$

for $(t,y,\eta) \in \Gamma, t \neq 0$.

PROOF. Since $a_\xi(x,\xi)$ is bounded in R^{2n}, integrating the first equations of (2.1) we get the first inequality of (2.6). Noting that (2.1) implies

$$a(x(t,y,\eta), \xi(t,y,\eta)) = a(y,\eta) \tag{2.9}$$

for any t and that from the assumption (1.2)

$$C^{-1}|\eta| \leq a(y,\eta) \leq C|\eta| \tag{2.10}$$

for any $(y,\eta) \in R^{2n}$, we obtain (2.7). Besides we have from (2.1)

$$(d/dt)\{x(t)\xi(t)\} = \tilde{a}(x(t), \xi(t)),$$

where $\tilde{a}(x,\xi) = \sum_j(\partial_{\xi_j}a)(x,\xi)\xi_j - (\partial/\partial_{x_j}a)(x,\xi)x_j$. Integrating this equation we have

$$x(t)\xi(t) = \sum_j x_j(t,y,\eta)\xi_j(t,y,\eta) = \int_0^t \tilde{a}(x(s), \xi(s))\,ds \tag{2.11}$$

for $(t, y, \eta) \in \Gamma$. The assumptions (1.7) and (1.8) yield

$$C^{-1}|\xi| \leq \tilde{a}(x, \xi) \leq C|\xi| \tag{2.12}$$

for $(x, \xi) \in R^{2n}$. Hence we obtain (2.8) from (2.7), (2.11) and (2.12). Moreover it follows from (2.11) that we have $|x(t)\xi(t)| \geq C^{-1}|t|\,|\eta|$ and consequently $C|x(t)| \geq C^{-1}|t|$. Hence we get the second inequality of (2.6) by integrating (2.1). Q.E.D.

LEMMA 2.3. *Let* $\{X(t, y, \eta), \Xi(t, y, \eta)\}$ *be the solution of* (1.4). *Then we can express*

$$\{X(t, y, \eta), \Xi(t, y, \eta)\} = \{x(a(y, \eta)t, y, \eta), \xi(a(y, \eta)t, y, \eta)\}, \tag{2.13}$$

where $(x(t), \xi(t))$ *is the solution of* (2.1). *Moreover if the condition* (1.6) *is verified, then* (1.5) *holds.*

PROOF. Recalling (2.9) and $a_2 = a^2/2$, we have

$$\begin{aligned}(d/dt)X(t) &= (dx/dt)(at, y, \eta)a(y, \eta) \\ &= (\partial a/\partial \xi)(x(at), \xi(at))a(y, \eta) \\ &= (\partial a_2/\partial \xi)(x(at), \xi(at)) \\ &= (\partial a_2/\partial \xi)(X(t), \Xi(t)).\end{aligned}$$

Similarly $\Xi(t)$ satisfies $(d\Xi(t)/dt = -(\partial a_2/\partial x)(X(t), \Xi(t))$. Therefore $\{X(t), \Xi(t)\}$ satisfies (1.4). We next prove (1.5). Since $(d/dt)\{X(t)\Xi(t)\} = \tilde{a}_2(X(t), \Xi(t))$, integrating this equation we have

$$X(t)\Xi(t) = y\eta + \int_0^t \tilde{a}_2(X(s), \Xi(s))\, ds. \tag{2.14}$$

Hence taking account of the estimate $C^{-1} \leq |\Xi(t)| \leq C$ for $|\eta| = 1$ and $(t, y) \in \boldsymbol{R}^{n+1}$, we get

$$\begin{aligned}\langle X(t, y, \eta)\rangle &\geq C^{-1}\langle X(t, y, \eta)\Xi(t, y, \eta)\rangle \\ &= C^{-1}\left\langle y\eta + \int_0^t \tilde{a}_2(X(s), \Xi(s))\, ds\right\rangle.\end{aligned} \tag{2.15}$$

Hence noting that $C^{-1} \leq \tilde{a}_2(X(t), \Xi(t)) \leq C$ for $|\eta| = 1$ and $(t, y) \in \boldsymbol{R}^{n+1}$ we can see from the condition (1.6) and (2.15)

$$\begin{aligned}&\left|\sum_{j=1}^n \int_0^\rho \operatorname{Im} b_j(X(t, y, \eta))\Xi_j(t, y, \eta)\, dt\right| \\ &\leq C \int_0^\rho \left\langle y\eta + \int_0^t \tilde{a}_2(X(s), \Xi(s))\, ds\right\rangle^{-\sigma} dt \\ &\leq C' \int_0^{\tilde{\rho}} \langle y\eta + s\rangle^{-\sigma}\, ds \begin{cases} \leq C'' & \text{if } \sigma > 1 \\ \leq C'' \log(1 + \rho) & \text{if } \sigma = 1\end{cases}\end{aligned}$$

for $\rho \geq 0$, $y \in \boldsymbol{R}^n$ and $|\eta| = 1$, where we write $\tilde{\rho} = \int_0^\rho \tilde{a}_2(X(s), \Xi(s))\, ds$. Q.E.D.

For $\{x(t),\xi(t)\}$ the solution of (2.1) with (y,η) satisfying $y\eta=0$, we denote

$$A(t)=\begin{bmatrix} a_{\xi x}(x(t),\xi(t)) & a_{\xi\xi}(x(t),\xi(t)) \\ -a_{xx}(x(t),\xi(t)) & -a_{x\xi}(x(t),\xi(t)) \end{bmatrix} \tag{2.16}$$

which is a $2n\times 2n$ matrix.

LEMMA 2.4. *Let $w(t)={}^t(w_1(t),w_2(t))$ be in $C^1([0,\infty);\boldsymbol{R}^{2n})$ and $F(t)={}^t(F_1(t),F_2(t))$ in $C^0([0,\infty);\boldsymbol{R}^{2n})$ and satisfy*

$$dw(t)/dt=A(t)w(t)+F(t) \tag{2.17}$$

for $t\geq 0$. Then there is $C>0$ such that

$$\begin{aligned} |w_1(t)| &\leq C\left\{(1+t\langle y\rangle^{-1-\delta})|w_1(0)|+|\eta|^{-1}|w_2(0)|(1+t)+t\sup_{0\leq s\leq t}|F_1(s)|+|\eta|^{-1}\int_0^t|F_2(s)|\,ds\right\} \\ |w_2(t)| &\leq C\left\{\langle y\rangle^{-1-\delta/2}|w_1(0)|\,|\eta|+|w_2(0)|+\sup_{0\leq s\leq t}|F_1(s)|\,|\eta|+\int_0^t|F_2(s)|\,ds\right\} \end{aligned} \tag{2.18}$$

for $t\geq 0$ and $y,\eta\in R^n$ with $y\eta=0$.

PROOF. Put $u_1(t)=|w_1(t)|\,|\eta|$, $f_1(t)=|F_1(t)|\,|\eta|$, $u_2(t)=|w_2(t)|$ and $f_2(t)=|F_2(t)|$. Then the assumption (1.8) and (2.7) yield

$$\begin{aligned} |a_{x\xi}(x(t),\xi(t))| &\leq C\langle x(t)\rangle^{-1-\delta}, \\ |a_{\xi\xi}(x(t),\xi(t))| &\leq C|\eta|^{-1} \\ |a_{xx}(x(t),\xi(t))| &\leq C\langle x(t)\rangle^{-2-\delta}|\eta| \end{aligned} \tag{2.19}$$

for $(t,y)\in\boldsymbol{R}^{n+1}$ and $\eta\in\boldsymbol{R}^n\backslash 0$. It follows from (2.6) that $\langle x(t)\rangle\geq c_0(t^2+\langle y\rangle^2)^{1/2}$. Therefore $u(t)=(u_1(t),u_2(t))$ satisfies (2.2) with $\rho_1(t)=c_0^{-1-\delta}C(t^2+\langle y\rangle^2)^{-(1+\delta)/2}$, $\rho_2(t)=c_0^{-1-\delta/2}C(t^2+\langle y\rangle^2)^{-(1+\delta/2)/2}$ and $M=2\max\{C,C^{1/2}c_0^{-1-\delta/2}\}$. Hence we obtain from (2.5)

$$\begin{aligned} u_1(t) &\leq C\left\{(1+t\langle y\rangle^{-1-\delta/2})u_1(0)+(1+t)u_2(0)+t\left(\sup_{0\leq s\leq t}f_1(s)+\int_0^t f_2(s)\,ds\right)\right\} \\ u_2(t) &\leq C\left\{\langle y\rangle^{-1-\delta/2}u_1(0)+u_2(0)+\sup_{0\leq s\leq t}f_1(s)+\int_0^t f_2(s)\,ds\right\} \end{aligned} \tag{2.20}$$

for $t\geq 0$.

LEMMA 2.5. *Let $\Gamma_i=\{(t,y,\eta)\in\Gamma;\eta_i\neq 0\}$. There is $C>0$ such that*

$$\sum_{j+|\beta|=1}|\partial_t^j\partial_y^\beta x(t,y,\eta)|\leq C\{|\eta_i|^{-1}|\eta|(1+|t|\langle y\rangle^{-1})\}^{|\beta|} \tag{2.21}$$

$$\sum_{j+|\beta|=1}|\partial_t^j\partial_y^\beta \xi(t,y,\eta)|\leq C\{|\eta_i|^{-1}|\eta|^2\langle y\rangle^{-1}\}^{|\beta|}\langle x(t)\rangle^{-j} \tag{2.22}$$

$$\sum_{|\alpha|=1} |\partial_\eta^\alpha x(t,y,\eta)| \le C\{|\eta_i|^{-1}|y|(1+|t|\langle y\rangle^{-1}) + |\eta|^{-1}\} \tag{2.23}$$

$$\sum_{|\alpha|=1} |\partial_\eta^\alpha \xi(t,y,\eta)| \le C(|\eta_i|^{-1}\,|\eta| + 1) \tag{2.24}$$

for $(t,y,\eta) \in \Gamma_i$, *where we write* $\tilde{y} = (y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n)$.

Proof. It follows from (2.1) that $\partial_t x$ and $\partial_t \xi$ satisfy (2.21) and (2.22) with $(j,\beta) = (1,0)$ respectively. For $|\beta| = 1$ we put $w_1(t) = \partial_{\tilde{y}}^\beta x(t)$ and $w_2(t) = \partial_{\tilde{y}}^\beta \xi(t)$. Then $w(t) = {}^t(w_1(t), w_2(t))$ satisfies (2.17) with $F(t) = 0$. Therefore noting that $|w_1(0)| \le C(|\eta_i|^{-1}|\eta| + 1)$ and $w_2(0) = 0$, we get (2.21) and (2.22) with $(j,|\beta|) = (0,1)$ from (2.18). Analogously put $w_1(t) = \partial_\eta^\alpha x(t)$ and $w_2(t) = \partial_\eta^\alpha \xi(t)$ for $|\alpha| = 1$. Since $|w_1(0)| \le C|\eta_i|^{-1}|y|$ and $|w_2(0)| \le C$, we obtain (2.23) and (2.24) from (2.18). Q.E.D.

For simplicity we denote

$$\begin{aligned} &\nabla_{t,\tilde{y}} x(t,y,\eta) = (\partial_t x, \partial_{y_1} x, \ldots, \partial_{y_{i-1}} x, \partial_{y_{i+1}} x, \ldots, \partial_{y_n} x) \\ &\nabla_\eta x(t,y,\eta) = (\partial_{\eta_1} x, \ldots, \partial_{\eta_n} x) \\ &\nabla x(t,y,\eta) = (\nabla_{t,\tilde{y}} x, \nabla_\eta x). \end{aligned} \tag{2.25}$$

Set

$$T(t) = T(t,y,\eta) = \begin{pmatrix} \nabla x(t,y,\eta) \\ \nabla \xi(t,y,\eta) \end{pmatrix}. \tag{2.26}$$

Then differentiating (2.1) we have

$$(d/dt)T(t) = A(t)T(t), \tag{2.27}$$

where $A(t)$ is defined in (2.16). Besides, we see

(2.28) $T(0) =$

$$\begin{bmatrix}
a_{\eta_1} & 1 & 0 & \cdots & 0 & & \cdots & 0 \;\cdots & 0 & 0 \;\cdots & 0 \\
a_{\eta_2} & 0 & 1 & & \vdots & & & \vdots & \vdots & \vdots & \vdots \\
 & & 0 & & & & & & & & \\
\vdots & \vdots & \vdots & \ddots & 0 & & & & & & \\
 & 0 & 0 & & 1 & 0 & \cdots & 0 \;\cdots & 0 \;\cdots & 0 & \cdots\; 0 \\
a_{\eta_i} & -\eta_i^{-1}\eta_1 & -\eta_i^{-1}\eta_2 & \cdots & -\eta_i^{-1}\eta_{i-1} & -\eta_i^{-1}\eta_{i+1} & \cdots & -\eta_i^{-1}\eta_n & -\eta_i^{-1}y_1 & \cdots & -\eta_i^{-1}y_n \\
\vdots & 0 & & & 0 & 1 & & 0 & 0 & \cdots & 0 \\
 & & & & & & \ddots & \vdots & \vdots & & \\
 & & & & & & & 0 & & & \vdots \\
a_{\eta_n} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 \\
-a_{y_1} & 0 & \cdots & \cdots & \cdots & \cdots & & 0 & 1 & 0\;\cdots & \\
\vdots & \vdots & & & & & & \vdots & & \ddots & 0 \\
-a_{y_n} & 0 & \cdots & \cdots & \cdots & \cdots & & 0 & & 0 & 1
\end{bmatrix}$$

for $(t, y, \eta) \in \Gamma_i$. Moreover a simple calculation yields

$$\det T(0) = (-1)^{i-1}\eta_i^{-1}\tilde{a}(y,\eta), \tag{2.29}$$

where

$$\begin{aligned}\tilde{a}(y,\eta) &= \sum_{j=1}^{n}\{\eta_j(\partial/\partial\eta_j)a(y,\eta) - y_j(\partial/\partial y_j)a(y,\eta)\}\\ &= (a_2(y,\eta))^{-1/2}\sum_{j=1}^{n}\{\eta_j(\partial/\partial\eta_j)a_2(y,\eta) - y_j(\partial/\partial y_j)a_2(y,\eta)\}.\end{aligned}$$

Therefore we obtain from the assumption (1.7)

$$\det T(0) \neq 0 \quad \text{for } (0, y, \eta) \in \Gamma_i. \tag{2.30}$$

On the other hand it follows from the well known fact (for example see [1]) that the equation (2.27) implies

$$(d/dt)\det T(t) = \{\text{trace } A(t)\}\det T(t). \tag{2.31}$$

Recalling (2.16) we have trace $A(t) = 0$. Hence we get

$$\det T(t, y, \eta) = \det T(0, y, \eta) \neq 0 \quad \text{for}(t, y, \eta) \in \Gamma_i. \tag{2.32}$$

Set

$$S(t, y, \eta) = T(t, y, \eta)^{-1}. \tag{2.33}$$

Then we can see easily from (2.28)

$$S(0, y, \eta) = \tilde{a}(y,\eta)^{-1}\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{bmatrix}, \tag{2.34}$$

$$S_{11} = \begin{bmatrix} \eta_1 & \eta_2 & & \cdots & \cdots & & \eta_n \\ b_1 & -a_{\eta_1}\eta_2 & & \cdots & \cdots & & -a_{\eta_1}\eta_n \\ -a_{\eta_2}\eta_1 & b_2 & -a_{\eta_2}\eta_3 & \cdots & \cdots & & -a_{\eta_2}\eta_n \\ \vdots & & \ddots & & & & \\ -a_{\eta_{i-1}}\eta_1 & \cdots & \cdots & b_{i-1} - a_{\eta_{i-1}}\eta_i & \cdots & & -a_{\eta_{i-1}}\eta_n \\ -a_{\eta_{i+1}}\eta_1 & \cdots & \cdots & - a_{\eta_{i+1}}\eta_i & b_{i+1} & \cdots & \\ \vdots & & & & & \ddots & \vdots \\ -a_{\eta_n}\eta_1 & \cdots & \cdots & \cdots & \cdots & - a_{\eta_n}\eta_{n-1} & b_n \end{bmatrix} \tag{2.35}$$

$$S_{12} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ -a_{\eta_1}y_1 & -a_{\eta_1}y_2 & \cdots & -a_{\eta_1}y_n \\ \vdots & & & \\ -a_{\eta_{i-1}}y_1 & -a_{\eta_{i-1}}y_2 & \cdots & -a_{\eta_{i-1}}y_n \\ -a_{\eta_{i+1}}y_1 & -a_{\eta_{i+1}}y_2 & \cdots & -a_{\eta_{i+1}}y_n \\ \vdots & \vdots & & \vdots \\ -a_{\eta_n}y_1 & -a_{\eta_n}y_2 & \cdots & -a_n y_n \end{bmatrix}$$

$$S_{21} = \begin{bmatrix} a_{y_1}\eta_1 & a_{y_1}\eta_2 & \cdots & a_{y_1}\eta_n \\ a_{y_2}\eta_1 & a_{y_2}\eta_2 & \cdots & a_{y_2}\eta_n \\ \cdots & & & \\ a_{y_n}\eta_1 & a_{y_n}\eta_2 & \cdots & a_{y_n}\eta_n \end{bmatrix}$$

$$S_{22} = \begin{bmatrix} c_1 & a_{y_1}y_2 & \cdots & \cdots & a_{y_1}y_n \\ a_{y_2}y_1 & c_2 & a_{y_2}y_3 & \cdots & a_{y_2}y_n \\ \vdots & & \ddots & & \vdots \\ a_{y_n}y_1 & \cdots & \cdots & a_{y_n}y_{n-1} & c_n \end{bmatrix}$$

$$b_k = \tilde{a}(y,\eta) - \eta_k a_{\eta_k}(y,\eta) \quad \text{and} \quad c_k = y_k a_{y_k}(y,\eta) + \tilde{a}(y,\eta).$$

Let us denote

$$S(t) = \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{bmatrix}.$$

Since $T(t)S(t) = I$ is the identity matrix, differentiating this with respect to t we have

$$(d/dt)S(t) = -S(t)A(t), \tag{2.36}$$

where $A(t)$ is given by (2.16). Put $u_1(t) = |\eta|\,|S_{12}(t)|$ and $u_2(t) = |S_{11}(t)|$. Then taking account of (2.36) and (2.19) we can see that $u(t) = (u_1(t), u_2(t))$ satisfies (2.2) with $\rho_1(t) = (t^2 + \langle y\rangle^2)^{-(1+\delta)/2}$, $\rho_2(t) = (t^2 + \langle y\rangle^2)^{-(1+\delta/2)/2}$ and $f_1 = f_2 = 0$ and consequently $u(t)$ satisfies (2.20) with $f_1 = f_2 = 0$. Since (2.34) and (2.35) yield that $u_1(0) \le C|y|$ and $u_2(0) \le C$, (2.20) implies

$$\begin{aligned} &|S_{11}(t)| \le C \\ &|S_{12}(t)| \le C(\langle y\rangle + |t|)|\eta|^{-1} \end{aligned} \tag{2.37}$$

for $(t, y, \eta) \in \Gamma_i$. Moreover noting that $|S_{21}(0)| \le |\eta|\langle y\rangle^{-1}$ and $|S_{22}(0)| \le C\langle y\rangle^{-1}$, we get similarly

$$\begin{aligned} &|S_{21}(t)| \le C|\eta|\langle y\rangle^{-1}, \\ &|S_{22}(t)| \le C(1 + |t|\langle y\rangle^{-1}) \end{aligned} \tag{2.38}$$

for $(t, y, \eta) \in \Gamma_i$.

LEMMA 2.6. *Assume that* (1.2), (1.7) *and* (1.8) *are valid and* $(x(t,y,\eta), \xi(t,y,\eta))$ *be the solution of* (2.1). *Then there are the functions* $t(x,\xi)$, $y(x,\xi) \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n\backslash 0)$ *homogenuous in* ξ *of degree 0 and* $\eta(x,\xi) \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n\backslash 0)$ *homogenuous in* ξ *of degree* 1 *such that* $(t(x,\xi), y(x,\xi), \eta(x,\xi))$ *belongs to* $\Gamma = \{(t,y,\eta)\,|\, y\eta = 0\}$ *for* $(x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^0\backslash 0$ *and satisfies*

$$\begin{aligned} &x(t(x,\xi), y(x,\xi), \eta(x,\xi)) = x \\ &\xi(t(x,\xi), y(x,\xi), \eta(x,\xi)) = \xi \end{aligned} \tag{2.39}$$

for $(x,\xi)\in \boldsymbol{R}^n\times \boldsymbol{R}^n\backslash 0$ *and moreover*

$$\begin{aligned}&|\nabla_x t(x,\xi)|+|\nabla_x y(x,\xi)|\le C\\ &|\nabla_\xi t(x,\xi)|+|\nabla_\xi y(x,\xi)|\le C(|t(x,\xi)|+\langle y(x,\xi)\rangle)|\xi|^{-1}\\ &|\nabla_x \eta(x,\xi)|\le C\langle y(x,\xi)\rangle^{-1}|\xi|\\ &|\nabla_\xi \eta(x,\xi)|\le C(1+|t(x,\xi)|\langle y(x,\xi)\rangle^{-1})\end{aligned}\tag{2.40}$$

for $(x,\xi)\in \boldsymbol{R}^n\times \boldsymbol{R}^n\backslash 0$.

PROOF. Since the Jacobian $J(t,y,\eta)=\det T(t,y,\eta)\neq 0$ for $(t,y,\eta)\in\Gamma_i$ and $\Gamma=\bigcup_{i=1}^n\Gamma_i$, the local implicit function theorem of the mapping (3.39) holds evidently. We can prove the global version of implicit function theorem of the mapping (3.39) following the proof of Theorem 1.22 in [**12**]. To do so it suffices to show that for any compact set K in $\boldsymbol{R}^n\times \boldsymbol{R}^n\backslash 0$ the set $\{(t,y,\eta)\in\Gamma;\ (x(t,y,\eta),\xi(t,y,\eta))=(x,\xi)$ for $(x,\xi)\in K\}$ is also compact in Γ. In fact, it follows from Lemma 2.2 that we have

$$\begin{aligned}C^{-1}|\xi|\le|\eta|\le C|\xi|,\\ |t|\le C|x\xi|\,|\eta|^{-1}\le C'|x|.\end{aligned}\tag{2.41}$$

Moreover the integration of (2.1) with respect to t and (2.41) yield

$$|y|\le|x|+\int_0^{|t|}|\nabla_\xi a(x(s),\xi(s))|\,ds\le C|x|.\tag{2.42}$$

Thus the invese image of K of the mapping $(x(t,y,\eta),\xi(t,y,\eta))=(x,\xi)$ is compact in Γ. Therefore (2.37) and (2.38) yield that $|T(t,y,\eta)^{-1}|\le C$ if (t,y,η) varies in a compact set in Γ_i and consequently we obtain the global implicit function theorem applying Theorem 1.22 in [**12**]. We next prove the estimates (2.40). Let $(t(x,\xi),$ $y(x,\xi),\eta(x,\xi))$ be in Γ_i. We note that the local implicit function theorem implies that $(t(x,\xi),y(x,\xi),\eta(x,\eta))$ are in $C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0)$. Differentiating (2.39) we have

$$\begin{bmatrix}\nabla_x t & \nabla_\xi t\\ \nabla_x\tilde y & \nabla_\xi\tilde y\\ \nabla_x\eta & \nabla_\xi\eta\end{bmatrix}=S(t(x,\xi),y(x,\xi),\eta(x,\xi)).\tag{2.43}$$

Hence noting $\Gamma=\bigcup_i\Gamma_i$, we obtain (2.40) by virtue of (2.37) and (2.38). Q.E.D.

LEMMA 2.7. *Let* $T(t,y,\eta)=\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}$ *be defined by* (2.26), *where* $T_{jk}(j,k=1,2)$ *are* $n\times n$ *matrices. Then* $T_{jk}(t,y,\eta)$ *are in* $C^\infty(\Gamma_i)$ *and satisfy*

$$\begin{aligned}&|\partial_t^j\partial_y^\beta\partial_\eta^\alpha T_{11}(t,y,\eta)|\le C_{\alpha\beta j}(1+|t|\langle y\rangle^{-1})\langle x(t)\rangle^{-j}\langle y\rangle^{-|\beta|}|\eta|^{-|\alpha|}\\ &|\partial_t^j\partial_y^\beta\partial_\eta^\alpha T_{21}(t,y,\eta)|\le C_{\alpha\beta j}|\eta|^{-|\alpha|+1}\langle x(t)\rangle^{-j}\langle y\rangle^{-|\beta|-1}\end{aligned}\tag{2.44}$$

$$\begin{aligned}&|\partial_t^j\partial_y^\beta\partial_\eta^\alpha T_{12}(t,y,\eta)|\le C_{\alpha\beta j}(\langle y\rangle+|t|)|\eta|^{-1-|\alpha|}\langle x(t)\rangle^{-j}\langle y\rangle^{-|\beta|}\\ &|\partial_t^j\partial_y^\beta\partial_\eta^\alpha T_{22}(t,y,\eta)|\le C_{\alpha\beta j}|\eta|^{-|\alpha|}\langle x(t)\rangle^{-j}\langle y\rangle^{-|\beta|}\end{aligned}\tag{2.45}$$

for $(t,y,\eta)\in\Gamma_i\cap\{\eta;|\eta_i|\ge|\eta|(2n)^{-1/2}\}$ *and* $(j,\alpha,\beta)\in\boldsymbol{N}^{2n}$.

PROOF. Since $x(t,y,\eta)$ and $\xi(t,y,\eta)$ are in $C^\infty(\boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0)$, it is evident that $T(t,y,\eta)$ is in $C^\infty(\Gamma_i)$. We next try to prove (2.44) and (2.45). Since $T(t,y,\eta)$ is homogenous in η, it siffices to prove (2.44) and (2.45) when $|\eta|=1$. We prove these by induction of $j+|\alpha|+|\beta|$. Lemma 2.5 implies (2.44) and (2.45) for $j+|\alpha|+|\beta|=0$. Assume that (2.44) and (2.45) are valid for $j+|\alpha|+|\beta|\le k-1$, where $k\ge 1$. We first prove (2.44) for $j=0$ and $|\alpha|+|\beta|=k$. For simplicity we write $\partial^\gamma=\partial_{\tilde{y}}^\beta\partial_\eta^\alpha$. Differentating (2.27) we get

$$(d/dt)\partial^\gamma T(t)=A(t)\partial^\gamma T(t)+F_\gamma(t), \tag{2.46}$$

where

$$F_\gamma(t)=\begin{bmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{bmatrix}=\sum_{0\ne\gamma'\le\gamma}\binom{\gamma}{\gamma'}\partial^{\gamma'}A(t)\partial^{\gamma-\gamma'}T(t). \tag{2.47}$$

Then we have from (2.46)

$$(d/dt)\begin{bmatrix}\partial^\gamma T_{11}\\ \partial^\gamma T_{21}\end{bmatrix}=A(t)\begin{bmatrix}\partial^\gamma T_{11}\\ \partial^\gamma T_{21}\end{bmatrix}+\begin{bmatrix}F_{11}\\ F_{21}\end{bmatrix}. \tag{2.48}$$

Then it follows from (2.18) in Lemma 2.4 that

$$\begin{aligned}|\partial^\gamma T_{11}(t)|&\le C\left\{(1+|t|\langle y\rangle^{-1})|\partial^\gamma T_{11}(0)|+|\partial^\gamma T_{21}(0)|+|t|\sup_{0\le s\le|t|}|F_{11}(s)|+\int_0^{|t|}|F_{21}(s)|\,ds\right\}\\ |\partial^\gamma T_{21}(t)|&\le C\left\{\langle y\rangle^{-1}|\partial^\gamma T_{11}(0)|+|\partial^\gamma T_{21}(0)|+\sup_{0\le s\le|t|}|F_{11}(s)|+\int_0^{|t|}|F_{21}(s)|\,ds\right\}.\end{aligned} \tag{2.49}$$

Moreover we can estimate F_{11} and F_{21} as follow

$$|F_{p1}(t)|\le C_\gamma\langle y\rangle^{-|\beta|-1},\quad p=1,2. \tag{2.50}$$

In fact we have from (2.47)

$$\begin{aligned}F_{11}(t)|&\le C_\gamma\sum_{0\ne\gamma'\le\gamma}\{|\partial^{\gamma'}a_{\xi x}(x(t),\xi(t))|\,|\partial^{\gamma-\gamma'}T_{11}(t)|+|\partial^{\gamma'}a_{\xi\xi}(x(t),\xi(t))|\,|\partial^{\gamma-\gamma'}T_{21}(t)|\}\\ |F_{21}(t)|&\le C_\gamma\sum_{0\ne\gamma'\le\gamma}\{|\partial^{\gamma'}a_{xx}(x(t),\xi(t))|\,|\partial^{\gamma-\gamma'}T_{11}(t)|+|\partial^{\gamma'}a_{x\xi}(x(t),\xi(t))|\,|\partial^{\gamma-\gamma'}T_{21}(t)|\}.\end{aligned} \tag{2.51}$$

Here we need the following lemma.

LEMMA 2.8. *Assume that* (2.44) *and* (2.45) *hold for* $j+|\alpha|+|\beta|\le k-1$ *and the condition* (1.8) *are valid. Then there is* $C_k>0$ *such that*

$$
(2.52)\qquad
\begin{aligned}
&|\partial_t^j\partial_{\tilde{y}}^\beta\partial_\eta^\alpha a_{\xi x}(x(t),\xi(t))| \le C_k\langle x(t)\rangle^{-1-\delta}\langle y\rangle^{-|\beta|-j}|\eta|^{-|\alpha|}\\
&|\partial_t^j\partial_{\tilde{y}}^\beta\partial_\eta^\alpha a_{xx}(x(t),\xi(t))| \le C_k\langle x(t)\rangle^{-2-\delta}\langle y\rangle^{-|\beta|-j}|\eta|^{1-|\alpha|}\\
&|\partial_t^j\partial_{\tilde{y}}^\beta\partial_\eta^\alpha a_{\xi\xi}(x(t),\xi(t))| \le C_k\langle y\rangle^{-|\beta|-j}|\eta|^{-1-|\alpha|}
\end{aligned}
$$

for $(t,y,\eta)\in\Gamma_i\cap\{|\eta_i|\ge|\eta|(2n)^{-1/2}\}$ *and* $j+|\alpha|+|\beta|=k$.

The proof of this lemma will be given in the appendix.

We continue to prove (2.50). By the assumption of induction we get from (2.52)

$$
(2.53)\qquad
\begin{aligned}
&|\partial^{\gamma-\gamma'}T_{11}(t)| \le C_\gamma(1+|t|\langle y\rangle^{-1})\langle y\rangle^{-|\beta-\beta'|}\\
&|\partial^{\gamma-\gamma'}T_{21}(t)| \le C_\gamma\langle y\rangle^{-1-|\beta-\beta'|}
\end{aligned}
$$

for $(t,y,\eta)\in\Gamma_i\cap\{|\eta_i|\ge|\eta|(2n)^{-1/2}\}\cap\{|\eta|=1\}$ and $0\ne\gamma'=(j',\alpha',\beta')\le\gamma$. Hence it follows from (2.51) and (2.53) that we get (2.50) taking account of the inequality $|t|+\langle y\rangle\le C\langle x(t)\rangle$. Besides, (2.28) yields that $|\partial^\gamma T_{11}(0)|\le C_\gamma\langle y\rangle^{-|\beta|}$ and $|\partial^\gamma T_{21}(0)|\le C_\gamma\langle y\rangle^{-1-|\beta|}$ and consequently we obtain (2.44) for $|\gamma|=k$ by virtue of (2.49). We next prove (2.44) in the case of $j+|\alpha|+|\beta|=k$ and $j\ne0$. In this case we get from (2.27)

$$
\begin{aligned}
\partial_t^j\partial^\gamma T(t) &= \partial_t^{j-1}\partial^\gamma\{A(t)T(t)\}\\
&=\sum\binom{j-1}{j'}\binom{\gamma}{\gamma'}\partial_t^{j'}\partial^{\gamma'}A(t)\partial_t^{j-1-j'}\partial^{\gamma-\gamma'}T(t),
\end{aligned}
$$

where $\gamma=(\alpha,\beta)$. Therefore we get (2.44) by use of (2.52) and the assumption of induction. We get (2.45) by the same way. Q.E.D.

LEMMA 2.9. *Let* $S(t)=\begin{bmatrix}S_{11}(t,y,\eta) & S_{12}(t,y,\eta)\\ S_{21}(t,y,\eta) & S_{22}(t,y,\eta)\end{bmatrix}$ *be the inverse of* $T(t,y,\eta)$, *where* S_{jk} *are* $n\times n$ *matrices. Then* $S(t,y,\eta)$ *is in* $C^\infty(\Gamma_i)$ *and satisfies*

$$
(2.54)\qquad
\begin{aligned}
&|\partial_t^j\partial_{\tilde{y}}^\beta\partial_\eta^\alpha S_{11}(t,y,\eta)| \le C_{\alpha\beta j}|\eta|^{-|\alpha|}\langle y\rangle^{-|\beta|-j}\\
&|\partial_t^j\partial_{\tilde{y}}^\beta\partial_\eta^\alpha S_{12}(t,y,\eta)| \le C_{\alpha\beta j}(|t|+\langle y\rangle)|\eta|^{-1-|\alpha|}\langle y\rangle^{-|\beta|-j}
\end{aligned}
$$

$$
(2.55)\qquad
\begin{aligned}
&|\partial_t^j\partial_{\tilde{y}}^\beta\partial_\eta^\alpha S_{21}(t,y,\eta)| \le C_{\alpha\beta j}\langle y\rangle^{-1}|\eta|^{1-|\alpha|}\langle y\rangle^{-|\beta|-j}\\
&|\partial_t^j\partial_{\tilde{y}}^\beta\partial_\eta^\alpha S_{22}(t,y,\eta)| \le C_{\alpha\beta j}(1+|t|\langle y\rangle^{-1})|\eta|^{-|\alpha|}\langle y\rangle^{-|\beta|-j}
\end{aligned}
$$

for $(t,y,\eta)\in\Gamma_i\cap\{|\eta_i|\ge|\eta|(2n)^{-1/2}\}$ *and* $(j,\alpha,\beta)\in\boldsymbol{N}^{2n}$, *where* $\tilde{y}=(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)$.

PROOF. Since $S(t)$ satisfies (2.36), (2.37) and (2.38) and $S(0)$ is given by (2.34), repeating the same argument as one in Lemma 2.7 we can get (2.54) and (2.55). Q.E.D.

LEMMA 2.10. *The implicit function* $(t(x,\xi),y(x,\xi),\eta(x,\xi))$ *of* (2.39) *is in* $C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash0)$ *and satisfies*

(2.56) $$|\partial_x^\beta \partial_\xi^\alpha t(x,\xi)| + |\partial_x^\beta \partial_\xi^\alpha y(x,\xi)| \\ \le C_{\alpha\beta}\langle y(x,\xi)\rangle^{-|\beta|}|\xi|^{-|\alpha|}(|t(x,\xi)| + \langle y(x,\xi)\rangle)(1 + |t(x,\xi)|\langle y(x,\xi)\rangle^{-1})^{|\alpha+\beta|}$$

(2.57) $$|\partial_x^\beta \partial_\xi^\alpha \eta(x,\xi)| \le C_{\alpha\beta}\langle y(x,\xi)\rangle^{-|\beta|}|\xi|^{1-|\alpha|}(1 + |t(x,\xi)|\langle y(x,\xi)\rangle^{-1})^{|\alpha+\beta|}$$

for $x, \xi \in \boldsymbol{R}^n(|\xi| \ge 1)$ *and* $\alpha, \beta \in \boldsymbol{N}^n$.

Proof. It is evident that the local implicit function theorem yields that $(t,y,\eta)(x,\xi)$ is in $C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n\backslash 0)$. We try to prove (2.56) and (2.57). Since $\Gamma = \bigcup_{i=1}^n \Gamma_i$, we may assume $(t,y,\eta)(x,\xi)$ is in Γ_i for some i. Recalling (2.43) we have proved (2.56) and (2.57) for $(\alpha,\beta) = 0$ in Lemma 2.6. Let $k \ge 1$. Assume (2.56) and (2.57) are valid for $|\alpha+\beta| \le k-1$. Noting that $(\nabla_x t, \nabla_x \tilde{y}) = S_{11}(t(x,\xi), y(x,\xi), \eta(x,\xi))$, $(\nabla_\xi t, \nabla_\xi \tilde{y}) = S_{12}(t(x,\xi), y(x,\xi), \eta(x,\xi))$, $\nabla_x \eta = S_{21}(t(x,\xi), y(x,\xi), \eta(x,\xi))$ and $\nabla_\xi \eta = S_{22}(t(x,\xi), y(x,\xi), \eta(x,\xi))$ hold, the following lemma implies (2.56) and (2.57).

Lemma 2.11. *Assume* (2.56) *and* (2.57) *are valid for* $|\alpha+\beta| \le k-1$. *Then we have*

$$|\partial_x^\beta \partial_\xi^\alpha S_{11}(x,\xi)| \le C_{\alpha\beta}\langle y(x,\xi)\rangle^{-|\beta|}|\xi|^{-|\alpha|}(1 + |t(x,\xi)|\langle y(x,\xi)\rangle^{-1})^{|\alpha+\beta|}$$

(2.58) $$|\partial_x^\beta \partial_\xi^\alpha S_{12}(x,\xi)| \\ \le C_{\alpha\beta}\langle y(x,\xi)\rangle^{-|\beta|}|\xi|^{-1-|\alpha|}(|t(x,\xi)| + \langle y(x,\xi)\rangle)(1 + |t(x,\xi)|\langle y(x,\xi)\rangle^{-1})^{|\alpha+\beta|}$$

(2.59) $$|\partial_x^\beta \partial_\xi^\alpha S_{21}(x,\xi)| \le C_{\alpha\beta}\langle y(x,\xi)\rangle^{-1-|\beta|}|\xi|^{1-|\alpha|}(1 + |t(x,\xi)|\langle y(x,\xi)\rangle^{-1})^{|\alpha+\beta|} \\ |\partial_x^\beta \partial_\xi^\alpha S_{22}(x,\xi)| \le C_{\alpha\beta}\langle y(x,\xi)\rangle^{-|\beta|}|\xi|^{-|\alpha|}(1 + |t(x,\xi)|\langle y(x,\xi)\rangle^{-1})^{|\alpha+\beta|}$$

for $|\alpha+\beta| \le k$ *and* $(x,\xi)(|\xi| \ge 1)$ *in the inverse image of* Γ_i *of the mapping* (2.39), *where we write* $S_{j\ell}(x,\xi) = S_{j\ell}(t(x,\xi), y(x,\xi), \eta(x,\xi))$.

The proof of this lemma will be given in the appendix.

§3. Proof of main Theorem

For $f(x,\xi)$ a function in $C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n\backslash 0)$ we denote the Hamilton vector field of f by $H_f = \sum_{j=1}^n \{\partial_{\xi_j} f \partial_{x_j} - \partial_{x_j} f \partial_{\xi_j}\}$. Let $a_2(x,\xi)$ be given by (1.3) and put $a(x,\xi) = (2a_2(x,\xi))^{1/2}$. In this section we assume (1.2), (1.7) and (1.8) are valid.

Let $g(x,\xi)$ be a real valued function in $C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n\backslash 0)$ and consider the following equation

(3.1) $$H_{a_2}\lambda(x,\xi) = a(x,\xi)g(x,\xi), \quad (x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n\backslash 0, \\ \lambda|_{\Gamma_0} = 0,$$

where $\Gamma_O = \{(y,\eta) \in \boldsymbol{R}^n \times \boldsymbol{R}^n\backslash 0;\ y\eta = \sum_{j=1}^n y_j\eta_j = 0\}$. Then we have

Lemma 3.1. *The solution* $\lambda(x,\xi)$ *of* (3.1) *is given as follows*

(3.2) $$\lambda(x,\xi) = \int_0^{t(x,\xi)} g(x(s, y(x,\xi), \eta(x,\xi)), \xi(s, y(x,\xi), \eta(x,\xi))\,ds$$

where $(x(t,y,\eta),\xi(t,y,\eta))$ *is the solution of* (2.1) *and* $(t(x,\xi),y(x,\xi),\eta(x,\xi))$ *is the implicit function of* (2.39).

PROOF. Since $H_{a_2} = aH_a$, the equation (3.1) is equivalent to

$$(3.1)' \qquad \begin{aligned} H_a\lambda(x,\xi) &= g(x,\xi), \\ \lambda\,|_{\Gamma_0} &= 0. \end{aligned}$$

Solving this equation we obtain (3.2), noting that $t(x,\xi)|_{\Gamma_0} = 0$. Q.E.D.

Recalling (2.11) we can see

$$(3.3) \qquad H_a x\xi = \tilde{a}(x,\xi) > 0$$

for $(x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \backslash 0$. Take $\chi(t) \in C_0^\infty(\boldsymbol{R})$ such that $\chi(t) = 1$ when $|t| \le 1$, $\chi(t) = 0$ when $|t| \ge 2, 0 \le \chi(t) \le 1$ for all t and $\chi'(t)t \le 0$ for all t. Put

$$(3.4) \qquad \begin{aligned} g_1(x,\xi) &= M_1\langle x\rangle^{-\sigma}\chi(C^2\langle x\rangle/a(x,\xi)), \\ g_2(x,\xi) &= M_2\langle Cx\xi/a(x,\xi)\rangle^{-\sigma}\chi(\langle Cx\xi/a(x,\xi)\rangle/a(x,\xi))\tilde{a}(x,\xi)/a(x,\xi) \end{aligned}$$

where $\langle x\rangle = (1+|x|^2)^{1/2}, \langle Cx\xi/a(x,\xi)\rangle = (1 + C^2|x\xi|^2 a(x,\xi)^{-2})^{1/2}$ and M_i, positive constants determined later on and C is a positive constant such that $\langle Cx\xi/a(x,\xi)\rangle \le C^2\langle x\rangle$ and $|t(x,\xi)| \le C|x\xi/a(x,\xi)|$. Set

$$(3.5) \qquad \begin{aligned} \lambda_1(x,\xi) &= \int_0^{t(x,\xi)} g_1(x(s,y(x,\xi),\eta(x,\xi)),\xi(s,y(x,\xi),\eta(x,\xi))\,ds \\ \lambda_2(x,\xi) &= M_2\int_0^{Cx\xi/a(x,\xi)} \langle s\rangle^{-\sigma}\chi(s/a(x,\xi))\,ds. \end{aligned}$$

Then $H_a\lambda_i(x,\xi) = g_i(x,\xi)(i=1,2)$ hold. Taking account of $\langle Cx\xi/a(x,\xi)\rangle \le C^2\langle x\rangle$ and consequently $\chi(C^2\langle x\rangle/a(x,\xi)) \le \chi(\langle Cx\xi/a(x,\xi)\rangle/a(x,\xi))$ we can choose M_1 and M_2 such that

$$(3.6) \qquad g_1(x,\xi) \le g_2(x,\xi)$$

for $(x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n\backslash 0$. Moreover since $x\xi$ and $t(x,\xi)$ have the same sign and $|t(x,\xi)| \le \langle Cx\xi/a(x,\xi)\rangle$ and $g_1(x,\xi) \le M_2\langle t(x,\xi)\rangle^{-\sigma}\chi(\langle t(x,\xi)\rangle/a(x,\xi))$ are valid, we have from (3.5) and from the fact $t(x(s),\xi(s)) = s$

$$(3.7) \qquad (x\xi)\{\lambda_1(x,\xi) - \lambda_2(x,\xi)\} \le 0$$

for $(x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n\backslash 0$. For $\varepsilon > 0$ we define

$$(3.8) \qquad \lambda(x,\xi) = -\lambda_1(x,\xi)\chi(x\omega/\varepsilon\langle x\rangle) - \lambda_2(x,\xi)\{1 - \chi(x\omega/\varepsilon\langle x\rangle)\}$$

where $x\omega = \sum_{j=1}^n x_j\omega_j$ and $\omega = \xi/|\xi|$. Then we have

LEMMA 3.2. *The function* $\lambda(x,\xi)$ *satisfies*

$$(3.9) \qquad H_a\lambda(x,\xi) \le -g_1(x,\xi)$$

and

$$|\lambda(x,\xi)| \le \begin{cases} \ell_0 \log(1+\min(\langle x\rangle, |\xi|)), & \text{if } \sigma = 1 \\ C & \text{if } \sigma > 1 \end{cases} \tag{3.10}$$

for $(x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \backslash 0$.

Proof. Noting that $(x\omega)\chi'(x\omega/\varepsilon\langle x\rangle) \le 0$ and $H_a(x\omega/\varepsilon\langle x\rangle) = \varepsilon^{-1}\langle x\rangle^{-1}[\tilde{a}(x,\omega) + (x\omega)\sum_{j=1}^n \{\partial_{x_j} a(x,\omega)\omega_j - \partial_{\xi_j} a(x,\xi) x_j \langle x\rangle^{-2}\}] \ge 0$ on $\operatorname{supp}\{\chi'(x\omega/\varepsilon\langle x\rangle)\}$ if $\varepsilon > 0$ is small enough, we obtain from (3.6) and (3.7)

$$\begin{aligned} H_a\lambda(x,\xi) &= -(H_a\lambda_1)\chi - H_a\lambda_2(1-\chi) - (\lambda_1 - \lambda_2)H_a\chi \\ &= -g_1 + (g_1 - g_2)(1-\chi) - (\lambda_1 - \lambda_2)\chi'(x\omega/\varepsilon\langle x\rangle)H_a(x\omega/\varepsilon\langle x\rangle) \\ &\le -g_1, \end{aligned}$$

where we write $\chi = \chi(x\omega/\varepsilon\langle x\rangle)$. This proves (3.9). To show (3.10) it is enough to prove that λ_1 and λ_2 satisfy (3.10) on $\operatorname{supp}\{\chi(x\omega/\varepsilon\langle x\rangle)\}$ and on $\operatorname{supp}\{1 - \chi(x\omega/\varepsilon\langle x\rangle)\}$ respectively. We see that (2.8) implies $\langle x(s,y,\eta)\rangle \ge c_0(1+|s|)$. Moreover we have $C^{-1}|s| \le |x(s,y,\eta)| \le C|\xi|$ on $\operatorname{supp}\{\chi(x(s,y,\eta)/a(x,\xi))\}$ and $|s| \le C|\xi|$ on $\operatorname{supp}\{\chi(s/a(x,\xi))\}$ when $y = y(x,\xi)$ and $\eta = \eta(x,\xi)$. Hence we have from (3.5)

$$|\lambda_i(x,\xi)| \le C\int_0^{\min\{\langle x\rangle, |\xi|\}} (1+s)^{-\sigma}\,ds \quad (i = 1,2).$$

This yields (3.10). Q.E.D.

Lemma 3.3. *For any* α *and* β *in* $\boldsymbol{N}^n$ *with* $|\alpha+\beta| \ne 0$ *there is* $C_{\alpha\beta} > 0$ *such that*

$$|\partial_x^\beta \partial_\xi^\alpha \lambda(x,\xi)| \le \begin{cases} C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|}\log\{1+\min(\langle x\rangle, |\xi|)\} & \text{if } \sigma = 1 \\ C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|} & \text{if } \sigma > 1 \end{cases} \tag{3.11}$$

for $(x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \backslash 0$.

Proof. Noting

$$|\partial_x^\beta \partial_\xi^\alpha \{\chi(\langle x\rangle/a(x,\xi))\}| \le C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|}, \tag{3.12}$$

we obtain

$$|\partial_x^\beta \partial_\xi^\alpha g_1(x,\xi)| \le C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|}. \tag{3.13}$$

Moreover integrating (2.1) with respect to t we have

$$y(x,\xi) = x - \int_0^{t(x,\xi)} a_\xi(x(s,y(x,\xi),\eta(x,\xi)), \xi(s,y(x,\xi),\eta(x,\xi)))\,ds.$$

This implies $\langle y(x,\xi)\rangle \ge \langle x\rangle - C|t(x,\xi)|$. On the other hand taking account of (2.8) we have $|t(x,\xi)| \le C|x\omega|$. Therefore we obtain

$$\langle x\rangle + |t(x,\xi)| \le C\langle y(x,\xi)\rangle \tag{3.14}$$

for $(x,\xi)\in\operatorname{supp}\{\chi(x\omega/\varepsilon\langle x\rangle)\}$ if $\varepsilon>0$ is small enough. Therefore it follows from (2.56) and (2.57) that

$$
\begin{aligned}
&|\partial_x^\beta\partial_\xi^\alpha t(x,\xi)|+|\partial_x^\beta\partial_\xi^\alpha y(x,\xi)|\le C_{\alpha\beta}\langle x\rangle^{1-|\beta|}|\xi|^{-|\alpha|}\\
&|\partial_x^\beta\partial_\xi^\alpha \eta(x,\xi)|\le C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{1-|\alpha|}
\end{aligned}
\tag{3.15}
$$

for $(x,\xi)\in\operatorname{supp}\{\chi(x\omega/\varepsilon\langle x\rangle)\}$ and consequently we have from Lemma 2.7

$$
\begin{aligned}
&|\partial_x^\beta\partial_\xi^\alpha x(s,y(x,\xi),\eta(x,\xi))|\le C_{\alpha\beta}\langle x\rangle^{1-|\beta|}|\xi|^{-|\alpha|}\\
&|\partial_x^\beta\partial_\xi^\alpha \xi(s,y(x,\xi),\eta(x,\xi))|\le C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{1-|\alpha|}
\end{aligned}
\tag{3.16}
$$

for $(x,\xi)\in\operatorname{supp}\{\chi(x\omega/\varepsilon\langle x\rangle)\}$. Hence from (3.13) and (3.16) we get by use of Lemma A.1 in the appendix

$$
\begin{aligned}
&|\partial_x^\beta\partial_\xi^\alpha\{g_1(x(s,y(x,\xi),\eta(x,\xi)),\xi(s,y(x,\xi),\eta(x,\xi))\}|\\
&\qquad\le C_{\alpha\beta}\langle x(s,y(x,\xi),\eta(x,\xi))\rangle^{-\sigma}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|}
\end{aligned}
\tag{3.17}
$$

for $(x,\xi)\in\operatorname{supp}\{\chi(x\omega/\varepsilon\langle x\rangle)\}$. Taking account of the following estimates

$$
\begin{aligned}
&|\partial_x^\beta\partial_\xi^\alpha\{\chi(\langle x(s,y(x,\xi),\eta(x,\xi))\rangle/a(x,\xi))\}|\le C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|}\\
&|\partial_x^\beta\partial_\xi^\alpha\{\chi(x\omega/\varepsilon\langle x\rangle)\}|\le C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|}
\end{aligned}
\tag{3.18}
$$

for $(x,\xi)\in\operatorname{supp}\{\chi(x\omega/\varepsilon\langle x\rangle)\}$. We obtain from (3.17) and (3.5)

$$
|\partial_x^\beta\partial_\xi^\alpha\{\lambda_1(x,\xi)\chi(x\omega/\varepsilon\langle x\rangle)\}|\le\begin{cases} C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|}\log(1+|\xi|), & \sigma=1\\ C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|}, & \sigma>1\end{cases}
\tag{3.19}
$$

for $(x,\xi)\in \boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0$. Noting that we have from (3.5)

$$
\lambda_2(x,\xi)=\int_0^{Cx\xi/a(x,\xi)}\langle s\rangle^{-\sigma}\chi(s/a(x,\xi))\,ds,
\tag{3.20}
$$

we can see easily

$$
|\partial_x^\beta\partial_\xi^\alpha\lambda_2(x,\xi)|\le C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|}
\tag{3.21}
$$

for $(x,\xi)\in\operatorname{supp}\{1-\chi(x\omega/\varepsilon\langle x\rangle)\}$ and $|\alpha+\beta|\neq 0$. Therefore we get from (3.18), (3.10) and (3.21)

$$
\begin{aligned}
|\partial_x^\beta\partial_\xi^\alpha\{\lambda_2(x,\xi)(1-\chi(x\omega/\varepsilon\langle x\rangle))\}|&\le C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|}\log\{1+\min(\langle x\rangle,|\xi|)\},\quad\sigma=1\\
&\le C_{\alpha\beta}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|},\quad\sigma>1
\end{aligned}
$$

for $(x,\xi)\in \boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0$. This together with (3.19) proves (3.11). Q.E.D.

LEMMA 3.4. *Assume* $b_j(t,x)(j=1,\dots,n)$ *satisfy* (1.6) *with* $\sigma \geq 1$. *Then there are* $C>0, M_i(i=1,2)>0$ *and* $\varepsilon>0$ *such that* $\lambda(x,\xi)$ *defined in* (3.8) *is verified with*

$$H_{a_2}\lambda(x,\xi)-\sum_{j=1}^{n}\operatorname{Im} b_j(t,x)\xi_j \leq C \tag{3.22}$$

for $(x,\xi)\in \boldsymbol{R}^n\times\boldsymbol{R}^n$.

PROOF. It follows from (3.4), (3.9) and (1.6) that

$$\begin{aligned}H_{a_2}\lambda(x,\xi)-\sum_{j=1}^{n}\operatorname{Im} b_j(t,x)\xi_j &\leq -M_1\langle x\rangle^{-\sigma}a(x,\xi)\chi(C^2\langle x\rangle/a(x,\xi))+M_0\langle x\rangle^{-\sigma}|\xi| \\ &\leq -M_1\langle x\rangle^{-\sigma}c_0|\xi|\chi(C^2\langle x\rangle/a(x,\xi))+M_0\langle x\rangle^{-\sigma}|\xi| \\ &\leq M_0|\xi|\langle x\rangle^{-\sigma}\{1-\chi(C^2\langle x\rangle/a(x,\xi))\}\leq C\end{aligned}$$

for $x,\xi\in\boldsymbol{R}^n$, if we take $M_1\geq M_0/c_0$. Q.E.D.

For $h\geq 1$ we define

$$\Lambda(x,\xi)=-\lambda(x,\xi)(1-\chi(|\xi|/h)). \tag{3.23}$$

Noting that $\Lambda(x,\xi)=0$ for $|\xi|\leq h$ and

$$|\partial_\xi^\alpha\chi(|\xi|/h)|\leq C_\alpha\langle\xi\rangle_h^{-|\alpha|} \tag{3.24}$$

for $\xi\in\boldsymbol{R}^n$ and $\alpha\in\boldsymbol{N}^n$, where $\langle\xi\rangle_h=(h^2+|\xi|^2)^{1/2}$, we have from (3.11)

$$|\partial_x^\beta\partial_\xi^\alpha\Lambda(x,\xi)|\leq\begin{cases}C_{\alpha\beta}\langle x\rangle^{-|\beta|}\langle\xi\rangle_h^{-|\alpha|}\log(1+\min(\langle x\rangle,\langle\xi\rangle_h), & \sigma=1,\\ C_{\alpha\beta}\langle x\rangle^{-|\beta|}\langle\xi\rangle_h^{-|\alpha|}, & \sigma>1\end{cases} \tag{3.25}$$

for $x,\xi\in\boldsymbol{R}^n, h\geq 1$ and $\alpha,\beta\in\boldsymbol{N}^n$, where $C_{\alpha\beta}$ are independent of h. Moreover (3.22) and (3.23) yield

$$H_{a_2}\Lambda(x,\xi)+\sum_{j=1}^{n}\operatorname{Im} b_j(t,x)\xi_j\geq -C \tag{3.26}$$

for $x,\xi\in\boldsymbol{R}^n$. Moreover it follows from (3.25) that

$$|\partial_x^\beta\partial_\xi^\alpha\{e^{\Lambda(x,\xi)}\}|\leq\begin{cases}C_{\alpha\beta}\langle x\rangle^{-|\beta|}\langle\xi\rangle_h^{-|\alpha|} & \text{if } \sigma>1\\ C_{\alpha\beta}\langle x\rangle^{-|\beta|}\langle\xi\rangle_h^{-|\alpha|}e^{\Lambda(x,\xi)} & \\ \quad\times\{\log(1+\min(\langle x\rangle,\langle\xi\rangle_h))\}^{|\alpha+\beta|}, & \text{if } \sigma=1\end{cases} \tag{3.27}$$

for $x,\xi\in\boldsymbol{R}^n, h\geq 1$ and $\alpha,\beta\in\boldsymbol{N}^n$, where $C_{\alpha\beta}$ are independent of h.

Denote by $e^\Lambda(x,D)$ the pseudodifferential operator with its symbol $e^{\Lambda(x,\xi)}$.

LEMMA 3.5. *Let* $p(x,\xi)$ *be a symbol satisfying*

$$|\partial_x^\beta\partial_\xi^\alpha p(x,\xi)|\leq C_{\alpha\beta}\langle\xi\rangle_h^{m-|\alpha|} \tag{3.28}$$

for $x, \xi \in \boldsymbol{R}^n, h \geq 1$ *and* $\alpha, \beta \in \boldsymbol{N}^n$, *where* $C_{\alpha\beta}$ *are independent of* h. *Then* $p_\Lambda(x,D) = e^{-\Lambda}(x,D)p(x,D)e^{\Lambda}(x,D)$ *is also a pseudodifferential operator of which symbol is given as follows*

$$
\begin{aligned}
p_\Lambda(x,\xi) &= \sum_{|\alpha|\leq 1} \omega_\alpha^0(-\Lambda) \sum_{\alpha'\leq\alpha} \omega_0^{\alpha'}(\Lambda) D_x^{\alpha-\alpha'} p(x,\xi) + \sum_{|\alpha|=2} \alpha!^{-1}\omega_\alpha^0(-\Lambda)D_x^\alpha p(x,\xi) + r(x,\xi) \\
&= (1 + i\sum_{j=1}^n \Lambda_{x_j}\Lambda_{\xi_j})p(x,\xi) + i^{-1}H_p\Lambda(x,\xi) + \sum_{|\alpha|=2} \alpha!^{-1}\omega_\alpha^0(-\Lambda)D_x^\alpha p(x,\xi) + r(x,\xi)
\end{aligned} \tag{3.29}
$$

where $\omega_\alpha^\beta(\Lambda) = e^{-\Lambda}\partial_\xi^\alpha D_x^\beta e^\Lambda, D_x = i^{-1}\partial_x, H_p\Lambda = \sum_{j=1}^n\{\partial_{\xi_j}p\partial_{x_j}\Lambda - \partial_{x_j}p\partial_{\xi_j}\Lambda\}$ *and* $r(x,\xi)$ *satisfies*

$$
|\partial_x^\beta\partial_\xi^\alpha r(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle_h^{m-2-|\alpha|} \tag{3.30}
$$

for $x, \xi \in \boldsymbol{R}^n, h \geq 1$ *and* $\alpha, \beta \in \boldsymbol{N}^n$, *where* $C_{\alpha\beta}$ *are independent of* h.

PROOF. Since it follows from (3.10) and (3.23) that $e^{\pm\Lambda(x,\xi)} \leq C(1+\min(\langle x\rangle, \langle\xi\rangle_h))^{\ell_0}$ if $\sigma = 1$ and $e^{\pm\Lambda} \leq C$ if $\sigma > 1$ from (3.10), from (3.27) we obtain

$$
\begin{aligned}
\sigma(p(x,D)e^\Lambda(x,D))(x,\xi) &= \sum_{|\alpha|\leq 2\ell_0+2} \alpha!^{-1}\partial_\xi^\alpha p(x,\xi)D_x^\alpha e^{\Lambda(x,\xi)} + r_1(x,\xi) \\
&= e^\Lambda\left\{p + i^{-1}\sum_{|\alpha|=1}\partial_\xi^\alpha p\partial_x^\alpha\Lambda + p_1(x,\xi)\right\} \\
&= e^{\Lambda(x,\xi)}\tilde{p}(x,\xi)
\end{aligned} \tag{3.31}
$$

where

$$
|\partial_x^\beta\partial_\xi^\alpha r_1(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle_h^{m-\ell_0-2}
$$

and $p_1 = \sum_{2\leq|\alpha|\leq 2\ell_0+2} \alpha!^{-1}\partial_\xi^\alpha p\omega_0^\alpha(\Lambda) + e^{-\Lambda}r_1$ satisfies

$$
|\partial_x^\beta\partial_\xi^\alpha p_1(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-2-|\alpha|} \tag{3.32}
$$

for $x, \xi \in \boldsymbol{R}^n$. Besides, we can see from (3.31)

$$
\sigma(e^{-\Lambda}(x,D)p(x,D)e^\Lambda(x,D))(x,\xi) = \sum_{|\alpha|\leq 2\ell_0} \alpha!^{-1}\omega_\alpha^0(-\Lambda)D_x^{\alpha-\alpha'}\tilde{p} + r_2
$$

where $\tilde{p}$ is defined in the right hand of (3.31) and $r_2(x,\xi)$ satisfied (3.32). Noting that $\tilde{p} = p + i^{-1}\sum_{j=1}^n\{\partial_{\xi_j}p\partial_{x_j}\Lambda - \partial_{x_j}p\partial_{\xi_j}\Lambda\} + p_1$ and that (3.27) yields

$$
\begin{aligned}
|\omega_\alpha^0(-\Lambda)\omega_0^{\alpha'}(\Lambda)| &\leq C_\alpha\{\log(1+\min(\langle x\rangle, \langle\xi\rangle_h))\}^{|\alpha|+|\alpha'|}\langle\xi\rangle_h^{-|\alpha|}\langle x\rangle^{-|\alpha'|} \\
&\leq C_\alpha\langle\xi\rangle_h^{-2}
\end{aligned} \tag{3.33}
$$

if $|\alpha| \geq 2$ and $\alpha' \neq 0$ or $|\alpha| \geq 3$ and $\alpha' = 0$, we obtain (3.29) and (3.30). Q.E.D.

Taking $p(x,\xi)=1$ we have from (3.29)

$$\begin{aligned}\sigma(e^{-\Lambda}(x,D)e^{\Lambda}(x,D))(x,\xi) &= 1+i\sum_{j=1}\partial_{\xi_j}\Lambda\partial_{x_j}\Lambda + r \\ &= 1+j(x,\xi),\end{aligned}\tag{3.34}$$

where $j(x,\xi)$ satisfies

$$\begin{aligned}|\partial_x^\beta\partial_\xi^\alpha j(x,\xi)| &\le C_{\alpha\beta}(\log(1+\min(\langle x\rangle,\langle\xi\rangle_h)))^2\langle x\rangle^{-1}\langle\xi\rangle_h^{-1-|\alpha|} \\ &\le C_{\alpha\beta}h^{-1}\langle\xi\rangle^{-|\alpha|}\end{aligned}\tag{3.35}$$

for $x,\xi\in \boldsymbol{R}^n$ and $h\ge 1$. Hence we have the inverse of $I+j(x,D)$ and consequently we obtain

$$(e^{\Lambda}(x,D))^{-1} = (I+j(x,D))^{-1}e^{-\Lambda}(x,D)\tag{3.36}$$

if h is large enough. Moreover we can see that $(e^{\Lambda}(x,D))^{-1}$ becomes a pseudo differential operator (c.f. [**3**]).

LEMMA 3.6. *Let $p(x,\xi)$ be a symbol satisfying* (3.28). *Then*

$$p(\Lambda;x,D) = (e^{\Lambda}(x,D)^{-1}p(x,D)e^{\Lambda}(x,D))\tag{3.37}$$

is a pseudodifferential operator of which symbol $p(\Lambda;x,\xi)$ is given by

$$p(\Lambda;x,\xi) = p(x,\xi) + i^{-1}H_p\Lambda(x,\xi) + \sum_{|\alpha|=2}\alpha!^{-1}\omega_\alpha^0(-\Lambda)D_x^\alpha p + r(x,\xi)\tag{3.38}$$

where $r(x,\xi)$ satisfies (3.30).

PROOF. It follows from Lemma 3.5 and (3.36) that

$$p(\Lambda;x,D) = (I+j(x,D))^{-1}p_\Lambda(x,D).\tag{3.39}$$

Moreover noting that we have from (3.34)

$$\sigma((I+j(x,D))^{-1})(x,\xi) = 1 - i\sum \Lambda_{\xi_j}\Lambda_{x_j} + \tilde r(x,\xi)$$

where $\tilde r$ satisfies (3.30) with $m=0$, we obtain (3.38) by virtue of (3.39), (3.29) and (3.35). Q.E.D.

If there is $\mu>0$ such that $p(x,\xi)$ satisfies

$$|\partial_x^\beta\partial_\xi^\alpha p(x,\xi)| \le C_{\alpha\beta}\langle x\rangle^{-\mu}\langle\xi\rangle_h^{m-|\alpha|}\tag{3.40}$$

for $x,\xi\in\boldsymbol{R}^n$ and $\alpha,\beta\in \boldsymbol{N}^n$ with $|\beta|\ge 2$, the third term in the right side of (3.38) satisfies

(3.30). Moreover if $p(x,\xi)$ satisfies (3.40) for $|\beta| \geq 1$ then we have

$$|\partial_x^\beta \partial_\xi^\alpha H_p \Lambda(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle_h^{m-1-|\alpha|} \tag{3.41}$$

for $x, \xi \in \boldsymbol{R}^n$ and $\alpha, \beta \in \boldsymbol{N}^n$. Hence we get

$$p(\Lambda; x, \xi) = p(x,\xi) + i^{-1} H_p \Lambda(x,\xi) + r(x,\xi) \tag{3.42}$$

where r satisfies (3.30).

Since $a_2(x,\xi) = \sum a_{jk}(x)\xi_j\xi_k$ satisfies (3.40) with $\mu = 2+\delta$ and $m = 2$, applying (3.41) to $p = a_2(x,\xi)$ we obtain

$$a_2(\Lambda; x, \xi) = a_2(x,\xi) + i^{-1} H_{a_2} \Lambda(x,\xi) + r(x,\xi) \tag{3.43}$$

where r satisfies

$$|\partial_x^\beta \partial_\xi^\alpha r(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle_h^{-|\alpha|} \tag{3.44}$$

for $x, \xi \in \boldsymbol{R}^n$ and $\alpha, \beta \in \boldsymbol{N}^n$. Therefore we get

$$\begin{aligned} &e^\Lambda(x,D)^{-1} \sum_{j,k} \partial_{x_j}\{a_{jk}(x)\partial_{x_k}(e^\Lambda(x,D)u(x))\} \\ &= -e^\Lambda(x,D)^{-1}\left\{a_2(x,D) + \sum_{j=1}^n (D_j a_{jk}(x) D_k\right\} e^\Lambda(x,D)u(x) \\ &= \left\{-a_2(x,D) - i^{-1} H_{a_2}\Lambda(x,D) - \sum_{j=1}^n (D_j a_{jk}(x) D_k + r(x,D)\right\} u(x) \\ &= \sum_{j,k} \partial_{x_j}(a_{jk}(x)\partial_{x_k} u(x)) - i^{-1}(H_{a_2}\Lambda)(x,D)u(x) + r(x,D)u(x) \end{aligned} \tag{3.45}$$

where r satisfies (3.44). Moreover we can see

$$\begin{aligned} e^\Lambda(x,D)^{-1} \sum_j b_j(t,x)\partial_{x_j} e^\Lambda(x,D) &= i e^\Lambda(x,D)^{-1} b(t,x,D) e^\Lambda(x,D) \\ &= i\left\{\sum_j b_j D_{x_j} + i^{-1} H_b \Lambda(x,D) + r_1(x,D)\right\}, \end{aligned} \tag{3.46}$$

where r_1 satisfies (3.44). We note that $H_b\Lambda = \sum_j \{\partial_{\xi_j} b \partial_{x_j}\Lambda - \partial_{x_j} b \partial_{\xi_j}\Lambda\}$ does not necessarily satisfy (3.44) when $\sigma = 1$. It follows from (3.25) that

$$|\mathrm{Im}(H_b\Lambda(x,\xi)| = |H_{\mathrm{Im}\, b}\Lambda(x,\xi)| \leq C \log\{1 + \min(\langle x \rangle, \langle \xi \rangle_h)\} \tag{3.47}$$

for $x, \xi \in \boldsymbol{R}^n$ if $\sigma = 1$. Thus we get

$$\begin{aligned} L(\Lambda; t, x, D) &= e^\Lambda(x,D)^{-1} L e^\Lambda(x,D) \\ &= L + H_{a_2}\Lambda(x,D) - H_b\Lambda(x,D) + r(x,D) \end{aligned} \tag{3.48}$$

where r satisfies (3.44).

Since $\operatorname{Im} H_b\Lambda$ is not bounded when $\sigma = 1$, to eliminate this term we must transform again the operator $L(\Lambda; x, D)$. Put

$$\Lambda_0(t,x,\xi) = tM_3(1-\chi(|\xi|/h))\log(1+a(x,\xi)), \tag{3.49}$$

where $a = (2a_2)^{1/2}$. Noting $H_{a_2}\log(1+a) = 0$, we can take $M_3 > 0$ such that

$$\partial_t\Lambda_0(t,x,\xi) + H_{a_2}\Lambda_0(t,x,\xi) - \operatorname{Im} H_b\Lambda(x,\xi) \geq -C \tag{3.50}$$

for $x, \xi \in \boldsymbol{R}^n$ and $t \in [0,T]$. Moreover we can see from (3.49)

$$|\partial_x^\beta\partial_\xi^\alpha\Lambda_0(t,x,\xi)| \leq C_{\alpha\beta}\langle x\rangle^{-|\beta|}\langle\xi\rangle_h^{-|\alpha|}\log(1+h) \tag{3.51}$$

for $x, \xi \in \boldsymbol{R}^n$ and $\alpha, \beta \in \boldsymbol{N}^n$ with $|\alpha+\beta| \neq 0$. Therefore we have the inverse $e^{\Lambda_0}(t,x,D)^{-1}$ if h is large enough. Furthermore we note that $H_b\Lambda_0(t,x,\xi)$ satisfies (3.44). Repeating the same argment as above we can get

$$\begin{aligned} &e^{\Lambda_0}(t,x,D)^{-1}e^{\Lambda}(x,D)^{-1}Le^{\Lambda}(x,D)e^{\Lambda_0}(t,x,D) \\ &\qquad = L + H_{a_2}\Lambda(x,D) - H_b\Lambda(x,D) + \partial_t\Lambda_0(t,x,D) + H_{a_2}\Lambda_0(t,x,D) \\ &\qquad\quad + r(t,x,D), \end{aligned} \tag{3.52}$$

where r satisfies (3.44).

Now we can prove our main theorem. Set

$$u(t,x) = e^{\Lambda}(x,D)e^{\Lambda_0}(t,x,D)w(t,x).$$

Then we get from (1.1) the following equation of w

$$\begin{aligned} &\{L + H_{a_2}\Lambda(x,D) + H_b\Lambda(x,D) + \partial_t\Lambda_0(\tau,x,D) + H_{a_2}\Lambda_0(t,x,D) + r(t,x,D)\}w \\ &\qquad = e^{\Lambda_0}(t,x,D)^{-1}e^{\Lambda}(x,D)^{-1}f(t,x) = g(t,x), (t,x) \in (0,T) \times \boldsymbol{R}^n, \\ &\qquad w(0,x) = e^{\Lambda_0}(0,x,D)^{-1}e^{\Lambda}(x,D)^{-1}u_0(x) = w_0(x), x \in \boldsymbol{R}^n, \end{aligned} \tag{3.53}$$

The Cauchy problem (3.53) is L^2-well posed in $[0,T]$. In fact it follows from (3.26) and (3.50) that we can see

$$\begin{aligned} &(1/2)(d/dt)\|w(t)\|_{L^2}^2 = (1/2)\operatorname{Re}(w'(t), w(t))_{L^2} \\ &\qquad = ((-\operatorname{Im} b(t,x,D) - H_{a_2}\Lambda - \partial_t\Lambda_0 - H_{a_2}\Lambda_0 + H_b\Lambda - r)w(t) - g(t), w(t))_{L^2} \\ &\qquad \leq C\|w(t)\|_{L^2}^2 + \|g(t)\|_{L^2}\|w(t)\|_{L^2} \end{aligned} \tag{3.54}$$

for $w(t) \in C^1([0,T]; L^2) \cap C^0([0,T]; H^1)$. This yields

$$\|w(t)\|_{L^2} \leq C\left\{\|w(0)\|_{L^2} + \int_0^t \|g(s)\|_{L^2}\,ds\right\} \tag{3.55}$$

for $t \in [0,T]$. Besides we can see similarly that for any $q \in \boldsymbol{R}$ there is $C_q > 0$ such that

$$\|w(t)\|_{H^q} \leq C_q\left\{\|w(0)\|_{H^q} + \int_0^t \|g(s)\|_{H^q}\,ds\right\} \tag{3.55}$$

for $t[0,T]$. Therefore recalling $u(t,x) = e^{\Lambda}(x,D)e^{\Lambda_0}(x,D)w(t,x)$, we obtain

$$\|u(t)\|_{H^q} \le C_q\left\{\|u(0)\|_{H^{q+\ell_0}} + \int_0^t \|f(s)\|_{H^{q+\ell_0}}\,ds\right\} \tag{3.56}$$

for $t \in [0,T]$. This energy estimate shows that the Cauchy problem (1.1) is H^∞-well posed in $[0,T]$. Thus we have completed the proof of our main theorem.

Appendix

Here we shall prove Lemma 2.8 and Lemma 2.11. We first estimate the derivatives of composite functions. Let $f(z,\zeta)$ be a real valued function in $C^\infty(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}\backslash 0)$ satisfying

$$|\partial_z^\beta \partial_\zeta^\alpha f(z,\zeta)| \le C_{\alpha\beta}\rho(z,\zeta)\langle z\rangle^{-|\beta|}|\zeta|^{-|\alpha|} \tag{A.1}$$

for $z \in \boldsymbol{R}^{n_1}, \zeta \in \boldsymbol{R}^{n_2}$ and $\alpha \in \boldsymbol{N}^{n_2}, \beta \in \boldsymbol{N}^{n_1}$, where $\rho(z,\zeta)$ is a potive function, and let $\varphi(x,\xi)$ and $\psi(x,\xi)$ in $C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n\backslash 0; \boldsymbol{R}^{n_2})$ satisfying

$$|\partial_x^\beta \partial_\xi^\alpha \varphi(x,\xi)| \le C_{\alpha\beta}\rho_1(x,\xi)\eta_1(x,\xi)^{-|\beta|}\eta_2(x,\xi)^{-|\alpha|} \tag{A.2}$$

$$|\partial_x^\beta \partial_\xi^\alpha \psi(x,\xi)| \le C_{\alpha\beta}\rho_2(x,\xi)\eta_1(x,\xi)^{-|\beta|}\eta_2(x,\xi)^{-|\alpha|} \tag{A.3}$$

for $x \in \boldsymbol{R}^n, \xi \in \boldsymbol{R}^n\backslash 0$ and $\alpha \in \boldsymbol{N}^n, \beta \in \boldsymbol{N}^n$ with $|\alpha+\beta| \le k$, where k is a positive integer and $\rho_i(x,\xi)$ and $\eta_i(x,\xi)(i=1,2)$ are positive functions. Then we have

Lemma A.1. *For $|\alpha+\beta| \le k$ there are $C_{\alpha\beta} > 0$ such that*

(A.4)
$$|\partial_x^\beta \partial_\xi^\alpha f(\varphi(x,\xi),\psi(x,\xi))| \le C_{\alpha\beta}\rho(\varphi,\psi)\{1+\rho_1/\langle\varphi\rangle + \rho_2/(2|\psi|)\}^{|\alpha+\beta|}\eta_1(x,\xi)^{-|\beta|}\eta_2(x,\xi)^{-|\alpha|}$$

for $x \in \boldsymbol{R}^n, \xi \in \boldsymbol{R}^n\backslash 0$.

Proof. Denote

$$X_j = \sum_{i=1}^{n_1} \partial\varphi_i/\partial x_j \partial/\partial z_i + \sum_{i=1}^{n_2} \partial\psi_i/\partial x_j \partial/\partial\zeta_i + \partial/\partial x_j$$
$$Y_j = \sum_{i=1}^{n_1} \partial\varphi_i/\partial\xi_j \partial/\partial z_i + \sum_{i=1}^{n_2} \partial\psi_i/\partial\xi_j \partial/\partial\zeta_i + \partial/\partial\xi_j$$

for $j = 1,\dots,n$. Then for $g(x,\xi,z,\zeta) \in C^\infty(\boldsymbol{R}^n \times \{\boldsymbol{R}^n\backslash 0\} \times \boldsymbol{R}^{n_1} \times \{\boldsymbol{R}^{n_2}\backslash 0\})$ we can write

$$(\partial/\partial x_j)\{g(x,\xi,\varphi(x,\xi),\psi(x,\xi))\} = X_j g\,|_{z=\varphi,\zeta=\psi}$$
$$(\partial/\partial\xi_j)\{g(x,\xi,\varphi(x,\xi),\psi(x,\xi))\} = Y_j g\,|_{z=\varphi,\zeta=\psi}.$$

Therefore we obtain

$$\partial_x^\beta \partial_\xi^\alpha\{f(\varphi(x,\xi),\psi(x,\xi))\} = (X^\beta Y^\alpha f)|_{z=\varphi,\zeta=\psi'}$$

where we write $X^\beta = X_1^{\beta_1}\cdots X_n^{\beta_n}$ and $Y^\alpha = Y_1^{\alpha_1}\cdots Y_n^{\alpha_n}$. To prove Lemma A.1 it is enough to show the following lemma.

LEMMA A.2. *Assume* (A.1), (A.2) *and* (A.3) *are valid. Then we have*

$$
\begin{aligned}
&|\partial_x^\beta\partial_\xi^\alpha\partial_z^\gamma\partial_\zeta^\lambda X(x,\xi,D)^{\tilde\beta}Y(x,\xi,D)^{\tilde\alpha}f(z,\zeta)| \\
\text{(A.5)}\qquad &\le C_{\alpha\beta\gamma\lambda\tilde\alpha\tilde\beta}\rho(z,\zeta)\{\rho_1(x,\xi)/\langle z\rangle+\rho_2(x,\xi)/\langle\zeta\rangle+\eta_1(x,\xi)^{-1}\}^{|\tilde\alpha|+|\tilde\beta|} \\
&\quad\times\eta_1(x,\xi)^{-|\beta|-|\tilde\beta|}\eta_2(x,\xi)^{-|\alpha|-|\tilde\alpha|}\langle z\rangle^{-|\gamma|}|\zeta|^{-|\lambda|}
\end{aligned}
$$

for $|\alpha|+|\beta|+|\gamma|+|\lambda|+|\tilde\alpha|+|\tilde\beta|\le k, (x,\xi)\in \boldsymbol{R}^n\times\boldsymbol{R}^n\backslash 0$ *and* $(z,\zeta)\in\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}\backslash 0$.

PROOF. We prove (A.5) by induction of $|\tilde\alpha|+|\tilde\beta|$. (A.1) implies (A.5) with $|\tilde\alpha|+|\tilde\beta|=0$. Assume (A.5) are valid for $|\tilde\alpha|+|\tilde\beta|\le\ell-1<k$. For simplicity we denote $\partial_x^\alpha\partial_\xi^\beta=\partial^q$ and $q=(\alpha,\beta)$. Then we have

$$
\begin{aligned}
&|\partial^q\partial_z^\gamma\partial_\zeta^\lambda X_jX^{\tilde\beta}Y^{\tilde\alpha}f(z,\zeta)| \\
&=\left|\sum_{q'\le q}\binom{q}{q'}\left\{\sum_{i=1}^{n_1}((\partial^{q-q'}\partial\varphi_i/\partial x_j)\partial_{z_i}+\sum_{i=1}^{n_2}(\partial^{q-q'}\partial\psi_i/\partial x_j)\partial_{\zeta_i}\right\}\right. \\
&\qquad\left.\times\,\partial^{q'}\partial_z^\gamma\partial_\zeta^\lambda X^{\tilde\beta}Y^{\tilde\alpha}f(z,\zeta)+\partial^q\partial_{x_j}\partial_z^\gamma\partial_\zeta^\lambda X^{\tilde\beta}Y^{\tilde\alpha}f(z,\zeta)\right| \\
&\le C_{q\gamma\lambda\tilde\alpha\tilde\beta}\{\rho_1(x,\xi)\langle z\rangle^{-1}+\rho_2(x,\xi)\langle\zeta\rangle^{-1}+\eta_1(x,\xi)^{-1}\}\rho(z,\zeta)\eta_1{}^{-|\beta|-|\tilde\beta|} \\
&\qquad\times\eta_2{}^{-|\alpha|-|\tilde\alpha|}\langle z\rangle^{-|\gamma|}|\zeta|^{-|\gamma|}\{\rho_1\langle z\rangle^{-1}+\rho_2|\zeta|^{-1}+\eta_1^{-1}\}^{|\tilde\alpha|+|\tilde\beta|}.
\end{aligned}
$$

Analogously we can see that $Y_jX^{\tilde\beta}Y^{\tilde\alpha}f$ satisfies the above estimate. Thus we have (A.5) for $|\tilde\alpha|+|\tilde\beta|=\ell$. Q.E.D.

PROOF OF LEMMA 2.8 AND LEMMA 2.11. We prove (2.52) for $a_{\xi x}$. In this case we take $\rho=\langle z\rangle^{-1-\delta}, \rho_1=\langle y\rangle+|t|, \rho_2=|\eta|, \varphi=x(t), \psi=\xi(t), \eta_1=\langle x\rangle$ and $\eta_2=|\xi|$. Then replacing (z,ζ) by (x,ξ) and (x,ξ) by (t,y,η) in (A.1) respectively, we can obtain (2.52) for $a_{\xi x}$ by use of Lemma A.1. By the similar way we obtain the other eatimates in (2.52) and in Lemma 2.11.

References

[1] V. Arnold, Equations différentielles ordinaires, Mir, Moscow, 1974.

[2] A. Baba, The H^∞-wellposedness Cauchy problem for Schrödinger type equations, Tsukuba J. Math., **18**, 1994, 101–117.

[3] R. Beals, Weighted distribution spaces and pseudodifferential operators, J. d'Analyse Math., **39**, 1981, 131–187.

[4] S. Doi, On the Cauchy problem for Schrödinger type equations and the regularity of solutions, J. Math. Kyoto Univ., **34**, 1994, 319–328.

[5] S. Hara, A necessary condition for H^∞-well posed Cauchy problem of Schrödinger type equations with variable coefficients, J. Math. Kyoto Univ., **32**, 1992, 287–305.

[6] W. Ichinose, Sufficient condition on H^∞ well posedness for Schrödinger type equations, Comm. Partial Differential Equations, **9**, 1984, 33–48.

[7] W. Ichinose, The Cauchy problem for Schrödinger type equations with variable coefficients, Osaka J. Math., **24**, 1987, 853–886.
[8] W. Ichinose, A note on the Cauchy problem for Schrödinger type equations on the Riemann Manifold, Math. Japonica, **35**, 1990, 205–213.
[9] K. Kajitani and A. Baba, The Cauchy problem for Schrödinger type equations, Bull. Sc. math., 2e série, **119**, 1995.
[10] S. Mizohata, On the Cauchy problem, Notes and Reports in Math., 3. 1985, Academic Press.
[11] J. Takeuchi, Le problème de Cauchy pour certaines équations aux dérivées partielles du type de Schrödinger, III, C. R. Acad. Sci. Paris, 312, Série I, 1991, 341–344.
[12] J. T. Schwartz, Nonlinear functional analysis, Notes on Mathematics and its Applications, 1969, Gordon and Breach Science Publishers.

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