

Carathéodory extremal maps of ellipsoids

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1. Introduction.

Let M be a domain in C^n and $p \in M$. Let M_p denote the couple (M, p) , a “pointed domain”. For two pointed domains M_p and N_q , let $\text{Hol}(M_p, N_q)$ denote the set of holomorphic mappings from M to N that send p to q . A map $f \in \text{Hol}(M_p, N_q)$ is said to be Carathéodory extremal, or C-extremal, if

$$|\det df(p)| = \sup \{ |\det dg(p)| : g \in \text{Hol}(M_p, N_q) \}.$$

In case both M and N contain the origin of C^n , we say f is C-extremal in $\text{Hol}(M, N)$ if f is C-extremal in $\text{Hol}(M_0, N_0)$. The C-extremal maps were first studied by Carathéodory in [CAR], and have been studied in, to name a few, [HAR], [SAD], [KUB], [TRA], [RAB] and [DIT]. In general it is very difficult or impossible to obtain explicit formulas for C-extremal maps. C-extremal maps between the ball and the polydisc were known to Carathéodory. Explicit formulas for C-extremal maps between the ball and symmetric domains were obtained by Kubota ([KUB]) and Travaglini ([TRA]). In this article we will give explicit formulas for the C-extremal maps between generalized ellipsoids (see Paragraph 2 for the definition) and the ball (Theorems 3.11 and 4.5). The generalized ellipsoids we consider are not necessarily convex. These results can be considered as an extension of the classical Schwarz lemma.

In Paragraph 2 we prove some basic properties of the C-extremal maps and of the extremal metric, and gather some known results which are needed for later paragraphs. In Paragraph 3 we give and prove the explicit formulas for the extremal maps from convex generalized ellipsoids to the ball. In Paragraph 4 we give formulas for the C-extremal maps from the ball to generalized ellipsoids, which may not be convex. In Paragraph 5 we discuss the geodesics and isolated points of the space of equivalent classes of pointed taut manifolds.

2. Basic properties.

We first give the definition of the extremal distance between pointed domains. Though we are mainly interested in bounded domains in C^n , it is more natural to give the definition for complex manifolds.

Let Q_n denote the set of all couples (M, p) , where M is a complex manifold of dimension n and $p \in M$. So Q_n is the set of pointed complex manifolds; (M, p) is the manifold M with “distinguished point” p . For convenience, write $M_p \equiv (M, p)$. Let T_n denote the subset of Q_n consisting of all pointed taut manifolds (see [WUH]). Let H_n denote the set of pointed hyperbolic manifolds (see [KOB]) of dimension n . We have $T_n \subset H_n \subset Q_n$.

DEFINITION 2.1. Let $M_p, N_q \in Q_n$. We say M_p and N_q are *biholomorphically equivalent* and write $M_p \sim N_q$ if there is a map $f \in \text{Hol}(M_p, N_q)$ which is a biholomorphism from M to N .

Obviously, \sim is an equivalence relation. Let $\mathcal{Q}_n = Q_n / \sim$, $\mathcal{T}_n = T_n / \sim$ and $\mathcal{H}_n = H_n / \sim$. If $M_p \equiv (M, p) \in Q_n$, let $\tilde{M}_p \equiv (M, p)^\sim$ denote the equivalence class to which (M, p) belongs. Sometimes we do not distinguish \tilde{M}_p from M_p if no ambiguity can arise. When M is homogeneous, we may not specify a distinguished point and, abusing notation, we may use M to denote M_p , where p is any point in M .

For $f \in \text{Hol}(M_p, N_q)$, $df(p)$ denotes the induced map between the tangent spaces: $df(p) : T_p M \rightarrow T_q N$. Consider the map $\det df(p) : \wedge^n(T_p M) \rightarrow \wedge^n(T_q N)$ induced by $df(p)$. If M, N are domains in C^n , then M, N have standard global coordinates and $\det df(p)$ is canonically represented by a complex number. If $f \in \text{Hol}(M_p, M_p)$, the complex number that represents $\det df(p)$ is independent of the choice of coordinates. Let $J_f(p)$ denote this number.

DEFINITION 2.2. Let $M_p, N_q \in Q_n$. A map $f \in \text{Hol}(M_p, N_q)$ is said to be *C-extremal* if

$$|\det df(p)| = \sup \{ |\det dg(p)| : g \in \text{Hol}(M_p, N_q) \}.$$

Obviously, this definition is independent of the choice of coordinates.

DENFINITION 2.3. Define the map $\mu : \mathcal{Q}_n \times \mathcal{Q}_n \rightarrow [-\infty, +\infty]$ by

$$\mu(\tilde{M}_p, \tilde{N}_q) = \inf \{ -\log |J_{g \circ f}(p)| : f \in \text{Hol}(M_p, N_q), g \in \text{Hol}(N_q, M_p) \}.$$

REMARK. One may also consider the map $\mu_1 : \mathcal{Q}_n \times \mathcal{Q}_n \rightarrow [-\infty, +\infty]$ defined by

$$\mu_1(\tilde{M}_p, \tilde{N}_q) = \inf \{ -\log |J_{g \circ f}(p)| : f \in \text{Emb}(M_p, N_q), g \in \text{Emb}(N_q, M_p) \}.$$

Here $\text{Emb}(M_p, N_q)$ denotes the set of injective holomorphic maps from M to N that sends p to q . Obviously, $\mu_1 \geq \mu$.

Shikata studied a somewhat similar distance function on the set of differentiable structures on a compact closed orientable topological manifold ([SHI]). Fridman studied a holomorphic invariant of hyperbolic manifolds which amounts to measure the distance between the domain and a homogeneous domain ([FRI]).

PROPOSITION 2.4. *The map μ is well defined and satisfies*

- i) $\mu(\tilde{M}_p, \tilde{N}_q) = \mu(\tilde{N}_q, \tilde{M}_p)$,
- ii) $\mu(\tilde{M}_p, \tilde{L}_r) \leq \mu(\tilde{M}_p, \tilde{N}_q) + \mu(\tilde{N}_q, \tilde{L}_r)$.

If $M_p \in H_n$ and $N_q \in Q_n$, then

- iii) $\mu(\tilde{M}_p, \tilde{N}_q) \geq 0$.

If $M_p \in T_n$ and $N_q \in Q_n$, then

- iv) $\mu(\tilde{M}_p, \tilde{N}_q) = 0$ implies $\tilde{M}_p = \tilde{N}_q$.

Consequently, (\mathcal{I}_n, μ) is a metric space.

An assertion equivalent to iv) is observed in [GMW]. In the case of $M_p = B_0^n$, iv) is equivalent to Theorem 1 in [GMW]. For the proof of Proposition 2.4, we need two preliminary results.

PROPOSITION 2.5 ([KOB]). *Let M be a hyperbolic manifold and $p \in M$. Let $f \in \text{Hol}(M_p, M_p)$. Then*

- 1) *The eigenvalues of $df(p)$ have modules not exceeding 1;*
- 2) $|\det df(p)| \leq 1$;
- 3) *If $|\det df(p)| = 1$, then $f \in \text{Aut}(M)$;*
- 4) *If $df(p) = \text{id}$, then $f = \text{id}$.*

PROPOSITION 2.6 ([GMW]). *Let M, N be complex manifolds of dimension n and let $f_j : M \rightarrow N$ and $g : N \rightarrow M$ be holomorphic mappings ($j=1, 2, \dots$) such that for some points $p \in M$, $q \in N$, $f_j(p) = q$ for all j . Furthermore, suppose $g \circ f_j$ converges uniformly on compact sets to an automorphism $\tau : M \rightarrow M$. Then g is biholomorphic.*

PROOF OF PROPOSITION 2.4. Suppose that $\phi : M_p \rightarrow M'_{p'}$ and $\psi : N_q \rightarrow N'_{q'}$ are biholomorphisms. Suppose $f : M_p \rightarrow N_q$ and $g : N_q \rightarrow M_p$ are holomorphic maps. Let $f_1 = \phi \circ f \circ \phi^{-1}$ and $g_1 = \psi \circ g \circ \psi^{-1}$. Then $f_1 : M'_{p'} \rightarrow N'_{q'}$ and $g_1 : N'_{q'} \rightarrow M'_{p'}$ are holomorphic maps, and $|J_{g \circ f}(p)| = |J_{g_1 \circ f_1}(p')|$. This implies that

$$\begin{aligned} -\log |J_{g \circ f}(p)| &= -\log |J_{g_1 \circ f_1}(p')| \\ &\geq \inf \{-\log |J_{g_1 \circ f_1}(p')| : f_1 \in \text{Hol}(M'_{p'}, N'_{q'}), g_1 \in \text{Hol}(N'_{q'}, M'_{p'})\}. \end{aligned}$$

This holds for any choice of f and g . Thus

$$\begin{aligned} &\inf \{-\log |J_{g \circ f}(p)| : f \in \text{Hol}(M_p, N_q), g \in \text{Hol}(N_q, M_p)\} \\ &\geq \inf \{-\log |J_{g_1 \circ f_1}(p')| : f_1 \in \text{Hol}(M'_{p'}, N'_{q'}), g_1 \in \text{Hol}(N'_{q'}, M'_{p'})\}. \end{aligned}$$

The reversed inequality is also true for similar reasons. Thus μ is well defined.

Since $|J_{g \circ f}(p)| = |J_{f \circ g}(q)|$, μ is symmetric.

Now we prove the triangle inequality. Consider $M_p, N_q, L_r \in Q_n$ and holomorphic maps $f : M_p \rightarrow N_q, g : N_q \rightarrow M_p, h : N_q \rightarrow L_r$ and $k : L_r \rightarrow N_q$. Let $\phi = h \circ f, \psi = g \circ k$. Then $|J_{\phi \circ \psi}(p)| = |J_{g \circ f}(p)| \cdot |J_{k \circ h}(q)|$. So

$$\mu(\tilde{M}_p, \tilde{L}_r) \leq -\log |J_{\phi \circ \psi}(p)| = -\log |J_{g \circ f}(p)| - \log |J_{k \circ h}(q)|.$$

Taking the infimum of the right hand side, we obtain

$$\mu(\tilde{M}_p, \tilde{L}_r) \leq \mu(\tilde{M}_p, \tilde{N}_q) + \mu(\tilde{N}_q, \tilde{L}_r).$$

If M is hyperbolic and $f \in \text{Hol}(M_p, N_q), g \in \text{Hol}(N_q, M_p)$, then $|J_{g \circ f}(p)| \leq 1$ by Proposition 2.5. Thus $\mu(\tilde{M}_p, \tilde{N}_q) \geq 0$.

Now suppose that $M_p \in T_n$ and $\mu(\tilde{M}_p, \tilde{N}_q) = 0$. By the tautness of M , there exists a C-extremal map $g \in \text{Hol}(N_q, M_p)$. There are $f_j : M_p \rightarrow N_q, j=1, 2, \dots$, such that $\lim_{j \rightarrow \infty} |\det df_j(p)| = \sup \{|\det df(p)| : f \in \text{Hol}(M_p, N_q)\}$. Thus $\lim_{j \rightarrow \infty} |J_{g \circ f_j}(p)| = 1$. Again the tautness of M implies the existence of a subsequence of $\{g \circ f_j\}$ which converges to some $\tau \in \text{Hol}(M_p, M_p)$. Obviously, $|\tau(p)| = 1$. By Proposition 2.5, $\tau \in \text{Aut}(M_p)$. Then Proposition 2.6 implies that g is a biholomorphism. Thus $\tilde{M}_p = \tilde{N}_q$. This concludes the proof. \square

REMARK. We may prove an analogue of Proposition 2.4 for μ_1 . Thus the restrictions of μ and μ_1 to T_n are metrics. We call them the *extremal metric* and the *univalent extremal metric* respectively.

DEFINITION 2.7. For $M_p \in Q_n$ the *indicatrix* of M at p in the (infinitesimal) Kobayashi metric (see [ROY]) is

$$I_p(M) \equiv \{\xi \in T_p M : K(p, \xi) < 1\}.$$

Let Δ denote the unit disc in \mathbf{C} . A domain $D \subset \mathbf{C}^n$ is said to be *balanced* if $cz \in D$ whenever $c \in \bar{\Delta}$ and $z \in D$.

PROPOSITION 2.8 ([SIB]). If $D \subset \mathbf{C}^n$ is a balanced domain of holomorphy, then $\{f'(0) : f \in \text{Hol}(\Delta_0, D_0)\} = \bar{D}$. Consequently $I_0(D) = D$. If D is a balanced domain (not necessarily a domain of holomorphy), then $I_0(D) \supset D$.

For two pointed domains $M_p, N_q \subset \mathbf{C}^n$, let

$$J_{\max}(M_p, N_q) = \sup \{|\det dg(p)| : g \in \text{Hol}(M_p, N_q)\}.$$

If both M and N contain 0, we write $J_{\max}(M, N) = J_{\max}(M_0, N_0)$. Obviously,

$$\mu(M_0, N_0) = -\log [J_{\max}(M, N) J_{\max}(N, M)].$$

The following proposition is a consequence of Proposition 2.8. Different forms of the proposition can be found in [RUD, p. 161] and [SAD].

PROPOSITION 2.9. If D_1, D_2 are balanced domains and if D_2 is a domain of holomorphy, then any holomorphic map $f \in \text{Hol}((D_1, 0), (D_2, 0))$ satisfies $df(0)(D_1) \subset D_2$. Hence

$$J_{\max}(D_1, D_2) = \sup \{ |\det(m \circ l)| : l \text{ complex linear map, } l(D_1) \subset D_2 \}.$$

If both D_1 and D_2 are balanced domains of holomorphy, then

$$\begin{aligned} \mu((D_1, 0), (D_2, 0)) &= \mu_1((D_1, 0), (D_2, 0)) \\ &= \inf \{ -\log |\det(m \circ l)| : l, m \text{ complex linear maps, } l(D_1) \subset D_2, m(D_2) \subset D_1 \}. \end{aligned}$$

PROOF. Since the Kobayashi metric decreases under the holomorphic maps, the differential of a holomorphic map sends indicatrix into indicatrix. Thus the proposition follows from Proposition 2.8. \square

DEFINITION 2.10. An *ellipsoid* is a domain of the form

$$(2.11) \quad E(m) = E(m_1, \dots, m_n) \equiv \left\{ z \in \mathbf{C}^n : \sum_{j=1}^n |z_j|^{2m_j} < 1 \right\}.$$

A generalized ellipsoid is a domain of the form

$$\begin{aligned} (2.12) \quad E(p, m) &= E(p_1, \dots, p_k; m_1, \dots, m_k) \\ &= \left\{ z = (z_1, \dots, z_k) \in \mathbf{C}^{p_1} \times \dots \times \mathbf{C}^{p_k} = \mathbf{C}^n : \sum_{j=1}^k |z_j|^{2m_j} < 1 \right\}, \\ n &= p_1 + \dots + p_k. \end{aligned}$$

An Hermitian ellipsoid is a domain of the form

$$(2.13) \quad \left\{ z \in \mathbf{C}^n : \sum_{i,j=1}^n a_{ij} z_i \bar{z}_j < 1 \right\},$$

where (a_{ij}) is a positive definite Hermitian matrix. \square

In (2.11) and (2.12), $0 < m_j \leq \infty$. If some of m_j 's are ∞ , the definition is to be understood in the sense of limit. For instance, (2.11) is to be understood as

$$E = \left\{ z \in \mathbf{C}^n : |z_j| < 1 \forall j \in \mathcal{I}, \sum_{j \in \mathcal{F}} |z_j|^{2m_j} < 1 \right\},$$

where

$$\mathcal{I} = \{j : m_j = \infty\}, \quad \mathcal{F} = \{1, \dots, n\} \setminus \mathcal{I}.$$

Let D be a domain of dimension n , containing 0. If l is a complex linear map such that $l(D) \subset B^n$, then $l^{-1}(B^n)$ is an Hermitian ellipsoid containing D . If l is a solution of the extremal problem

$$\sup \{ |\det l| : l \text{ complex linear map, } l(D) \subset B^n \},$$

then $l^{-1}(B^n)$ is a circumscribed Hermitian ellipsoid of D of least volume, or a *minimal circumscribed Hermitian ellipsoid*. If m is a solution of the extremal problem

$$\sup \{ |\det m| : m \text{ complex linear map, } m(B^n) \subset D \},$$

then $m(B^n)$ is an inscribed Hermitian ellipsoid of D of greatest volume, or a *maximal inscribed Hermitian ellipsoid*.

PROPOSITION 2.14. *Let D be a bounded domain. Then D has minimal circumscribed and maximal inscribed Hermitian ellipsoids, and the minimal circumscribed Hermitian ellipsoid of D is unique. If, in addition, D is convex and balanced, then the maximal inscribed Hermitian ellipsoid is also unique.*

PROOF. The existence of extremal Hermitian ellipsoids is proved in [JOH]. It follows from the fact that we may consider only Hermitian ellipsoids in a compact family (in the Euclidean topology: an Hermitian ellipsoid is defined by a positive definite Hermitian matrix, hence is represented by a point in some Euclidean space). The uniqueness of the minimal circumscribed Hermitian ellipsoid is implicitly used in [JOH]. For completeness we give a simple proof here. We proceed as follows.

Seeking a contradiction, suppose that D has two minimal circumscribed Hermitian ellipsoids P_1 and P_2 . Minimal circumscribed Hermitian ellipsoids are invariant under complex linear maps in the sense that if P is a minimal circumscribed Hermitian ellipsoid of D and if l is a complex linear map then $l(P)$ is a minimal circumscribed Hermitian ellipsoid of $l(D)$. Therefore we may assume that $P_1 = B^n$ and

$$P_2 = \left\{ z \in \mathbf{C}^n : \sum_{j=1}^n a_j |z_j|^2 < 1 \right\},$$

where all $a_j > 0$. Since $V(P_1) = V(P_2)$, it follows that $\prod_{j=1}^n a_j = 1$. Let $b_j = (1 + a_j)/2$ and

$$P_3 = \left\{ z \in \mathbf{C}^n : \sum_{j=1}^n b_j |z_j|^2 < 1 \right\}.$$

Then $P_3 \supset P_1 \cap P_2 \supset D$ and $V(P_3) = \omega_n (\prod_{j=1}^n b_j)^{-2}$, where $\omega_n = V(B^n)$. Since the a_j 's are not all 1,

$$\prod_{j=1}^n b_j = \prod_{j=1}^n [(1 + a_j)/2] > \prod_{j=1}^n a_j^{1/2} = 1.$$

Thus $V(P_3) < V(B^n)$, which contradicts the minimality of P_1 . Thus the minimal circumscribed Hermitian ellipsoid is unique.

Now suppose D is convex and balanced. For a convex and balanced domain Ω define $J(\Omega)$ by

$$J(\Omega) = \left\{ z \in \mathbb{C}^n : \sup_{w \in \Omega} \left| \sum_{j=1}^n z_j w_j \right| < 1 \right\}.$$

Then $J(\Omega)$ is also convex and balanced; $J^2(\Omega) = \Omega$; and $\Omega_1 \subset \Omega_2$ implies $J(\Omega_1) \supset J(\Omega_2)$. If $P = \{z \in \mathbb{C}^n : \sum_{i,j=1}^n a_{ij} z_i \bar{z}_j < 1\}$ is an Hermitian ellipsoid, then $J(P) = \{z \in \mathbb{C}^n : \sum_{i,j=1}^n b_{ij} z_i \bar{z}_j < 1\}$ is also an Hermitian ellipsoid, where the matrix (b_{ij}) is the inverse of the transpose of the matrix (a_{ij}) . Moreover $(V(P) \cdot V(J(P))) = \omega_n^2$. Thus P is a maximal inscribed Hermitian ellipsoid of D if and only if $J(P)$ is a minimal circumscribed Hermitian ellipsoid of $J(D)$. Since the minimal circumscribed Hermitian ellipsoid of $J(D)$ is unique, so is the maximal inscribed Hermitian ellipsoid of D . \square

For a convex balanced domain D , let $P(D)$, $Q(D)$ denote its minimal circumscribed Hermitian ellipsoid and maximal inscribed Hermitian ellipsoid respectively.

The uniqueness of the extremal Hermitian ellipsoids implies that the extremal Hermitian ellipsoids of D have whatever “linear symmetry” that D has. That is, if $l \in \mathrm{GL}(n, \mathbb{C})$ and $l(D) = D$, then the extremal Hermitian ellipsoids are invariant under l . This is a useful observation for determining the C-extremal maps between the ball and a balanced domain of holomorphy.

The image of a Reinhardt domain $G \subset \mathbb{C}^n$ under the map $z \mapsto (\log |z_1|, \dots, \log |z_n|)$ is called the logarithmic image of G .

DEFINITION 2.15. A Reinhardt domain G is said to be *strictly logarithmically convex* if the following conditions are satisfied.

- 1) The logarithmic image of G is a strictly convex domain in \mathbb{R}^n .
- 2) The image of an arbitrary section $\{z \in G : z_{i_1} = \dots = z_{i_k} = 0\}$, $1 \leq k \leq n-2$, under the mapping $z \mapsto (\log |z_{j_1}|, \dots, \log |z_{j_{n-k}}|)$, where (j_1, \dots, j_{n-k}) is the multi-index conjugate to (i_1, \dots, i_k) , is a strictly convex domain in \mathbb{R}^{n-k} .

PROPOSITION 2.16 ([RAB]). A. *If D is a balanced domain of holomorphy, then each C-extremal linear mapping in $\mathrm{Hol}(B^n, D)$ has the form*

$$(2.17) \quad f = \mathrm{diag}(a_1, \dots, a_n) \circ U,$$

where *diag* is the diagonal matrix with specified entries, $a_1, \dots, a_n > 0$, and U is a unitary transformation.

B. *If D is a strictly logarithmically convex domain, then the mapping $\mathrm{diag}(a_1, \dots, a_n)$ in (2.17) is defined uniquely.* \square

THEOREM 2.18 ([HAR, RAB]). Let G be a balanced domain, let D be a Reinhardt domain such that its logarithmic image is strictly convex, and let $f \in \mathrm{Hol}(G_0, D_0)$ with $df(0) = \mathrm{diag}(a_1, \dots, a_n)$, $a_1, \dots, a_n > 0$. If $df(0)(\partial G) \cap \partial D \setminus$

$\{w : w_1 \cdots w_n = 0\} \neq \emptyset$, then $f = df$. □

3. C-extremal maps from generalized ellipsoids to the ball.

In this paragraph we will give explicit formulas for $J_{\max}(E(p, m), B^n)(E(p, m))$ is the generalized ellipsoid defined in (2.12)), for $m_1, \dots, m_k \geq 1$. The following fact will be useful. Its proof is straightforward.

PROPOSITION 3.1. *Let $f \in \text{Hol}(G_p, D_q)$ and $g \in \text{Hol}(D_q, Q_r)$. If $g \circ f$ is C-extremal in $\text{Hol}(G_p, Q_r)$, then f is C-extremal in $\text{Hol}(G_p, D_q)$ and g is C-extremal in $\text{Hol}(D_q, Q_r)$.* □

We begin with the case where $m_j = \infty$, i.e., $E \equiv E(n, \infty) = B^{p_1} \times \cdots \times B^{p_k}$. Since E is invariant under $U(p_1) \times \cdots \times U(p_k)$, so is its minimal circumscribed Hermitian ellipsoid $P(E)$. Thus $P(E)$ must have the form

$$P(E) = \left\{ z = (z_1, \dots, z_k) \in \mathbb{C}^n : \sum_{i=1}^k b_i |z_i|^2 < 1 \right\}.$$

By the definition of minimal circumscribed Hermitian ellipsoid the numbers b_i satisfy

$$b_1^{p_1} \cdots b_k^{p_k} = \max \{c_1^{p_1} \cdots c_k^{p_k} : c_1 + \cdots + c_k = 1\}.$$

Direct computations give that $b_i = p_i/n$. Hence a C-extremal linear map in $\text{Hol}(E, B^n)$ is given by

$$f(z_1, \dots, z_k) = (\sqrt{p_1/n} z_1, \dots, \sqrt{p_k/n} z_k).$$

Similar arguments show that the maximal inscribed Hermitian ellipsoid $Q(E) = B^n$, hence the inclusion map from B^n to E is C-extremal in $\text{Hol}(B^n, E)$. Thus we obtain

PROPOSITION 3.2. *For the domain*

$$E(p, \infty) = B^{p_1} \times \cdots \times B^{p_k},$$

we have

$$(3.3) \quad J_{\max}(E(p, \infty), B^n) = n^{-n/2} p_1^{p_1/2} \cdots p_k^{p_k/2}.$$

In the extremal metric the distance between $E(p, \infty)$ and the ball is

$$\mu(B^n, E(p, \infty)) = \frac{1}{2} \log(n^n p_1^{-p_1} \cdots p_k^{-p_k}).$$

If all p_i are 1, then $E = \Delta^n$ and (3.3) becomes $J_{\max}(\Delta^n, B^n) = n^{-n/2}$. This last was known to Carathéodory ([CAR]).

We now consider $E=E(p, m)$ with $p_1 + \dots + p_k = n$ and $1 \leq m_i \leq \infty$, $i=1, \dots, k$. Consider the inclusion maps $I: B^n \rightarrow E(p, m)$ and $J: E(p, m) \rightarrow E(p, \infty)$. Since $J \circ I$ is C-extremal in $\text{Hol}(B^n, E(p, \infty))$, both I and J are C-extremal.

Now we turn to find a C-extremal map in $\text{Hol}(E(p, m), B^n)$. Let

$$\begin{aligned} s(p, m) &= \sum_{j=1}^k p_j/m_j, \quad h_j(p, m) = (m_j s(p, m)/p_j)^{-1/m_j}, \\ a_j(p, m) &= (p_j/n)h_j(p, m)^{-1}, \end{aligned}$$

for $j=1, \dots, k$, with the convention that $\alpha^{-1/\alpha}=1$ when $\alpha=+\infty$.

PROPOSITION 3.4. *Let*

$$\begin{aligned} f: E(p, m) &\longrightarrow C^n \\ f_j(z) &= a_j(p, m)^{1/2}z_j, \quad j=1, \dots, n. \end{aligned}$$

Then $f(E(p, m)) \subset B^n$.

PROOF. We first treat the case $1 < m_j < \infty$ for $j=1, \dots, k$. Write $a_j=a_j(p, m)$. It suffices to show that the quantity

$$T \equiv \sup \left\{ \sum_{j=1}^k a_j |z_j|^2 : \sum_{j=1}^k |z_j|^{2m_j} = 1 \right\}$$

does not exceed 1. Now T is the maximal value of the function

$$(3.5) \quad F(q) = F(q_1, \dots, q_n) = \sum_{j=1}^k a_j q_j$$

with the constraint

$$(3.6) \quad G(q) \equiv \sum_{j=1}^k q_j^{m_j} = 1.$$

First we claim that the maximal value cannot be attained at a point q with some $q_j=0$. It is sufficient to see that it cannot be attained at a point q with $q_1=0$. Consider $q^0=(0, q_2, \dots, q_n)$. Let

$$q^t = (t, (1-t^{m_1})^{1/m_2}q_2, \dots, (1-t^{m_1})^{1/m_n}q_n), \quad t \in (0, 1).$$

Then $G(q^t)=1$. One may verify that

$$\lim_{t \rightarrow 0^+} \frac{F(q^t) - F(q^0)}{t} = a_1 > 0.$$

Thus the maximal value cannot be attained at q^0 . The maximal value is therefore attained at a point q such that $q_j \neq 0$ for each j . By the method of Lagrange's multipliers, this q and some λ satisfy (3.6) and

$$(3.7) \quad a_j = \lambda m_j q_j^{m_j-1}, \quad j = 1, \dots, n.$$

If one solves for q_j 's in (3.7) and substitutes into (3.6), one obtains a decreasing function of λ on the left hand side. The resulting equation determines λ uniquely. Thus (3.6) and (3.7) has a unique solution. One may verify that

$$\lambda = s(p, m)/n, \quad q_j = h_j(p, m)$$

is the solution. Thus

$$T = F(h_1(p, m), \dots, h_n(p, m)) = \sum_{j=1}^k a_j h_j(p, m) = 1.$$

This concludes the proof for the case where each $m_j \in (1, \infty)$.

We turn now to the case where $m_j \in [1, +\infty)$ and some m_j 's are 1. Without loss of generality, we may assume that $1 < m_j < +\infty$ for $j = 1, \dots, r$ and $m = (m_1, \dots, m_r, 1, \dots, 1)$. Choose $\varepsilon > 0$ and let

$$m^\varepsilon = (m_1, \dots, m_r, 1 + \varepsilon, \dots, 1 + \varepsilon).$$

Since $E(p, m) \subset E(m^\varepsilon)$, what we have proved gives

$$(3.8) \quad \sum_{j=1}^k a_j(m^\varepsilon) |z_j|^2 \leq 1, \quad z \in E(p, m).$$

Letting $\varepsilon \rightarrow 0$ in (3.8) gives

$$\sum_{j=1}^k a_j(p, m) |z_j|^2 \leq 1, \quad z \in E(p, m).$$

This proves the assertion for the case where $m_j \in [1, +\infty)$ for each j .

We have proved that if $m_j \in [1, +\infty)$ for each j , then

$$(3.9) \quad \sum_{j=1}^k p_j n^{-1} h_j(p, m)^{-1} |z_j|^2 \leq 1 \quad \text{if } \sum_{j=1}^k |z_j|^{2m_j} < 1.$$

Now consider the case where some of m_j 's are ∞ . We assume

$$m = (m_1, \dots, m_r, \infty, \dots, \infty),$$

where $m_j \in [1, +\infty)$ for $j = 1, \dots, r$. By (3.9),

$$(3.10) \quad \sum_{j=1}^r p_j r^{-1} h_j(p, m)^{-1} |z_j|^2 \leq 1 \quad \text{if } \sum_{j=1}^r |z_j|^{2m_j} < 1.$$

It follows easily from (3.10) that

$$\sum_{j=1}^k p_j n^{-1} h_j(p, m)^{-1} |z_j|^2 \leq 1 \quad \text{if } \sum_{j=1}^r |z_j|^{2m_j} < 1, \quad |z_i| < 1, \quad i = r+1, \dots, k.$$

This implies that $f(E(p, m)) \subset B^n$. \square

REMARK. Note that in the proof we conclude that (3.6) and (3.7) have a unique solution, and we give the solution explicitly. Though one can check that the given solution indeed satisfies (3.6) and (3.7), the proof does not indicate how the solution is found. The author found the solution by trial and luck. It is still a mystery to the author how to find the solution by an argument or an algorithm.

THEOREM 3.11. Consider the generalized ellipsoid $E(p, m)$. Suppose that $m = (m_1, \dots, m_k)$, $1 \leq m_j \leq \infty$. Then the map f in Proposition 3.4 is C-extremal in $\text{Hol}(E(p, m), B^n)$. Hence

$$(3.12) \quad J_{\max}(E(p, m), B^n) = \left[n^{-n} s^s \prod_{1 \leq j \leq k} p_j^{p_j} \prod_{\substack{1 \leq j \leq k \\ m_j < \infty}} \left(\frac{p_j}{m_j} \right)^{-p_j/m_j} \right]^{1/2},$$

and in the extremal metric the distance between the ball and $E(p, m)$ with distinguished point 0 is

$$(3.13) \quad \mu(B^n, E(p, m)_0) = \frac{1}{2} \log \left[n^n s^{-s} \prod_{1 \leq j \leq k} p_j^{-p_j} \prod_{\substack{1 \leq j \leq k \\ m_j < \infty}} \left(\frac{p_j}{m_j} \right)^{p_j/m_j} \right],$$

where $s = \sum_{j=1}^k (p_j/m_j)$.

PROOF. Let $f: E(p, m) \rightarrow B^n$ be the map in Proposition 3.4. Consider the map

$$g: E(p, \infty) \longrightarrow \mathbf{C}^n = \mathbf{C}^{p_1} \times \cdots \times \mathbf{C}^{p_k}$$

$$g_j(z) = h_j(p, m)^{1/2} z_j, \quad j = 1, \dots, k.$$

It is easy to check that $g(E(p, \infty)) \subset E(p, m)$. We have

$$\det(f \circ g)(0) = \prod_{j=1}^k (a(p, m)h(p, m))^{p_j/2} = n^{-n/2} \prod_{j=1}^k p_j^{p_j/2}.$$

By Proposition 3.2, $f \circ g$ is C-extremal. Thus both f and g are C-extremal. Now (3.12) and (3.13) follow from direct computations. \square

COROLLARY 3.14. Let $p_j \in \mathbf{N}$ and $n = p_1 + \cdots + p_k$. Let $1 \leq m_j \leq \infty$, $j = 1, \dots, k$. Then

$$\sup \left\{ \prod_{j=1}^k a_j^{p_j} : \sum_{j=1}^k a_j q_j \leq 1 \quad \text{if } q_j \geq 0, \sum_{j=1}^k q_j^{m_j} = 1 \right\}$$

equals

$$n^{-n} s^s \prod_{1 \leq j \leq k} p_j^{p_j} \prod_{\substack{1 \leq j \leq k \\ m_j < \infty}} \left(\frac{p_j}{m_j} \right)^{-p_j/m_j}, \quad s = \sum_{j=1}^k p_j/m_j,$$

and is attained at

$$a_j = \frac{p_j}{n} \left(\frac{m_j}{p_j} \sum_{i=1}^k \frac{p_i}{m_i} \right)^{1/m_j}, \quad j = 1, \dots, k.$$

COROLLARY 3.15. Let $p_j \in N$ and $n = p_1 + \dots + p_k$. Let $1 \leq m \leq \infty$, $j = 1, \dots, k$. Then

$$\sup \left\{ \prod_{j=1}^k a_j^{p_j} : \sum_{j=1}^k a_j q_j \leq 1 \text{ if } q_j \geq 0, \sum_{j=1}^k q_j^m = 1 \right\}$$

equals

$$n^{n(1/m-1)} \prod_{1 \leq j \leq k} p_j^{p_j(1-1/m)},$$

and is attained at

$$a_j = (p_j/n)^{1-1/m}, \quad j = 1, \dots, k.$$

4. C-extremal maps from the ball to generalized ellipsoids.

We now consider $E(p, m)$, where $m = (m_1, \dots, m_k)$ and $m_1, \dots, m_k < 1$. It follows from Proposition 2.14 that the inclusion map from $E(p, m)$ to B^n is C-extremal and $J_{\max}(E(p, m), B^n) = 1$. We now proceed to obtain formulas for C-extremal maps in $\text{Hol}(B^n, E(p, m))$. One can check that $E(p, m)$ is strictly logarithmically convex.

We first consider $E(p, m_0)$, where $m_0 = (u, \dots, u)$, $0 < u < 1$. By Proposition 2.16, there are uniquely determined (e_1, \dots, e_k) such that the map

$$f(z_1, \dots, z_k) = (e_1 z_1, \dots, e_k z_k)$$

is C-extremal in $\text{Hol}(B^n, E(p, m))$. Now

$$\begin{aligned} e_1^{2p_1} \cdots e_k^{2p_k} &= \sup \left\{ \prod_{j=1}^k b_j^{p_j} : \sum_{j=1}^k (b_j q_j)^u \leq 1 \text{ if } q_j \geq 0, \sum_{j=1}^k q_j = 1 \right\} \\ &= \sup \left\{ \prod_{j=1}^k c_j^{p_j/u} : \sum_{j=1}^k (c_j l_j) \leq 1 \text{ if } l_j \geq 0, \sum_{j=1}^k l_j^{1/u} = 1 \right\}. \end{aligned}$$

By Corollary 3.15,

$$e_j = c_j^{1/(2u)} = (p_j/n)^{(1/u-1)/2}, \quad j = 1, \dots, k.$$

Thus we obtain

PROPOSITION 4.1. Let

$$(4.2) \quad e_j = (p_j/n)^{(1/u-1)/2}, \quad j = 1, \dots, k.$$

Then the map $f: B^n \rightarrow E(p, m_0)$ defined by

$$(4.3) \quad f(z_1, \dots, z_k) = (e_1 z_1, \dots, e_k z_k)$$

is C-extremal in $\text{Hol}(B^n, E(p, m))$.

We turn to consider a general $E(p, m)$, where $m=(m_1, \dots, m_k)$, $m_j < 1$. Choose $m_0=(u, \dots, u)$ so that $u < m_j$, $j=1, \dots, k$. Let

$$(4.4) \quad a_j = (n/p_j)^{1/2} (s^{-1} p_j/m_j)^{1/(2m_j)}, \quad h_j = e_j/a_j, \quad j = 1, \dots, k.$$

THEOREM 4.5. *Let*

$$g : B^n \longrightarrow \mathbf{C}^n = \mathbf{C}^{p_1} \times \cdots \times \mathbf{C}^{p_k}$$

$$g(z_1, \dots, z_k) = (a_1 z_1, \dots, a_k z_k).$$

Then $g(B^n) \subset E(p, m)$ and g is C-extremal in $\text{Hol}(B^n, E(p, m))$. Thus

$$(4.6) \quad J_{\max}(B^n, E(p, m)) = \prod_{j=1}^k a_j^{p_j} = \left[n^n s^{-s} \prod_{1 \leq j \leq k} p_j^{-p_j} \prod_{\substack{1 \leq j \leq k \\ m_j < \infty}} \left(\frac{p_j}{m_j} \right)^{p_j/m_j} \right]^{1/2},$$

and

$$\mu(B^n, E(p, m)) = \frac{1}{2} \log \left[n^{-n} s^s \prod_{1 \leq j \leq k} p_j^{p_j} \prod_{\substack{1 \leq j \leq k \\ m_j < \infty}} \left(\frac{p_j}{m_j} \right)^{-p_j/m_j} \right],$$

where $s = \sum_{j=1}^k (p_j/m_j)$.

PROOF. Let $t : E(p, m) \rightarrow \mathbf{C}^n$ be defined by $t(z_1, \dots, z_k) = (h_1 z_1, \dots, h_k z_k)$. By arguments very similar to the proof of Proposition 3.4, we can show that $g(B^n) \subset E(p, m)$ and that $t(E(p, m)) \subset E(p, m_0)$. Now $t \circ g$ equals the C-extremal map f in Proposition 4.1. Thus g is C-extremal map in $\text{Hol}(B^n, E(p, m))$ by Proposition 3.1. Similarly, the inclusion map from $E(p, m)$ to B^n is C-extremal in $\text{Hol}(E(p, m), B^n)$. \square

5. Geodesic segments and isolated points.

DEFINITION 5.1. Let (S, μ) be a metric space. A continuous curve $\gamma : [c, d] \rightarrow S$ is said be a *geodesic segment* in S if there is an $\epsilon > 0$ such that whenever $c \leq t_1 < t_2 < \cdots < t_k \leq d$, $k \in N$, and $\mu(\gamma(t_1), \gamma(t_k)) < \epsilon$, we have

$$(5.2) \quad \mu(\gamma(t_1), \gamma(t_k)) = \mu(\gamma(t_1), \gamma(t_2)) + \mu(\gamma(t_2), \gamma(t_3)) + \cdots + \mu(\gamma(t_{k-1}), \gamma(t_k)).$$

Recall that \mathcal{T}_n is the space of equivalent classes of pointed taut manifolds. We will construct geodesics in \mathcal{T}_n to exhibit a phenomenon that there are two arbitrarily close points in \mathcal{T}_n such that there are infinitely many geodesics connecting them. To construct these geodesics we need to have formulas for the extremal distances between generalized ellipsoids.

Consider the generalized ellipsoid $E(p, m)$ defined in Definition 2.10. Let

$$(5.3) \quad W(p, m) = \left[s^s \prod_{\substack{1 \leq j \leq k \\ m_j < \infty}} p_j^{p_j} \prod_{\substack{1 \leq j \leq k \\ m_j < \infty}} \left(\frac{p_j}{m_j} \right)^{-p_j/m_j} \right]^{1/2},$$

$$s = \sum_{j=1}^k (p_j/m_j).$$

For $m=(m_1, \dots, m_k)$ and $l=(l_1, \dots, l_k)$ we say $m \geqq l$ if $m_j \geqq l_j$ for each j . We say $m > l$ if $m \geqq l$ and $m \neq l$. Let $e=(1, \dots, 1)$. Thus $E(p, e)=B^n$.

THEOREM 5.4. *If $m \geqq l \geqq e$ or $e \geqq m \geqq l$, then*

$$J_{\max}(E(p, m), E(p, l)) = W(p, m)/W(p, l), \quad J_{\max}(E(p, l), E(p, m)) = 1,$$

$$\mu(E(p, m), E(p, l)) = \log W(p, l) - \log W(p, m).$$

PROOF. Suppose that $m \geqq l \geqq e$. Let f_m (resp. f_l) be the extremal map from $E(p, m)$ (resp. $E(p, l)$) to B^n in Proposition 3.4. Let $h=f_l^{-1} \circ f_m$. By an argument very similar to the proof of Proposition 3.14, we can show that $h(E(p, m)) \subset E(p, l)$. Since $f_l \circ h=f_m$ is C-extremal in $\text{Hol}(E(p, m), B^n)$, h is C-extremal in $\text{Hol}(E(p, m), E(p, l))$ by Proposition 3.1. Similarly, the inclusion map from $E(p, l)$ to $E(p, m)$ is C-extremal. So the formulas in the statement of the theorem follow from direct computations. The case $e \geqq m \geqq l$ is similar. \square

COROLLARY 5.5. *Consider ellipsoids*

$$E(m) = E(m_1, \dots, m_n) \equiv \left\{ z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2m_j} < 1 \right\}.$$

Let

$$(5.6) \quad W(m) = \left[s^s \prod_{\substack{1 \leq j \leq n \\ m_j < \infty}} m_j^{1/m_j} \right]^{1/2}, \quad s = \sum_{j=1}^n (1/m_j).$$

If $m \geqq l \geqq e$ or $e \geqq m \geqq l$, then

$$(5.7) \quad J_{\max}(E(m), E(l)) = W(p, m)/W(p, l), \quad J_{\max}(E(l), E(m)) = 1,$$

$$\mu(E(m), E(l)) = \log W(p, l) - \log W(p, m).$$

PROPOSITION 5.8. *Suppose that $m(t)=(m_1(t), \dots, m_n(t))$ and each $m_j: [0, 1] \rightarrow [1, +\infty]$ is a continuous nondecreasing function such that $m_j(0)=1$ and $m_j(1)=+\infty$, and m is injective. Then the curve $\gamma: [0, 1] \rightarrow \mathcal{T}_n$ defined by $\gamma(t)=(E(m(t)), 0)$ is a geodesic segment connecting $(B^n, 0)$ and $(\Delta^n, 0)$.*

PROOF. Both the continuity of γ and (5.2) follow from (5.7). \square

PROPOSITION 5.9. *Let $m=(m_1, \dots, m_n)$ and $m_j \in [1, +\infty]$. Then for each $\varepsilon > 0$ there is a $(D, p) \in \mathcal{T}_n$ such that $(E(m), 0) \sim (D, p)^\sim$ and $\mu((E(m), 0), (D, p)) < \varepsilon$ and such that there exist infinitely many geodesic segments connecting $(E(m), 0)$*

and (D, p) .

PROOF. Choose $l=(l_1, \dots, l_n)$ so that $l \neq m$, $l_j \geq m_j$ (or $l_j \leq m_j$ if $m_j = +\infty$ for each j) and $\mu((E(m), 0), (E(l), 0)) < \varepsilon$. Let $D=E(l)$ and $p=0$. Then for each continuous monotonous map $\nu: [0, 1] \rightarrow [1, +\infty]^n$ such that $\nu(0)=m$ and $\nu(1)=l$, $\gamma(t)=(E(\nu(t)), 0)$ is a geodesic segment connecting $(E(m), 0)$ and $(E(l), 0)$. This assertion follows from (5.4). These geodesic segments are generically different. \square

We turn now to the question whether \mathcal{T}_n is bounded. Consider the domain $E^\alpha = \{z \in \mathbf{C}^n : \sum_{j=1}^n |z_j|^{2\alpha} < 1\}$ with $\alpha < 1$. Since the defining function is plurisubharmonic, E^α is a balanced domain of holomorphy. Now E^α is taut since it satisfies the condition (*) defined in [KER]. We have

$$(5.10) \quad \begin{aligned} \mu(E_0^\alpha, B^n) &= \frac{1}{2} \log \frac{V(P(E^\alpha))}{V(Q(E^\alpha))}; \\ \mu(E_0^\alpha, B^n) &\geq \frac{1}{2} \log \frac{V(B^n)}{V(E^\alpha)}. \end{aligned}$$

Some computations give

$$V(E^\alpha) = \frac{\pi^n (\Gamma(1+1/\alpha))^n}{\Gamma(1+n/\alpha)}.$$

Then it is not difficult to see that $\lim_{\alpha \rightarrow 0} V(E^\alpha) = 0$. This and (5.10) give

$$\lim_{\alpha \rightarrow 0} \mu(E_0^\alpha, B^n) = +\infty.$$

Thus we have

PROPOSITION 5.11. *The metric space (\mathcal{T}_n, μ) is unbounded.*

Proposition 5.9 indicates, in some vague sense, that some parts of the metric space (\mathcal{T}_n, μ) are very dense. We will now show that some other parts of (\mathcal{T}_n, μ) are “sparse” by exhibiting certain isolated points.

PROPOSITION 5.12 ([CAR]). *Let Ω be a bounded multiply connected domain in \mathbf{C} and $p \in \Omega$. Then there exists a constant $c=c(\Omega, p) \in (0, 1)$ such that each $f \in \text{Hol}(\Omega_p, \Omega_p)$ with $|f'(p)| > c$ is an automorphism.*

DEFINITION 5.13. Let M be a complex manifold and $p \in M$. The complex manifold M is said to be *rigid in the sense of Carathéodory at p* if there is a constant $c \in (0, 1)$ such that each $f \in \text{Hol}(M_p, M_p)$ with $|J_f(p)| > c$ is an automorphism.

PROPOSITION 5.14. *If $M_p \in \mathcal{T}_n$ and M is rigid in the sense of Carathéodory at p , then M_p is an isolated point in \mathcal{T}_n .*

PROPOSITION 5.15. *Let M be a bounded taut domain in C^n such that $0 < \dim H_n(\Omega, R) < \infty$. Then M is rigid in the sense of Carathéodory at each point $p \in M$. Consequently, \tilde{M}_p is an isolated point in \mathcal{T}_n for each $p \in M$.*

REMARK. A related result can be found in [WIN].

PROOF. Seeking a contradiction, suppose that M is not rigid in the sense of Carathéodory at some p . Then there is a sequence $f_j \subset \text{Hol}(M_p, M_p)$ such that for each j , $|J_{f_j}(p)| > 1 - 1/j$ and f_j is not an automorphism. Thus $|J_{f_j}(p)| \in (1 - 1/j, 1)$ for each j . Without loss of generality we may assume that $f_j \rightarrow \phi$ as $j \rightarrow \infty$. Obviously $|J_\phi| = 1$, hence $\phi \in \text{Aut}(M)$. Thus the map $\phi_*: H_n(M, R) \rightarrow H_n(M, R)$ induced by ϕ is an isomorphism.

Let $\gamma_1, \dots, \gamma_k$ be generators of $H_n(M, R)$, represented by chains in a compact subset K of M . Since $f_j \rightarrow \phi$ uniformly on K , $f_j^*(\gamma_i) = \phi_*(\gamma_i)$ for each i and each sufficiently large j . Thus, for j sufficiently large, $f_j^* = \phi_*$ and f_j^* is an isomorphism from $H_n(M, R)$ to $H_n(M, R)$. Fix such a j , and write $g = f_j$ for convenience.

By Theorem 1.1 in [BED], a subsequence of the sequence $\{g^j\}$ of iterates of g converges uniformly on compact subsets of M to a map $\psi \in \text{Hol}(M_p, M_p)$ such that $\psi = \lambda \circ \tau$, where τ is a holomorphic retraction to a complex submanifold V of M , and $\lambda \in \text{Aut}(V)$. Since $g \notin \text{Aut}(M)$, it follows that $\psi \notin \text{Aut}(M)$. Thus V is a proper submanifold of M and $\dim V < n$. Since V is a Stein manifold, V can be imbedded in some C^N as a closed submanifold. By Theorem 7.2 in [MIL], V has the homotopy type of a CW-complex of dimension not exceeding $\dim_C V$. Hence $H_n(V, R) = 0$. Since τ is a retraction to V , the map $\tau_*: H_n(M, R) \rightarrow H_n(V, R)$ is 0. Thus $\psi_* = 0$.

Now for sufficiently large j , $(g^j)_* = \phi_* = 0$, which is contradictory to the assertion that g_* is an isomorphism. This concludes the proof. \square

It seems appropriate to mention that there exist topologically contractible strongly pseudoconvex domains with real analytic boundary in C^2 which are rigid in the sense of Carathéodory at some point ([ZL1, ZL2]). For such a domain D and some point $p \in D$, D_p is an isolated point in \mathcal{T}_n .

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