J. Math. Soc. Japan
Vol. 50, NO. 1, 1998

A topology on the semigroup of endomorphisms on a von Neumann algebra

Dedicated to Professor Richard V. Kadison on the occasion of his 70th birthday.

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(Received Jan. 26, 1996)

1. Introduction.

1.1. The automorphism group of a von Neumann algebra reflects the structure of the algebra in many ways, and this idea was a cornerstone in the first pioneering works of A. Connes (cf. e.g. [**1**]). In fact the classification of (groups of) automorphisms is essential to most known classification theorems in von Neumann algebra. A main reason for the usefulness of this approach is that the automorphism group $\mathrm{Aut}(M)$ of a von Neumann algebra M can be topologized in a very nice way. Of course, several natural locally convex topologies on $\mathrm{Aut}(M)$ can be defined, but one of them is particularly useful, namely the u-topology (cf. [**4**]). A net (α_n) in $\mathrm{Aut}(M)$ converges to $\alpha \in \mathrm{Aut}(M)$ in the u-topology if and only if

$$\varphi \circ \alpha_n \longrightarrow \varphi \circ \alpha, \quad \varphi \in M_*,$$

and we then write $\alpha_n \xrightarrow{u} \alpha$. One nice property is that, in this topology, $\mathrm{Aut}(M)$ is a Polish topological group, in fact in a standard representation $M \subseteq \mathscr{B}(H)$ there is a multiplicative homeomorphism of $\mathrm{Aut}(M)$ onto a closed subgroup of the unitary group of H [**4**].

1.2. Motivated by striking applications in mathematical physics, as well as by quite surprising connections with mathematical disciplines such as knot theory and quantum Lie groups, the classification of *inclusions* of von Neumann algebras has become an object of intense study over the last 12 years. As an attempt to mimic the automorphism approach for single factors, given an inclusion $M \supseteq N$, one can study the group $\mathrm{Aut}(M,N)$ of automorphisms of M that preserve N globally. This has been quite successful in an indirect way, but of course we do not get any direct information on the subalgebras of M since we actually fixed N. In terms of endomorphisms on M, this corresponds to limit attention to those endomorphisms that map a fixed subalgebra into itself, or, even more specialized, onto the subalgebra. We shall denote the semigroup of endomorphisms of M by $\mathrm{End}(M)$, and by $\mathrm{End}(M,N)$ we mean the elements of

*Supported in part by a Carlsberg Foundation travel grant and the Danish Natural Science Research Council.

End(M) with N as the range algebra. Note that this notation is not consistent with [9, p. 221].

1.3. The aim of this paper is to consider the natural topological structure of endomorphism semigroups, and to relate it to some important invariants and constructions for endomorphisms, such as Longo's canonical endomorphisms and Izumi's Connes-Takesaki type module. In this study we are naturally inspired by the deep theory of automorphisms on von Neumann algebras. In particular, we have the following natural choice of a topology on End(M):

DEFINITION (Cf. [4]). *The u-topology on* End(M) *is given by pointwise convergence in the predual of M, i.e. if (ρ_n) is a net in* End(M), *then to say that it converges to $\rho \in \mathrm{End}(M)$ means that*

$$\|\varphi \circ \rho_n - \varphi \circ \rho\| \to 0$$

for all $\varphi \in M_*$. In this situation we write $\rho_n \xrightarrow{u} \rho$ for short.

1.4. Note that if $\rho_n \xrightarrow{u} \rho$ as above then it is trivial from the definition that ρ_n will converge pointwise to ρ in the σ-weak topology of M, and from this we get in fact pointwise convergence in σ-strong* topology, because the unitary group of M is left globally invariant by End(M), and spans M. This weaker pointwise (say, $p-$) convergence of (ρ_n) to ρ is denoted $\rho_n \xrightarrow{p} \rho$. As we shall see (cf. ex. 3.3), the p-topology is less useful for our purposes.

1.5. To produce examples of u-convergent nets of endomorphisms on $M \subseteq \mathscr{B}(H)$, consider the normalizer

$$\mathscr{N}(M) = \{U \in \mathscr{B}(H) : U^*U = UU^* = 1, UMU^* \subseteq M\}$$

and the endomorphisms $\rho_U = \mathrm{Ad}(U)|_M$ that it induces. Then, by the same argument as in [4, 3.8], the map $U \mapsto \rho_U$ is continuous when $\mathscr{N}(M)$ is provided with strong operator topology. (For a study of the 'inverse' of this map, see [13].) A warning is in place here, however: the unitary group is not closed, and if the limit of a net is just an isometry, then the net of the corresponding endomorphisms is not u-convergent in End(M).

ACKNOWLEDGEMENTS. It is a pleasure to thank U. Haagerup for a discussion concerning example 3.4. A part of this work was done at the Fields Institute, and we wish to thank G. Elliott and the staff for the hospitality extended to us. This paper is a revised version of our previous preprint 'Topologizing the semigroup ...', and we are indebted to M. Izumi for pointing out some errors in that version.

2. Considering a fixed inclusion.

In this section, we study the meaning of the u-topology on a fixed inclusion $M \supseteq N$ of von Neumann algebras admitting a conditional expectation E of M onto N.

We begin by observing that the restriction of the u-topology to $\mathrm{End}(M,N)$ is just the natural 'relative' version of it.

2.1. LEMMA. *Let* $(\rho_k) \subseteq \mathrm{End}(M,N)$ *be a net, and* $\rho \in \mathrm{End}(M,N)$. *Then* $\rho_k \xrightarrow{u} \rho$ *if and only if* $\varphi \circ \rho_k \to \varphi \circ \rho$ *for all* $\varphi \in N_*$.

PROOF. If $\rho_k \xrightarrow{u} \rho$ and $\varphi \in N_*$, then $\varphi \circ E \in M_*$, so that

$$\varphi \circ \rho_k = \varphi \circ E \circ \rho_k \to \varphi \circ E \circ \rho = \varphi \circ \rho.$$

The converse is equally obvious from restriction of M_*. Q.E.D.

Moreover, if $(\rho_k) \subseteq \mathrm{End}(M,N)$ is a convergent net, its limit is automatically in $\mathrm{End}(M,N)$; an important special case is $\mathrm{Aut}(M) = \mathrm{End}(M,M)$.

2.2. PROPOSITION. *The set* $\mathrm{End}(M,N)$ *is a closed subset of* $\mathrm{End}(M)$.

PROOF. Assume $(\rho_k) \subseteq \mathrm{End}(M,N)$ is a net with $\rho_k \xrightarrow{u} \rho \in \mathrm{End}(M)$. Let $y \in \rho(M)$ and put $x = \rho^{-1}(y) \in M$. Then $\rho_k(x) \in N$ and, for all $\varphi \in M_*$,

$$|\varphi \circ \rho_k(x) - \varphi(y)| \leq \|\varphi \circ \rho_k - \varphi \circ \rho\| \, \|x\| \to 0,$$

proving $y \in N$. Thus $\rho(M) \subseteq N$. Conversely, if $y \in N$, let $x_k = \rho_k^{-1}(y)$ for all k. Then

$$\begin{aligned} |\varphi(\rho(x_k) - y)| &= |\varphi(\rho(x_k) - \rho_k(x_k))| \\ &\leq \|\varphi \circ \rho_k - \varphi \circ \rho\| \, \|x_k\| \\ &= \|\varphi \circ \rho_k - \varphi \circ \rho\| \, \|y\| \to 0 \end{aligned}$$

for all $\varphi \in M_*$, so that $y = \sigma\text{-weak-}\lim_k \rho(x_k) \in \rho(M)$. Q.E.D.

A main tool in modern analysis of $\mathrm{Aut}(M)$ is the standard implementation (due to Araki, Connes and Haagerup independently), cf. [**4**]. In the simple case of $\mathrm{End}(M,N)$ it generalizes without difficulty, assuming that M and N are properly infinite. Namely, using the Dixmier-Maréchal theorem as in [**9**], we may then assume that M and N act standardly on the same Hilbert space H, with a common separating and cyclic vector in H. Let J_M and J_N denote the corresponding modular conjugations on H, and let $\mathcal{P}^M$ and $\mathcal{P}^N$ be the associated positive cones in H. Then, from [**4**, 2.18], there exists for any $\rho \in \mathrm{End}(M,N)$ exactly one unitary v_ρ on H satisfying

$$\begin{gathered} \rho(x) = v_\rho x v_\rho^*, \quad x \in M, \\ J_N = v_\rho J_M v_\rho^* \quad \text{and} \quad \mathcal{P}^N = v_\rho \mathcal{P}^M. \end{gathered}$$

Using lemma 2.1 and the fact that $H = \overline{\mathrm{span}}\, \mathcal{P}^N$, the same proof as in [**4**, 3.6] gets us the following

2.3. PROPOSITION. *The map* $\rho \mapsto v_\rho$ *is a homeomorphism of* $\mathrm{End}(M,N)$ *onto a closed subset of the unitary group* $\mathcal{U}(H)$ *of* H, *equipped with the weak operator topology.*

In particular $\mathrm{End}(M,N)$ is a Polish space, which can also be proved directly and more generally (cf. e.g. [**13**, 2.6]).

Using prop. 2.3 and the fact [**3**, 2.1] that, in the notation above, the unitary $J_M v_\rho^* J_M$ induces a conjugate endomorphism $\bar{\rho}$ of ρ, it follows easily that conjugation in $\mathrm{End}(M,N)$ can be chosen u-u-continuous. A different proof is based on the following simple observation: if we fix an endomorphism $\rho_0 \in \mathrm{End}(M,N)$, then any $\rho \in \mathrm{End}(M,N)$ is of the form $\rho = \rho_0 \circ \alpha_\rho$ where $\alpha_\rho \in \mathrm{Aut}(M)$; in fact $\alpha_\rho = \rho_0^{-1}\rho$. The following is obvious.

2.4. LEMMA. *The map $\rho \mapsto \alpha_\rho$ is u-u-continuous.*

Now we get a continuous choice of conjugates (thus continuity of the conjugation of sectors); this can also be derived from Cor. 3.6.

2.5. PROPOSITION. *Fix a conjugate $\overline{\rho_0}$ of ρ_0, say $\overline{\rho_0} = \rho_0^{-1}\,\mathrm{Ad}(J_N J_M)$, and let $\bar{\rho} = \alpha_\rho^{-1}\overline{\rho_0}$ for all $\rho \in \mathrm{End}(M,N)$. Then the map $\rho \mapsto \bar{\rho}$ is u-u-continuous.*

PROOF. Follows immediately from the lemma above. Q.E.D.

The same idea gives a slick proof of the continuity of Izumi's Connes-Takesaki type module on $\mathrm{End}(M,N)$ (see [7]; we first recall its definition. Assuming now $M \supseteq N$ to be an inclusion of (infinite) factors with finite index, the module is defined as follows: let $E_0 : M \to N$ be the minimal conditional expectation, let $d = \mathrm{Index}(E_0)^{1/2}$ and let ψ be a normal semifinite weight on M. Put

$$\tilde{M} = M \rtimes_{\sigma^\psi} \boldsymbol{R} = (\pi(M) \cup \lambda(\boldsymbol{R}))'' \subseteq B(L^2(\boldsymbol{R}, H))$$

where H is the Hilbert space on which M acts, λ is the left regular representation of $\boldsymbol{R}$ and π is the corresponding equivariant representation of (M, σ^ψ). Then there is by [7] a canonical extension of any $\rho \in \mathrm{End}(M,N)$ to a $\tilde{\rho} \in \mathrm{End}(\tilde{M})$ satisfying

$$\tilde{\rho}(\pi(x)) = \pi(\rho(x)), \quad x \in M,$$
$$\tilde{\rho}(\lambda_t) = d^{it}\pi((D\psi \circ \rho^{-1} \circ E_0 : D\psi)_t)\lambda_t, \quad t \in \boldsymbol{R}.$$

In this setup, ρ is said to have Connes-Takesaki module if $Z(\tilde{\rho}(\tilde{M})) = Z(\tilde{M})$. We denote the set of such endomorphisms by $\mathrm{End}_{CT}(M)$. In this case the module

$$\mathrm{mod} : \mathrm{End}_{CT}(M) \to \mathrm{Aut}(Z(\tilde{M}))$$

is defined by

$$\mathrm{mod}(\rho) = \tilde{\rho}|_{Z(\tilde{M})}, \quad \rho \in \mathrm{End}_{CT}(M)$$

(cf. [7],[**5**]).

Notice that the property of having a module is a property of inclusions (sometimes called the *common flow property*) rather than of endomorphisms, which makes sense of the following:

2.6. PROPOSITION. *Assume* $M \supseteq N$ *has Connes-Takesaki module. Then* mod *is continuous on* $\mathrm{End}(M,N)$.

PROOF. For $\rho \in \mathrm{End}(M,N)$, we have, in the above notation

$$\mathrm{mod}(\rho) = \mathrm{mod}(\rho_0\alpha_0) = \mathrm{mod}(\rho_0)\,\mathrm{mod}(\alpha_\rho).$$

By [**5**], $\mathrm{mod}(\alpha_\rho)$ is the usual Connes-Takesaki module, and this is continuous by [**2**]. Now the result follows from lemma 2.4. Q.E.D.

A key concept in the theory of (infinite) subfactors is that of the canonical endomorphisms as defined by Longo (see e.g. [**9**]). We shall see that, in our present context, the closure of these endomorphisms play the role of approximately inner automorphisms in the classical automorphism theory. Assuming now that $M \supseteq N$ is an inclusion of properly infinite factors, we let

$$\mathrm{Can}(M,N) = \{\mathrm{Ad}(u)\,\mathrm{Ad}(J_N J_M) : u \in \mathcal{U}(N)\}.$$

Then we have the following

2.7. THEOREM. *Let* $M \supseteq N$ *be an irreducible finite index inclusion of infinite AFD factors* M *and* N. *Assume that* $M \supseteq N$ *has Connes-Takesaki module. Then for any downward basic construction* $M \supseteq N \supseteq N_1$ *of* $M \supseteq N$, *the inclusion* $M \supseteq N_1$ *has Connes-Takesaki module. Let* $\mathscr{B}$ *be the set of factors* L *contained in* N *and such that* $M \supseteq N \supseteq L$ *is a downward basic construction of* $M \supseteq N$. *Then, for* $\rho \in \mathrm{End}(M)$, *the following are equivalent:*

(i) $\rho \in \overline{\mathrm{Can}(M,N)}$ *(closure in u-topology).*

(ii) $\rho(M) \in \mathscr{B}$ *and* $\mathrm{mod}(\rho) = 1$.

PROOF. Let $\gamma_0 = \mathrm{Ad}(J_N J_M) \in \mathrm{Can}(M,N)$ and put $N_1 = \gamma_0(M)$. Then by, [**12**, 1.8] and [**10**, 4.14], we have $\mathscr{B} = \{uN_1u^* : u \in \mathcal{U}(N)\}$. As $\tilde{M} \supseteq \tilde{N}_1$ is conjugate to $\tilde{M}_2 \supseteq \tilde{M}$, the inclusion $M \supseteq N_1$ has Connes-Takesaki module. For $u \in \mathcal{U}(N)$,

$$(\tilde{M} \supseteq (uN_1u^*)^{\sim}) \cong (\tilde{M} \supseteq u\tilde{N}_1u^*)$$

and

$$Z(u\tilde{N}_1u^*) = uZ(\tilde{N}_1)u^* = uZ(\tilde{M})u^* = Z(\tilde{M})$$

so that all members of $\mathscr{B}$ have Connes-Takesaki module.

Assume now $\rho \in \overline{\mathrm{Can}(M,N)}$ with $\rho = \lim_k \mathrm{Ad}(u_k)\gamma_0$ in u-topology, where all $u_k \in \mathcal{U}(N)$. Then with $\alpha = \lim_k \mathrm{Ad}(u_k) \in \mathrm{Aut}(M,N)$, we have $\rho(M) \in \mathscr{B}$ because

$$\alpha^{-1}(\rho(M) \subseteq N \subseteq M) = (\gamma_0(M) \subseteq N \subseteq M)$$

and $\gamma_0 \in \mathscr{B}$. Further,

$$\mathrm{mod}(\rho) = \mathrm{mod}(\lim_k \mathrm{Ad}(u_k))\,\mathrm{mod}(\gamma_0) = \mathrm{mod}(\gamma_0)$$

by continuity of mod. So to get $\mathrm{mod}(\rho) = 1$ it suffices to see $\mathrm{mod}(\gamma_0) = 1$. However, by [7, 5.1] there is an isometry $v \in N$ satisfying

$$v^* \gamma_0(x) v = x, \quad x \in M,$$

and by [7, 2.3.3] we then get

$$v^* \tilde{\gamma}_0(x) v = x, \quad x \in \tilde{M}.$$

It follows that

$$\tilde{\gamma}_0(x) = x, \quad x \in Z(\tilde{M}),$$

i.e. $\mathrm{mod}(\gamma_0) = 1$.

Conversely, assume $u \in \mathscr{U}(N)$ and $\gamma \in \mathrm{End}(M, uN_1u^*)$ has $\mathrm{mod}(\gamma) = 1$. Then $\gamma' = \mathrm{Ad}(u^*)\gamma \in \mathrm{End}(M, N)$. Put $\beta_\gamma = \gamma'\gamma_0^{-1} \in \mathrm{Aut}(N_1)$. As $\mathrm{mod}(\gamma_0) = 1$ by the first part of the proof, we get $\mathrm{mod}(\beta_\gamma) = 1$ so that $\beta_\gamma \in \overline{\mathrm{Int}}(N_1)$ by [1],[8]. Hence there is a net $(u_k) \subseteq \mathscr{U}(N_1)$ so that $\beta_\gamma = \lim_k \mathrm{Ad}(u_k)$ in u-topology, and therefore

$$\gamma = \mathrm{Ad}(u)\gamma' = \mathrm{Ad}(u)\beta_\gamma\gamma_0 = \lim(\mathrm{Ad}(uu_k)\gamma_0) \in \overline{\mathrm{Can}(M, N)},$$

finishing the proof. Q.E.D.

2.8. REMARK. The general problem of determining $\mathrm{Ker(mod)} \subseteq \mathrm{End}_{CT}(M)$ is left open. Note however that this kernel can contain endomorphisms of arbitrary allowed dimensions by composing with an automorphism of opposite Connes-Takesaki module. However, it seems likely that, in general, the canonical endomorphisms are not dense in Ker(mod).

3. Martingale type convergence.

When we consider the full endomorphism semigroup $\mathrm{End}(M)$ of a von Neumann algebra M, convergence of a net in the right topology should imply the convergence of the range algebras in some sense. In order to maintain some control over the subalgebras involved, it is natural to focus a first discussion on monotone nets, that is nets with increasing or decreasing range. We shall write $\rho_k \overset{u}{\nearrow} \rho$ if a net $(\rho_k) \subseteq \mathrm{End}(M)$ converges to $\rho \in \mathrm{End}(M)$ in u-topology and $(\rho_k(M))$ is an increasing net. Similarly we define $\rho_k \overset{u}{\searrow} \rho$.

At the basis of our discussion in this section is the following

3.1. THEOREM. *Let M be a von Neumann algebra, let $\rho \in \mathrm{End}(M)$ and let $(\rho_k)_{k \in I} \subseteq \mathrm{End}(M)$ be a net.*

(i) *If $\rho_k \overset{u}{\nearrow} \rho$ then $\rho(M) = \left(\bigcup_{k \in I} \rho_k(M)\right)''$.*

(ii) *If $\rho_k \overset{u}{\searrow} \rho$ then $\rho(M) = \bigcap_{k \in I} \rho_k(M)$.*

PROOF. By the standard argument we may assume M is σ-finite, i.e. that M possesses a faithful normal state φ. Then

$$d_\varphi(x, y) = \varphi((x - y)^*(x - y))^{1/2}, \quad x, y \in M$$

defines a metric on M which induces the σ-strong topology on bounded subsets of M.

(i): The inclusion $\subseteq$ follows clearly from just pointwise convergence of the net in the σ-weak topology, so we work on the opposite inclusion.

Let $k_0 \in I$ be arbitrary; to prove the inclusion $\supseteq$ of (i), it suffices to prove that any unitary element of $\rho_{k_0}(M)$ is contained in $\rho(M)$. Let y be such an element, and let $\varepsilon > 0$. We denote by φ_{y^*} the normal functional $x \mapsto \varphi(y^* x)$. Choose a majorant $k_1 \in I$ of k_0 such that

$$\|\varphi_{y^*} \circ \rho_{k_1} - \varphi_{y^*} \circ \rho\| \leq \frac{\varepsilon^2}{2}.$$

As $y \in \rho_{k_0}(M) \subseteq \rho_{k_1}(M)$ we put $x = \rho_{k_1}^{-1}(y)$. Also let $z = \rho(x)$. Then

$$\begin{aligned}
d_\varphi(y,z)^2 &= \varphi((y-z)^*(y-z)) \\
&= \varphi(1 - y^* z - z^* y + 1) \\
&\leq |\varphi(1 - y^* z)| + |\varphi(1 - z^* y)| \\
&= 2|\varphi(1 - y^* z)| \\
&= 2|\varphi_{y^*}(y - z)| \\
&= 2|\varphi_{y^*} \circ \rho_{k_1}(x) - \varphi_{y^*} \circ \rho(x)| \\
&\leq 2\|\varphi_{y^*} \circ \rho_{k_1} - \varphi_{y^*} \circ \rho\| \leq \varepsilon^2.
\end{aligned}$$

This proves $y \in \rho(M)'' = \rho(M)$.

(ii): The inclusion $\subseteq$ follows from choosing, for given $x \in M$ and $\varepsilon > 0$, a $k_0 \in I$ such that

$$d_\varphi(\rho_k(x), \rho(x)) \leq \varepsilon \quad \text{whenever } k \geq k_0,$$

since this shows

$$\rho(x) \in \bigcap_{k \geq k_0}{}' \rho_k(M) = \bigcap_{k \in I} \rho_k(M).$$

To prove the reverse inclusion in the decreasing case, an argument similar to what we did in the increasing case works. Q.E.D.

3.2. Corollary. *The set of irreducible endomorphisms on M is closed under increasing convergence in the u-topology.*

3.3. Remark. If M is a σ-finite factor and $(\rho_k) \subseteq \mathrm{End}(M)$ is an increasing net with $d(\rho_{k_0}) < \infty$ for some k_0, then $d(\rho_k) < \infty$ for all $k \geq k_0$. In this situation, $\rho_k(M)$ becomes eventually constant because, if $d(\rho_{k_0}) < 2^n$, there can be at most $n-1$ factors sitting in an increasing proper sequence between $\rho_{k_0}(M)$ and M. Thus the study of increasing finite dimensional nets on factors reduces to the case considered in Sec. 2. We thank M. Izumi for pointing this out to us.

3.4. EXAMPLE. Let F be a von Neumann algebra with a state ϕ. Let $F_n = F$, $\phi_n = \phi$ for all $n \in \mathbf{N}$, and consider the infinite tensor product

$$M = \bigotimes_{n=1}^{\infty}(F_n, \phi_n).$$

Recall that this von Neumann algebra is a factor if F is so, and by definition it is the von Neumann algebra generated by the algebraic tensor product in the GNS-representation corresponding to the product state, hence it is generated by the elementary tensors

$$x_1 \otimes x_2 \otimes \cdots x_k \otimes 1 \otimes 1 \otimes \cdots$$

where k runs through $\mathbf{N}$ and each $x_j \in F_j$.

Define a sequence $(\rho_k) \subseteq \mathrm{End}(M)$ by its action on elementary tensors:

$$\rho_k(x_1 \otimes x_2 \otimes \cdots) = x_k \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_{k-1} \otimes x_{k+1} \otimes x_{k+2} \otimes \cdots$$

and let $\rho \in \mathrm{End}(M)$ be similarly given by

$$\rho(x_1 \otimes x_2 \otimes \cdots) = 1 \otimes x_1 \otimes x_2 \otimes \cdots.$$

Then $\rho_k \xrightarrow{p} \rho$, and as in fact $(\rho_k) \subseteq \mathrm{Aut}(M)$ the sequence is trivially increasing and decreasing. Hence

$$\rho(M) \neq M = \left(\bigcup_{k \in \mathbf{N}} \rho_k(M)\right)''$$

showing that the conclusion of theorem 3.1 does not hold assuming just pointwise convergence; and in fact (ρ_k) does not converge to ρ in u-topology (this is also immediate from Prop. 2.2).

Note also from this example that, although u-topology coincides with p-topology on $\mathrm{Aut}(M)$ for M a finite factor (see [**4** ,3.13]), this is not the case on all of $\mathrm{End}(M)$ when M is the hyperfinite factor of type II_1.

It is natural to investigate the behaviour of canonical endomorphisms in the present picture (cf. also Sec. 2), because they depend only on range algebras, and these converge nicely by theorem 3.1. In order for canonical endomorphisms to exist, we assume that M is separable and properly infinite. Both assumptions are very mild, in particular the essential objects in subfactor theory, as well as in its applications to quantum physics, are hyperfinite and of type III_1. In our context, the basic observation that justifies this claim is the observation that, for a net $(\rho_k) \in \mathrm{End}(M)$ and N another von Neumann algebra (say of type III_1, the net $(\rho_k \otimes 1) \in \mathrm{End}(M \otimes N)$ converges to $\rho \otimes 1$ (where $\rho \in \mathrm{End}(M)$) if and only if $\rho_k \xrightarrow{u} \rho$. This is because $M_* \otimes N_*$ is total in $(M \otimes N)_*$.

3.5. DEFINITION. A family of endomorphisms on M is called *standard* if M can be represented on a Hilbert space that contains a common separating cyclic vector for M and all the range algebras associated to the family. The canonical endomorphisms associated to a standard family and a choice of common separating cyclic vector are called a *standard choice of canonical endomophisms* for the family. (A similar notion of standard nets was considered in [**11**].)

As an application of theorem 3.1 and non-commutative martingale theory we now have, as a stronger version of prop. 2.5:

3.6. Corollary. *Let* $(\rho_k)_{k \in I} \subseteq \mathrm{End}(M)$ *be a net and assume that* $\rho_k \overset{u}{\nearrow} \rho \in \mathrm{End}(M)$. *Then there is a standard choice* γ_ρ, γ_{ρ_k} *of canonical endomorphisms for* ρ *and a tail* $(\rho_k)_{k \geq k_0}$ *of the net, which satisfies* $\gamma_{\rho_k} \overset{u}{\to} \gamma_\rho$.

Proof. In our present assumptions, any increasing net in $\mathrm{End}(M)$ has a standard subnet, because we can choose a common separating cyclic vector for M and the range algebra of any particular member of the net, and then this vector is automatically separating and cyclic for all subsequent range algebras, so that in fact any 'tail' of the net is standard. Our result now follows from theorem 3.1 and the Hiai-Tsukada martingale theorem [**6**, Thm. 3] because the generalized conditional expectations corresponding to the common standard vector are in fact just canonical endomorphisms.

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NOTE ADDED IN PROOF.

The result of Theorem 3.1 has been considerably generalized in Cor. 2.14 of "The Effros-Maréchal topology in the space of von Neumann-algebras" by Uffe Haagerup and the author (to appear in Amer. Math. J.).