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Topological regularity theorems for Alexandrov spaces

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§ 1. Introduction.

Since Gromov gave in [**G1**], [**G2**] an abstract definition of Hausdorff distance between two compact metric spaces, the Gromov-Hausdorff convergence theory has played an important role in Riemannian geometry. Usually, Gromov-Hausdorff limits of Riemannian manifolds are almost never Riemannian manifolds. This motivates the study of Alexandrov spaces which are more singular than Riemannian manifolds since it is observed in [**GP1**] that the limit spaces are Alexandrov spaces if the manifolds in the sequence have curvature bounded uniformly from below. Alexandrov spaces are finite dimensional inner metric spaces with a lower curvature bound in the sense of distance comparison. It is now well known that the topological and geometric properties of Gromov-Hausdorff limits will reveal those of Riemannian manifolds considered in the sequence. For a discussion of this viewpoint, see [**W1**]. In view of this, the investigation of the topological and geometric properties of Alexandrov spaces has recently attracted a lot of attention; see for example [**BGP**], [**FY**], [**GP1**], [**Pe**], [**Sh**] and [**Pt**]. The structure of Alexandrov spaces is studied in [**BGP**], [**Pe**] and [**Pt**]. In particuar, if p is a point in an Alexandrov space X, then the space of directions $\sum_p$ at p is an Alexandrov space of one less dimension and with curvature $\geqq 1$. Moreover, a neighborhood of p in X is homeomorphic to the linear cone over $\sum_p$. One important implication of this local structure result is that if $\sum_p$ is a sphere then the point p is a manifold point. However, the converse is not true. This can be seen from the following example from [**GP2**].

EXAMPLE 1. Let M be a non-simply connected homology sphere which is also a space form of constant curvature 1. For instance, M is the Poincare's 3-dimensional homology sphere. The spherical suspension $\sum M$ is also an Alexandrov space with curvature bounded below by 1. It is not a topological manifold since the two vertices of the suspension are not manifold points. The Edward double suspension theorem ([**D**]) states that the double suspension of a

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homology sphere is a manifold and hence a sphere. Thus the double suspension $\Sigma\Sigma M$ is a manifold, but the spaces of directions of the two vertices are both ΣM which is not a sphere.

In [**GP2**] and [**Sh**], Grove, Peteresen and Shen give some geometric characterizations of sphere to investigate manifold points in an Alexandrov space. The main purpose of this paper is to give some topological characterization of manifolds for Alexandrov spaces. In this paper, topological manifolds stand for topological manifolds without boundary.

THEOREM 1.1. *An n-dimensional Alexandrov space X is a topological manifold if and only if it satisfies the following two properties:*

(1) *X is a homology manifold. That is, $H_*(X, X-p)\cong H_*(\boldsymbol{R}^n, \boldsymbol{R}^n-o)$ for any $p\in X$, and*

(2) *in case that $n\geqq 4$, the space of directions Σ_p at p is simply connected for any $p\in X$.*

Example 1 shows that our result is best as one can expect. A local version of this theorem to characterize manifold points in an Alexandrov space is the following.

THEOREM 1.2. *Let X be an Alexandrov space of dimension n. Then a point $p\in X$ is a manifold point if and only if it satisfies the following two properties:*

(1) *the point p is a homology manifold point. That is, there exists an open neighborhood U of p such that $H_*(X, X-x)\cong H_*(\boldsymbol{R}^n, \boldsymbol{R}^n-o)$ for any $x\in U$, and*

(2) *in case that $n\geqq 4$ the space of directions Σ_p at p in X is simply connected.*

This result parallels a theorem of R. D. Edwards, which characterizes polyhedra which are topological manifolds.

THEOREM (EDWARDS). *A finite polyhedron P is a closed topological n-manifold if and only if*

(1) *the link of every vertex of P is simply connected if $n\geqq 3$, and*

(2) *the link of every point $p\in P$ has the homology of an $(n-1)$-sphere.*

In geometric topology, it is an interesting open problem that if the product space $X\times\boldsymbol{R}^2$ is a topological manifold, is $X\times\boldsymbol{R}$ a manifold? In section three, we shall establish this for Alexandrov spaces and use it to prove Theorem 1.1. To be more precise, we shall obtain.

THEOREM 1.3. *Let X be an Alexandrov space, then the following three statements are equivalent.*

(1) *X is a homology manifold,*

(2) *$X\times\boldsymbol{R}^k$ is a topological manifold for some $k\geqq 1$, and*
(3) *$X\times\boldsymbol{R}$ is a topological manifold.*

In section three, we shall also discuss some applications of these results to Riemannian manifolds. In particular, we shall show Gromov-Hausdorff limit spaces of certain class of Riemannian manifolds are indeed topological manifolds. This will allow one to obtain some finiteness and structure results for Riemannian manifolds

Consider the class $\mathcal{M}$ of compact Riemannian n-manifolds M with sectional curvature $K_M\geqq k$, diameter $d(M)\leqq D$ and $C(M)\geqq r_0>0$ where $C(M)$ denotes the contractibility radius of M. See section three for the definition of $C(M)$. Grove, Petersen and the auther showed in [**GPW**] that the class $\mathcal{M}$ contains at most finitely many homeomorphism (resp. homotopy) types for $n\neq 3$ (resp. $n=3$). Let X be a metric space in $\partial\mathcal{M}$ with respect to the Gromov-Hausdorff topology. It is conjectured that X must be a topological manifold. As an application of Theorem 1.1, we shall prove this regularity property for the limit space X.

THEOREM 1.4. *Each limit space $X\in\partial\mathcal{M}$ is a topological manifold.*

Using the stability theorem in [**Pe**] one can also obtain this result. However, the proof we shall present here is quite different from that in [**Pe**] and appeals to our characterization theorem. The stability theorem of Perelman also shows that limit spaces of the class of Riemannian n-manifolds M satifying $K_M\geqq k$, $d(D)\leqq D$, and $vol(M)\geqq v>0$ are also topological manifolds. The technique here also allows one to verify this when $n\leqq 3$. For $n\geqq 4$, the author, so far, can only obtain that each limit space is a topological manifold with at most finitely many singular points.

In section four, we shall also investigate the structure of compact nonnegatively curved Alexandrov spaces. Our main result concerning this is

THEOREM 1.5. *Let X be an n-dimensional Alexandrov space with curvature $K_X\geqq 0$. Then the fundamental group of X is, up to a finite quotient, an almost abelian group of rank $k(1\leqq k\leqq n)$, i.e., $\pi_1(X)$ contains a finite normal subgroup Φ such that $\pi_1(X)/\Phi$ is an almost abelian group of rank k. Moreover, a finite covering $\hat{X}$ of X is homeomorphic to a product space $V^*\times T^k$, where V^* is an Alexandrov space with $K_{V^*}\geqq 0$ and finite fundamental group, and T^k is the k-dimensional torus. In case that X is a topological manifold, then* (1) *V^* is a topological manifold for $n-k\leqq 3$, and* (2) *X is homeomorphic to $N\times T^k$ for $n-k\geqq 4$ where N is a topological manifold homotopic to V^*.*

REMARK. Recall that a group is called an almost abelian group of rank k if it contains a free abelian group of rank k and finite index. In case that $n-k\geqq 4$, one can not hope that V^* itself will be a topological manifold even when X is a topological manifold. We give an example to illustrate this.

EXAMPLE 2. Let M be a non-simply connected homology manifold as in Example 1. We take X to be the metric product space of the suspension $\sum M$ of M and the unit circle S^1. Then in this case V^* is $\sum M$ which is not a topological manifold. However, the space X is by Theorem 1.1 or Theorem 1.3, a topological manifold.

§ 2. Main tools.

Since we shall need several results in geometric topology and about Alexandrov spaces to prove theorems mentioned in the previous section, we list in this section some important ones. For basic notation and results of Alexandrov spaces, we refer to Burago, Gromov and Perelman [BGP] [Pe] and Plaut [Pt]; for the definitions and basic results about resolutions, homology manifolds and local index of Quinn we refer to Daverman [D] and Quinn [Q1]-[Q4]. Recall that a map f from a topological manifold M to a space X is called a resolution if it has point inverses compact, nonempty, and contractible inside any neighborhood (cf. [Q1]-[Q4]). We shall say that such a space X has a resolution.

Let X be an n-dimensional Alexandrov space and p a point in X. It is shown in [BGP] that the space of directions $\sum_p$ at p is an $(n-1)$-dimensional Alexandrov space with curvature $\geqq 1$. The tangent cone at p is defined as the topological cone over $\sum_p$ with the following standard cone metric

$$d((v, s), (w, t))=\sqrt{s^2+t^2-2ts\cos(\sphericalangle(v, w))}$$

for any $v, w \in \Sigma_p$ and $s, t \geqq 0$. We denote the tangent cone at p by $K(\Sigma_p)$. The vertex of $K(\Sigma_p)$ is still denoted by p. For $r>0$, let $K_r(\Sigma_p)$ be the open metric r-ball in $K(\Sigma_p)$ around p. A fundamental result about the local structure of an Alexandrov space is due to Perelman.

THEOREM 2.1 (Perelman [**Pe**]). *Let X be an Alexandrov space and $p \in X$. Then there exists a positive number $r_p>0$ such that $(\bar{B}(p, r), S(p, r), p)$ is homeomorphic to $(\bar{K}_r(\Sigma_p), \Sigma_p, p)$ for each $r \in (0, r_p)$.*

The next three results concern the characterization of topological manifolds.

THEOREM 2.2 (Quinn [**Q1**]-[**Q4**]). *Let X and Y be two metric spaces with dimensions $\geqq 2$. Then one has*

(I) *the product space $X \times Y$ is a topological manifold if and only if X and Y are ANR homology manifold of local index* 1 *in the sense of Quinn;*

(II) *if X is a metric space of finite dimension $\geqq 4$, the following three statements are equivalnet:*

(1) *X has a resolution,*

(2) *$X\times\boldsymbol{R}^k$ is a topological manifold for some $k\geqq 2$, and*

(3) *$X\times\boldsymbol{R}^2$ is a topological maniofld.*

(III) *if $f: M\to X$ is a resolution and X is a manifold, then for any $\varepsilon>0$, f can be ε-approximated by a homeomorphism.*

A characterization of topological manifolds takes the following form

THEOREM 2.3 (Edwards-Quinn [**D**], p. 288). *A finite dimensional space X is a topological n-manifold, $n\geqq 5$, if and only if*

(1) *X is an ANR homology manifold of local index* 1 *in the sense of Quinn, and*

(2) *X satifies the disjoint disk property (DDP).*

Let A be a subset of a metric space X and $x\in A\cap(\overline{X-A})$. We say that $X-A$ is locally 1-connected at x provided that each neighborhood U of x contains another neighborhood V such that each map of the circle S^1 into $V-A$ can be extended to a map of the disk D^2 into $U-A$. The subset A is said to be locally 1-co-connected (1-LCC) provided that $X-A$ is locally 1-connected at each $x\in A\cap(\overline{X-A})$. For related notions, please see Daverman [**D**].

Let X be a metric space and $S(X)$ denote the set of non-manifold points in X. Cannon, Bryant and Lacher gave a characterization of topological manifolds when the dimension $S(X)$ has large co-dimensions.

THEOREM 2.4 (Cannon, Bryant and Lacher [**D**], p. 286). *If X is an n-dimensional ANR homology manifold whose singular set $S(X)$ is 1-LCC embedded and has dimension k where $2k+3\leqq n$, then X is a topological manifold.*

The next result is about the splitting property of open nonnegatively curved Alexandrov spaces.

THEOREM 2.5 (Grove and Petersen [**GP3**], Yamaguchi [**Y**]). *Let (X, d) be an Alexandrov space with curvature $K_X\geqq 0$. Then (X, d) is isometric to a metric product $(V\times\boldsymbol{R}^k, d\oplus\|\cdot\|)$ for some totally convex subset (V, d) containing no lines where $\|\cdot\|$ is the standard flat metric of $\boldsymbol{R}^k$.*

Theorem 2.1 implies that an Alexandrov space X is a WCS set in the sense of Siebenmann [**S**]. Thus main results in [**S**] imply

THEOREM 2.6 (Siebenmann [**S**]). *The topological group* Hom(X) *of homeomorphisms of a compact Alexandrov space X onto itself is locally contractible.*

§ 3. Regularity theorems.

In this section we shall prove Theorems 1.1–1.3. First of all we point out that an Alexandrov space is always an ANR since it is locally compact, locally path connected and, by Theorem 2.1, locally contractible. Next we show that if an Alexandrov space is a homology manifold, then so are all of its spaces of directions.

PROPOSITION 3.1. *Let X be an Alexandrov space of dimension n and $p\in X$. If p is a homology manifold point, then one has* (1) $H_*(\Sigma_p)\cong H_*(S^{n-1})$, *and* (2) Σ_p *is a homology manifold.*

PROOF. (1) According to Theorem 2.1, one can find an $r>0$ such that (a) $B(p, 2r)$ is a homology manifold, and (b) $(\bar{B}(p, r), S(p, r), p)$ is homeomorphic to $(\bar{K}_r(\Sigma_p), \Sigma_p, p)$. The excision theorem in homology theory then gives $H_*(\boldsymbol{R}^n, \boldsymbol{R}^n-o)\cong H_*(X, X-p)\cong H_*(\bar{B}(p, r), \bar{B}(p, r)-p)\cong H_*(\bar{H}_r(\Sigma_p), \bar{K}_r(\Sigma_p)-p)$. On the other hand the exact sequences for pairs $(\bar{K}_r(\Sigma_p), \bar{K}_r(\Sigma_p)-p)$ and $(\boldsymbol{R}^n, \boldsymbol{R}^n-o)$ give

$$H_j(\bar{K}_r(\Sigma_p)-p)\longrightarrow H_j(\bar{K}_r(\Sigma_P))\longrightarrow H_j(\bar{K}_r(\Sigma_p), \bar{K}_r(\Sigma_p)-p)\longrightarrow H_{j-1}(\bar{K}_r(\Sigma_p)-p)$$

and

$$\longrightarrow H_j(\boldsymbol{R}^n-o)\longrightarrow H_j(\boldsymbol{R}^n)\longrightarrow H_j(\boldsymbol{R}^n, \boldsymbol{R}^n-o)\longrightarrow H_{j-1}(\boldsymbol{R}^n-o)\longrightarrow\cdots$$

Since $K_r(\Sigma_p)$ is a cone over Σ_p, one can easily conclude from these exact sequences that $H_*(\Sigma_p)\cong H_*(S^{n-1})$.

(2) To see that Σ_p is also a homology manifold, we know from (1) that $B(p, r)-p$ is homeomorphic to $\Sigma_p\times\boldsymbol{R}$. We endow $\Sigma_p\times\boldsymbol{R}$ with the product metric. Thus $\Sigma_p\times\boldsymbol{R}$ is an Alexandrov space with curvature $\geqq 0$ and it is a homology manifold. For any $v\in\Sigma_p$ and $o\in\boldsymbol{R}$ the space of directions $\Sigma_{(v,o)}$ at (v, o) in $\Sigma_p\times\boldsymbol{R}$ is the spherical suspension $\Sigma(\Sigma_v)$ of the space of directions Σ_v at v in Σ_p. Note that the dimension of Σ_v is $n-2$. From (1) and the suspension theorem of homology groups [**Wh**], we have for $j\geqq 2$

$$H_j(S^{n-1})\cong H_j(\Sigma_{(v,o)})\cong H_j(\Sigma(\Sigma_v))\cong H_{j-1}(\Sigma_v).$$

In particular, $H_*(\Sigma_v)\cong H_*(S^{n-2})$. Then Theorem 2.1 for the Alexandrov space Σ_p at $v\in\Sigma_p$ and the argument in (1) imply that $H_*(\Sigma_p, \Sigma_p-v)\cong H_*(\boldsymbol{R}^{n-1}, \boldsymbol{R}^{n-1}-o)$. Since v is arbitrary, Σ_p is a homology manifold. □

Recall that an Alexandrov space X is said to have no boundary if all of its spaces of directions have no boundary. Thus a two-dimensional Alexandrov space has no boundary if and only if all of its spaces of directions are circles. Theorem 2.1 then shows that all two-dimensional Alexandrov spaces without boundary are topological manifolds. It is also well known (cf. [**D**]) that all two dimensional ANR homology manifolds are topological manifolds. An easy corollary of Proposition 3.1 is

COROLLARY 3.2. *Let X be an Alexandrov space. Then X has no boundary provided that X is a homology manifold.*

From now on all of Alexandrov spaces discussed in this paper will be assumed to have no boundary. Before we investigate the structure of the singular set $S(X)$ of an Alexandrov space X, we discuss the lower dimensional cases.

PROPOSITION 3.3. *Let X be an Alexandrov space of dimension* 3 *and $p \in X$. If p is a homology manifold point, then it is a manifold point.*

PROOF. Proposition 3.1 and Corollary 3.2 imply that the space of direction Σ_p is a two-dimensional homology manifold, and hence a topological manifold, with $H_*(\Sigma_p) \cong H_*(S^2)$. Since the sphere S^2 is the only 2-manifold with this property, Σ_p is homeomorphic to S^2. Thus Theorem 2.1 implies that for small $r>0$ the open ball $B(p, r)$ is homeomorphic to the 3-disk D^3. This means that p is a manifold point. □

Next we use some results of Quinn to give a rough estimation of the dimension of $S(X)$.

PROPOSITION 3.4. *Let X be an Alexandrov space of dimension $n \geqq 4$. If X is a homology manifold, then one has* $\dim S(X) \leqq 1$.

PROOF. Given any point $p \in X$, Theorem 2.1 implies that the space of directions Σ_p at p is an Alexandrov space with curvature $\geqq 1$ and for small $r>0$, $B(p, r)-p$ is homeomorphic to the product space $\Sigma_p \times \boldsymbol{R}$. The distance comparison theorem gives that the diameter of Σ_p is at most π and hence compact. Applying Theorem 2.1 to the Alexandrov space Σ_p, we can conclude that there are finite points $\{v_i\}_{i=1}^m$ of Σ_p such that $\Sigma_p \subset \bigcup_{i=1}^m B(v_i, r_i)$ and each open set $B(v_i, r_i)-v_i$ in Σ_p is homeomorphic to a product space $\Sigma_{v_i} \times \boldsymbol{R}$. Here Σ_{v_i} is the space of directions of v_i in Σ_p and has dimension $n-2 \geqq 2$. Theorem 2.1 implies in particular that the set of manifold points is dense in X. Since the local index of Quinn is locally defined and locally constant (cf. [**Q4**]), we know that the Σ_{v_i} has the local index 1 in the sense of Quinn. Proposition 3.1 also gives that Σ_{v_i} is an ANR homology manifold. Thus Theorem 2.2 (I) yields that $\Sigma_{v_i} \times \boldsymbol{R}^2$ is a topological manifold for each v_i. From this one can easily conclude that $S(X) \cap (B(p, r)-p)$ is contained in $\bigcup_{i=1}^m v_i \times \boldsymbol{R}$ if we identify $B(p, r)-p$ with $\Sigma_p \times \boldsymbol{R}$. Thus $S(X)$ is locally like the Paris railway, hence locally contractible, and has $\dim S(X) \leqq 1$. □

We are now in a position to investigate the product space $X\times\boldsymbol{R}$.

THEOREM 3.5. *Let X be an Alexandrov space. If X is a homology manifold, then* $X\times\boldsymbol{R}$ *is a topological manifold and* $\dim S(X)\leqq 0$.

PROOF. If the space X has $\dim X\leqq 3$, then we know from Proposition 3.3 that X itself is a topological manifold and so is $X\times\boldsymbol{R}$. Thus we can assume $\dim X\geqq 4$. Proposition 3.4 shows in particular that the set of manifold points is dense in X. Hence X and $X\times\boldsymbol{R}$ has the local index 1 in the sense of Quinn and is an ANR homology manifold.

In view of Theorem 2.3, to prove the $X\times\boldsymbol{R}$ is a topological manifold we only need to verify that $X\times\boldsymbol{R}$ has the disjoint disk property. To do so, we shall imitate the argument in the proof of Corollary 3C in [**D**], p. 184. Consider two maps $f_1, f_2: D^2\to X\times\boldsymbol{R}$. Let $\pi_1: X\times\boldsymbol{R}\to X$ and $\pi_2: X\times\boldsymbol{R}\to\boldsymbol{R}$ denote the projection maps. By Proposition 3.4, we know that $\dim S(X)\leqq 1$, so $S(X)$ must be nowhere dense and 0-LCC in X. Thus, the maps $\pi_1 f_1$ and $\pi_1 f_2$ can be modified to send increasingly dense 1-skeleta of D^2 into $X-S(X)$ and obtain in the limit approximations $m_1, m_2: D^2\to X$ such that $m_i^{-1}(S(X))$ $(i=1, 2)$ is a compact 0-dimensional set K_i. To be more precise, the maps m_i can be constructed as follows. Take a large compact metric ball K in X which contains $\pi_1 f_1(D^2)$ and $\pi_1 f_2(D^2)$. Since $S(X)$ is nowhere dense and 0-LCC in X, given any $\eta>0$ there exists a positive number $\tau(\eta)$ with $\lim_{\eta\to 0}\tau(\eta)=0$ such that any two points x, y in $K\cap(X-S(X))$ with $d(x, y)<\tau(\eta)$ can be connected by a curve lying in the η-neighborhood of x in $X-S(X)$.

Now consider a sequence of increasingly dense 1-skeleta K_n of D^2 with the properties (1) the image of each cell of K_n under $\pi_1 f_i$ has mesh $<\tau(2^{-n}\varepsilon)/4$; (2) $|K_n|$ is $2^{-n}\varepsilon$ dense in D^2 with the usual metric, and (3) a refinement of K_n is a subcomplex of K_{n+1}. Then the maps $f_{i,n}: K_n\to X-S(X)$ can be constructed by induction. If the image of some vertex of K_1 under $\pi_1 f_i$ is in $S(X)$ then replace it by a point in $X-S(X)$ within the $\tau(2^{-1}\varepsilon/4)$ neighborhood of the image of that vertex. Then by the choice of the number $\tau(2^{-n}\varepsilon)$, we can obtain a map $f_{i,1}: K_1\to X-S(X)$. Suppose the map $f_{i,n-1}: K_n\to X-S(X)$ is defined. We set $f_{i,n}|_{|K_{n-1}|}=f_{i,n-1}$. Then consider the image of the vertice of K_n which is in $S(X)$. As before, replace them by $\tau(2^{-n}\varepsilon)/4$-nearby points in $X-S(X)$. Again, the map $f_{i,n-1}$ can be extended to a map $f_{i,n}: K_n\to X-S(X)$. According to our construction, it is easy to see that $f_{i,n}$ converges to a continuous map $m_i: D^2\to X$ which is ε-close to the map $\pi_1 f_i$ and $m_i^{-1}(S(X))$ is a compact 0-dimensional set. Now the maps $\pi_2 f_i: K_i\to\boldsymbol{R}$ can be approximated by disjoint embeddings $g_i: K_i\to\boldsymbol{R}$, where g_i is so close to $\pi_2 f_i|_{K_i}$ that g_i extends to a map $\alpha_i: D^2\to\boldsymbol{R}$ close to $\pi_2 f_i: D^2\to\boldsymbol{R}$. Define the approximations $h_1, h_2: D^2\to X\times\boldsymbol{R}$ to f_1, f_2 by $h_i(x)=(m_i(x), \alpha_i(x))$. Hence one has $h_1(D^2)\cap h_2(D^2)\subset(X-S(X))\times\boldsymbol{R}$. Since

$(X-S(X))\times \boldsymbol{R}$ is a topological manifold of dimension at least 5, one can further more produce disjoint approximations and thus the space $X\times \boldsymbol{R}$ satisfies the disjoint disk property. Hence $X\times \boldsymbol{R}$ is a topological manifold. To see that $\dim S(X)\leqq 0$ we consider any point $p\in X$. We have for sufficiently small $r>0$ that $B(p, r)-p$ is homeomorphic to the product space $\Sigma_p\times \boldsymbol{R}$ and Σ_p is also a homology manifold. Since $\Sigma_p\times \boldsymbol{R}$ is a topological manifold, we can conclude that the point p is the only possible non-manifold point in $B(p, r)$. Hence $S(X)$ is either empty or a discrete subset of X. Thus $\dim S(X)\leqq 0$ and the theorem follows. □

Theorem 1.3 follows basically from Theorem 3.5 and Theorem 2.2.

PROOF OF THEOREM 1.3. Let X be an Alexandrov space. From Theorem 3.5 we have (1)⇒(2). To see that (2)⇒(3), we know from Theorem 2.2 (I) that if $X\times \boldsymbol{R}^k$ is a topological manifold for some $k\geqq 2$, then X is a homology manifold. Theorem 3.5 then gives that $X\times \boldsymbol{R}$ is a topological manifold. The conclusion (3)⇒(1) again follows easily from Theorem 2.2 (I) and this completes the proof of Theorem 1.3. □

Since we know $\dim S(X)\leqq 0$, Theorem 2.4 tells us that to prove Theorems 1.1 and 1.2 we only need to check the 1-LCC condition for each point p in $S(X)$.

PROPOSITION 3.6. *Let X be an Alexandrov space of dimension $n\geqq 3$ and $p\in X$. Then $\{p\}$ is a* 1-LCC *subset of X if and only if the space of directions Σ_p at p is simply connected.*

PROOF. Suppose that $\{p\}$ is a 1-LCC subset of X. This means that each neighborhood U of p contains another neighborhood V such that each map of the circle S^1 into $V-p$ can be extended to a map of the disk D^2 into $U-p$. Theorem 2.1 implies for sufficiently small $r>0$ that $(B(p, r), p)$ is homeomorphic to $(K_r(\Sigma_p), p)$. Let U be any neighborhood of p with $U\subset B(p, r)$. We identify $B(p, r)$ with its image $K_r(\Sigma_p)$. Then there is an $\varepsilon>0$ such that $K_\varepsilon(\Sigma_p)\subset V$. Note that $\partial K_\varepsilon(\Sigma_p)$ is homeomorphic to Σ_p and there is a natural map π from $K_r(\Sigma_p)-p$ onto $\partial K_\varepsilon(\Sigma_p)$. Let f be any map from S^1 into $\partial K_\varepsilon(\Sigma_p)$. We can extend f to a map $\hat{f}$ from D^2 into $K_r(\Sigma_p)-p$. The composition of $\hat{f}$ and π gives a map from D^2 into $\partial K_\varepsilon(\Sigma_p)$ and it extends the original map f. This shows that $\partial K_\varepsilon(\Sigma_p)$ is simply connected and so is Σ_p.

Next suppose that Σ_p is simply connected. Theorem 2.1 implies that there is an $r>0$ such that $B(p, r)-p$ is homeomorphic to the product space $\Sigma_p\times \boldsymbol{R}$. Thus $B(p, r)-p$ is also simply connected and any map from S^1 into $B(p, r)$ extends to a map from D^2 into $B(p, r)-p$. Thus $\{p\}$ is a 1-LCC subset of X. □

Now Theorems 1.1 and 1.2 follow easily from Theorem 2.4, Proposition 3.3, Theorem 3.5 and Proposition 3.6. Using Theorem 1.2 and Theorem 1.3, we can extend the double suspension theorem of Edwards to Alexandrov spaces. An n-dimensional Alexandrov space X with curvature $\geqq 1$ is called an Alexandrov sphere if it is a homology manifold with $H_*(X)\cong H_*(S^n)$. Theorem 1.2 tells us that a point p in an Alexandrov space X with $\dim X\geqq 4$ is a manifold point if and only if the space of directions Σ_p at p is a simply connected Alexandrov sphere.

PROPOSITION 3.7. *The suspension ΣX of a simply connected Alexandrov sphere X is a topological manifold and hence a sphere.*

PROOF. Since X is an Alexandrov space with $K_X\geqq 1$, the suspension ΣX is also an Alexandrov space with curvatures $\geqq 1$. The suspension theorem for homology groups then gives that ΣX is an Alexandrov sphere. Let p and q denote the two vertices of the suspension ΣX of X. Thus $\Sigma X-\{p, q\}$ is homeomorphic to $X\times \boldsymbol{R}$ and hence, by Theorem 1.3, a topological manifold. To see that p and q are also manifold points, we know from Theorem 1.2 that it is sufficient to check the simply connectness of their spaces of directions in ΣX. However, it is easy to see that their spaces of directions are both the space X. Since X is simply connected, Theorem 1.2 implies that p and q are both manifold points. Finally, since the sphere is the only manifold which is also a suspension, ΣX is a sphere. □

Since the suspension of a connected space is always simply connected, this proposition gives

DOUBLE SUSPENSION THEOREM. *The double suspension $\Sigma\Sigma X$ of an Alexandrov sphere X is a sphere.*

Example 1 in the introduction shows that this proposition is optimal. Next we shall use these regularity results to investigate Gromov-Haudorff limits under curvature bounded.

Let M be a compact Riemannian n-manifold. For a point $p\in M$, we define the distance function at p by $d_p(x)=d(p, x)$. A point $q(\neq p)$ is called a critical point of p if there is, for any non-zero vector $v\in TM_q$, a minimal geodesic γ from q to p making an angle $\sphericalangle(v, \gamma'(0))\leqq \pi/2$ with v. Let $C(p)$ denote the distance between p and its nearest critical point and let $C(M)=\inf_{p\in M} C(p)$. Note that $C(M)$ is bounded below by the injectivity radius of M and hence is always positive.

The importance of the critical points of the distance function lies in the fact that the isotopy-type lemma of Morse theory still holds.

ISOTOPY LEMMA ([**G3**]). *Let M be a compact Riemannian manifold of dimension n. One has that $(B(p, r), p)$ is homeomorphic to (D^n, o) for any $r \in (0, C(M))$ where D^n is the unit disk in $\boldsymbol{R}^n$. Moreover, for $0<r_1<r_2<C(M)$ the annulus $B(p, r_2)-B(p, r_1)$ is homeomorphic to $S^{n-1} \times \boldsymbol{R}$.*

Recall that $\boldsymbol{M}$ denotes the class of compact Riemannian n-manifolds M with sectional curvature $K_M \geqq k$, diameter $d(M) \leqq D$ and $C(M) \geqq r_0 > 0$. Let X be a compact metric space in $\partial \boldsymbol{M}$ with respect to the Gromov-Hausdorff topology. That is, there exists a sequence of manifolds M_i in $\boldsymbol{M}$ such that $\lim_{i \to \infty} d_{GH}(M_i, X)=0$.

Our next aim is to obtain a regularity property for limiting spaces X of $\boldsymbol{M}$. Grove, Petersen and the author show in [**GP1**] and [**GPW**] that the limit space X has the following important properties:

(1) X is an Alexandrov space of dimension n and with $K_X \geqq k$ and $d(X) \leqq D$.
(2) X is a $LGC(\rho)$ space with the function $\rho(\varepsilon)=\varepsilon$, $\varepsilon \in (0, r_0)$. In particular, there are $\varepsilon_0 > 0$ and $m \geqq 1$ such that if h_0 and h_1 are maps from S^1 into X with $d(f_0(s), f_1(s)) \leqq \varepsilon \leqq \varepsilon_0$ for all $s \in S^1$, then there exists a homotopy h_t, $t \in [0, 1]$ from h_0 to h_1 with $d(h_0(s), h_t(s)) \leqq m\varepsilon$ for any $s \in S^1$ and $t \in [0, 1]$.
(3) X is an ANR homology manifold with local index 1 in the sense of Quinn.
(4) There exist for large i homotopy equivalences $f_i : X \to M_i$ and $g_i : M_i \to X$ which have for any $x, y \in X$ and $\bar{x}, \bar{y} \in M_i$

$$|d(x, y)-d(f_i(x), f_i(y))| < \varepsilon_i,$$

$$|d(\bar{x}, \bar{y})-d(g_i(\bar{x}), g_i(\bar{y}))| < \varepsilon_i,$$

with $\varepsilon_i \to 0$ as $i \to \infty$.

THEOREM 1.4. *Each limit space $X \in \partial \boldsymbol{M}$ is a topological manifold.*

PROOF. In view of Theorem 1.1 we only need to show that the space of directions Σ_p of any point $p \in X$ is simply connected if $n \geqq 4$. Proposition 3.6 tells us that this is equivalent to check the 1-LCC condition for the subset $\{p\}$.

Given any neighborhood U of p in X, choose $r>0$ so small that $100r<r_0$ and $B(p, 100r) \subset U$ and set V to be the open ball $B(p, r)$. Let h be any map from S^1 into $V-p$. Choose $\eta>0$ so that $d(p, h(s)) \geqq \eta$ for all $s \in S^1$ and choose also an i so large that $100m\varepsilon_i < \eta$ and $d_{GH}(M_i, X) < \eta/100$ where $m \geqq 1$ is the number as in Property (2) of X.

Fix an admissible metric d on the disjoint union $M_i \amalg X$ such that the Hausdorff distance $d_H(M_i, X) \leqq \eta/100$ and choose a point $p_i \in M_i$ with $d(p, p_i) \leqq \eta/100$.

Next we push the map h to M_i by setting $h_i = f \circ h : S^1 \to M_i$. The triangle inequality gives that $h_i(S^1) \subset B(p_i, 2r) - B(p, (\eta/2))$. Since $C(M_i) \geqq r_0 \geqq 100r$ and $n \geqq 4$, the isotopy lemma gives that h_i can be extended to a map, still denoted by h_i from D^2 into $B(p_i, 2r) - B(p, (\eta/2))$. Pull the map h_i back to the space X by setting $\hat{h} = g_i \circ h_i$. Note that f_i and g_i are the maps as in Property (4) of X. The triangle inequality again gives $\hat{h}(D^2) \subset B(p, 3r) - B(p, (\eta/4))$. Moreover, the restriction of $\hat{h}$ to S^1 $(=\partial D^2)$ is $(10\varepsilon_i)$-close to the original map $h : S^1 \to X$. That is, $d(\hat{h}(s), h(s)) \leqq 10\varepsilon_i$ for all $s \in S^1$. Thus Property (2) of X implies that $\hat{h}$ can be deformed to h through a homotopy inside $B(p, 4r) - B(p, (\eta/8))$ since $10m\varepsilon_i < (\eta/10)$.

This homotopy along with the map $\hat{h} : D^2 \to X$ provides the extension of the map $h : S^1 \to V - p$ to a map from D^2 into $B(p, 4r) - B(p, (\eta/8))$ which is contained in $U - p$. This shows that $\{p\}$ is a 1-LCC subset of X and the theorem follows. □

One can use this regularity theorem to obtain some finiteness and structure results for Riemannian manifolds. However, we shall not pursue them here and leave them to interested readers.

§ 4. Compact nonnegatively curved Alexandrov spaces.

In this section we investigate the fundamental group and the topological structure of an Alexandrov space X with $K_X \geqq 0$. In [**Pe**] Perelman constructed an example to indicate that the soul theorem of Cheeger and Gromoll [**CG**] does not hold for Alexandrov spaces. However, we shall show in the compact category that the results of Cheeger and Gromoll still hold to some extent. To prove the structural part of Theorem 1.5, we shall not follow the argument of Cheeger and Gromoll but, instead, use a method developed in [**W2**]. See also [**SW**] for a discussion about Riemannian manifolds with almost nonnegative curvature.

Let X be a compact Alexandrov space with $K_X \geqq 0$ and let $\tilde{X}$ be its universal covering with the pull-back metric. Thus $\tilde{X}$ is still an Alexandrov space with $K_{\tilde{X}} \geqq 0$ and the fundamental group $\pi_1(X)$ acts isometrically on $\tilde{X}$ as deck transformations.

Theorem 2.5 implies that $\tilde{X}$ splits isometrically as a product space $V \times \boldsymbol{R}^k$ where V is a totally convex subset of $\tilde{X}$ and contains no lines. Thus V is also an Alexandrov space with $K_V \geqq 0$. The space $\tilde{X}$ is simply connected and so is V. Since $\tilde{X}$ covers a compact space, we can obtain that V is compact.

LEMMA 4.1. *The space V is compact.*

PROOF. Indeed, if V is not compact, there exists a sequence of points p_i in V such that $d(p_1, p_i)=l_i \to \infty$. Let $\gamma_i : [0, l_i] \to V$ be a sequence of minimal geodesics from p_1 to p_i. Set $y_i=\gamma_i(l_i/2)$ and fix a point $z \in \boldsymbol{R}^k$. Note that the points (p_i, z) and (y_i, z) are in $\tilde{X}$. Since X is compact, one can find, for each i, an element h_i in $\pi_1(X)$ so that $d(h_i(y_i, z), (p_1, z)) \leqq d(X)$ where $d(X)$ denotes the diameter of X. Let π_1 and π_2 denote the projections of $\tilde{X}=V \times \boldsymbol{R}^k$ onto the first and second factors, V and $\boldsymbol{R}^k$, respectively. Set $\bar{y}_i=\pi_1 h_i(y_i, z)$. Thus $d(\bar{y}_i, p_1) \leqq d(X)$. Now consider the minimal geodesics $\bar{\gamma}_i=\pi_1 h_i \gamma_i$ in V. A subsequence of these minimal geodesics $(\bar{\gamma}_i, \bar{y}_i)$ converge to a line (γ, y) in V with $d(p_1, y) \leqq d(X)$. Since V contains no lines, this leads to a contradiction. Therefore, V is compact. □

To show that $\pi_1(X)$ is, up to a finite quotient, almost abelian, we need another lemma.

LEMMA 4.2. *Let Y be a compact Alexandrov space with $K_Y \geqq 0$ and $\boldsymbol{R}^k$ be the Euclidean k-space. Let X be the product space of Y and $\boldsymbol{R}^k$ with the product metric. Then any isometry Φ of X can be written as (Φ_1, Φ_2) where Φ_1 is an isometry of Y and Φ_2 is an isometry of $\boldsymbol{R}^k$; in terms of isometry groups, $Isom(X) = Isom(Y) \times Isom(\boldsymbol{R}^k)$.*

PROOF. We note first that X is an Alexandrov space with $K_X \geqq 0$. For any isometry Φ on X one can write $\Phi(y, v)=(\Phi_1(y, v), \Phi_2(y, v))$ for any $(y, v) \in X$. To prove the lemma, we only need to check that $\Phi_1(y, v)$ and $\Phi_2(y, v)$ are independent of v and y respectively. Given any non-zero vector v, consider the line $l(t)=(v/\|v\|)t$ in $\boldsymbol{R}^k$. Since Φ is an isometry, $\Phi(y, l(t))$ is also a line in X. Theorem 2.5 gives that this line is perpendicular to the factor Y, and hence $\Phi_1(y, l(t))$ is independent of t and $\Phi_2(y, l(t))$ is again a line in $\boldsymbol{R}^k$. This shows that $\Phi_1(y, v)$ is independent of v.

Next fix a point $z \in Y$. For any $y \in Y$ choose a minimal geodesic $\gamma(s)$ from z to y. For any $v \in \boldsymbol{R}^k$ consider a line $l(t)$ in $\boldsymbol{R}^k$ with $l(0)=v$. Hence the geodesics $(\gamma(s), v)$ and $(z, l(t))$ in X are perpendicular, and so are $\Phi(\gamma(s), v)$ and $\Phi(z, l(t))$. As discussed above, $\Phi(z, l(t))$ is still a line in X and lies in $\Phi(z, v) \times \boldsymbol{R}^k$. Since the line l with $l(0)=v$ is arbitrary, $\Phi(\gamma(s), v)$ lies in $Y \times \Phi(z, v)$. This gives that $\Phi_2(\gamma(s), v)$ is independent of s and in particular $\Phi_2(y, v)=\Phi_2(z, v)$. Hence, $\Phi_2(y, v)$ is independent of y and the lemma follows. □

The kernel of the natural projection $\Phi : \pi_1(X) \subset Isom(V) \times Isom(\boldsymbol{R}^k) \to Isom(\boldsymbol{R}^k)$ is finite since $Isom(V)$ is compact. Consider the covering $\tilde{X}=V \times \boldsymbol{R}^k \to X^*=\tilde{X}/\ker \Phi = V^* \times \boldsymbol{R}^k$ and the corresponding isometry group $Isom(V^* \times \boldsymbol{R}^k)= Isom(V^*) \times Isom(\boldsymbol{R}^k)$. Note that V^* is an Alexandrov space with curvature $\geqq 0$ and finite fundamental group $\pi_1(V^*)=\ker \Phi$. This is the space we need for Theorem 1.5.

The projection $\Phi^*: Isom(V^*)\times Isom(\boldsymbol{R}^k)\to Isom(\boldsymbol{R}^k)$ maps $G^*=\pi_1(X)/\ker\Phi\subset Isom(V^*\times\boldsymbol{R}^k)$ isomorphically into a discrete uniform subgroup of $Isom(\boldsymbol{R}^k)$. Note that $V^*\times\boldsymbol{R}^k\to\boldsymbol{R}^k$ induces a continuous map $X=X^*/G^*\to\boldsymbol{R}^k/\Phi^*G^*$, hence $\boldsymbol{R}^k/\Phi^*G^*$ is compact. Thus the Bieberbach theorem implies that G^* contains a normal free abelian rank k subgroup Γ of finite index where $\Phi^*\Gamma$ is a lattice in the subgroup T of translations in $Isom(\boldsymbol{R}^k)$. Therefore, $\pi_1(X)$ is, up to finite quotient, almost abelian and this prove the first part of Theorem 1.5 concerning the fundamental group $\pi_1(X)$.

Next we verify the structural part of Theorem 1.5. Let $\boldsymbol{Z}^k=\boldsymbol{Z}e_1\oplus\boldsymbol{Z}e_2\oplus\cdots\oplus\boldsymbol{Z}e_k$ be the normal free abelian subgroup Γ. Note that $\boldsymbol{Z}^k$ can be viewed as a lattice in the subgroup T of translations on $\boldsymbol{R}^k=\boldsymbol{R}e_1\oplus\boldsymbol{R}e_2\oplus\cdots\boldsymbol{R}e_k$. Set $\bar{X}=X^*/\boldsymbol{Z}^k$. The space $\bar{X}$ is a finite covering of X.

The action of $\boldsymbol{Z}^k$ on X^* gives the fibration $V^*\to\bar{X}\to T^k$ where T^k is the flat torus $\boldsymbol{R}^k/\boldsymbol{Z}^k$. This fibration is, in general, not trivial. Namely, the action of $\boldsymbol{Z}^k$ on the first factor V^* of X^* is, in general, not trivial. Thus we investigate the fibration in a more detailed way.

Since $\boldsymbol{Z}^k$ is a lattice in the subgroup of translations on $\boldsymbol{R}^k$, we can decompose this fibration into k small fibrations:

$$\begin{cases} W_{k-1}\to\bar{X}\to S^1_k \\ W_{k-2}\to W_{k-1}\to S^1_{k-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ W_1\to W_2\to S^1_2 \\ V^*\to W_1\to S^1_1, \end{cases}$$

where W_i inherits a metric from $\bar{X}$ and S^1_i is the circle $\boldsymbol{R}e_i/\boldsymbol{Z}_i$, induced by the action of $\boldsymbol{Z}_i$ for $i=1, 2, \cdots, k$. Note that W_i's are also Alexandrov spaces with nonnegative curvature.

In what follows, we shall show that there are finite coverings $\hat{W}_1$ and $\hat{S}^1_1$ of W_1 and S^1_1 such that $\hat{W}_1$ is homeomorphic to $V^*\times\hat{S}^1_1$. If the action of $\boldsymbol{Z}e_1$ on V^* is trivial, then W_1 is isometric to the metric product space $V^*\times S^1_1$. However, this may not be the case. Hence, we need to deform the action of $\boldsymbol{Z}e_1$ to a trivial action, and then this will imply that W_1 is, topologically, a product space.

Since the isometry group $Isom(V^*)$ is compact, there is, for any $\varepsilon_1>0$, an integer $m(\varepsilon_1)$ such that the isometry $m(\varepsilon_1)e_1$ of V^* is ε_1-close to the identity map, Id, on V^*. On the other hand, $Isom(V^*)\subset Hom(V^*)$, and $Hom(V^*)$ is, by Theorem 2.6, locally contractible. Hence, we can choose ε_1 so small that there is a path $\theta:[0, 1]\to Hom(V^*)$ inside the ε_1-neighborhood of the identity map Id of V^* in $Hom(V^*)$ with $\theta(0)=m(\varepsilon_1)e_1$ and $\theta(1)=Id$. We now consider the $m(\varepsilon_1)$-fold coverings $\hat{W}_1$ and $\hat{S}^1_1$ of W_1 and S^1_1, respectively. Endow these two coverings with the pull-back metrics and then $\hat{W}_1$ is also an Alexandrov spaces.

By using these homeomorphisms $\theta(t)$, $t\in[0, 1]$, one can deform the fibration

$$V^* \longrightarrow \hat{W}_1 \longrightarrow \hat{S}_1^1$$

to a trivial fibration.

Now we move to the second level:

$$W_1 \longrightarrow W_2 \longrightarrow S_2^1.$$

First, we take the $m(\varepsilon_1)$-fold covering $\overline{W}_2$ of W_2 and endow it with the pull back metric. Thus it is again an Alexandrov space with curvature $\geqq 0$. We have the fibration

$$\hat{W}_1 \longrightarrow \overline{W}_2 \longrightarrow S_2^1.$$

Once again, we can proceed with the same argument as above. Since the isometry group $Isom(\hat{W}_1)$ is compact, there is, for any $\varepsilon_2>0$, an integer $m(\varepsilon_2)$ such that the isometry $m(\varepsilon_2)e_2$ of $\hat{W}_1$ is ε_2-close to the identity map, Id, of $\hat{W}_1$. Since $Hom(\hat{W}_1)$ is locally contractible, we can choose ε_2 so small that there is a path $\theta:[0, 1]\rightarrow Hom(\hat{W}_1)$ inside the ε_2-neighborhood of the identity map Id in $Hom(\hat{W}_1)$ with $\theta(0)=m(\varepsilon_2)e_2$ and $\theta(1)=Id$. Consider the $m(\varepsilon_2)$-fold covering $\hat{W}_2$ (resp. $\hat{S}_2^1$) of $\overline{W}_2$ (resp. S_2^1) associated to $m(\varepsilon_2)e_2$.

The homeomorphisms $\theta(t)$, $t\in]0, 1]$, allow us to deform the fibration

$$\hat{W}_1 \longrightarrow \hat{W}_2 \longrightarrow \hat{S}_2^1$$

to a trivial fibration.

Now we can proceed with the same argument on the third level, then the fourth, and so on. Finally, we shall reach the top level and obtain a trivial fibration

$$\hat{W}_{k-1} \longrightarrow \hat{X} \longrightarrow \hat{S}_k^1$$

where $\hat{X}$ is a $(m(\varepsilon_1), m(\varepsilon_2), \cdots, m(\varepsilon_k))$-fold covering of $\overline{X}$. Therefore, we have

$$\begin{aligned}\hat{X} &\overset{\text{homeo}}{\approx} \hat{W}_{k-1}\times\hat{S}_k^1\\ &\overset{\text{homeo}}{\approx} \hat{W}_{k-2}\times\hat{S}_{k-1}^1\times\hat{S}_k^1\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &\overset{\text{homeo}}{\approx} V^*\times\hat{S}_1^1\times\hat{S}_2^1\times\cdots\hat{S}_k^1.\end{aligned}$$

This means that $\hat{X}$ is homeomorphic to the product space $V^*\times T^k$. We denote this homeomorphism by $H: \hat{X}\rightarrow V^*\times T^k$.

Next we investigate the regularity part in Theorem 1.5. If the Alexandrov space X is a topological manifold, then so is $\hat{X}$. Hence if $\dim V^*=n-k\geqq 4$, Theorem 2.2 (II) implies that V^* has a resolution. Let $f: N\rightarrow V^*$ be a resolution of V^*. Hence N is a manifold with $\dim N = n-k$ and homotopic to V^*. Therefore $F = f \times Id : N \times T^k \to V^* \times T^k$ is again a resolution of $V^* \times T^k$ where Id denotes the identity map of T^k. Theorem 2.2 (III) implies for any positive ε that there exists a homeomorphism $G : N \times T^k \to V^* \times T^k$ which is ε-close to F. Thus the composition of G with H^{-1} provides us a homeomorphism from $N \times T^k$ onto $\hat{X}$ and this completes the proof of Theorem 1.5.

Let us conclude this note by a corollary of Theorem 1.3 and 1.5.

COROLLARY 4.3. *Let X be a compact Alexandrov space with $K_X \geqq 0$. If X is a homology manifold and $\pi_1(X)$ is infinite, then X is a topological manifold.*

PROOF. We know from Theorem 1.4 that a finite covering $\hat{X}$ of X is homeomorphic to a product space $V^* \times T^k$ where V^* is an Alexandrov space with curvature $\geqq 0$ and its fundamental group is finite. Since $\pi_1(X)$ is infinite, then so is $\pi_1(\hat{X})$. Hence one has $k \geqq 1$. Since X is a homology manifold, Theorem 2.2 (I) implies that V^* is also a homology manifold. Then Theorem 1.3 gives that $\hat{X} = V^* \times T^k$ is a topological manifold. In turn, X is a topological manifold. □

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