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Semiinvariant vectors associated to decompositions of monomial representations of exponential Lie groups

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Introduction.

Let G be a connected Lie group of type I, H be a closed subgroup, and χ be a unitary character of H. Writing Δ_G and Δ_H for the modular functions of G and H, respectively, let $\Delta_{H,G}^{1/2}(h)=(\Delta_H(h)/\Delta_G(h))^{1/2}$ for $h\in H$. For a unitary representation $(\pi, \mathcal{H}_\pi)$ of G, letting $\mathcal{H}_\pi^\infty$ be the space of C^∞ vectors and $\mathcal{H}_\pi^{-\infty}$ be its antidual, we extend π to $\mathcal{H}_\pi^{-\infty}$ and denote the space of $(H, \chi\Delta_{H,G}^{1/2})$-semiinvariant vectors by

$$(\mathcal{H}_\pi^{-\infty})^{H,\chi\Delta_{H,G}^{1/2}} = \{a\in\mathcal{H}_\pi^{-\infty};\ \pi(h)a=\chi(h)\Delta_{H,G}^{1/2}(h)a,\ {}^\forall h\in H\}.$$

We consider a representation $\sigma=\mathrm{ind}_H^G\chi$ of G induced from χ and its direct integral decomposition: $\sigma=\int_{\hat{G}}^{\oplus} m(\pi)\pi\, d\mu(\pi)$, where $\hat{G}$ is the unitary dual of G with usual Borel structure, $d\mu$ is a Borel measure and m is a multiplicity function defined μ-almost everywhere. Realize σ in a space $\mathcal{H}_\sigma=L^2(\chi, G)$ of functions v on G such that $v(gh)=\chi(h^{-1})\Delta_{H,G}^{1/2}(h)v(g)$ for all $g\in G$ and $h\in H$, and $\mu_{G,H}(|v|^2)<\infty$, where $\mu_{G,H}$ is the positive G-invariant form on functions ϕ satisfying $\phi(gh)=\Delta_{H,G}(h)\phi(g)$ for all $g\in G$, $h\in H$ (see [2, Ch. V]). In $\mathcal{H}_\sigma$, $x\in G$ acts by $\sigma(x)v(g)=v(x^{-1}g)$, $g\in G$. By Penney's Plancherel theorem [11], the canonical cyclic vector a_σ of σ defined by $\langle a_\sigma, v\rangle=\overline{v(e)}$, where $e\in G$ is the unit element, decomposes into the direct integral of $(H, \chi\Delta_{H,G}^{1/2})$-semiinvariant vectors. For the Plancherel theorem of this type, see [4], [5], [6], [8], [10], [11]. Here we will treat the following:

PROBLEM. Is $\dim(\mathcal{H}_\pi^{-\infty})^{H,\chi\Delta_{H,G}^{1/2}}=m(\pi)$?

Note that $\dim(\mathcal{H}_\pi^{-\infty})^{H\chi\Delta_{H,G}^{1/2}}\geqq m(\pi)$ holds by Theorem (II.6) of Penney [11].

We are concerned with exponential groups G, that is, solvable Lie groups G for which exponential mappings are diffeomorphisms of their Lie algebras $\mathfrak{g}$ to G. For such G, several cases are treated in [1], [4], [5] and [6]. We will find upper bounds of dimensions of semiinvariant vectors for those irreducible representations which satisfy the condition (C) below and which occur in σ with at most finite multiplicities, and we give an affirmative answer to the above problem.

1. Statement of the result.

Let G be an exponential Lie group and H be a connected subgroup whose Lie algebras are $\mathfrak{g}$ and $\mathfrak{h}$, respectively. For a unitary character χ of H, we find $f\in\mathfrak{g}^*$ (the dual vector space of $\mathfrak{g}$) satisfying $f([\mathfrak{h}, \mathfrak{h}])=\{0\}$ and $\chi(\exp X)=\chi_f(\exp X)=e^{\sqrt{-1}f(X)}$ for $X\in\mathfrak{h}$.

Regarding $\sigma=\mathrm{ind}_H^G\chi_f$, let us recall the description of the direct integral decomposition $\int_{\hat{G}}^{\oplus}m(\pi)\pi d\mu(\pi)$ in terms of the orbit method [3], [6], [9]: Writing $\mathfrak{h}^\perp=\{l\in\mathfrak{g}^*; l|_\mathfrak{h}=0\}$, we obtain the measure μ as the image of the Lebesgue measure of the affine space $\mathfrak{h}^\perp+f$ by the Kirillov-Bernat mapping $\theta_0: \mathfrak{g}^*\to\hat{G}$. The multiplicity $m(\pi)$ is determined as follows: $m(\pi)$ is the number of connected components of $\theta_0^{-1}(\pi)\cap(\mathfrak{h}^\perp+f)$ if each component is a single H-orbit, and $m(\pi)=\infty$ if this condition is not satisfied. That is, $m(\pi)$ is the number of H-orbits included in $\theta_0^{-1}(\pi)\cap(\mathfrak{h}^\perp+f)$ for μ-almost all π [6].

Let Ω be a coadjoint orbit, and for $l\in\Omega$ let $\mathfrak{g}(l)=\{X\in\mathfrak{g}; l([X, \mathfrak{g}])=\{0\}\}$. For a connected component C of $\Omega\cap(\mathfrak{h}^\perp+f)$, the following (i) and (ii) are equivariant:

(i) $\mathfrak{h}+\mathfrak{g}(l)$ is a Lagrangian subspace for the bilinear form $(X, Y)\to l([X, Y])$ for each $l\in C$.

(ii) C is a single H-orbit.

(See [6].) Let us remark that (i) and (ii) are necessary conditions for the above $m(\pi)$ to be finite, but that they are not sufficient.

We investigate irreducible representations π satisfying the following condition for the corresponding coadjoint orbit Ω.

CONDITION (C). There exists an ideal $\mathfrak{p}$ such that $\Omega+\mathfrak{p}^\perp=\Omega$ and $l([\mathfrak{p}, \mathfrak{p}])=\{0\}$ for $l\in\Omega$.

REMARK 1. Let $\mathfrak{p}_\Omega$ be the intersection of all subspaces $L\subset\mathfrak{g}$ satisfying $\Omega+L^\perp=\Omega$. Then $\Omega+\mathfrak{p}_\Omega^\perp=\Omega$ and $\mathfrak{p}_\Omega$ is an ideal of $\mathfrak{g}$. The condition (C) means that $\mathfrak{p}_\Omega$ satisfies $l([\mathfrak{p}_\Omega, \mathfrak{p}_\Omega])=\{0\}$ for $l\in\Omega$. It can be proved by the standard induction that $\mathfrak{p}_\Omega\supset\mathfrak{g}(l)$ for all $l\in\Omega$ and if $\mathfrak{g}$ is nilpotent, $\mathfrak{p}_\Omega=i_\Omega$: the ideal generated by $\mathfrak{g}(l)$, $l\in\Omega$. But if $\mathfrak{g}$ is general exponential, $\mathfrak{p}_\Omega\supsetneqq i_\Omega$ may happen.

EXAMPLE 1. Suppose $\mathfrak{g}$ is nilpotent. Then a coadjoint orbit Ω satisfies the condition (C) if and only if $l([i_\Omega, i_\Omega])=\{0\}$.

EXAMPLE 2. Let $\mathfrak{g}$ be a normal j-algebra treated in [7] and Ω be an open coadjoint orbit. Then Ω satisfies the condition (C) with $\mathfrak{p}=\mathfrak{g}_1$ with the notation in [7]. (Thus the result of [7] is obtained from our theorem below.)

Now, our result is the following:

THEOREM. *Let $G=\exp\mathfrak{g}$ be an exponential Lie group and $H=\exp\mathfrak{h}$ be a connected subgroup, and let $f\in\mathfrak{g}^*$ satisfying $f([\mathfrak{h},\mathfrak{h}])=\{0\}$. Define a unitary character χ_f of H by $\chi_f(\exp X)=e^{\sqrt{-1}f(X)}$ for $X\in\mathfrak{h}$.*

Suppose that $\pi\in\hat{G}$ corresponds to a coadjoint orbit Ω satisfying the condition (C). Then

1. *If $\Omega\cap(\mathfrak{h}^{\perp}+f)=\emptyset$, then*

$$(\mathcal{H}_{\pi}^{-\infty})^{H,\chi_f\Delta_{H,G}^{1/2}}=\{0\}.$$

2. *Suppose that $\Omega\cap(\mathfrak{h}^{\perp}+f)\neq\emptyset$ and each of its connected components is a single H-orbit and the number $m(\Omega)$ of H-orbits included in $\Omega\cap(\mathfrak{h}^{\perp}+f)$ is finite. Then*

$$\dim(\mathcal{H}_{\pi}^{-\infty})^{H,\chi_f\Delta_{H,G}^{1/2}}\leqq m(\Omega).$$

For the decomposition of $\sigma=\mathrm{ind}_H^G\chi_f=\int_{\hat{G}}^{\oplus}m(\pi)\pi d\mu(\pi)$, the above statements give a certain reciprocity. For example, suppose μ-almost all π satisfy the condition (C) with the corresponding orbit. Then the statement 2 and the known inequality: $\dim(\mathcal{H}_{\pi}^{-\infty})^{H,\chi\Delta_{H,G}^{1/2}}\geqq m(\pi)$ imply that the dimension is equal to $m(\pi)=m(\Omega)$.

In section 2, we will prove the theorem by realizing π in a space of suitable functions on G. We will use fundamental arguments in [**4**], [**5**], [**6**] to treat distribution vectors.

2. Proof of the theorem.

For an element $l\in\Omega$, we can take a polarization $\mathfrak{b}$ at l satisfying the Pukanszky condition (i.e. $\mathfrak{b}^{\perp}+l=B\cdot l$, where $B=\exp\mathfrak{b}$) and $\mathfrak{p}\subset\mathfrak{b}$ since $\mathfrak{p}$ is an ideal satisfying $l([\mathfrak{p},\mathfrak{p}])=\{0\}$ [**2**, Ch. IV, 4.3]. In the sequel, we realize the irreducible representation π corresponding to Ω as $\mathrm{ind}_B^G\chi_l$, where $\chi_l(\exp X)=e^{\sqrt{-1}l(X)}$ for $X\in\mathfrak{b}$, in a space $\mathcal{H}_{\pi}$ of functions ϕ on G such that $\phi(gb)=\chi_l(b^{-1})\Delta_{B,G}^{1/2}(b)\phi(g)$ for all $b\in B$ and $g\in G$ [**2**, Ch. V].

PROOF OF 1. Let us remark that the assumption $\Omega\cap(\mathfrak{h}^{\perp}+f)=\emptyset$ implies $\Omega\cap((\mathfrak{h}\cap\mathfrak{p})^{\perp}+f)=\emptyset$. In fact, if there exists $m\in\Omega\cap((\mathfrak{h}\cap\mathfrak{p})^{\perp}+f)=\Omega\cap(\mathfrak{h}^{\perp}+\mathfrak{p}^{\perp}+f)$, then $m+m_0\in\mathfrak{h}^{\perp}+f$ for some $m_0\in\mathfrak{p}^{\perp}$. By the condition (C), $m+m_0\in\Omega$ holds, too.

For $h\in H\cap\exp\mathfrak{p}$, $v\in\mathcal{H}_{\pi}^{\infty}$ and $g\in G$, a semiinvariant vector a satisfies

$$\langle a,((\chi_f\Delta_{H,G}^{-1/2})(h)-\chi_l(g^{-1}hg))v(g)\rangle=0$$

since $\pi(h)v(g)=\chi_l(g^{-1}hg)v(g)$. If $\Delta_{H,G}(\exp X)\neq1$ for an $X\in\mathfrak{h}\cap\mathfrak{p}$, we get $a=0$ considering the semiinvariance for $\exp\boldsymbol{R}X$. Suppose $\Delta_{H,G}(\exp X)=1$ for all

$X \in \mathfrak{h} \cap \mathfrak{p}$, then the support of a is

$$\mathrm{supp}(a) \subset \{g \in G\,;\, g \cdot l(X) = f(X) \text{ for all } X \in \mathfrak{h} \cap \mathfrak{p}\} = \emptyset$$

since $G \cdot l \cap ((\mathfrak{h} \cap \mathfrak{p})^{\perp} + f) = \emptyset$, this proves the claim 1.

PROOF OF 2. Taking $l \in \Omega$ and realizing $\pi = \mathrm{ind}_B^G \chi_l$ in a suitable function space $\mathcal{H}_\pi$ on G/B as described before, we note that $\mathcal{H}_\pi^\infty$ includes the space $C_c^\infty(G/B)$ of smooth functions of compact support on G/B. Thus it is sufficient to prove that the dimension of the space of $(H, \chi_f \Delta_{H,G}^{1/2})$-semiinvariant distributions is at most the number of H-orbits in $\Omega \cap (\mathfrak{h}^{\perp} + f)$.

We will prove the claim by induction on $\dim G$. For $G = \boldsymbol{R}$, it is clearly verified. Let G, H, f, Ω be as in the statement of the theorem, $\dim G > 1$, and suppose that the claim is verified for exponential groups of lower dimensions. Let $l \in \Omega \cap (\mathfrak{h}^{\perp} + f)$, and note that $\mathfrak{h}^{\perp} + f = \mathfrak{h}^{\perp} + l$.

CASE 1. *$l = 0$ on an abelian ideal $\mathfrak{a} \neq 0$.* Then $A = \exp \mathfrak{a} \subset \ker \pi$ and the conclusion is deduced from the induction hypothesis for $(G/A, HA/A, \dot{\pi}, \dot{l})$, where $\dot{\pi} \in \widehat{G/A}$ and $\dot{l} \in (\mathfrak{g}/\mathfrak{a})^*$ are obtained from π and l, respectively, by the quotient map $G \to G/A$.

Let us suppose that there are no such ideals as in case 1. Then the dimension of the center $\mathfrak{z}$ of $\mathfrak{g}$ is at most one.

CASE 2. *$\dim \mathfrak{z} = 1$, and $\ker l$ includes no non-zero abelian ideals for $l \in \Omega$.* If $\mathfrak{p} \neq \mathfrak{z}$, taking a minimal subspace of $\mathfrak{p}/\mathfrak{z}$ which is invariant under the action of $\mathfrak{g}/\mathfrak{z}$, we get an ideal $\mathfrak{g}_2$, $\mathfrak{p} \supset \mathfrak{g}_2 \supset \mathfrak{z}$, with $\dim \mathfrak{g}_2/\mathfrak{z} = 1$ or 2. If $\mathfrak{p} = \mathfrak{z}$, let $\mathfrak{g}_2$ be an ideal of $\mathfrak{g}$, $\mathfrak{g}_2 \supset \mathfrak{z}$, obtained from a minimal ideal of $\mathfrak{g}/\mathfrak{z}$. Then $\mathfrak{g}_2$ is an abelian ideal satisfying the condition (C), so that we can skip this case by taking $\mathfrak{g}_2$ anew as $\mathfrak{p}$. Writing $\mathfrak{g}_2^l = \{X \in \mathfrak{g}\,;\, l([X, \mathfrak{g}_2]) = \{0\}\}$, we will separately treat case 2.1: $\mathfrak{h} \subset \mathfrak{g}_2^l$ for all $l \in \Omega \cap (\mathfrak{h}^{\perp} + f)$ and case 2.2: otherwise.

RRMARK 2. (1) By the assumption of case 2, $[\mathfrak{g}, \mathfrak{g}_2] \supset \mathfrak{z}$ and $\mathfrak{g}_2 \cap \mathfrak{g}(l) \neq \mathfrak{g}_2$. For $X \in \mathfrak{g}_2$, define $l_X \in \mathfrak{g}^*$ by $l_X(Y) = l([X, Y])$, $Y \in \mathfrak{g}$. Then the mapping $X \mapsto l_X$ induces an isomorphism of $\mathfrak{g}_2/(\mathfrak{g}_2 \cap \mathfrak{g}_2(l))$ to $(\mathfrak{g}/\mathfrak{g}_2^l)^*$.

(2) Let $\mathfrak{k} = \mathfrak{g}_2^l$. Since G is an exponential group, the stabilizer of $l|_{\mathfrak{g}_2}$ is $K = \exp \mathfrak{k}$, and $G \cdot l \cap (\mathfrak{g}_2^{\perp} + l) = K \cdot l$. By the assumption $G \cdot l + \mathfrak{p}^{\perp} = G \cdot l$ and $\mathfrak{p} \supset \mathfrak{g}_2$, we get $\mathfrak{p} \subset \mathfrak{k}$, $K \cdot l + \mathfrak{p}^{\perp} = K \cdot l$ and the K-orbit $K \cdot l_0$, where $l_0 = l|_{\mathfrak{k}}$, in $\mathfrak{k}^*$ satisfies the condition (C) with $\mathfrak{p}$.

(3) Since $\mathfrak{k}(l_0) = \mathfrak{g}(l) + \mathfrak{g}_2$, $\tau = \mathrm{ind}_B^K \chi_l$ corresponds to the orbit $K \cdot l_0$. We can also regard $\pi = \mathrm{ind}_B^G \chi_l$ as induced from τ.

CASE 2.1. Suppose that $\mathfrak{h} \subset \mathfrak{g}_2^l$ for all $l \in \Omega \cap (\mathfrak{h}^{\perp} + f)$. For $l \in \Omega \cap (\mathfrak{h}^{\perp} + f)$, this means $g^{-1} \cdot \mathfrak{h} \subset \mathfrak{g}_2^l$ for all $g \in G$ such that $g \cdot l \in \mathfrak{h}^{\perp} + l$. We also have its

H-orbit $Hg\cdot l\subset\mathfrak{g}_2^{\perp}+g\cdot l$. Fix $l\in\Omega\cap(\mathfrak{h}^{\perp}+f)$, and let $\mathfrak{k}=\mathfrak{g}_2^{l}$ and $K=\exp\mathfrak{k}$, and realize π using a polarization $\mathfrak{b}$ at l, $\mathfrak{p}\subset\mathfrak{b}$.

As in the proof of 1, it is sufficient to consider cases of $\Delta_{H,G}(\exp X)=1$ for all $X\in\mathfrak{h}\cap\mathfrak{p}$. A semiinvariant vector a satisfies

$$\langle a,\ (\chi_l(\exp X)-\chi_l(g^{-1}\exp Xg))v(g)\rangle = 0$$

for $X\in\mathfrak{h}\cap\mathfrak{p}$, $v\in C_c^{\infty}(G/B)$ and $g\in G$, and

$$\mathrm{supp}(a)\subset\{g\in G\,;\,g\cdot l(X)-l(X)=0,\ \text{for all } X\in\mathfrak{h}\cap\mathfrak{p}\}.$$

By the condition (C), we have $G\cdot l\cap((\mathfrak{h}\cap\mathfrak{p})^{\perp}+l)=G\cdot l\cap(\mathfrak{h}^{\perp}+l)+\mathfrak{p}^{\perp}$. Thus, for $g\cdot l\in((\mathfrak{h}\cap\mathfrak{p})^{\perp}+l)\cap G\cdot l$, its connected component $C(g\cdot l)$ including $g\cdot l$ satisfies $C(g\cdot l)\subset\mathfrak{g}_2^{\perp}+g\cdot l$, and $\{x\in G\,;\,x\cdot l\in C(g\cdot l)\}\subset(\exp\mathfrak{g}_2^{g\cdot l})g=gK$. It follows that the number of cosets gK satisfying $gK\cdot l\cap((\mathfrak{h}\cap\mathfrak{p})^{\perp}+l)\neq\emptyset$ is bounded by the number of connected components of $G\cdot l\cap(\mathfrak{h}^{\perp}+l)$.

By remark 2(1), $\dim\mathfrak{g}/\mathfrak{k}=1$ or 2. We first treat the case of $\dim\mathfrak{g}/\mathfrak{k}=1$. Taking a suitable vector $S\in\mathfrak{g}$, we get $G=(\exp\boldsymbol{R}S)K$ and identify G with $\boldsymbol{R}\times K$. We write $\boldsymbol{U}=\{g\in G\,;\,g\cdot l\in((\mathfrak{h}\cap\mathfrak{p})^{\perp}+l)\}$, $\boldsymbol{S}=\{s\in\boldsymbol{R}\,;\,\exp(sS)K\cdot l\cap((\mathfrak{h}\cap\mathfrak{p})^{\perp}+l)\neq\emptyset\}$.

Then a is described as a linear combination of distributions $\{a_s\,;\,\mathrm{supp}(a_s)\subset\boldsymbol{U}\cap\exp(sS)K,\ s\in\boldsymbol{S}\}$. For $\phi\otimes w\in C_c^{\infty}(\boldsymbol{R})\otimes C_c^{\infty}(K/B)$, each a_s ($s\in\boldsymbol{S}$) is of the form

$$\langle a_s,\ \phi\otimes w\rangle=\sum_{i\geq 0}\overline{\frac{d^i\phi}{dx^i}(s)}\langle a_K^i,\ w\rangle,$$

where a_K^i is a distribution on K/B. Now, let j be the maximum index such that $a_K^j\neq 0$. Suppose $j\geq 1$, and choose test functions $\phi\in C_c^{\infty}(\boldsymbol{R})$ such that $\phi^{(i)}(s)=0$ for $1\leq i\leq j-2$. Then the semiinvariance

$$\langle a,\ (\chi_{\exp(xS)k\cdot l}(\exp Y)-\chi_l(\exp Y))\phi(x)w(k)\rangle=0$$

for $w\in C_c^{\infty}(K/B)$ and $Y\in\mathfrak{h}\cap\mathfrak{p}$ implies that

$$\begin{aligned} &j\langle a_K^j,\ \sqrt{-1}\exp(sS)k\cdot l([Y,\ S])\chi_{\exp(sS)k\cdot l}(\exp Y)w(k)\rangle\\ &=-\langle a_K^{j-1},\ (\chi_{\exp(xS)k\cdot l}(\exp Y)-\chi_l(\exp Y))w(k)\rangle. \end{aligned}$$

Let $k_0\in K$ such that $g_s=\exp(sS)k_0\in\boldsymbol{U}$. Then $g_s\cdot l+m\in G\cdot l\cap(\mathfrak{h}^{\perp}+l)$ with some $m\in\mathfrak{p}^{\perp}$, and $\mathfrak{h}+\mathfrak{g}(g_s\cdot l+m)$ is a polarization at $g_s\cdot l+m$. We note that $\mathfrak{g}(g_s\cdot l)=\mathfrak{g}(g_s\cdot l+m)$ since $\mathfrak{g}(l)\subset\mathfrak{p}$ and $\mathfrak{p}^{\perp}+G\cdot l=G\cdot l$. By remark 2(1), there exists $X\in\mathfrak{g}_2$ satisfying $g_s\cdot l([X,\ S])\neq 0$ and noting that $\mathfrak{h}\subset\mathfrak{g}_2^{g_s\cdot l}$, we find $Y=X+g_s\cdot V\in\mathfrak{h}$ with $V\in\mathfrak{g}(l)$. Considering test functions w supported in a neighborhood $\mathcal{V}_{k_0}$ of k_0 such that $\exp(sS)k\cdot l([Y,\ S])e^{\exp(sS)k\cdot l(Y)}\neq 0$ for $k\in\mathcal{V}_{k_0}$ and the above semiinvariance obtained by $\boldsymbol{R}Y$, we get $\langle a_K^j,\ w\rangle=0$. It follows that a_s is a linear combination of

$$\overline{\delta(s)}\otimes a_K,$$

where a_K satisfies $\langle a_K, (\tau\Delta_{K,G}^{-1/2})(\exp(-sS)h\exp(sS))w\rangle=\langle a_K, (\chi_l\Delta_{H,G}^{-1/2})(h)w\rangle$ for all $h\in H$. Noting that $\Delta_{H,G}(h)=\Delta_{\exp(-sS)H\exp(sS),G}(\exp(-sS)h\exp(sS))$, we get

$$\langle a_K, \tau(h_s)w\rangle = \langle a_K, \chi_{\exp(-sS)\cdot l}(h_s)\Delta_{H_s,K}^{-1/2}(h_s)w\rangle$$

for all $h_s\in H_s=\exp(-sS)H\exp sS$.

Each H-orbit of $G\cdot l\cap(\mathfrak{h}^\perp+l)$ is included in a subset $\exp(sS)K\cdot l\cap(\mathfrak{h}^\perp+l)$, $s\in \boldsymbol{S}$, and $m(\Omega)$ equals $\sum_{s\in S}\#(H_s\text{-orbits in } K\cdot l\cap(\exp(-sS)\cdot\mathfrak{h}^\perp+\exp(-sS)\cdot l))=\sum_{s\in S}\#(H_s\text{-orbits in } K\cdot l_0\cap(\exp(-sS)\cdot\mathfrak{h}^\perp+l_s))$ in $\mathfrak{k}^*$, where $l_s=\exp(-sS)\cdot l|\mathfrak{k}$. By the induction hypothesis for (K, H_s, τ, l_s), $s\in \boldsymbol{S}$, the dimension of semiinvariant vectors is bounded by $m(\Omega)$.

We can similarly treat the case of $\dim\mathfrak{g}/\mathfrak{k}=2$. Taking vectors $S_1, S_2\in\mathfrak{g}$ such that $G=(\exp \boldsymbol{R}S_1)(\exp \boldsymbol{R}S_2)K$, we identify G with $\boldsymbol{R}^2\times K$. Let $\boldsymbol{U}$ be as in the previous case and $\boldsymbol{S}=\{s=(s_1, s_2)\in \boldsymbol{R}^2; \exp(s_1S_1)\exp(s_2S_2)K\cdot l\cap((\mathfrak{h}\cap\mathfrak{p})^\perp+l)\neq\emptyset\}$. Then a semiinvariant distribution a is a linear combination of $\{a_s; s=(s_1, s_2)\in \boldsymbol{S}\}$, where a_s is a distribution with $\operatorname{supp}(a_s)\subset \boldsymbol{U}\cap\exp(s_1S_1)\exp(s_2S_2)K$, and is of the form

$$\langle a_s, \phi\otimes w\rangle = \sum_{(i_1, i_2)\in I}\left\langle a_K^{(i_1, i_2)}, \overline{\frac{\partial^{i_1+i_2}\phi}{\partial x_1^{i_1}\partial x_2^{i_2}}(s_1, s_2)w}\right\rangle,$$

where $a_K^{(i_1, i_2)}$ is a distribution on K/B with an index set I. Let I be ordered lexicographically, that is, $(i_1, i_2)<(j_1, j_2)$ if $i_1<j_1$ or both $i_1=j_1$ and $i_2<j_2$ are satisfied, and let (j_1, j_2) be the maximum element of I, and suppose $j_2>0$. Taking test functions $\phi=\phi_1\otimes\phi_2\in C_c^\infty(\boldsymbol{R})\otimes C_c^\infty(\boldsymbol{R})$ satisfying $\phi_1^{(i)}(s_1)=0$ for $0<i<j_1$ and $\phi_2^{(i)}(s_2)=0$ for $0<i<j_2-1$, we get $a_K^{(j_1, j_2)}=0$ as in the previous case. Thus a_s is of the form

$$a_s = \overline{\delta(s_1, s_2)}\otimes a_K,$$

and we can also verify the claim of the dimension of the space of semiinvariant distributions.

CASE 2.2. Suppose that there exists an element $l\in\Omega\cap(\mathfrak{h}^\perp+f)$ such that $\mathfrak{h}\not\subset\mathfrak{g}_2^l$. Take such l to realize π, and let $\mathfrak{k}=\mathfrak{g}_2^l$.

For the case of $\dim\mathfrak{g}_2=2$, we take a basis $\{X_1, X_2\}$ of $\mathfrak{g}_2$ satisfying $X_1\in\mathfrak{z}$, $l(X_1)=1$ and $l(X_2)=0$, and describe the action of $\mathfrak{g}$ as follows. For $X\in\mathfrak{g}$,

$$[X, X_2] = \lambda(X)X_2+\gamma(X)X_1,$$

where $\lambda, \gamma\in\mathfrak{g}^*$, $\gamma\neq0$ (by the assumption of case 2). For the case of $\dim\mathfrak{g}_2=3$, noting that $\mathfrak{g}$ is exponential [2, Ch. I], we take a basis $\{X_1, X_2, Y_2\}$ of $\mathfrak{g}_2$, where X_1 is as above and $l(X_2)=l(Y_2)=0$, and for $X\in\mathfrak{g}$,

$$[X, X_2] = \lambda(X)(X_2-\alpha Y_2)+\gamma_1(X)X_1,$$

$$[X, Y_2] = \lambda(X)(\alpha X_2+Y_2)+\gamma_2(X)X_1,$$

where $\alpha\in\boldsymbol{R}-\{0\}$, λ, γ_1, $\gamma_2\in\mathfrak{g}^*$, $\lambda\neq 0$, $rank(\gamma_1, \gamma_2)\neq 0$ (by the assumption of case 2).

Denote the centralizer of $\mathfrak{g}_2$ by $\mathfrak{k}_0$, and let $K_0=\exp\mathfrak{k}_0$. Then $\dim\mathfrak{g}_2=2$ or 3, and $1\leqq\dim\mathfrak{g}/\mathfrak{k}_0\leqq\dim\mathfrak{g}_2$.

(1) $\dim\mathfrak{g}/\mathfrak{k}_0=1$ and $\dim\mathfrak{g}_2=2$ or 3. In this case, $\mathfrak{k}_0=\mathfrak{k}=\mathfrak{g}_2^m$ for all $m\in G\cdot l$. Taking $T\in\mathfrak{h}\backslash(\mathfrak{h}\cap\mathfrak{k})$, we identify $G=(\exp\boldsymbol{R}T)K$ with $\boldsymbol{R}\times K$.

For $v=\phi\otimes w\in C_c^\infty(\boldsymbol{R})\otimes C_c^\infty(K/B)$, the action of H is described as follows: for all $(x, \dot{k})\in\boldsymbol{R}\times K/B$, $t\in\boldsymbol{R}$ and $y\in H\cap K$,

$$\pi(\exp tT)\phi(x)w(\dot{k}) = \phi(x-t)w(\dot{k}),$$

$$\pi(h)\phi(x)w(\dot{k}) = \phi(x)\tau(\exp(-xT)h\exp(xT))w(\dot{k}).$$

Thus the semiinvariant vector a satisfies

$$\begin{aligned}&\langle a, (\pi(\exp tT)-(\chi_l\Delta_{H,G}^{-1/2})(\exp tT))\phi(x)w(\dot{k})\rangle\\ &= \langle a, (\phi(x-t)-(\chi_l\Delta_{H,G}^{-1/2})(\exp tT)\phi(x))w(\dot{k})\rangle = 0\end{aligned}$$

for all $t\in\boldsymbol{R}$. Using the uniqueness of the Haar measure for $\boldsymbol{R}$, we get

$$\langle a, \phi(x)w(\dot{k})\rangle = \int_{\boldsymbol{R}}(\bar{\chi}_l\Delta_{H,G}^{-1/2})(\exp xT)\overline{\phi(x)}\langle a_K, w\rangle dx,$$

where a_K is a distribution on K/B satisfying

$$\int_{\boldsymbol{R}}(\bar{\chi}_l\Delta_{H,G}^{-1/2})(\exp xT)\overline{\phi(x)}\langle a_K, (\tau(\exp(-xT)h\exp(xT))-(\chi_l\Delta_{H,G}^{-1/2})(h))w\rangle dx = 0$$

for all $\phi\in C_c^\infty(\boldsymbol{R})$ and $h\in H\cap K$. By the continuity of the representation (especially for the variable x), the above condition deduces that

$$\langle a_K, (\tau(h)-(\chi_l\Delta_{H,G}^{-1/2})(h))w\rangle = 0$$

for all $h\in\exp(\mathfrak{h}\cap\mathfrak{k})$. Noting that $\exp\boldsymbol{R}X_2\subset\ker\tau$ and $\Delta_{H,G}(h)=\Delta_{H\cap K,K}(h)$ for $h\in H\cap K$, we get

$$\langle a_K, (\tau(h)-(\chi_l\Delta_{H\cap K,K}^{-1/2})(h))w\rangle = 0$$

for all $h\in H\cap K$.

For an H-orbit C in $G\cdot l\cap(\mathfrak{h}^\perp+l)$, let $C_0=C\cap(\mathfrak{g}_2^\perp+l)$. Then, considering the action of $\exp\boldsymbol{R}T\subset H$, we get that C_0 is an $H\cap K$-orbit, and the mapping $C\mapsto C_0$ from the set of H-orbits in $G\cdot l\cap(\mathfrak{h}^\perp+l)$ to the set of $H\cap K$-orbits in $G\cdot l\cap(\mathfrak{h}^\perp+l)\cap(\mathfrak{g}_2^\perp+l)=K\cdot l\cap(\mathfrak{h}^\perp+l)$ is bijective. Since $K\cdot l+\mathfrak{k}^\perp=K\cdot l$, the number of H-orbits of $G\cdot l\cap(\mathfrak{h}^\perp+l)$ coincides with that of $H\cap K$-orbits in $K\cdot l_0\cap((\mathfrak{h}\cap\mathfrak{k})^\perp+l_0)$.

Therefore, using the induction hypothesis for $(K, H\cap K, \tau, l_0)$, we verify the claim.

(2) $\dim \mathfrak{g}/\mathfrak{k}_0=2$ and $\dim \mathfrak{g}_2=2$. In this case, choose $S, T\in\mathfrak{g}$ such that $[S, X_2]=\beta_S X_1$, $[T, X_2]=\beta_T X_2$, $(\beta_S, \beta_T\neq 0)$. Then $\mathfrak{g}=\boldsymbol{R}S+\mathfrak{k}$, $\mathfrak{k}=\boldsymbol{R}T+\mathfrak{k}_0$, and by the Jacobi identity, $[[T, S], X_2]=-\beta_S\beta_T X_1$, that is, $[T, S]\in -\beta_T S+\mathfrak{k}_0$. Thus we get $G=(\exp \boldsymbol{R}S)K$, and we identify G with $\boldsymbol{R}\times K$.

(i) If $\mathfrak{h}+\mathfrak{k}_0=\mathfrak{g}$, or $\mathfrak{h}\subset\ker\lambda$, we can take the above S so that $S\in\mathfrak{h}$. As subcase (1), a semiinvariant vector a is described as follows. For $v=\phi\otimes w\in C_c^\infty(\boldsymbol{R})\otimes C_c^\infty(K/B)$,

$$\langle a, v\rangle = \int_{\boldsymbol{R}} (\bar{\chi}_l\Delta_{H,G}^{-1/2})(\exp xS)\overline{\phi(x)}\langle a_K, w\rangle dx\,,$$

where a_K is a distribution on K/B. For $h\in H\cap K$ and $x\in\boldsymbol{R}$, let $k_h(x)=\exp(-\Delta_{K,G}^{-1}(h)xS)h^{-1}\exp(xS)$. Then $k_h(x)\in K$ and

$$\begin{aligned}
&\langle a, \pi(h)v\rangle \\
&= \int_{\boldsymbol{R}} (\bar{\chi}_l\Delta_{H,G}^{-1/2})(\exp xS)\overline{\phi(\Delta_{K,G}^{-1}(h)x)}\langle a_K, (\tau^{-1}\Delta_{K,G}^{1/2})(k_h(x))w\rangle dx \\
&= \Delta_{K,G}(h)\int_{\boldsymbol{R}} (\bar{\chi}_l\Delta_{H,G}^{-1/2})(\exp(\Delta_{K,G}(h)xS))\overline{\phi(x)}\langle a_K, (\tau^{-1}\Delta_{K,G}^{1/2})(k_h(\Delta_{K,G}(h)x))w\rangle dx \\
&= \langle a, (\chi_l\Delta_{H,G}^{-1,2})(h)v\rangle = \int_{\boldsymbol{R}} (\bar{\chi}_l\Delta_{H,G}^{-1/2})(\exp xS)\overline{\phi(x)}\langle a_K, (\chi_l\Delta_{H,G}^{-1/2})(h)w\rangle dx\,.
\end{aligned}$$

By the semiinvariance for $h\in H\cap K$,

$$\langle a_K, \tau(h)w\rangle = \langle a_K, \chi_l(h)\Delta_{H,G}^{-1/2}(h)\Delta_{K,G}^{-1,2}(h)w\rangle$$

for all $h\in H\cap K$. We note that this holds for $h\in\exp\boldsymbol{R}X_2$, and writing $\mathfrak{h}_1=(\mathfrak{h}\cap\mathfrak{k})+\boldsymbol{R}X_2$ and $H_1=\exp\mathfrak{h}_1$, we get $\Delta_{H,G}(h)\Delta_{K,G}(h)=\Delta_{H\cap K,K}(h)\Delta_{K,G}(h)=\Delta_{H_1,K}(h)$ for $h\in H_1$ by simple calculations. Thus the equality

$$\langle a_K, \tau(h)w\rangle = \langle a_K, \chi_l(h)\Delta_{H_1,K}^{-1/2}(h)w\rangle$$

holds for $h\in H_1$.

For an H-orbit C in $G\cdot l\cap(\mathfrak{h}^\perp+l)$, $C_1=C\cap\boldsymbol{R}X_2^\perp$ is a $H\cap K$-orbit since $\exp sS\cdot X_2=X_2-sX_1$, and the mapping $C\mapsto C_1$ from the set of H-orbits in $G\cdot l\cap(\mathfrak{h}^\perp+l)$ to that of $H\cap K$-orbits in $G\cdot l\cap\boldsymbol{R}X_2^\perp\cap(\mathfrak{h}^\perp+l)=K\cdot l\cap(\mathfrak{h}^\perp+l)$ is bijective. Thus the number of H-orbits in $G\cdot l\cap(\mathfrak{h}^\perp+l)$ coincides with that of $H\cap K$-orbits in $K\cdot l_0\cap(\mathfrak{h}_1^\perp+l_0)$. We can use the induction hypothesis for (K, H_1, τ, l_0), and verify the claim.

(ii) If $\mathfrak{h}+\mathfrak{k}_0\neq\mathfrak{g}$ and $\mathfrak{h}\not\subset\ker\lambda$, we get $U\in\mathfrak{h}$ such that $\mathfrak{h}=\boldsymbol{R}U+(\mathfrak{h}\cap\mathfrak{k}_0)$, and $[U, X_2]=X_2+\beta X_1$, $\beta\in\boldsymbol{R}\setminus\{0\}$. Take T, S such that $U=T+S$, $[T, X_2]=X_2$, $[S, X_2]=\beta X_1$. Then $[T, S]\in -S+\mathfrak{k}_0$ and $G=(\exp\boldsymbol{R}S)K$. Since $\exp uU=\exp((1-e^{-u})S)\exp(uT)k_0$, where $k_0\in K_0$,

$$HK=(\exp \boldsymbol{R}U)K=\{\exp(sS)k\,;\,s<1,\ k\in K\},$$

$$H(\exp 2S)K=(\exp \boldsymbol{R}U)(\exp 2S)K=\{\exp(sS)k\,;\,s>1,\ k\in K\}.$$

We first consider functions v such that $\mathrm{supp}(v)\subset HK$. Identifying $HK=(\exp \boldsymbol{R}U)K$ with $\boldsymbol{R}\times K$, let $v=\phi\otimes w\in C_c^\infty(\boldsymbol{R})\otimes C_c^\infty(K/B)$. For $(x, \dot{k})\in \boldsymbol{R}\times K/B$, $t\in \boldsymbol{R}$ and $h\in H\cap K=H\cap K_0$, we have

$$\pi(\exp tU)\phi(x)w(\dot{k})=\phi(x-t)w(\dot{k}),$$

$$\pi(h)\phi(x)w(\dot{k})=\phi(x)\tau(\exp(-xU)h\exp(xU))w(\dot{k}).$$

As subcase (1), a semiinvariant vector a of $\mathrm{supp}(a)\subset HK$ is of the form

$$(*)\qquad \langle a,\ \phi\otimes w\rangle=\int_{\boldsymbol{R}}(\bar{\chi}_l\Delta_{H,G}^{-1/2})(\exp xU)\overline{\phi(x)}\langle a_K,\ w\rangle dx,$$

where a_K is a distribution on K/B satisfying

$$\langle a_K,\ (\tau(h)-(\chi_l\Delta_{H\cap K,K}^{-1/2})(h))w\rangle=0$$

for all $h\in H\cap K$. (Note that $\Delta_{H,G}(h)=\Delta_{H\cap K,K}(h)$ for $h\in H\cap K\subset K_0$.)

Next, we treat functions v of $\mathrm{supp}(v)\subset H(\exp 2S)K=(\exp \boldsymbol{R}U)(\exp 2S)K$, which we identify with $\boldsymbol{R}\times K$. Let $v=\phi\otimes w\in C_c^\infty(\boldsymbol{R})\otimes C_c^\infty(K/B)$. For $(x, \dot{k})\in\boldsymbol{R}\times K/B$, $t\in R$ and $h\in H\cap K$, we have

$$\pi(\exp tU)\phi(x)w(\dot{k})=\phi(x-t)w(\dot{k}),$$

$$\pi(h)\phi(x)w(\dot{k})=\phi(x)\tau(\exp(-2S)\exp(-xU)h\exp(xU)\exp(2S))w(\dot{k}).$$

Thus a semiinvariant distribution a of $\mathrm{supp}(a)\subset H(\exp 2S)K$ is of the form $(*)$, where a_K is a distribution on K/B satisfying

$$\langle a_K,\ (\tau(\exp(-2S)h\exp(2S))-(\chi_l\Delta_{H\cap K,G}^{-1/2})(h))w\rangle=0,$$

for all $h\in H\cap K$. In other words, letting $l_2=(\exp(-2S)\cdot l)|_{\mathfrak{k}}$, $\mathfrak{h}_2=\exp(-2S)\cdot\mathfrak{h}\cap\mathfrak{k}_0$ and $H_2=\exp\mathfrak{h}_2$, we have

$$\langle a_K,\ (\tau(y)-(\chi_{l_2}\Delta_{H_2,K}^{-1/2})(y)w\rangle=0,$$

for all $y\in H_2$ noting that $\Delta_{H\cap K,G}(h)=\Delta_{H_2,K}(\exp(-2S)h\exp(2S))$ for $h\in H\cap K$.

Here let us observe coadjoint orbits. Since $\exp(uU)\cdot m(X_2)=e^{-u}(m(X_2)+\beta)-\beta$ for $m\in G\cdot l$, an H-orbit in $G\cdot l\cap(\mathfrak{h}^\perp+l)$ is included in one of the following: $\{m\in\mathfrak{g}^*\,;\,m(X_2)>-\beta\}$, $\{m\in\mathfrak{g}^*\,;\,m(X_2)<-\beta\}$, $\{m\in\mathfrak{g}^*\,;\,m(X_2)=-\beta\}$. For every H-orbit C in $G\cdot l\cap(\mathfrak{h}^\perp+l)\cap\{m\,;\,m(X_2)>-\beta\}$, $C_1=C\cap\boldsymbol{R}X_2^{\frac{1}{2}}\neq\emptyset$ is a $H\cap K$-orbit and the mapping $C\mapsto C_1$ from the set of H-orbits in $G\cdot l\cap(\mathfrak{h}^\perp+l)\cap\{m\,;\,m(X_2)>-\beta\}$ to that of $H\cap K$-orbits in $G\cdot l\cap(\mathfrak{h}^\perp+l)\cap\boldsymbol{R}X_2^{\frac{1}{2}}=K\cdot l\cap(\mathfrak{h}^\perp+l)$ is bijective. Since $K\cdot l+\mathfrak{k}^\perp=K\cdot l$, the number of H-orbits in $G\cdot l\cap(\mathfrak{h}^\perp+l)\cap\{m\,;\,m(X_2)>-\beta\}$ coincides with that of $H\cap K$-orbits in $K\cdot l_0\cap((\mathfrak{h}\cap\mathfrak{k}_0)^\perp+l_0)$.

For $G\cdot l\cap(\mathfrak{h}^{\perp}+l)\cap\{m\,;\,m(X_2)<-\beta\}$, the set of H-orbits corresponds to the set of $H\cap K$-orbits in $G\cdot l\cap(\mathfrak{h}^{\perp}+l)\cap\{m\,;\,m(X_2)=-2\beta\}=\exp(2S)\cdot(K\cdot l\cap\exp(-2S)\cdot(\mathfrak{h}^{\perp}+l))$. And it follows that the number of H-orbits in $G\cdot l\cap(\mathfrak{h}^{\perp}+l)\cap\{m\,;\,m(X_2)<-\beta\}$ coincides with that of H_2-orbits in $K\cdot l_0\cap(\mathfrak{h}_2^{\perp}+l_2)$.

For treating H-orbits included in $\{m\,;\,m(X_2)=-\beta\}$, remark that $\exp(-S)\cdot\mathfrak{h}\subset\mathfrak{k}$ and $G\cdot l\cap(\mathfrak{h}^{\perp}+l)\cap\{m\,;\,m(X_2)=-\beta\}=(\exp S)K\cdot l\cap(\mathfrak{h}^{\perp}+l)=\exp S\cdot(K\cdot l\cap\exp(-S)\cdot(\mathfrak{h}^{\perp}+l))$. Thus the number of H-orbits equals that of $\exp(\exp(-S)\cdot\mathfrak{h})$-orbits in $K\cdot l_0\cap(\exp(-S)\cdot\mathfrak{h}^{\perp}+\exp(-S)\cdot l)|_{\mathfrak{k}}$.

Now, let a be a semiinvariant distribution of $\mathrm{supp}(a)\subset(\exp S)K$. Identifying $G=(\exp \boldsymbol{R}S)K$ with $\boldsymbol{R}\times K$, for functions $v=\phi\otimes w\in C_c^\infty(\boldsymbol{R})\otimes C_c^\infty(K/B)$, we can describe a as follows:

$$\langle a,\, v\rangle=\sum_{i\geq 0}\overline{\frac{d^i\phi}{dx^i}(1)}\langle a_K^i,\, w\rangle.$$

Let $\mathfrak{h}_S=\exp(-S)\cdot\mathfrak{h}$, $H_S=\exp\mathfrak{h}_S$, $l_S=(\exp(-S)\cdot l)|_{\mathfrak{k}}$. We first treat the case of $(\exp S)K\cdot l\cap(\mathfrak{h}^{\perp}+l)=\emptyset$. Let j be the maximum index with $a_K^i\neq 0$, and suppose $\phi\in C_c^\infty(\boldsymbol{R})$ satisfies $\phi^{(i)}(1)=0$ for $1\leq i\leq j-1$ and $\phi(1)\neq 0$. Then for $h\in H\cap K=H\cap K_0$,

$$\begin{aligned}\langle a,\, \pi(h)v\rangle &= \langle a,\, \phi(x)(\tau\Delta_{K,G}^{-1/2})(\exp(-xS)h\exp(xS))w\rangle\\ &=\overline{\phi^{(j)}(1)}\langle a_K^j,\, (\tau\Delta_{K,G}^{-1/2})(\exp(-S)h\exp S)w\rangle\\ &\quad+\overline{\phi(1)}\sum_{i=0}^{i=j}\left\langle a_K^i,\, \left(\frac{d^i}{dx^i}(\tau\Delta_{K,G}^{-1/2})(\exp(-xS)h\exp(xS))w\right)(1)\right\rangle\\ &=\langle a,\, (\chi_l\Delta_{H,G}^{-1/2})(h)v\rangle\\ &=\overline{\phi^{(j)}(1)}\langle a_K^j,\, (\chi_l\Delta_{H,G}^{-1/2})(h)w\rangle+\overline{\phi(1)}\langle a_K^0,\, (\chi_l\Delta_{H,G}^{-1/2})(h)w\rangle.\end{aligned}$$

Taking a test function ϕ satisfying $\phi^{(j)}(1)=0$, from the semiinvariance, we get $\sum_{i=0}^{i=j}\langle a_K^i,\, (d^i/dx^i(\tau\Delta_{K,G}^{-1/2})(\exp(-xS)h\exp(xS))w)(1)\rangle-\langle a_K^0,\, (\chi_l\Delta_{H,G}^{-1/2})(h)w\rangle=0$ for all $w\in C_c^\infty(K/B)$. Thus

$$\langle a_K^j,\, (\tau\Delta_{K,G}^{-1/2})(\exp(-S)h\exp S)w\rangle=\langle a_K^j,\, (\chi_l\Delta_{H,G}^{-1/2})(h)w\rangle.$$

This means

$$\langle a_K^j,\, \tau(h_S)w\rangle=\langle a_K^j,\, (\chi_{l_S}\Delta_{H_S,K}^{-1/2})(h_S)w\rangle$$

for all $h_S\in\exp(\exp(-S)\cdot(\mathfrak{h}\cap\mathfrak{k}))\supset\exp(\mathfrak{h}_S\cap\mathfrak{p})$. Since $K\cdot l\cap(\mathfrak{h}_S^{\perp}+l_S)=\emptyset$, we find that $K\cdot l\cap((\mathfrak{h}_S\cap\mathfrak{p})^{\perp}+l_S)=\emptyset$ by the condition (C), and $a_K^j=0$, and thus $a=0$.

We next suppose $(\exp S)K\cdot l\cap(\mathfrak{h}^{\perp}+l)\neq\emptyset$. Then we can treat it as in the case 2.1, and get

$$a=\overline{\delta(1)}\otimes a_K,$$

where a_K satisfies

$$\langle a_K, (\tau\Delta_{K,G}^{-1/2})(\exp(-S)h\exp S)w\rangle = \langle a_K, (\chi_l\Delta_{H,G}^{-1/2})(h)w\rangle$$

for all $h\in H$, in other words,

$$\langle a_K, \tau(h_S)w\rangle = \langle a_K, (\chi_{l_S}\Delta_{H_S,K}^{-1/2})(h_S)w\rangle$$

for all $h_S\in H_S$.

Using the induction hypothesis for $(K, H\cap K, \tau, l_0)$, (K, H_2, τ, l_2), (K, H_S, τ, l_S), we verify the claim.

(3) $\dim \mathfrak{g}_2/\mathfrak{z}=2$ and $\dim \mathfrak{g}/\mathfrak{k}_0=2$, i.e., $rank(\lambda, \gamma_1, \gamma_2)=2$. Then $rank(\gamma_1, \gamma_2)=1$, that is $\mathfrak{k}_0\subsetneqq\mathfrak{k}\subsetneqq\mathfrak{g}$. (The case $\lambda\neq 0$, $\alpha\neq 0$ and $rank(\gamma_1, \gamma_2)=2$, i.e., $\mathfrak{k}_0=\mathfrak{k}$ cannot happen. In fact, suppose $\lambda=p\gamma_1+q\gamma_2$, where $p, q\in\boldsymbol{R}$, $p^2+q^2\neq 0$ and $S_i\in\mathfrak{g}$, $\gamma_j(S_i)=\delta_{ij}$, $i, j=1, 2$. Then by the Jacobi identity, $[[S_1, S_2], X_2]=(q+\alpha p)X_1$, $[[S_1, S_2], Y_2]=(q\alpha-p)X_1$, and $[S_1, S_2]\in(q+\alpha p)S_1+(q\alpha-p)S_2+\mathfrak{k}$. But $\lambda([S_1, S_2])=\lambda((q+\alpha p)S_1+(q\alpha-p)S_2)=(q+\alpha p)p+(q\alpha-p)q=\alpha(p^2+q^2)\neq 0$, which is a contradiction.)

We may assume $rank(\lambda, \gamma_1)=2$, and let $\gamma_2=c\gamma_1$, $c\in\boldsymbol{R}$. Take $S, T\in\mathfrak{g}$ such that $\gamma_1(S)=1$, $\lambda(S)=0$, $\gamma_1(T)=0$, $\lambda(T)=1$. Then by the Jacobi identity, $[[T, S], X_2]=(-1+c\alpha)X_1$, $[[T, S], Y_2]=(-\alpha-c)X_1$, and we get $-\alpha-c=c(-1+\alpha)$, which implies $\alpha=0$. Thus this case cannot happen.

(4) $\dim \mathfrak{g}_2/\mathfrak{z}=2$ and $\dim \mathfrak{g}/\mathfrak{k}_0=3$, i.e., $rank(\lambda, \gamma_1, \gamma_2)=3$. Let $S_1, S_2\in\mathfrak{g}\backslash\mathfrak{k}$ and $T\in\mathfrak{k}\backslash\mathfrak{k}_0$ such that $\gamma_i(S_j)=\delta_{ij}$, $\lambda(S_i)=0$, $\gamma_i(T)=0$, $\lambda(T)=1$, $i, j=1, 2$. Then by the Jacobi identity,

$$[T, S_1] \in -S_1-\alpha S_2+\mathfrak{k}_0,$$

$$[T, S_2] \in \alpha S_1-S_2+\mathfrak{k}_0,$$

$$[S_1, S_2] \in \mathfrak{k}_0$$

and thus $\mathfrak{k}$ acts irreducibly on $\mathfrak{g}/\mathfrak{k}$.

(i) $\mathfrak{h}+\mathfrak{k}=\mathfrak{g}$. In this case, either $\mathfrak{h}+\mathfrak{k}_0=\mathfrak{g}$ or $\mathfrak{h}\subset\ker\lambda$ holds since $T\in\mathfrak{g}\backslash\ker\lambda$ acts irreducibly on $\ker\lambda/\mathfrak{k}_0$. We can take the above S_1, S_2 so that $S_1, S_2\in\mathfrak{h}\cap\ker\lambda$. Then $G=(\exp\boldsymbol{R}S_1)(\exp\boldsymbol{R}S_2)K$ and we identify G with $\boldsymbol{R}^2\times K$. As case 2.2 (2) (i), a semiinvariant distribution a is described as follows: For $v=\phi\otimes w\in C_c^\infty(\boldsymbol{R}^2)\otimes C_c^\infty(K/B)$, we have

$$\langle a, v\rangle = \iint_{\boldsymbol{R}^2}(\bar{\chi}_l\Delta_{H,G}^{-1/2})(\exp(x_1S_1)\exp(x_2S_2))\overline{\phi(x_1, x_2)}\langle a_K, w\rangle dx_1dx_2,$$

where a_K is a distribution on K/B, and we find

$$\langle a_K, \tau(h)-(\chi_l\Delta_{H_1,K}^{-1/2})(h)w\rangle = 0 \quad \text{for all } h\in H_1,\ w\in C_c^\infty(K/B),$$

where $\mathfrak{h}_1=\mathfrak{h}\cap\mathfrak{k}+\boldsymbol{R}X_2+\boldsymbol{R}Y_2$ and $H_1=\exp\mathfrak{h}_1$.

The number of H-orbits in $G\cdot l\cap(\mathfrak{h}^{\perp}+l)$ equals that of $H\cap K$-orbits in $G\cdot l\cap(\mathfrak{h}^{\perp}+l)\cap(\boldsymbol{R}X_2+\boldsymbol{R}Y_2)^{\perp}=K\cdot l\cap(\mathfrak{h}^{\perp}+l)$, and thus coincides with that of H_1-orbits in $K\cdot l_0\cap(\mathfrak{h}_1^{\perp}+l_0)$. We can verify the claim of the theorem applying the induction hypothesis for (K, H_1, τ, l_0).

(ii) $\mathfrak{k}\subsetneqq\mathfrak{k}+\mathfrak{h}\subsetneqq\mathfrak{g}$ and $\mathfrak{h}\subset\ker\lambda$. Let $S\in\mathfrak{h}\backslash(\mathfrak{h}\cap\mathfrak{k})$, $X=l([S, Y_2])X_2-l([S, X_2])Y_2$ and $Y=l([S, X_2])X_2+l([S, Y_2])Y_2$. Then $l([\mathfrak{h}, X])=\{0\}$ and $l([S, Y])\neq 0$. Thus $\mathfrak{g}_2\cap(\mathfrak{h}+\mathfrak{g}(l))=\boldsymbol{R}X+\mathfrak{z}$, $[\mathfrak{h}, X]=\{0\}$ and for each $m\in G\cdot l$, its H-orbit satisfies $H\cdot m\subset(\boldsymbol{R}X+\mathfrak{z})^{\perp}+m$. Taking $S'\in\ker\lambda$ such that $G=(\exp\boldsymbol{R}S)(\exp\boldsymbol{R}S')K$, we identify G with $\boldsymbol{R}^2\times K$. Let $\mathbf{S}'=\{s\in\boldsymbol{R}\,;\,\exp(sS')K\cdot l\cap(\mathfrak{h}^{\perp}+l)\neq\emptyset\}$, whose number is bounded by the number of H-orbits in $G\cdot l\cap(\mathfrak{h}^{\perp}+l)$. Then by the semiinvariance for $\exp\boldsymbol{R}S$ and $\exp\boldsymbol{R}X'$, where $X'\in(X+\mathfrak{g}(l))\cap\mathfrak{h}$, a is described by a linear combination of a_s, $s\in\mathbf{S}'$, defined as follows: for $v=\phi\otimes w\in C_c^{\infty}(\boldsymbol{R}^2)\otimes C_c^{\infty}(K/B)$,

$$\langle a_s, v\rangle=\int_{\boldsymbol{R}}(\bar{\chi}_l\Delta_{H,G}^{-1/2})(\exp x_1S)\overline{\phi(x_1, s)}dx_1\langle a_K, w\rangle,$$

where a_K is a distribution on K/B satisfying

$$\langle a_K, \tau(\exp(-sS')h\exp(sS'))w\rangle=\langle a_K, (\chi_l\Delta_{H\cap K,G}^{-1/2})(h)w\rangle$$

for $h\in H\cap K=H\cap K_0$. In other words, writing $\mathfrak{h}_s=\exp(-sS')\cdot(\mathfrak{h}\cap\mathfrak{k}_0)$, $H_s=\exp\mathfrak{h}_s$ and $l_s=(\exp(-sS')\cdot l)|_{\mathfrak{k}}$, we have

$$\langle a_K, \tau(h_s)w\rangle=\langle a_K, (\chi_l\Delta_{H_s,G}^{-1/2})(h_s)\rangle$$

for all $h_s\in H_s$.

By the above observation, the number of H-orbits in $G\cdot l\cap(\mathfrak{h}^{\perp}+l)$ equals $\sum_{s\in\mathbf{S}'}$ (number of $H\cap K$-orbits in $\exp(sS')K\cdot l\cap(\mathfrak{h}^{\perp}+l)$). For each $s\in\mathbf{S}'$, the number of $H\cap K$-orbits equals the number of H_s-orbits in $K\cdot l_0\cap(\mathfrak{h}_s^{\perp}+l_s)$. Thus we can verify this case using the induction hypothesis for (K, H_s, τ, l_s).

(iii) $\mathfrak{k}\subsetneqq\mathfrak{k}+\mathfrak{h}\subsetneqq\mathfrak{g}$ and $\mathfrak{h}\not\subset\ker\lambda$. Let $\mathfrak{h}\backslash(\mathfrak{h}\cap\mathfrak{k})\ni U=\kappa T+\xi_1S_1+\xi_2S_2$, where S_1, S_2, and T are as at the beginning of (4), $\kappa, \xi_1, \xi_2\in\boldsymbol{R}$, $\kappa\neq 0$, $\xi_1^2+\xi_2^2=1$. Then $\mathfrak{h}=\boldsymbol{R}U+(\mathfrak{h}\cap\mathfrak{k})=\boldsymbol{R}U+(\mathfrak{h}\cap\mathfrak{k}_0)$. Take $\beta, \delta\in[0, 2\pi)$ such that $e^{\sqrt{-1}\beta}=\xi_1+\sqrt{-1}\xi_2$, $\sqrt{1+\alpha^2}e^{\sqrt{-1}\delta}=1-\sqrt{-1}\alpha$. Then we get $\exp uU=\exp(u_1S_1+u_2S_2)\exp(u\kappa T)k_0(u)$, where $k_0(u)\in K_0$ and

$$\begin{aligned}\begin{pmatrix}u_1\\u_2\end{pmatrix}&=\sum_{n\geqq 1}\frac{(u\kappa)^{n-1}}{n!}\begin{pmatrix}-1&\alpha\\-\alpha&-1\end{pmatrix}^{n-1}u\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}\\&=\frac{1}{\kappa\sqrt{1+\alpha^2}}\begin{pmatrix}\cos(\beta+\delta)-e^{-u\kappa}\cos(-u\kappa\alpha+\beta+\delta)\\\sin(\beta+\delta)-e^{-u\kappa}\sin(-u\kappa\alpha+\beta+\delta)\end{pmatrix}.\end{aligned}$$

Letting

$$s_1(s)=\frac{\cos s+\cos(\beta+\delta)}{\kappa\sqrt{1+\alpha^2}},\quad s_2(s)=\frac{\sin s+\sin(\beta+\delta)}{\kappa\sqrt{1+\alpha^2}},$$

we get

$$\exp(uU)\exp(s_1(s)S_1+s_2(s)S_2)=\exp(s_1(u,\ s)S_1+s_2(u,\ s)S_2)\exp(uT)k_0(u,\ s),$$

where $k_0(u,\ s)\in K_0$ and

$$s_1(u,\ s)=\frac{1}{\kappa\sqrt{1+\alpha^2}}(\cos(\beta+\delta)+e^{-u\kappa}\cos(-u\kappa\alpha+s))$$

$$s_2(u,\ s)=\frac{1}{\kappa\sqrt{1+\alpha^2}}(\sin(\beta+\delta)+e^{-u\kappa}\sin(-u\kappa\alpha+s)).$$

Thus we obtain a bijection

$$\Psi:\boldsymbol{R}\times[0,\ 2\pi)\times K\longrightarrow G_1=G\backslash\exp\Big(\frac{\cos(\beta+\delta)}{\kappa\sqrt{1+\alpha^2}}S_1+\frac{\sin(\beta+\delta)}{\kappa\sqrt{1+\alpha^2}}S_2\Big)K$$

by $\Psi(u,\ s,\ k)=\exp(uU)\exp(s_1(s)S_1+s_2(s)S_2)k$, and we also obtain a bijection

$$\Psi^*:\boldsymbol{R}\times[0,\ 2\pi)\longrightarrow \boldsymbol{R}X_2^*+\boldsymbol{R}Y_2^*+X_1^*\backslash\Big(\frac{-\cos(\beta+\delta)X_2^*-\sin(\beta+\delta)Y_2^*}{\kappa\sqrt{1+\alpha^2}}+X_1^*\Big),$$

where $\{X_2^*,\ Y_2^*,\ X_1^*\}$ is the dual basis of $\{X_2,\ Y_2,\ X_1\}$, by $\Psi^*(u,\ s)=\Psi(u,\ s,\ e)\cdot l|_{\mathfrak{g}_2}$.

Writing $p=\exp(1/\kappa\sqrt{1+\alpha^2})(\cos(\beta+\delta)S_1+\sin(\beta+\delta)S_2)$, we can treat distributions a of $\operatorname{supp}(a)\subset G_1=G\backslash pK$ and $\operatorname{supp}(a)\subset pK$ separately. Let us consider functions v of $\operatorname{supp}(v)\subset G_1$. Identifying G_1 with $\boldsymbol{R}\times[0,\ 2\pi)\times K$ through Ψ, let $v=\phi_1\otimes\phi_2\otimes w\in C_c^\infty(\boldsymbol{R})\otimes C_c^\infty([0,\ 2\pi))\otimes C_c^\infty(K/B)$. Then

$$\pi(\exp tU)\phi_1(x)\phi_2(s)w(\dot{k})=\phi_1(x-t)\phi_2(s)w(\dot{k}),$$

$$\pi(h)\phi_1(x)\phi_2(s)w(\dot{k})=\tau(\Psi(u,\ s,\ e)^{-1}h\Psi(u,\ s,\ e))\phi_1(x)\phi_2(s)w(\dot{k}),$$

for $t,\ x\in\boldsymbol{R}$, $s\in[0,\ 2\pi)$, $\dot{k}\in K/B$, $h\in H\cap K_0$. Writing $g(s)=\exp(s_1(u,\ s)S_1+s_2(u,\ s)S_2)$, let $\boldsymbol{S}=\{s\in[0,\ 2\pi)\,;\ g(s)K\cdot l\cap((\mathfrak{h}\cap\mathfrak{k}_0)^\perp+l)\neq\emptyset\}$. Then $\#\boldsymbol{S}\leqq\#(H\text{-orbits in } G\cdot l\cap(\mathfrak{h}^\perp+l))$. For $s\in\boldsymbol{S}$, let $Y=Y_s=g(s)\cdot l([U,\ Y_2])X_2-g(s)\cdot l([U,\ X_2])Y_2$, which satisfies $g(s)\cdot l([\mathfrak{h},\ Y])=\{0\}$, and take $Y'\in\mathfrak{g}(g(s)\cdot l)$ such that $Y+Y'\in\mathfrak{h}$. Then

$$\frac{d}{ds}\Big|_s(\chi_l(g(s)^{-1}\exp(Y+Y')g(s))-\chi_l(\exp(Y+Y')))$$

$$=\sqrt{-1}\,g(s)\cdot l\Big(\Big[Y+Y',\ \frac{1}{\kappa\sqrt{1+\alpha^2}}(-\sin sS_1+\cos sS_1)\Big]\Big)\chi_l(g(s)^{-1}\exp(Y+Y')g(s)),$$

and

$$g(s)\cdot l([Y+Y',\ -\sin sS_1+\cos sS_2])=-\frac{1}{\sqrt{1+\alpha^2}}\neq 0.$$

As case 2.1, we can obtain that a semiinvariant distribution a of $\operatorname{supp}(a)\subset G_1$ is a linear combination of a_s, $s\in\boldsymbol{S}$, such that

$$\langle a_s, \phi_1 \otimes \phi_2 \otimes w \rangle = \int_{\boldsymbol{R}} (\bar{\chi}_l \Delta_{H,G}^{-1/2})(\exp xU) \overline{\phi_1(x)\phi_2(s)} \langle a_K, w \rangle dx ,$$

where a_K is a distribution on K/B satisfying

$$\langle a_K, (\tau(g(s)^{-1} h g(s)) - (\chi_l \Delta_{H \cap K, K}^{-1/2}(h)))w \rangle = 0$$

for all $h \in H \cap K_0$.

As in case 2.2 (2) (ii), noting that $p^{-1} \cdot \mathfrak{h} \subset \mathfrak{k}$, we can treat distributions a of $\mathrm{supp}(a) \subset pK$. And from the above observations, we can verify the claim for this whole case similarly as case 2.2 (2) (ii).

CASE 3. *$\mathfrak{z} = \{0\}$, and $\ker l$ includes no non-zero abelian ideals for $l \in \Omega$.* Let $\mathfrak{g}_1$ be a minimal ideal of $\mathfrak{g}$ satisfying $\mathfrak{g}_1 \subset \mathfrak{p}$. Then $\dim \mathfrak{g}_1 = 1$ or 2. By the assumption, $l|_{\mathfrak{g}_1} \neq 0$ for all $l \in \Omega$, and $\mathfrak{g}_1^l$ is the centralizer of $\mathfrak{g}_1$. Fix $l \in \Omega \cap (\mathfrak{h}^\perp + f)$ and let $\mathfrak{k} = \mathfrak{g}_1^l$ for realizing π.

When $\dim \mathfrak{g}_1 = 1$, we take $X_1 \in \mathfrak{g}_1$ satisfying $l(X_1) = 1$, and for $X \in \mathfrak{g}$,

$$[X, X_1] = \lambda(X) X_1 ,$$

where $\lambda \in \mathfrak{g}^* \setminus \{0\}$. When $\dim \mathfrak{g}_1 = 2$, we take a base $\{X_1, Y_1\}$ such that $l(X_1) = 1$, $l(Y_1) = 0$ and for $X \in \mathfrak{g}$,

$$[X, X_1] = \lambda(X)(X_1 - \alpha Y_1)$$

$$[X, Y_1] = \lambda(X)(\alpha X_1 + Y_1) ,$$

where $\alpha \in \boldsymbol{R} \setminus \{0\}$, $\lambda \in \mathfrak{g}^* \setminus \{0\}$ since $\mathfrak{g}$ is exponential.

If $\dim \mathfrak{g}_1 = 1$, we have $\mathfrak{g}_1 \cap \mathfrak{g}(l) = \{0\}$, and if $\dim \mathfrak{g}_1 = 2$, we have $\mathfrak{g}_1 \cap \mathfrak{g}(l) = \boldsymbol{R}(\alpha X_1 - Y_1)$. It also holds that $\mathfrak{p} \subset \mathfrak{k}$, $K \cdot l_0 + \mathfrak{p}^\perp = K \cdot l_0$, where $l_0 = l|_{\mathfrak{k}}$, and $\mathfrak{k}(l_0) = \mathfrak{g}(l) + \mathfrak{g}_1$. We realize π as mentioned at the beginning of the proof using a polarization $\mathfrak{b}$ at l satisfying the Pukanszky condition and $\mathfrak{p} \subset \mathfrak{b}$, so that $\pi = \mathrm{ind}_B^G \chi_l = \mathrm{ind}_K^G \tau$, where $\tau \in \hat{K}$ corresponds to the coadjoint orbit $K \cdot l_0$.

Then the case $\mathfrak{h} \subset \mathfrak{k}$ can be treated as case 2.1. Next, suppose $\mathfrak{h} + \mathfrak{k} = \mathfrak{g}$. Then $\mathfrak{h} \cap \mathfrak{g}_1 = \{0\}$, $G = (\exp \boldsymbol{R}T)K$ taking $T \in \mathfrak{h} \setminus (\mathfrak{h} \cap \mathfrak{k})$, and the number of H-orbits in $G \cdot l \cap (\mathfrak{h}^\perp + l)$ coincides with the number of $H \cap K$-orbits in $G \cdot l \cap (\mathfrak{h}^\perp + l) \cap \{m ; m(X_1) = 1\} = K \cdot l \cap ((\mathfrak{h} + \mathfrak{g}_1)^\perp + l)$. Noting that $K \cdot l + \mathfrak{k}^\perp = K \cdot l$, we find that the number equals the number of H_1-orbits in $K \cdot l_0 \cap (\mathfrak{h}_1^\perp + l_0)$, where $\mathfrak{h}_1 = (\mathfrak{h} \cap \mathfrak{k}) + \mathfrak{g}_1$ and $H_1 = \exp \mathfrak{h}_1$. As in the case 2.2 (1), using the induction hypothesis for (K, H_1, τ, l_0), we verify the claim for this case. □

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