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The time slicing approximation of the fundamental solution for the Schrödinger equation with electromagnetic fields

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1. Introduction.

In this paper we construct the fundamental solution of the Schrödinger equation for a charged particle in electromagnetic fields:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}u &= \left[\frac{1}{2}\sum_{k=1}^{d}\left(-i\hbar\frac{\partial}{\partial x_k}-A_k(t, x)\right)^2+V(x)\right]u \\ &= H(\hbar, t)u, \quad t\in \boldsymbol{R},\ x=(x_1, \cdots, x_d)\in \boldsymbol{R}^d, \end{aligned} \tag{1.1}$$

where $0<\hbar\leqq 1$ is a parameter and $A(t, x)=(A_1(t, x), \cdots, A_d(t, x))$ and $V(x)$ are the vector and scalar potentials of the fields which satisfy the following assumption:

ASSUMPTION (A). For $k=1, \cdots, d$, $A_k(t, x)$ is a real-valued function of $(t, x)\in\boldsymbol{R}\times\boldsymbol{R}^d$, and for any multi-index α, $\partial_x^\alpha A_k(t, x)$ is C^1 in $(t, x)\in\boldsymbol{R}\times\boldsymbol{R}^d$. There exists $\varepsilon>0$ such that

$$|\partial_x^\alpha A_k(t, x)|+|\partial_x^\alpha\partial_t A_k(t, x)|\leqq C_\alpha, \quad |\alpha|\geqq 1,\ (t, x)\in\boldsymbol{R}\times\boldsymbol{R}^d, \tag{1.2}$$

$$|\partial_x^\alpha B(t, x)|\leqq C_\alpha(1+|x|)^{-1-\varepsilon}, \quad |\alpha|\geqq 1, \tag{1.3}$$

where $B(t, x)$ is a skew symmetric matrix with (k, l)-component $B_{kl}(t, x)=(\partial A_l/\partial x_k-\partial A_k/\partial x_l)(t, x)$ and $|B|$ denotes the norm of matrix B regarded as an operator on $\boldsymbol{R}^d$. $V(x)$ is a real-valued C^∞ function which satisfies

$$|\partial_x^\alpha V(x)|\leqq C_\alpha, \quad |\alpha|\geqq 2. \tag{1.4}$$

Let $q(\tau)$ be a classical path in the electromagnetic field joining $y\in\boldsymbol{R}^d$ at time s to $x\in\boldsymbol{R}^d$ at time t: $q(\tau)$ satisfies Lagrange's equation

$$\dot{q}(\tau)=v(\tau), \quad \dot{v}(\tau)=B(\tau, q(\tau))v(\tau)+F(\tau, q(\tau)), \quad s\leqq\tau\leqq t, \tag{1.5}$$

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with boundary condition

$$q(s)=y\,,\quad q(t)=x\,, \tag{1.6}$$

where $F(t, x)=-(\partial_t A)(t, x)-(\partial_x V)(x)$. When Assumption (A) is satisfied, if $|t-s|$ is sufficiently small, for any $x, y\in \boldsymbol{R}^d$ there exists a unique solution $q(\tau)$ of (1.5) and (1.6) (see [**12**, Proposition 2.6]).

For a path $\gamma:[s, t]\to \boldsymbol{R}^d$ satisfying $(d/d\tau)\gamma\in L^2([s, t]; \boldsymbol{R}^d)$, $\gamma(s)=y$ and $\gamma(t)=x$, we define the action along γ as

$$S(\gamma)=\int_s^t L(\tau, \dot{\gamma}(\tau), \gamma(\tau))d\tau\,, \tag{1.7}$$

where $L(\tau, \dot{\gamma}, \gamma)$ is the Lagrangian

$$L(\tau, \dot{\gamma}, \gamma)=\frac{|\dot{\gamma}|^2}{2}+A(\tau, \gamma)\dot{\gamma}-V(\gamma)\,. \tag{1.8}$$

We write the action $S(q)$ along the classical path $q(\tau)$ which is a function of (t, s, x, y) as

$$S(q)=S(t, s, x, y)\,. \tag{1.9}$$

Let $\Delta: s=\tau_0<\tau_1<\cdots<\tau_L=t$ be an arbitrary division of the interval $[s, t]$ into subintervals. Put $\Delta\tau_j=\tau_j-\tau_{j-1}$, $j=1, \cdots, L$, and $|\Delta|=\max_{1\leq j\leq L}\Delta\tau_j$. For this division Δ and for any $x_j\in \boldsymbol{R}^d$, $j=0, \cdots, L$, we denote by q_Δ the piecewise classical path joining the (τ_j, x_j), i.e., q_Δ satisfies

$$\ddot{q}_\Delta(\tau)=B(\tau, q_\Delta(\tau))\dot{q}_\Delta(\tau)+F(\tau, q_\Delta(\tau))\,,\quad \tau_{j-1}\leqq\tau\leqq\tau_j\,,\ j=1, \cdots, L, \tag{1.10}$$

with $q_\Delta(\tau_j)=x_j$, $j=0, \cdots, L$. The action $S(q_\Delta)$ along the picewise classical path q_Δ can be written as

$$S(q_\Delta)=S(x_L, \cdots, x_0)=\sum_{j=1}^{L}S(\tau_j, \tau_{j-1}, x_j, x_{j-1})\,. \tag{1.11}$$

What we mean by the time slicing approximation of Feynman path integral (see [**2**]) is

$$\begin{aligned}&K(\Delta; \hbar, t, s, x, y)\\&=\prod_{j=1}^{L}\left(\frac{1}{2\pi i\hbar\Delta\tau_j}\right)^{d/2}\int_{\boldsymbol{R}^{d(L-1)}}\exp\left(i\hbar^{-1}\sum_{j=1}^{L}S(\tau_j, \tau_{j-1}, x_j, x_{j-1})\right)\prod_{j=1}^{L-1}dx_j\,,\end{aligned} \tag{1.12}$$

with $x=x_L$ and $y=x_0$.

The construction of the fundamental solution of the Schrödinger equation based on the idea of Feynman path integral has been done by Fujiwara [**3**], [**4**], [**7**], Kitada [**9**] and Yajima [**12**].

When the vector potential A is absent and the scalar potential V, though time-dependent, satisfies (1.4), Fujiwara [**4**] constructed the fundamental solution for the Schrödinger equation by using the associated integral equation to show that $K(\Delta;\hbar, t, s, x, y)$ converges to the fundamental solution in a strong topology as $|\Delta|\rightarrow 0$ (cf. [**3**]). When A is present, Yajima [**12**] also constructed the fundamental solution in a similar way to [**4**], under Assumption (A), showing the convergence of $K(\Delta;\hbar, t, s, x, y)$ as $|\Delta|\rightarrow 0$ in the same topology as in [**4**].

On the other hand, again, in the case where A is absent and V satisfies (1.4), Fujiwara [**6**] applied the stationary phase method [**1**] to express the integral (1.12) as the main term plus a remainder term for L large (cf. [**5**]), and in [**7**] constructed the fundamental solution by not using the integral equation but directly showing the convergence of the obtained expression as $|\Delta|\rightarrow 0$. Tsuchida [**11**] has extended the results of Fujiwara [**6**] to the case where both A and V are present and satisfy Assumption (A). In this paper we want to use it to study the convergence of the integral (1.12) in this case where A and V satisfy Assumption (A) and to construct the fundamental solution for the Schrödinger equation (1.1).

In §2 we state our main results, Theorems 2.3, 2.4 and 2.5, which are proved in §3, 4 and 5.

2. Main results.

In this section we denote the k-th component of $x\in\boldsymbol{R}^d$ by $(x)_k$, $k=1, \cdots, d$. The next two propositions were proved in [**11**].

PROPOSITION 2.1. *There exists a positive constant δ such that if $|t-s|\leqq\delta$ then the action corresponding to the division Δ*

$$S(x_L, \cdots, x_0)=\sum_{j=1}^{L}S(\tau_j, \tau_{j-1}, x_j, x_{j-1}) \tag{2.1}$$

has the following properties:

(S) $S(\tau_j, \tau_{j-1}, x_j, x_{j-1})$ *is of the form*

$$S(\tau_j, \tau_{j-1}, x_j, x_{j-1})=\frac{|x_j-x_{j-1}|^2}{2(\tau_j-\tau_{j-1})}+\omega(\tau_j, \tau_{j-1}, x_j, x_{j-1}), \quad j=1, \cdots, L. \tag{2.2}$$

For notational simplicity we omit τ_j and τ_{j-1} to write

$$S_j(x_j, x_{j-1})=S(\tau_j, \tau_{j-1}, x_j, x_{j-1}) \tag{2.3}$$

and

$$\omega_j(x_j, x_{j-1})=\omega(\tau_j, \tau_{j-1}, x_j, x_{j-1}). \tag{2.4}$$

These $\omega_j(x_j, x_{j-1})$ have the following properties:

(i) *For any* $m\geqq 2$ *there exists a constant* $\kappa_m>0$ *independent of* j *such that*

$$\max_{2\leq|\alpha+\beta|\leq m}\sup_{x,y\in\boldsymbol{R}^d}|\partial_x^\alpha\partial_y^\beta\omega_j(x, y)|\leqq\kappa_m. \tag{2.5}$$

(ii) *Let* $(\bar{x}_L, \cdots, \bar{x}_0)$ *be an arbitrary solution of the system of the equation*

$$\partial_{x_j}S_{j+1}(\bar{x}_{j+1}, \bar{x}_j)+\partial_{x_j}S_j(\bar{x}_j, \bar{x}_{j-1})=0, \quad j=1, \cdots, L-1. \tag{2.6}$$

Then for any $m\geqq 1$, *there exists a constant* B_m *independent of* $(\bar{x}_L, \cdots, \bar{x}_0)$ *and the division* Δ, *but dependent on* d *such that*

$$\sum_{j=1}^{L-1}\sum_{\substack{1\leq|\alpha|\leq m\\|\beta|=1}}|[(\partial_{x_{j-1}}+\partial_{x_j}+\partial_{x_{j+1}})^\alpha\partial_{x_j}^\beta(\omega_j+\omega_{j+1})](\bar{x}_{j-1}, \bar{x}_j, \bar{x}_{j+1})|\leqq B_m, \tag{2.7}$$

where $(\partial_{x_{j-1}}+\partial_{x_j}+\partial_{x_{j+1}})^\alpha=\prod_{k=1}^d(\partial_{(x_{j-1})_k}+\partial_{(x_j)_k}+\partial_{(x_{j+1})_k})^{\alpha_k}$ *for a multi-index* $\alpha=(\alpha_1, \cdots, \alpha_d)$.

PROOF. For the proof see [**11**, Theorem 2.3]. □

When $S(x_L, \cdots, x_0)=\sum_{j=1}^L S(\tau_j, \tau_{j-1}, x_j, x_{j-1})$ satisfies (S) in Proposition 2.1, then if $|t-s|$ is small enough, for any $x_L, x_0\in\boldsymbol{R}^d$ there exists a unique critical point $(x_{L-1}^*, \cdots, x_1^*)$, i.e.,

$$\partial_{x_j}S_{j+1}(x_{j+1}^*, x_j^*)+\partial_{x_j}S_j(x_j^*, x_{j-1}^*)=0, \quad j=1, \cdots, L-1, \tag{2.8}$$

where $x_L^*=x_L$, $x_0^*=x_0$ (the proof is in [**11**, §3]). The piecewise classical path q_Δ^* corresponding to this critical point, joining the (τ_j, x_j^*), coincides with the classical path $q(\tau)$ so that $q(s)=x_0$ and $q(t)=x_L$. In particular we have $x_j^*=q(\tau_j)$, $j=1, \cdots, L-1$.

We write the Hessian matrix of $S(x_L, \cdots, x_0)$ with respect to $(x_{L-1}, \cdots, x_1)$, which is a $d(L-1)\times d(L-1)$ matrix, as

$$\begin{aligned}\operatorname{Hess}S(x_L, \cdots, x_0)&=H_\Delta+W_\Delta(X)\\&=\begin{pmatrix}h_\Delta & 0 & 0 & \cdots\\ 0 & h_\Delta & 0 & \cdots\\ \vdots & \vdots & \ddots & \cdots\\ 0 & \cdots & \cdots & h_\Delta\end{pmatrix}+\begin{pmatrix}W_{11}(X) & W_{12}(X) & \cdots\\ W_{21}(X) & W_{22}(X) & \cdots\\ \vdots & \ddots & \vdots\\ \cdots & \cdots & W_{dd}(X)\end{pmatrix}\end{aligned} \tag{2.9}$$

for $X=(x_L, \cdots, x_0)$, where h_Δ is an $(L-1)\times(L-1)$ matrix given by

$$h_\Delta=\begin{pmatrix}\dfrac{1}{\Delta\tau_1}+\dfrac{1}{\Delta\tau_2} & -\dfrac{1}{\Delta\tau_2} & 0 & \cdots & \cdots\\ -\dfrac{1}{\Delta\tau_2} & \dfrac{1}{\Delta\tau_2}+\dfrac{1}{\Delta\tau_3} & -\dfrac{1}{\Delta\tau_3} & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & \cdots & -\dfrac{1}{\Delta\tau_{L-1}} & \dfrac{1}{\Delta\tau_{L-1}}+\dfrac{1}{\Delta\tau_L}\end{pmatrix} \tag{2.10}$$

and $W_{kl}(X)$, $k, l=1, \cdots, d$, are $(L-1)\times(L-1)$ matrices with entries

$$(W_{kl}(X))_{j-1,j} = \partial_{(x_j)_l}\partial_{(x_{j-1})_k}\omega_j(x_j, x_{j-1}), \tag{2.11}$$

$$(W_{kl}(X))_{j,j} = \partial_{(x_j)_l}\partial_{(x_j)_k}(\omega_{j+1}(x_{j+1}, x_j)+\omega_j(x_j, x_{j-1})), \tag{2.12}$$

$$(W_{kl}(X))_{j,j+1} = \partial_{(x_{j+1})_l}\partial_{(x_j)_k}\omega_{j+1}(x_{j+1}, x_j), \tag{2.13}$$

$$(W_{kl}(X))_{i,j} = 0, \quad \text{if } |i-j| \geqq 2. \tag{2.14}$$

Let G_Δ be the inverse of H_Δ;

$$H_\Delta^{-1} = G_\Delta = \begin{pmatrix} g_\Delta & 0 & 0 & \cdots \\ 0 & g_\Delta & 0 & \cdots \\ \vdots & \vdots & \ddots & \cdots \\ 0 & \cdots & \cdots & g_\Delta \end{pmatrix}, \tag{2.15}$$

where g_Δ is an $(L-1)\times(L-1)$ matrix with (i, j) entry

$$\begin{aligned}(g_\Delta)_{ij} &= \frac{(\tau_i-s)(t-\tau_j)}{t-s}, \quad \text{if } 1\leqq i\leqq j\leqq L-1, \\ &= \frac{(\tau_j-s)(t-\tau_i)}{t-s}, \quad \text{if } 1\leqq j\leqq i\leqq L-1.\end{aligned}$$

We apply the stationary phase method to the right-hand side of (1.12).

PROPOSITION 2.2. *There exists a positive constant δ such that if $|t-s|\leqq\delta$ then $K(\Delta;\hbar, t, s, x, y)$ is of the form*

$$\begin{aligned}K(\Delta;\hbar, t, s, x, y) = \left(\frac{1}{2\pi i\hbar(t-s)}\right)^{d/2}&\exp(i\hbar^{-1}S(t, s, x, y)) \\ &\times D(\Delta; t, s, x, y)^{-1/2}(1+r(\Delta;\hbar, t, s, x, y)),\end{aligned} \tag{2.16}$$

where

$$D(\Delta; t, s, x, y) = \det(I+G_\Delta W_\Delta(X))|_{X=(x, x^*_{L-1},\cdots, x^*_1, y)}. \tag{2.17}$$

For any α and β there exists a positive constant $C_{\alpha\beta}$ independent of Δ such that

$$|\partial_x^\alpha\partial_y^\beta r(\Delta;\hbar, t, s, x, y)| \leqq C_{\alpha\beta}\hbar|t-s|^2. \tag{2.18}$$

PROOF. Since the phase function $\sum_{j=1}^L S(\tau_j, \tau_{j-1}, x_j, x_{j-1})$ satisfies (S), we can apply [**11**, Theorem 2]. Hence we have Proposition 2.2. □

The aim of this paper is to show the following three theorems.

THEOREM 2.3. *There exists a positive constant δ such that if $|t-s|\leqq\delta$ then there exist functions $d(t, s, x, y)$ and $r(\hbar, t, s, x, y)$ in $\mathcal{B}(\boldsymbol{R}^d\times\boldsymbol{R}^d)$ such that*

$$D(\Delta; t, s, x, y)\to 1+(t-s)d(t, s, x, y), \quad \textit{in } \mathcal{B}(\boldsymbol{R}^d\times\boldsymbol{R}^d), \quad \textit{as } |\Delta|\to 0 \tag{2.19}$$

and

$$r(\Delta;\hbar, t, s, x, y)\rightarrow r(\hbar, t, s, x, y), \quad in\ \mathscr{B}(\boldsymbol{R}^{d}\times\boldsymbol{R}^{d}), \quad as\ |\Delta|\rightarrow 0. \tag{2.20}$$

There exists a positive constant C *such that*

$$|d(t, s, x, y)|\leqq C(|t-s|+|x-y|) \tag{2.21}$$

and for any α *and* β *there exists a positive constant* $C_{\alpha\beta}$ *such that*

$$|\partial_{x}^{\alpha}\partial_{y}^{\beta}d(t, s, x, y)|\leqq C_{\alpha\beta} \tag{2.22}$$

and

$$|\partial_{x}^{\alpha}\partial_{y}^{\beta}r(\hbar, t, s, x, y)|\leqq C_{\alpha\beta}\hbar|t-s|^{2}. \tag{2.23}$$

In Theorem 2.4 below we will give an explicit formula for $1+(t-s)d(t, s, x, y)$. Let $\mathcal{L}$ be the Hilbert space $L^{2}([s, t]; \boldsymbol{R}^{d})$ with inner product

$$(u, v)_{\mathcal{L}}=\sum_{k=1}^{d}\int_{s}^{t}u_{k}(\tau)v_{k}(\tau)d\tau, \quad \text{for } u=(u_{1}, \cdots, u_{d}),\ v=(v_{1}, \cdots, v_{d})\in\mathcal{L},$$

and let $\mathscr{H}$ be the Sobolev space $H_{0}^{1}([s, t]; \boldsymbol{R}^{d})=\{u\,|\,u, \dot{u}\in\mathcal{L}, u(s)=u(t)=0\}$ which is a Hilbert space with inner product $(u, v)_{\mathscr{H}}=(\dot{u}, \dot{v})_{\mathcal{L}}$.

We write the unique solution $q(\tau)$ of (1.5) and (1.6) as

$$q(\tau)=q^{0}(\tau)+q^{1}(\tau) \ \text{ with } q^{0}(\tau)=\frac{t-\tau}{t-s}y+\frac{\tau-s}{t-s}x. \tag{2.24}$$

Then we have

$$\ddot{q}^{1}(\tau)=B(\tau, q(\tau))\dot{q}(\tau)+F(\tau, q(\tau)), \quad s\leqq\tau\leqq t, \text{ and } q^{1}(s)=q^{1}(t)=0. \tag{2.25}$$

The first variation of the functional $\mathscr{H}\ni u\rightarrow S(q^{0}+u)$ vanishes at the critical path q^{1}. The second variation at q^{1} is

$$\mathscr{H}\ni h\rightarrow\int_{s}^{t}\left(\left|\frac{d}{d\tau}h(\tau)\right|^{2}+{}^{t}h(\tau)(Yh)(\tau)\right)d\tau, \tag{2.26}$$

where

$$\begin{aligned}(Yh)(\tau)&=B(\tau, q(\tau))\frac{d}{d\tau}h(\tau)+Z(\tau)h(\tau)\\ &=\begin{pmatrix}0 & B_{12}(\tau, q(\tau)) & B_{13}(\tau, q(\tau)) & \cdots & \cdots\\ B_{21}(\tau, q(\tau)) & 0 & B_{23}(\tau, q(\tau)) & \cdots & \cdots\\ \vdots & \vdots & \vdots & \ddots & \cdots\\ B_{d1}(\tau, q(\tau)) & \cdots & \cdots & \cdots & 0\end{pmatrix}\frac{d}{d\tau}\begin{pmatrix}h_{1}(\tau)\\ h_{2}(\tau)\\ \vdots\\ h_{d}(\tau)\end{pmatrix}\\ &\quad+\begin{pmatrix}Z_{11}(\tau) & Z_{12}(\tau) & \cdots\\ Z_{21}(\tau) & Z_{22}(\tau) & \cdots\\ \vdots & \ddots & \cdots\\ \cdots & \cdots & Z_{dd}(\tau)\end{pmatrix}\begin{pmatrix}h_{1}(\tau)\\ h_{2}(\tau)\\ \vdots\\ h_{d}(\tau)\end{pmatrix}.\end{aligned} \tag{2.27}$$

Here $Z_{kl}(\tau)$, $k, l=1, \cdots, d$, are multiplication operators given by

$$Z_{kl}(\tau)=\sum_{m=1}^{d}(\partial_{(x)_l}B_{km})(\tau, q(\tau))\dot{q}_m(\tau)+(\partial_{(x)_l}F_k)(\tau, q(\tau)). \tag{2.28}$$

The operator Y is symmetric in $\mathcal{L}$ with domain $D(Y)=\mathcal{H}$, since $B(\tau, q(\tau))$ is skew symmetric and we have

$$\frac{d}{d\tau}B(\tau, q(\tau))=Z(\tau)-{}^tZ(\tau).$$

For each fixed $x, y\in \boldsymbol{R}^d$, Y is a bounded operator from $\mathcal{H}$ to $\mathcal{L}$, i.e., there exists a constant $C(x, y)$ such that $\|Yh\|_{\mathcal{L}}\leqq C(x, y)\|h\|_{\mathcal{H}}$. Z is a Hilbert-Schmidt operator from $\mathcal{H}$ to $\mathcal{L}$ because the $Z_{kl}(\tau)$ are bounded functions for fixed $x, y\in \boldsymbol{R}^d$.

Let G be the Green operator for the Dirichlet boundary value problem:

$$-\ddot{u}(\tau)=f(\tau), \quad s\leqq\tau\leqq t, \quad u(s)=u(t)=0.$$

Then for $f=(f_1, \cdots, f_d)\in\mathcal{L}$, we can write $Gf=((Gf)_1, \cdots, (Gf)_d)$ as

$$(Gf)_k(\tau)=\int_s^t g(\tau, \tau')f_k(\tau')d\tau', \quad k=1, \cdots, d, \tag{2.29}$$

where

$$\begin{aligned} g(\tau, \tau') &= \frac{(\tau'-s)(t-\tau)}{(t-s)}, \quad \text{if } s\leqq\tau'\leqq\tau\leqq t, \\ &= \frac{(\tau-s)(t-\tau')}{(t-s)}, \quad \text{if } s\leqq\tau\leqq\tau'\leqq t. \end{aligned} \tag{2.30}$$

Since G is a Hilbert-Schmidt operator from $\mathcal{L}$ to $\mathcal{H}$, GY is a Hilbert-Schmidt operator on $\mathcal{H}$ and GZ is trace class on $\mathcal{H}$.

THEOREM 2.4.

$$1+(t-s)d(t, s, x, y)\equiv\lim_{|\Delta|\to 0}D(\Delta; t, s, x, y)=\det{}_2(I+GY)e^{\mathrm{Tr}\, GZ}, \tag{2.31}$$

where $\det_2$ *is the regularized determinant and* $\mathrm{Tr}\, GZ$ *means the trace of the operator* GZ.

For the definitions of the regularized determinant $\det_n$ and the trace, for example, see [**10**, §9 and §3].

REMARK. The expression for $1+(t-s)d(t, s, x, y)$ in (2.31) differs from that in Fujiwara's result [7] where the vector potential A is absent. In this case GY was trace class so that $1+(t-s)d(t, s, x, y)=\det(I+GY)$, while in our case we have to modify its expression as in (2.31) since GY is not trace class.

Using the functions d and r given in Theorem 2.3, put

$$k(\hbar, t, s, x, y) = (1+(t-s)d(t, s, x, y))^{-1/2}(1+r(\hbar, t, s, x, y)). \tag{2.32}$$

THEOREM 2.5. *There exists a positive constant δ such that if $|t-s| \leqq \delta$ then the limit of $K(\Delta; \hbar, t, s, x, y)$ as $|\Delta| \to 0$*

$$\begin{aligned} K(\hbar, t, s, x, y) &= \lim_{|\Delta|\to 0} K(\Delta; \hbar, t, s, x, y) \\ &= \left(\frac{1}{2\pi i\hbar(t-s)}\right)^{d/2} \exp(i\hbar^{-1}S(t, s, x, y))k(\hbar, t, s, x, y), \end{aligned} \tag{2.33}$$

is the fundamental solution of the Schrödinger equation (1.1).

Theorems 2.3, 2.4, 2.5 will be proved in § 3, 4, 5, respectively.

3. Proof of Theorem 2.3.

Before proving Theorem 2.3, we prepare some lemmas about the classical path in electromagnetic fields satisfying Assumption (A). In this section also we denote the k-th component of x by $(x)_k$, $k=1, \cdots, d$.

LEMMA 3.1. *Let $|t-s| \leqq 1$.*

(i) *For any α with $|\alpha| \geqq 1$, there exists a constant C_α such that for any solution $(q(\tau), v(\tau))$, $s \leqq \tau \leqq t$, of* (1.5),

$$\int_s^t |(\partial_x^\alpha B)(\tau, q(\tau))|\,|v(\tau)|\,d\tau \leqq C_\alpha. \tag{3.1}$$

(ii) *There exists a positive constant δ_0 such that if $|t-s| \leqq \delta_0$, then for any $x, y \in \boldsymbol{R}^d$ there exists a unique solution $(q(\tau), v(\tau))$, $s \leqq \tau \leqq t$, of* (1.5) *with* (1.6).

PROOF. We refer the proof to Yajima [**12**, Lemma 2.1 and Proposition 2.6]. □

Let $q(\tau)=q^0(\tau)+q^1(\tau)$ be the classical path joining (s, y) to (t, x) defined by (2.24). Put $\|f\|_{L^1}=\int_s^t |f(\tau)|\,d\tau$ and $\|f\|_{L^\infty}=\sup_{s\leqq\tau\leqq t}|f(\tau)|$.

LEMMA 3.2. *There exists a positive constant $\delta_1 \leqq \delta_0$ such that if $|t-s| \leqq \delta_1$ then the following inequalities hold:*

$$\|\partial_x^\alpha \partial_y^\beta q^1\|_{L^\infty} \leqq \|\partial_x^\alpha \partial_y^\beta \dot{q}^1\|_{L^1} \leqq C_{\alpha\beta}|t-s|, \quad \text{for } |\alpha+\beta| \geqq 1; \tag{3.2}$$

$$\|\partial_x^\alpha \partial_y^\beta \dot{q}^1\|_{L^\infty} \leqq C_{\alpha\beta}, \quad \text{for } |\alpha+\beta| \geqq 1; \tag{3.3}$$

$$\|\dot{q}\|_{L^1} \leqq C(|t-s|+|x-y|). \tag{3.4}$$

For any τ_1 and τ_2 with $s \leqq \tau_1 < \tau_2 \leqq t$,

$$|\partial_x^\alpha \partial_y^\beta (q(\tau_2)-q(\tau_1))| \leqq \frac{|\tau_2-\tau_1|}{|t-s|}\delta_{|\alpha+\beta|,1}+C_{\alpha\beta}|\tau_2-\tau_1|, \quad for \ |\alpha+\beta| \geqq 1, \tag{3.5}$$

where $\delta_{j,k}$ is Kronecker's delta. In (3.2–3.5) all constants C, $C_{\alpha\beta}$ are independent of x, y.

PROOF. The proof of (3.2) is in [**11**, Lemma 2.2]. (3.3) can be proved in a similar way to (3.2).

(3.4). By Assumption (A) and (2.25), using $\|d(Gf)/d\tau\|_{L^1} \leqq |t-s|\|f\|_{L^1}$, we have

$$\begin{aligned}\|\dot{q}^1\|_{L^1} &\leqq |t-s|\|B\dot{q}+F\|_{L^1} \\ &\leqq C|t-s|\|\dot{q}\|_{L^1}+C|t-s|^2 \\ &\leqq C|t-s|(|x-y|+\|\dot{q}^1\|_{L^1})+C|t-s|^2.\end{aligned}$$

This yields that

$$\|\dot{q}^1\|_{L^1} \leqq C|t-s|(|t-s|+|x-y|), \tag{3.6}$$

if $|t-s|$ is sufficiently small. Therefore it follows that

$$\|\dot{q}\|_{L^1} \leqq |x-y|+\|\dot{q}^1\|_{L^1} \leqq C(|t-s|+|x-y|). \tag{3.7}$$

(3.5) is easily seen by (3.4) and (2.24). □

We will need the next lemma in this section and §5.

LEMMA 3.3. *The notation $\omega(t, s, x, y)$ given by (2.2) is used.*

(i) *Let $s<t$ with $|t-s| \leqq \delta_1$. There exists a positive constant C such that*

$$|(\Delta_y \omega)(t, s, x, y)+(\operatorname{div} A)(s, y)| \leqq C(|t-s|+|x-y|), \tag{3.8}$$

where Δ_y is the Laplacian with respect to y.

(ii) *Let $s<r<t$ with $|t-s| \leqq \delta_1$. For any α there exists a positive constant C_α such that*

$$\begin{aligned}&|[(\partial_x+\partial_z+\partial_y)^\alpha \Delta_z](\omega(t, r, x, z)+\omega(r, s, z, y))|_{z=z^*} \\ &= C_\alpha\Big(|t-s|+\int_s^t \sum_{1\leq|\beta|\leq|\alpha|+1} |(\partial_x^\beta B)(\tau, q(\tau))|\,|v(\tau)|\,d\tau\Big),\end{aligned} \tag{3.9}$$

where z^* is the critical point of $S(t, r, x, z)+S(r, s, z, y)$ with respect to z.

PROOF. (i). By the definition of $S(t, s, x, y)$ we have

$$\begin{aligned}S(t, s, x, y) &= \int_s^t \Big(\frac{|\dot{q}^0(\tau)+\dot{q}^1(\tau)|^2}{2}+A(\tau, q(\tau))\dot{q}(\tau)-V(q(\tau))\Big)d\tau \\ &= \frac{|x-y|^2}{2(t-s)}+\int_s^t \Big(\frac{|\dot{q}^1(\tau)|^2}{2}+A(\tau, q(\tau))\dot{q}(\tau)-V(q(\tau))\Big)d\tau,\end{aligned} \tag{3.10}$$

and hence

$$\omega(t, s, x, y)=\int_s^t\left(\frac{|\dot{q}^1(\tau)|^2}{2}+A(\tau, q(\tau))\dot{q}(\tau)-V(q(\tau))\right)d\tau. \tag{3.11}$$

Since q satisfies (2.25), we obtain

$$(\partial_{(y)_k}\omega)(t, s, x, y)=\int_s^t\partial_{(y)_k}q^0\cdot(B(\tau, q(\tau))\dot{q}(\tau)+F(\tau, q(\tau)))d\tau-A_k(s, y), \tag{3.12}$$

$$(\partial_{(x)_k}\omega)(t, s, x, y)=\int_s^t\partial_{(x)_k}q^0\cdot(B(\tau, q(\tau))\dot{q}(\tau)+F(\tau, q(\tau)))d\tau+A_k(t, x). \tag{3.13}$$

Since $\partial_{(y)_k}q_m^0=(t-\tau)(t-s)^{-1}\delta_{km}$, it follows that

$$\begin{aligned}\Delta_y\omega(t, s, x, y)=\int_s^t\frac{t-\tau}{t-s}\Big(\sum_{k,n=1}^d B_{kn}\partial_{(y)_k}\dot{q}_n^1+\sum_{k,n,m=1}^d(\partial_{(x)_n}B_{km})\dot{q}_m\cdot\partial_{(y)_k}q_n \\ +\sum_{k,n=1}^d(\partial_{(x)_n}F_k)\cdot\partial_{(y)_k}q_n\Big)d\tau-(\operatorname{div}A)(s, y).\end{aligned} \tag{3.14}$$

Therefore by (3.2) and (3.4), we have

$$\begin{aligned}&|(\Delta_y\omega)(t, s, x, y)+(\operatorname{div}A)(s, y)| \\ &\leqq\int_s^t\Big(\sum_{k,n=1}^d|B_{kn}\partial_{(y)_k}\dot{q}_n^1|+\sum_{k,n,m=1}^d|(\partial_{(x)_n}B_{km})\dot{q}_m\cdot\partial_{(y)_k}q_n| \\ &\quad+\sum_{k,n=1}^d|(\partial_{(x)_n}F_k)\cdot\partial_{(y)_k}q_n|\Big)d\tau \\ &\leqq C|t-s|+C(|t-s|+|x-y|)+C|t-s| \\ &\leqq C(|t-s|+|x-y|).\end{aligned} \tag{3.15}$$

(ii). Similarly to the above, we have

$$\begin{aligned}&\Delta_z(\omega(t, r, x, z)+\omega(r, s, z, y))|_{z=z^*} \\ &=\int_r^t\frac{t-\tau}{t-r}\Big(\sum_{k,n=1}^d B_{kn}\partial_{(z)_k}\dot{q}_n^1+\sum_{k,n,m=1}^d(\partial_{(x)_n}B_{km})\dot{q}_m\cdot\partial_{(z)_k}q_n \\ &\qquad+\sum_{k,n=1}^d(\partial_{(x)_n}F_k)\cdot\partial_{(z)_k}q_n\Big)d\tau \\ &\quad+\int_s^r\frac{\tau-s}{r-s}\Big(\sum_{k,n=1}^d B_{kn}\partial_{(z)_k}\dot{q}_n^1+\sum_{k,n,m=1}^d(\partial_{(x)_n}B_{km})\dot{q}_m\cdot\partial_{(z)_k}q_n \\ &\qquad+\sum_{k,n=1}^d(\partial_{(x)_n}F_k)\cdot\partial_{(z)_k}q_n\Big)d\tau.\end{aligned} \tag{3.16}$$

Hence it follows with (3.2) that

$$|\Delta_z(\omega(t, r, x, z)+\omega(r, s, z, y))|_{z=z^*}|$$
$$\leqq C\Big(|t-s|+\int_s^t \sum_{|\beta|=1} |(\partial_x^\beta B)(\tau, q(\tau))||v(\tau)|d\tau\Big). \tag{3.17}$$

Similar arguments are valid for the differentiations $(\partial_x+\partial_z+\partial_y)^\alpha$, $|\alpha|\geqq 1$. Thus (ii) has been proved. □

In the rest of this section we prove Theorem 2.3. Let $|t-s|$ be so small that Propositions 2.1 and 2.2 hold. We shall first estimate $D(\Delta; t, s, x, y)$ and next show the convergence of $D(\Delta; t, s, x, y)$ and $r(\Delta; \hbar, t, s, x, y)$.

Let x_1^* be the critical point of $S_2(x_2, x_1)+S_1(x_1, x_0)$ with respect to x_1. We define a function $D(S_2+S_1; x_2, x_0)$ through the Hessian determinant at x_1^* in the following way:

$$D(S_2+S_1; x_2, x_0) = \Big(\frac{\Delta\tau_1\Delta\tau_2}{\Delta\tau_1+\Delta\tau_2}\Big)^d \det \mathrm{Hess}_{x_1^*}(S_2+S_1). \tag{3.18}$$

Then $D(S_2+S_1; x_2, x_0)$ is of the form

$$\begin{aligned} &D(S_2+S_1; x_2, x_0) \\ &= \det\Big(I+\frac{\Delta\tau_1\Delta\tau_2}{\Delta\tau_1+\Delta\tau_2}\Omega(x_2, x_1, x_0)|_{x_1=x_1^*}\Big) \\ &= 1+\frac{\Delta\tau_1\Delta\tau_2}{\Delta\tau_1+\Delta\tau_2}\Delta_{x_1}(\omega_2+\omega_1)|_{x_1=x_1^*}+\Big(\frac{\Delta\tau_1\Delta\tau_2}{\Delta\tau_1+\Delta\tau_2}\Big)^2 c(x_2, x_0), \end{aligned} \tag{3.19}$$

with

$$\Omega(x_2, x_1, x_0) = \begin{pmatrix} \partial^2_{(x_1)_1}(\omega_2+\omega_1) & \partial_{(x_1)_1}\partial_{(x_1)_2}(\omega_2+\omega_1) & \cdots & \\ \partial_{(x_1)_2}\partial_{(x_1)_1}(\omega_2+\omega_1) & \partial^2_{(x_1)_2}(\omega_2+\omega_1) & \cdots & \\ \vdots & \vdots & \ddots & \\ \cdots & \cdots & \cdots & \partial^2_{(x_1)_d}(\omega_2+\omega_1) \end{pmatrix}$$

where Δ_{x_1} is the Laplacian with respect to x_1 and $c(x_2, x_0)$ is a function in $\mathscr{B}(\boldsymbol{R}^d\times\boldsymbol{R}^d)$ because of (S) (i).

We introduce the notations $S_{k,l}$ and $\omega_{k,l}$. For two integers k and l with $L\geqq k>l\geqq 0$, put

$$S_{k,l}(x_k, x_l) = S(\tau_k, \tau_l, x_k, x_l) \tag{3.20}$$

and define $\omega_{k,l}$ as

$$S_{k,l}(x_k, x_l) = \frac{|x_k-x_l|^2}{2(\tau_k-\tau_l)}+\omega_{k,l}(x_k, x_l). \tag{3.21}$$

Then we have

$$S_{k,k-1}(x_k, x_{k-1}) = S_k(x_k, x_{k-1})$$

and

$$S_{k,l}(x_k, x_l) = S_k(x_k, x_{k-1}^*)+\cdots+S_l(x_{l+1}^*, x_l)$$

with critical point $(x_{k-1}^*, \cdots, x_{l+1}^*)$ with respect to the variables $(x_{k-1}, \cdots, x_{l+1})$.

The following proposition estimates $D(\Delta; t, s, x, y)$.

THEOREM 3.4. *There exists a positive constant $\delta_2 \leqq \delta_1$ such that if $|t-s| \leqq \delta_2$ then $D(\Delta; t, s, x, y)$ is of the form*

$$\begin{aligned} D(\Delta; t, s, x, y) &= \prod_{k=2}^{L} D(S_k + S_{k-1,0}; x_k, x_0)|_{X=(x, x_{L-1}^*, \cdots, x_1^*, y)} \\ &= 1 + \sum_{k=2}^{L} \frac{\Delta\tau_k(\tau_{k-1}-s)}{(\tau_k - s)} \Delta_{x_{k-1}}(\omega_k + \omega_{k-1,0})|_{X=(x, x_{L-1}^*, \cdots, x_1^*, y)} \\ &\qquad + |t-s|^2 d_1(\Delta; t, s, x, y) \\ &= 1 + (t-s) d_2(\Delta; t, s, x, y). \end{aligned} \tag{3.22}$$

For any α and β there exists a positive constant $C_{\alpha\beta}$ independent of Δ such that

$$|\partial_x^\alpha \partial_y^\beta d_1(\Delta; t, s, x, y)| \leqq C_{\alpha\beta}, \tag{3.23}$$

$$|\partial_x^\alpha \partial_y^\beta d_2(\Delta; t, s, x, y)| \leqq C_{\alpha\beta}, \tag{3.24}$$

and there exists a positive constant C independent of Δ such that

$$|d_2(\Delta; t, s, x, y)| \leqq C(|t-s| + |x-y|). \tag{3.25}$$

PROOF. The first equality in (3.22) was proved in [**11**, Lemma 3.9]. The second equality and (3.23) follow from (3.19), (S) (i) and Lemma A (A.2) in Appendix. (S) (i) yields the third equality in (3.22) and (3.24). Next we show (3.25). By Lemma 3.3 (ii), (3.4) and (3.23) we have

$$\begin{aligned} &|d_2(\Delta; t, s, x, y)| \\ &\leqq \max_{2\leqq k\leqq L}(|\Delta_{x_{k-1}}(\omega_k + \omega_{k-1,0})|_{X=(x, x_{L-1}^*, \cdots, x_1^*, y)}|) + |t-s|\,|d_1(\Delta; t, s, x, y)| \\ &\leqq C\Big(|t-s| + \int_s^t \sum_{|\gamma|=1} |(\partial_x^\gamma B)(\tau, q(\tau))|\,|v(\tau)|\,d\tau\Big) + C|t-s| \\ &\leqq C(|t-s| + |x-y|). \end{aligned} \tag{3.26}$$

Thus we have proved (3.25). □

The next theorem shows that $\{D(\Delta; t, s, x, y)\}_\Delta$ forms a Cauchy net.

THEOREM 3.5. *Let Δ' be any refinement of Δ. Then if $|t-s| \leqq \delta_2$ there exists a function $d(\Delta', \Delta; t, s, x, y)$ such that*

$$\frac{D(\Delta'; t, s, x, y)}{D(\Delta; t, s, x, y)} = 1 + |\Delta|\, d(\Delta', \Delta; t, s, x, y). \tag{3.27}$$

For any α and β, there exists a positive constant $C_{\alpha\beta}$ independent of Δ and Δ' such that

$$|\partial_x^\alpha \partial_y^\beta d(\Delta', \Delta; t, s, x, y)| \leqq C_{\alpha\beta}. \tag{3.28}$$

We have the following corollary to this theorem.

COROLLARY 3.6. *If* $|t-s| \leqq \delta_2$ *then for any* α *and* β, *there exists a positive constant* $C_{\alpha\beta}$ *independent of* Δ *and* Δ' *such that*

$$|\partial_x^\alpha \partial_y^\beta (D(\Delta'; t, s, x, y) - D(\Delta; t, s, x, y))| \leqq C_{\alpha\beta} |\Delta|. \tag{3.29}$$

PROOF OF COROLLARY 3.6. From Theorem 3.5 we have

$$D(\Delta'; t, s, x, y) - D(\Delta; t, s, x, y) = |\Delta| d(\Delta', \Delta; t, s, x, y) D(\Delta; t, s, x, y).$$

This and Theorem 3.4 yield Corollary 3.6. □

PROOF OF THEOREM 3.5. Let $\Delta: s=\tau_0<\tau_1< \cdots <\tau_L=t$ be a division of the interval $[s, t]$. Let $1 \leqq j \leqq L$. Since Δ' is a refinement of Δ, Δ' divides the interval $[\tau_{j-1}, \tau_j]$. We denote this division of $[\tau_{j-1}, \tau_j]$ by Δ'_j, i.e.,

$$\Delta'_j: \tau_{j-1} = \tau_{j,0} < \tau_{j,1} < \cdots < \tau_{j, L_j} = \tau_j, \quad L_j > 0.$$

We introduce some notations. Corresponding to this subdivision we set $x_{j,l} = q_{\Delta'}(\tau_{j,l})$, $\Delta\tau_{j,l} = \tau_{j,l} - \tau_{j,l-1}$ and $T_{j,l} = \Delta\tau_{j,1} + \cdots + \Delta\tau_{j,l}$ for $l=1, \cdots, L_j$. For two integers k and l with $L_j \geqq k > l \geqq 1$, put

$$\omega_{k,l}^{(j)}(x_{j,k}, x_{j,l}) = \omega(\tau_{j,k}, \tau_{j,l}, x_{j,k}, x_{j,l}). \tag{3.30}$$

By [**6**, Proposition 2.6], we have

$$D(\Delta'; t, s, x, y) = D(\Delta; t, s, x, y) \prod_{j=1}^{L} D(\Delta'_j; \tau_j, \tau_{j-1}, x_j^*, x_{j-1}^*), \tag{3.31}$$

where $x_j^* = q(\tau_j)$. So to prove Theorem 3.5 we have to prove that

$$\prod_{j=1}^{L} D(\Delta'_j; \tau_j, \tau_{j-1}, x_j^*, x_{j-1}^*) = 1 + |\Delta| d(\Delta', \Delta; t, s, x, y), \tag{3.32}$$

and that $d(\Delta', \Delta; t, s, x, y)$ satisfies (3.28).

By the second equality in (3.22) we have with a function d_1 in $\mathscr{B}(\boldsymbol{R}^d \times \boldsymbol{R}^d)$

$$\begin{aligned} &\prod_{j=1}^{L} D(\Delta'_j; \tau_j, \tau_{j-1}, x_j^*, x_{j-1}^*) \\ &= \prod_{j=1}^{L} \Big[1 + \prod_{k=2}^{L_j} \frac{\Delta\tau_{j,k} T_{j,k-1}}{T_{j,k}} \Delta_{x_{j,k-1}} (\omega_{k,k-1}^{(j)} + \omega_{k-1,0}^{(j)}) |_{(x_{j,1}^*, \cdots, x_{j,L_j-1}^*)} \\ &\qquad + \Delta\tau_j^2 d_1(\Delta'_j; \tau_j, \tau_{j-1}, x_j^*, x_{j-1}^*) \Big]. \end{aligned} \tag{3.33}$$

To apply Lemma A (A.1) to the right-hand side of (3.33), we estimate the second term in each factor of the product on the right-hand side of (3.33).

First, from Lemma 3.3 (ii) it follows that

$$\left|\sum_{k=2}^{L_j}\frac{\Delta\tau_{j,k}T_{j,k-1}}{T_{j,k}}\Delta_{x_{j,k-1}}(\omega_{k,k-1}^{(j)}+\omega_{k-1,0}^{(j)})|_{(x_{j,1}^*,\cdots,x_{j,L_j-1}^*)}\right|$$
$$\leqq C\Delta\tau_j\Big(\Delta\tau_j+\int_{\tau_{j-1}}^{\tau_j}\sum_{|\gamma|=1}|(\partial_x^\gamma B)(\tau,q(\tau))||v(\tau)|d\tau\Big). \tag{3.34}$$

Second, we can show that

$$\left|\partial_x^\alpha\partial_y^\beta\Big[\sum_{k=2}^{L_j}\frac{\Delta\tau_{j,k}T_{j,k-1}}{T_{j,k}}\Delta_{x_{j,k-1}}(\omega_{k,k-1}^{(j)}+\omega_{k-1,0}^{(j)})|_{(x_{j,1}^*,\cdots,x_{j,L_j-1}^*)}\Big]\right|$$
$$\leqq C_{\alpha\beta}\Delta\tau_j\Big(\Delta\tau_j+\frac{\Delta\tau_j}{|t-s|}+\int_{\tau_{j-1}}^{\tau_j}\sum_{1\leq|\gamma|\leq2}|(\partial_x^\gamma B)(\tau,q(\tau))||v(\tau)|d\tau\Big),\quad |\alpha+\beta|\geqq1. \tag{3.35}$$

In fact, for instance, differentiating by $(x)_m$, $m=1,\cdots,d$, the summand of the summation on the left-hand side of (3.35), we have

$$\begin{aligned}
&\partial_{(x)_m}[\Delta_{x_{j,k-1}}(\omega_{k,k-1}^{(j)}+\omega_{k-1,0}^{(j)})|_{(x_{j,k}^*,x_{j,k-1}^*,x_{j,0}^*)}]\\
&=\partial_{(x)_m}x_{j,k}^*(\partial_{x_{j,k}}\Delta_{x_{j,k-1}})(\omega_{k,k-1}^{(j)}+\omega_{k-1,0}^{(j)})|_{(x_{j,k}^*,x_{j,k-1}^*,x_{j,0}^*)}\\
&\quad+\partial_{(x)_m}x_{j,k-1}^*(\partial_{x_{j,k-1}}\Delta_{x_{j,k-1}})(\omega_{k,k-1}^{(j)}+\omega_{k-1,0}^{(j)})|_{(x_{j,k}^*,x_{j,k-1}^*,x_{j,0}^*)}\\
&\quad+\partial_{(x)_m}x_{j,0}^*(\partial_{x_{j,0}}\Delta_{x_{j,k-1}})(\omega_{k,k-1}^{(j)}+\omega_{k-1,0}^{(j)})|_{(x_{j,k}^*,x_{j,k-1}^*,x_{j,0}^*)}\\
&=\partial_{(x)_m}x_{j,0}^*[(\partial_{x_{j,k}}+\partial_{x_{j,k-1}}+\partial_{x_{j,0}})\Delta_{x_{j,k-1}}](\omega_{k,k-1}^{(j)}+\omega_{k-1,0}^{(j)})|_{(x_{j,k}^*,x_{j,k-1}^*,x_{j,0}^*)}\\
&\quad+\partial_{(x)_m}(x_{j,k}^*-x_{j,0}^*)(\partial_{x_{j,k}}\Delta_{x_{j,k-1}})(\omega_{k,k-1}^{(j)}+\omega_{k-1,0}^{(j)})\\
&\quad+\partial_{(x)_m}(x_{j,k-1}^*-x_{j,0}^*)(\partial_{x_{j,k-1}}\Delta_{x_{j,k-1}})(\omega_{k,k-1}^{(j)}+\omega_{k-1,0}^{(j)}).
\end{aligned} \tag{3.36}$$

So by (S) (i) and Lemma 3.3 (ii), we obtain

$$\begin{aligned}
&|\partial_{(x)_m}\Delta_{x_{j,k-1}}(\omega_{k,k-1}^{(j)}+\omega_{k-1,0}^{(j)})|_{(x_{j,k}^*,x_{j,k-1}^*,x_{j,0}^*)}|\\
&\leqq C\Big(T_{j,k}+\int_{\tau_{j,0}}^{\tau_{j,k}}\sum_{1\leq|\gamma|\leq2}|(\partial_x^\gamma B)(\tau,q(\tau))||v(\tau)|d\tau\Big)+C\Big(\frac{T_{j,k}}{|t-s|}+T_{j,k}\Big)\\
&\leqq C\Big(\Delta\tau_j+\frac{\Delta\tau_j}{|t-s|}+\int_{\tau_{j-1}}^{\tau_j}\sum_{1\leq|\gamma|\leq2}|(\partial_x^\gamma B)(\tau,q(\tau))||v(\tau)|d\tau\Big),
\end{aligned} \tag{3.37}$$

using $|\partial_{(x)_m}x_{j,0}^*|\leqq C$ and $|\partial_{(x)_m}(x_{j,k}^*-x_{j,0}^*)|\leqq T_{j,k}/|t-s|+CT_{j,k}$ (see (3.2) and (3.6)). Hence together with $\sum_{k=2}^{L_j}\Delta\tau_{j,k}T_{j,k-1}/T_{j,k}\leqq\Delta\tau_j$, (3.37) yields (3.35) for $|\alpha|=1$. Arguments for other differentiation $\partial_x^\alpha\partial_y^\beta$ hold analogously. So applying Lemma A (A.1), we obtain by (3.34, 35) and Lemma 3.1 (i),

$$\Big|\partial_x^\alpha\partial_y^\beta\Big(\prod_{j=1}^{L}\Big[1+\sum_{k=2}^{L_j}\frac{\Delta\tau_{j,k}T_{j,k-1}}{T_{j,k}}\Delta_{x_{j,k-1}}(\omega_{k,k-1}^{(j)}+\omega_{k-1,0}^{(j)})|_{(x_{j,1}^*,\cdots,x_{j,L_j-1}^*)}$$
$$+\Delta\tau_j^2d_1(\Delta_j';\tau_j,\tau_{j-1},x_j^*,x_{j-1}^*)\Big]-1\Big)\Big|$$

$$\leqq \prod_{j=1}^{L}\left[1+C_{\alpha\beta}\Delta\tau_j\left(\Delta\tau_j+\frac{\Delta\tau_j}{|t-s|}+\int_{\tau_{j-1}}^{\tau_j}\sum_{1\leq|\gamma|\leq 2}|(\partial_x^\gamma B)(\tau, q(\tau))|\,|v(\tau)|\,d\tau\right)\right]-1$$

$$\leqq \sum_{j=1}^{L} C_{\alpha\beta}\Delta\tau_j\left(\Delta\tau_j+\frac{\Delta\tau_j}{|t-s|}+\int_{\tau_{j-1}}^{\tau_j}\sum_{1\leq|\gamma|\leq 2}|(\partial_x^\gamma B)(\tau, q(\tau))|\,|v(\tau)|\,d\tau\right)$$

$$\leqq C_{\alpha\beta}|\Delta|. \tag{3.38}$$

This with (3.31, 33) means (3.28, 32). Thus we have proved Theorem 3.5. □

Next we show the convergence of $r(\Delta; \hbar, t, s, x, y)$ as $|\Delta|\to 0$.

THEOREM 3.7. *Let Δ' be any refinement of Δ and let $|t-s|\leqq\delta_2$. Then for any α and β there exists a positive constant $C_{\alpha\beta}$ independent of Δ and Δ' such that*

$$|\partial_x^\alpha\partial_y^\beta(r(\Delta; \hbar, t, s, x, y)-r(\Delta'; \hbar, t, s, x, y))| \leqq C_{\alpha\beta}|\Delta|. \tag{3.39}$$

PROOF. We use the same notations as in the proof of Theorem 3.5. As Δ' is a refinement of Δ, we can use Proposition 2.2 to write $K(\Delta'; \hbar, t, s, x, y)$ as

$$K(\Delta'; \hbar, t, s, x, y)=\int_{R^{d(L-1)}}\prod_{j=1}^{L}K(\Delta'_j; \hbar, \tau_j, \tau_{j-1}, x_j, x_{j-1})\prod_{j=1}^{L-1}dx_j, \tag{3.40}$$

where

$$\begin{aligned}&K(\Delta'_j; \hbar, \tau_j, \tau_{j-1}, x_j, x_{j-1})\\&=\prod_{k=1}^{L_j}\left(\frac{1}{2\pi i\hbar\Delta\tau_{j,k}}\right)^{d/2}\int_{R^{d(L_j-1)}}\exp\left(i\hbar^{-1}\sum_{k=1}^{L_j}S(\tau_{j,k}, \tau_{j,k-1}, x_{j,k}, x_{j,k-1})\right)\prod_{k=1}^{L_j-1}dx_{j,k}\\&=\left(\frac{1}{2\pi i\hbar\Delta\tau_j}\right)^{d/2}\exp(i\hbar^{-1}S(\tau_j, \tau_{j-1}, x_j, x_{j-1}))\\&\quad\times D(\Delta'_j; \tau_j, \tau_{j-1}, x_j, x_{j-1})^{-1/2}(1+r(\Delta'_j; \hbar, \tau_j, \tau_{j-1}, x_j, x_{j-1})),\end{aligned} \tag{3.41}$$

with $x_{j,0}=x_{j-1}$ and $x_{j,L_j}=x_j$. And for any α and β there exists a positive constant $C_{\alpha\beta}$ such that

$$|\partial_x^\alpha\partial_y^\beta r(\Delta'_j; \hbar, \tau_j, \tau_{j-1}, x_j, x_{j-1})| \leqq C_{\alpha\beta}\hbar\Delta\tau_j^2. \tag{3.42}$$

So we have

$$\begin{aligned}&K(\Delta'; \hbar, t, s, x, y)-K(\Delta; \hbar, t, s, x, y)\\&\quad=\prod_{j=1}^{L}\left(\frac{1}{2\pi i\hbar\Delta\tau_j}\right)^{d/2}\int_{R^{d(L-1)}}\exp\left(i\hbar^{-1}\sum_{j=1}^{L}S(\tau_j, \tau_{j-1}, x_j, x_{j-1})\right)\\&\quad\quad\times b(\Delta', \Delta; \hbar, x_L, \cdots, x_0)\prod_{j=1}^{L-1}dx_j,\end{aligned} \tag{3.43}$$

where

$$
\begin{aligned}
& b(\Delta', \Delta; \hbar, x_L, \cdots, x_0) \\
& = \prod_{j=1}^{L} [D(\Delta'_j; \tau_j, \tau_{j-1}, x_j, x_{j-1})^{-1/2} \\
& \quad \times (1+r(\Delta'_j; \hbar, \tau_j, \tau_{j-1}, x_j, x_{j-1}))]-1 . \qquad (3.44)
\end{aligned}
$$

We can show that for any α and β there exists a positive constant $C_{\alpha\beta}$ such that

$$
|\partial_{x_L}^{\alpha}\partial_{x_0}^{\beta} b(\Delta', \Delta; \hbar, x_L, x_{L-1}^*, \cdots, x_1^*, x_0)| \leqq C_{\alpha\beta}|\Delta|. \qquad (3.45)
$$

Moreover, recalling here [**11**, Theorem 1], we can show, as we do below, that $b(\Delta', \Delta; \hbar, x_L, \cdots, x_0)$ satisfies the assumption (H.2) there. With the same notation as used in (H.2) we can put

$$
A_K = C_K|\Delta| \text{ and } X_K = [2(1+C_K|\Delta|)]^2. \qquad (3.46)
$$

Suppose for the moment that (3.45) and (3.46) are valid, which will be proved in Lemma 3.8 below. Then we can apply [**11**, Theorem 1] to the right-hand side of (3.43) and we have

$$
\begin{aligned}
& K(\Delta'; \hbar, t, s, x, y)-K(\Delta; \hbar, t, s, x, y) \\
& = \left(\frac{1}{2\pi i\hbar(t-s)}\right)^{d/2} \exp(i\hbar^{-1}S(t, s, x, y)) \\
& \quad \times D(\Delta; t, s, x, y)^{-1/2} p(\Delta', \Delta; \hbar, t, s, x, y), \qquad (3.47)
\end{aligned}
$$

where for any α and β there exists a positive constant $C_{\alpha\beta}$ such that

$$
|\partial_x^{\alpha}\partial_y^{\beta} p(\Delta', \Delta; \hbar, t, s, x, y)| \leqq C_{\alpha\beta}|\Delta|. \qquad (3.48)
$$

On the other hand, we have by (2.16)

$$
\begin{aligned}
& p(\Delta', \Delta; \hbar, t, s, x, y) \\
& = \left(\frac{D(\Delta'; t, s, x, y)}{D(\Delta; t, s, x, y)}\right)^{-1/2} (1+r(\Delta'; \hbar, t, s, x, y)) \\
& \quad -(1+r(\Delta; \hbar, t, s, x, y)). \qquad (3.49)
\end{aligned}
$$

Therefore (3.39) follows from Theorem 3.5 and (3.49). □

In the next lemma we check (3.45) and (3.46) which remain to be proved in the proof of Theorem 3.7 above. For two integers $L \geqq k > l \geqq 0$ and a function $a(x_L, \cdots, x_0)$, we use the notation:

$$
a(x_L, \cdots, \overline{x_k, x_l}, \cdots, x_0) = a(x_L, \cdots, x_k, x_{k-1}^*, \cdots, x_{l+1}^*, x_l, \cdots, x_0).
$$

LEMMA 3.8. (i) *For any sequence of positive integers with* $j_0=0<j_1-1<j_1<j_2-1<\cdots<j_n \leqq j_{n+1}-1=L$, $n=1, \cdots, L-1$, *if* $|\alpha_j| \leqq K$, $j=0, j_1-1, j_1, \cdots, j_n-1, j_n, L$, *there exists a positive constant* C_K *such that*

$$\left|\partial_{x_0}^{\alpha_0}\partial_{x_L}^{\alpha_L}\prod_{u=1}^{n}\partial_{x_{j_u-1}}^{\alpha_{j_u-1}}\partial_{x_{j_u}}^{\alpha_{j_u}} b(\Delta', \Delta; \hbar, \overline{x_L, x_{j_n}}, \overline{x_{j_n-1}, x_{j_{n-1}}}, \cdots, \overline{x_{j_1-1}, x_0})\right|$$
$$\leqq C_K|\Delta|[2(1+C_K|\Delta|)]^{2n}. \tag{3.50a}$$

If $j_n=L$, *we read the above inequality as*

$$\left|\partial_{x_0}^{\alpha_0}\prod_{u=1}^{n}\partial_{x_{j_u-1}}^{\alpha_{j_u-1}}\partial_{x_{j_u}}^{\alpha_{j_u}} b(\Delta', \Delta; \hbar, x_L, \overline{x_{j_n-1}, x_{j_{n-1}}}, \cdots \overline{x_{j_1-1}, x_0})\right|$$
$$\leqq C_K|\Delta|[2(1+C_K|\Delta|)]^{2n}. \tag{3.50b}$$

(ii) (3.45) *holds: for any* α *and* β *there exists a positive constant* $C_{\alpha\beta}$ *such that*

$$|\partial_{x_L}^{\alpha}\partial_{x_0}^{\beta} b(\Delta', \Delta; \hbar, \overline{x_L, x_0})| \leqq C_{\alpha\beta}|\Delta|. \tag{3.51}$$

PROOF. We prove only (3.50a). (3.50b) and (3.51) can be proved in a similar way. We can write

$$\begin{aligned} & b(\Delta', \Delta; \hbar, \overline{x_L, x_{j_n}}, \overline{x_{j_n-1}, x_{j_{n-1}}}, \cdots, \overline{x_{j_1-1}, x_0}) \\ &= \prod_{u=1}^{n+1}(1+|\Delta|d_u(x_{j_u-1}, x_{j_{u-1}}))\prod_{u=1}^{n}(1+|\Delta|d'_u(x_{j_u}, x_{j_u-1})) \\ &\quad \times \prod_{j=1}^{L}(1+\Delta\tau_j^2 r'_j(x_j^*, x_{j-1}^*))-1 \end{aligned} \tag{3.52}$$

with $x^*_{j_{u-1}}=x_{j_{u-1}}$ and $x^*_{j_u-1}=x_{j_u-1}$, $u=1, \cdots, n+1$. Here we put

$$1+|\Delta|d_u(x_{j_u-1}, x_{j_{u-1}}) = \prod_{m=j_{u-1}+1}^{j_u-1} D(\Delta'_m; \tau_m, \tau_{m-1}, x_m^*, x_{m-1}^*)^{-1/2}, \tag{3.53}$$

$$1+|\Delta|d'_u(x_{j_u}, x_{j_u-1}) = D(\Delta'_{j_u}; \tau_{j_u}, \tau_{j_u-1}, x_{j_u}, x_{j_u-1})^{-1/2} \tag{3.54}$$

and

$$1+\Delta\tau_j^2 r'_j(x_j^*, x_{j-1}^*) = 1+r(\Delta'_j; \hbar, \tau_j, \tau_{j-1}, x_j^*, x_{j-1}^*), \tag{3.55}$$

with $x^*_{j_{u-1}}=x_{j_{u-1}}$ and $x^*_{j_u-1}=x_{j_u-1}$, $u=1, \cdots, n+1$. So (3.32, 22) and (3.42) yield that d_u, d'_u and r'_j are in $\mathcal{B}(\boldsymbol{R}^d\times\boldsymbol{R}^d)$. Apply Lemma A (A.1) in Appendix to the right-hand side of (3.52), then we obtain that if $|\alpha_j|\leqq K$, $j=0, j_1-1, j_1, \cdots, j_n-1, j_n, L$, there exist positive constants C_K, C'_K, C''_K such that

$$\begin{aligned} &\left|\partial_{x_0}^{\alpha_0}\partial_{x_L}^{\alpha_L}\prod_{u=1}^{n}\partial_{x_{j_u-1}}^{\alpha_{j_u-1}}\partial_{x_{j_u}}^{\alpha_{j_u}} b(\Delta', \Delta; \hbar, \overline{x_L, x_{j_n}}, \overline{x_{j_n-1}, x_{j_{n-1}}}, \cdots, \overline{x_{j_1-1}, x_0})\right| \\ &\leqq (1+C_K|\Delta|)^{2n+1}\prod_{j=1}^{L}(1+C_K\Delta\tau_j^2)-1 \\ &= \Big(\prod_{j=1}^{L}(1+C_K\Delta\tau_j^2)-1\Big)(1+C_K|\Delta|)^{2n+1}+(1+C_K|\Delta|)^{2n+1}-1 \end{aligned}$$

$$\leqq C'_K|\Delta||T_L|(1+C_K|\Delta|)^{2n}+C_K|\Delta|(2n+1)(1+C_K|\Delta|)^{2n}$$
$$\leqq C''_K|\Delta|[2(1+C_K|\Delta|)]^{2n},$$

where in the second inequality we have used $(1+a)^{2n+1}-1\leqq a(2n+1)(1+a)^{2n}$, $a\geqq 0$. Therefore we have proved (3.50a). □

With Theorem 3.4, Corollary 3.6 and Theorem 3.7, we have completed the proof of Theorem 2.3.

4. Proof of Theorem 2.4.

In this section, to simplify notation we consider the case $d=2$. Let $q_\Delta(\tau)$ be a piecewise classical path given by (1.10) such as $q_\Delta(\tau_j)=x_j$, $j=0, \cdots, L$. Put $q_\Delta=q_\Delta^0+q_\Delta^1$ with $q_\Delta^0(\tau)=((\tau_j-\tau)/\Delta\tau_j)x_{j-1}+((\tau-\tau_{j-1})/\Delta\tau_j)x_j$, $\tau_{j-1}\leqq\tau\leqq\tau_j$, $j=1, \cdots, L$, and fix $x_0=y$ and $x_L=x$. Using the operator Y given by (2.27), we can write the entries of the matrix $W_\Delta^*=W_\Delta(X)|_{X=(x, x_{L-1}^*, \cdots, x_1^*, y)}$ as

$$\begin{aligned}&(W_{kl}(X))_{ij}|_{X=(x, x_{L-1}^*, \cdots, x_1^*, y)}\\ &=(\partial_{(x_j)_l}\partial_{(x_i)_k}\sum_{j'=1}^{L-1}\omega_{j'}(x_{j'}, x_{j'-1}))|_{X=(x, x_{L-1}^*, \cdots, x_1^*, y)}\\ &=(\partial_{(x_i)_k}q_\Delta^0, Y(\partial_{(x_j)_l}q_\Delta^0+\partial_{(x_j)_l}q_\Delta^{1*}))_{\mathcal{L}}\\ &=(W_{kl}^0)_{ij}+(W_{kl}^1)_{ij}, \quad k, l=1, 2, \ i, j=1, \cdots, L-1,\end{aligned} \tag{4.1}$$

where we put $\partial_{(x_j)_l}q_\Delta^{1*}=(\partial_{(x_j)_l}q_\Delta^1)|_{X=(x, x_{L-1}^*, \cdots, x_1^*, y)}$ and

$$(W_{kl}^0)_{ij}=(\partial_{(x_i)_k}q_\Delta^0, Y\partial_{(x_j)_l}q_\Delta^0)_{\mathcal{L}}, \quad (W_{kl}^1)_{ij}=(\partial_{(x_i)_k}q_\Delta^0, Y\partial_{(x_j)_l}q_\Delta^{1*})_{\mathcal{L}}. \tag{4.2}$$

So we write W_Δ^* as

$$W_\Delta^*=W_\Delta^0+W_\Delta^1=\begin{pmatrix}W_{11}^0 & W_{12}^0\\ W_{21}^0 & W_{22}^0\end{pmatrix}+\begin{pmatrix}W_{11}^1 & W_{12}^1\\ W_{21}^1 & W_{22}^1\end{pmatrix}.$$

Note that W_Δ^0 and W_Δ^1 are symmetric matrices. The next two propositions yield Theorem 2.4.

PROPOSITION 4.1. *For fixed* $x, y\in\boldsymbol{R}^2$,

$$\det(I+G_\Delta W_\Delta^*)=\det(I+G_\Delta W_\Delta^0)+o(1), \quad \textit{as } |\Delta|\to 0. \tag{4.3}$$

PROPOSITION 4.2. *For fixed* $x, y\in\boldsymbol{R}^2$,

$$\det(I+G_\Delta W_\Delta^0)\to \det{}_2(I+GY)\exp(\mathrm{Tr}\, GZ), \quad \textit{as } |\Delta|\to 0. \tag{4.4}$$

We prove first Proposition 4.2 and next Proposition 4.1.

PROOF OF PROPOSITION 4.2. Here we use a finite element method. Let P_Δ be an orthogonal finite rank projection in $\mathscr{H}$ onto the linear hull of $\{\partial_{(x_j)_k} q_\Delta^0;\ j=1, \cdots, L-1,\ k=1, 2\}$, which is a set of piecewise linear paths. It is clear that P_Δ converges to the identity operator I strongly on $\mathscr{H}$ as $|\Delta|\to 0$.

Since $G_\Delta W_\Delta^0$ has the same eigenvalues as $P_\Delta GY$, it follows that

$$\det(I+G_\Delta W_\Delta^0)=\det(I+P_\Delta GY), \tag{4.5}$$

where det on the right-hand side of (4.5) means the Fredholm determinant. Since P_Δ is of finite rank so that $P_\Delta GY$ is trace class on $\mathscr{H}$ and since $\mathrm{Tr}(P_\Delta GB(d/d\tau))=0$ by the skew-symmetricity of B, we have

$$\begin{aligned}\det(I+P_\Delta GY)&=\det{}_2(I+P_\Delta GY)\exp(\mathrm{Tr}\,P_\Delta GY)\\&=\det{}_2(I+P_\Delta GY)\exp(\mathrm{Tr}\,P_\Delta GZ).\end{aligned} \tag{4.6}$$

Let $\|A\|_1=\mathrm{Tr}|A|$ for A in the trace class and $\|A\|_2=(\mathrm{Tr}(A^*A))^{1/2}$ for a Hilbert-Schmidt operator A. By the inequalities in [**10**, Theorem 9.2 (c) and Theorem 3.1], we have

$$\begin{aligned}&|\det{}_2(I+P_\Delta GY)-\det{}_2(I+GY)|\\&\leqq\|(P_\Delta-I)GY\|_2\exp[C(\|P_\Delta GY\|_2+\|GY\|_2+1)^2]\end{aligned} \tag{4.7}$$

and

$$|\mathrm{Tr}\,P_\Delta GZ-\mathrm{Tr}\,GZ|\leqq\|(P_\Delta-I)GZ\|_1. \tag{4.8}$$

By the arguments in [**10**, p. 42, Example 3], for the factors in (4.7, 8) we have

$$\|(P_\Delta-I)GY\|_2,\ \|(P_\Delta-I)GZ\|_1\to 0,\quad\text{as } |\Delta|\to 0. \tag{4.9}$$

Therefore by (4.5-4.9) we have proved Proposition 4.2. □

PROOF OF PROPOSITION 4.1. We begin with three lemmas. To simplify the formulas, in Lemmas 4.3 and 4.4 and their proofs we write

$$\int_{\tau_{j-1}}^{\tau_{j+1}}|\partial B|\,|v|\,d\tau=\int_{\tau_{j-1}}^{\tau_{j+1}}\sum_{|\gamma|=1}|(\partial_x^\gamma B)(\tau, q(\tau))|\,|v(\tau)|\,d\tau.$$

LEMMA 4.3. *There exists a positive constant C independent of Δ such that for any $k, l=1, 2$,*

$$|(W_{kl}^0)_{j-1,j}|,\quad |(W_{kl}^0)_{j,j}|,\quad |(W_{kl}^0)_{j+1,j}|\leqq C, \tag{4.10}$$

$$\begin{aligned}&|(W_{kl}^0)_{j-1,j}+(W_{kl}^0)_{j,j}+(W_{kl}^0)_{j+1,j}|\\&\leqq C\Big(\int_{\tau_{j-1}}^{\tau_{j+1}}|\partial B|\,|v|\,d\tau+(\Delta\tau_j+\Delta\tau_{j+1})\Big),\end{aligned} \tag{4.11}$$

$$\begin{aligned}&|(W_{kl}^1)_{j-1,j}|\leqq C\Delta\tau_j,\quad |(W_{kl}^1)_{j,j}|\leqq C(\Delta\tau_j+\Delta\tau_{j+1}),\\&|(W_{kl}^1)_{j+1,j}|\leqq C\Delta\tau_{j+1},\end{aligned} \tag{4.12}$$

and

$$
\begin{aligned}
&|(W_{kl}^{1})_{j-1,j}+(W_{kl}^{1})_{j,j}+(W_{kl}^{1})_{j+1,j}| \\
&\leqq C(\Delta\tau_{j}+\Delta\tau_{j+1})\Big(\int_{\tau_{j-1}}^{\tau_{j+1}}|\partial B||v|d\tau+(\Delta\tau_{j}+\Delta\tau_{j+1})\Big). \qquad (4.13)
\end{aligned}
$$

PROOF. First we show (4.12) and (4.13). By the symmetricity of Y, Lemma 3.1 (i) and (3.2), we have

$$
\begin{aligned}
|(W_{kl}^{1})_{j-1,j}| &= |(\partial_{(x_{j-1})_k}q_{\Delta}^{0},\ Y\partial_{(x_j)_l}q_{\Delta}^{1*})_{\mathcal{L}}| = |(Y\partial_{(x_{j-1})_k}q_{\Delta}^{0},\ \partial_{(x_j)_l}q_{\Delta}^{1*})_{\mathcal{L}}| \\
&\leqq \|Y\partial_{(x_{j-1})_k}q_{\Delta}^{0}\|_{L^1(\tau_{j-1},\tau_j)}\|\partial_{(x_j)_l}q_{\Delta}^{1*}\|_{L^\infty(\tau_{j-1},\tau_j)} \leqq C\Delta\tau_j,
\end{aligned}
$$

since $\operatorname{supp}\partial_{(x_{j-1})_k}q_{\Delta}^{0}\cap\operatorname{supp}\partial_{(x_j)_l}q_{\Delta}^{1*}$ is included in $[\tau_{j-1},\ \tau_j]$. The second and third inequalities in (4.12) can be shown analogously. (4.13) follows from

$$
\begin{aligned}
&|(W_{kl}^{1})_{j-1,j}+(W_{kl}^{1})_{j,j}+(W_{kl}^{1})_{j+1,j}| \\
&= |((\partial_{(x_{j-1})_k}+\partial_{(x_j)_k}+\partial_{(x_{j+1})_k})q_{\Delta}^{0},\ Y\partial_{(x_j)_l}q_{\Delta}^{1*})_{\mathcal{L}}| \\
&= |(Y(\partial_{(x_{j-1})_k}+\partial_{(x_j)_k}+\partial_{(x_{j+1})_k})q_{\Delta}^{0},\ \partial_{(x_j)_l}q_{\Delta}^{1*})_{\mathcal{L}}| \\
&\leqq \|Y(\partial_{(x_{j-1})_k}+\partial_{(x_j)_k}+\partial_{(x_{j+1})_k})q_{\Delta}^{0}\|_{L^1(\tau_{j-1},\tau_{j+1})}\|\partial_{(x_j)_l}q_{\Delta}^{1*}\|_{L^\infty} \\
&\leqq C(\Delta\tau_{j}+\Delta\tau_{j+1})\Big(\int_{\tau_{j-1}}^{\tau_{j+1}}|\partial B||v|d\tau+(\Delta\tau_{j}+\Delta\tau_{j+1})\Big),
\end{aligned}
$$

since the support of $\partial_{(x_j)_l}q_{\Delta}^{1*}$ is in $[\tau_{j-1},\ \tau_{j+1}]$ and $(\partial_{(x_{j-1})_k}+\partial_{(x_j)_k}+\partial_{(x_{j+1})_k})(q_{\Delta}^{0})_m=\delta_{k,m}$ on $[\tau_{j-1},\ \tau_{j+1}]$.

We can show (4.10) and (4.11), replacing $\partial_{(x_j)_l}q_{\Delta}^{1*}$ by $\partial_{(x_j)_l}q_{\Delta}^{0}$ in the arguments above and using $\|\partial_{(x_j)_l}q_{\Delta}^{0}\|_{L^\infty}\leqq 1$. □

For an $N\times N$ matrix $A=(a_{ij})$, put $\|A\|_{2,R^N}=(\sum_{i,j=1}^{N}|a_{ij}|^2)^{1/2}$.

LEMMA 4.4. *There exists a positive constant C independent of Δ such that*

$$\|G_{\Delta}W_{\Delta}^{1}\|_{2,R^{2(L-1)}}^{2} \leqq CL|\Delta|^2, \qquad (4.14)$$

$$|\operatorname{Tr}G_{\Delta}W_{\Delta}^{1}| \leqq C|\Delta| \qquad (4.15)$$

and

$$|\operatorname{Tr}G_{\Delta}W_{\Delta}^{0}| \leqq C. \qquad (4.16)$$

PROOF. For the $(i,\ j)$ entry of $g_{\Delta}W_{kl}^{1}$, we have

$$
\begin{aligned}
(g_{\Delta}W_{kl}^{1})_{ij} &= (g_{\Delta})_{i,j-1}(W_{kl}^{1})_{j-1,j}+(g_{\Delta})_{i,j}(W_{kl}^{1})_{j,j}+(g_{\Delta})_{i,j+1}(W_{kl}^{1})_{j+1,j} \\
&= ((g_{\Delta})_{i,j-1}-(g_{\Delta})_{i,j})(W_{kl}^{1})_{j-1,j} \\
&\quad +(g_{\Delta})_{i,j}((W_{kl}^{1})_{j-1,j}+(W_{kl}^{1})_{j,j}+(W_{kl}^{1})_{j+1,j}) \\
&\quad +((g_{\Delta})_{i,j+1}-(g_{\Delta})_{i,j})(W_{kl}^{1})_{j,j+1}. \qquad (4.17)
\end{aligned}
$$

Using $|(g_\Delta)_{i,j-1}-(g_\Delta)_{i,j}|\leqq\Delta\tau_j$, $|(g_\Delta)_{i,j+1}-(g_\Delta)_{i,j}|\leqq\Delta\tau_{j+1}$ and Lemma 4.3, we obtain

$$|(g_\Delta W^1_{kl})_{ij}| \leqq C(\Delta\tau_j+\Delta\tau_{j+1})\Big(\int_{\tau_{j-1}}^{\tau_{j+1}}|\partial B|\,|v|\,d\tau+(\Delta\tau_j+\Delta\tau_{j+1})\Big). \tag{4.18}$$

Hence by Lemma 3.1 (i), we have

$$\begin{aligned}\|G_\Delta W^1_\Delta\|^2_{2,R^{2(L-1)}} &= \sum_{\substack{1\leq i,j\leq L-1\\ k,l=1,2}}|(g_\Delta W^1_{kl})_{ij}|^2\\ &\leqq 2^2CL|\Delta|^2\sum_{j=1}^{L-1}\Big(\int_{\tau_{j-1}}^{\tau_{j+1}}|\partial B|\,|v|\,d\tau+(\Delta\tau_j+\Delta\tau_{j+1})\Big)\\ &\leqq CL|\Delta|^2\end{aligned}$$

and

$$\begin{aligned}|\operatorname{Tr} G_\Delta W^1_\Delta| &\leqq \sum_{\substack{1\leq j\leq L-1\\ k=1,2}}|(g_\Delta W^1_{kk})_{jj}|\\ &\leqq 2C|\Delta|\sum_{j=1}^{L-1}\Big(\int_{\tau_{j-1}}^{\tau_{j+1}}|\partial B|\,|v|\,d\tau+(\Delta\tau_j+\Delta\tau_{j+1})\Big)\\ &\leqq C|\Delta|.\end{aligned}$$

Therefore we have shown (4.14) and (4.15). Similarly to the above, we have by Lemma 4.3

$$|(g_\Delta W^0_{kl})_{ij}| \leqq C\Big(\int_{\tau_{j-1}}^{\tau_{j+1}}|\partial B|\,|v|\,d\tau+(\Delta\tau_j+\Delta\tau_{j+1})\Big), \tag{4.19}$$

and so (4.16). □

Recall the Hilbert-Schmidt norm $\|\cdot\|_2$ and the orthogonal projection P_Δ defined in the proof of Proposition 4.2.

LEMMA 4.5. *We have*

$$\|\sqrt{G_\Delta}W^0_\Delta\sqrt{G_\Delta}\|_{2,R^{2(L-1)}} \leqq \|P_\Delta GY\|_2.$$

PROOF. Let $\{\phi_j\}_{j=1}^\infty$ be an ONB in $\mathcal{H}$ such that $\phi_1,\cdots,\phi_{2(L-1)}\in\operatorname{Range}(P_\Delta)$ and

$$\sqrt{G_\Delta}\begin{pmatrix}\partial_{(x_1)_1}q^0_\Delta\\ \partial_{(x_2)_1}q^0_\Delta\\ \vdots\\ \partial_{(x_{L-1})_2}q^0_\Delta\end{pmatrix}=\begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_{2(L-1)}\end{pmatrix}. \tag{4.20}$$

Then by (4.20) we have

$$\begin{aligned}\|P_\Delta GY\|_2^2 &= \sum_{i,j=1}^{\infty}|(\phi_i, P_\Delta GY\phi_j)_{\mathcal{H}}|^2\\ &\geqq \sum_{i,j=1}^{2(L-1)}|(\phi_i, Y\phi_j)_{\mathcal{L}}|^2 = \|\sqrt{G_\Delta}W^0_\Delta\sqrt{G_\Delta}\|^2_{2,R^{2(L-1)}}.\end{aligned}$$

□

REMARK. If (4.20) is satisfied, then $(\phi_i, \phi_j)_{\mathscr{H}}=\delta_{i,j}$, because $H_\Delta=((\partial_{(x_i)_k} q_\Delta^0, \partial_{(x_j)_l} q_\Delta^0)_{\mathscr{H}})$.

We are now in a position to show Proposition 4.1. For a division Δ, let $\Delta' : s<\tau_1'<\cdots<\tau_{L'-1}'<t$ be a refinement of Δ such that $L'|\Delta'|\leqq 2|t-s|$. Put $A_\Delta=\det(I+G_\Delta W_\Delta^*)$ and $B_\Delta=\det(I+G_\Delta W_\Delta^0)$. Then we have

$$\begin{aligned}
&|\det(I+G_\Delta W_\Delta^*)-\det(I+G_\Delta W_\Delta^0)| \\
&\leqq |A_\Delta - A_{\Delta'}|+|A_{\Delta'}-B_{\Delta'}|+|B_{\Delta'}-B_\Delta|. \qquad (4.21)
\end{aligned}$$

We have to estimate the three terms on the right-hand side of (4.21). First, we have obtained in Theorem 2.3 that $|A_\Delta-A_{\Delta'}|\leqq C|\Delta|$. Second, for the third term, we have shown in Proposition 4.2 that B_Δ is convergent. Third, noting

$$\det(I+G_\Delta W_\Delta^*) = \det(I+\sqrt{G_\Delta}W_\Delta^*\sqrt{G_\Delta}),$$

$$\det(I+G_\Delta W_\Delta^0) = \det(I+\sqrt{G_\Delta}W_\Delta^0\sqrt{G_\Delta})$$

and $\mathrm{Tr}\sqrt{G_\Delta}W_\Delta^0\sqrt{G_\Delta}=\mathrm{Tr}\, G_\Delta W_\Delta^0$, we have

$$\begin{aligned}
&|A_{\Delta'}-B_{\Delta'}| \\
&= |\det{}_2(I+\sqrt{G_{\Delta'}}W_{\Delta'}^*\sqrt{G_{\Delta'}})\exp(\mathrm{Tr}\, G_{\Delta'}W_{\Delta'}^*) \\
&\qquad -\det{}_2(I+\sqrt{G_{\Delta'}}W_{\Delta'}^0\sqrt{G_{\Delta'}})\exp(\mathrm{Tr}\, G_{\Delta'}W_{\Delta'}^0)| \\
&\leqq |\det{}_2(I+\sqrt{G_{\Delta'}}W_{\Delta'}^*\sqrt{G_{\Delta'}})-\det{}_2(I+\sqrt{G_{\Delta'}}W_{\Delta'}^0\sqrt{G_{\Delta'}})|\,|\exp(\mathrm{Tr}\, G_{\Delta'}W_{\Delta'}^*)| \\
&\quad +|\exp(\mathrm{Tr}\, G_{\Delta'}W_{\Delta'}^*)-\exp(\mathrm{Tr}\, G_{\Delta'}W_{\Delta'}^0)|\,|\det{}_2(I+\sqrt{G_{\Delta'}}W_{\Delta'}^0\sqrt{G_{\Delta'}})|. \qquad (4.22)
\end{aligned}$$

For the first term on the right-hand side of (4.22) we have

$$\begin{aligned}
&|\det{}_2(I+\sqrt{G_{\Delta'}}W_{\Delta'}^*\sqrt{G_{\Delta'}})-\det{}_2(I+\sqrt{G_{\Delta'}}W_{\Delta'}^0\sqrt{G_{\Delta'}})|\,|\exp(\mathrm{Tr}\, G_{\Delta'}W_{\Delta'}^*)| \\
&\leqq \|\sqrt{G_{\Delta'}}W_{\Delta'}^1\sqrt{G_{\Delta'}}\|_{2,R^{2(L-1)}}\exp[C(\|\sqrt{G_{\Delta'}}W_{\Delta'}^*\sqrt{G_{\Delta'}}\|_{2,R^{2(L-1)}} \\
&\qquad +\|\sqrt{G_{\Delta'}}W_{\Delta'}^0\sqrt{G_{\Delta'}}\|_{2,R^{2(L-1)}}+1)^2]\,|\exp(\mathrm{Tr}\, G_{\Delta'}W_{\Delta'}^*)| \\
&\leqq \|G_{\Delta'}W_{\Delta'}^1\|_{2,R^{2(L-1)}}\exp[C(\|P_{\Delta'}GY\|_2^2+\|G_{\Delta'}W_{\Delta'}^1\|_{2,R^{2(L-1)}}^2+1)] \\
&\leqq CL'^{1/2}|\Delta'| \leqq C|t-s|^{1/2}|\Delta'|^{1/2}, \qquad (4.23)
\end{aligned}$$

where we have used an inequality in [**10**, Theorem 9.2 (c)] in the first step,

$$\|\sqrt{G_{\Delta'}}W_{\Delta'}^1\sqrt{G_{\Delta'}}\|_{2,R^{2(L-1)}} \leqq \|G_{\Delta'}W_{\Delta'}^1\|_{2,R^{2(L-1)}},$$

(see [**10**, Theorem 8.1 (8.4′)]) and Lemma 4.5 in the second step, and Lemma 4.4 in the third step. For the second term on the right-hand side of (4.22) we have

$$
\begin{aligned}
&|\exp(\mathrm{Tr}\, G_{\Delta'}W^{*}_{\Delta'})-\exp(\mathrm{Tr}\, G_{\Delta'}W^{0}_{\Delta'})|\,|\det{}_2(I+\sqrt{G_{\Delta'}}W^{0}_{\Delta'}\sqrt{G_{\Delta'}})| \\
\leqq\ & |\mathrm{Tr}\, G_{\Delta'}W^{*}_{\Delta'}-\mathrm{Tr}\, G_{\Delta'}W^{0}_{\Delta'}|\exp(2(|\mathrm{Tr}\, G_{\Delta'}W^{*}_{\Delta'}|+|\mathrm{Tr}\, G_{\Delta'}W^{0}_{\Delta'}|)) \\
& \times\exp(C\|\sqrt{G_{\Delta'}}W^{0}_{\Delta'}\sqrt{G_{\Delta'}}\|^2_{2,R^{2(L-1)}}) \\
\leqq\ & |\mathrm{Tr}\, C_{\Delta'}W^{1}_{\Delta'}|\exp(C(|\mathrm{Tr}\, G_{\Delta'}W^{0}_{\Delta'}|+|\mathrm{Tr}\, G_{\Delta'}W^{1}_{\Delta'}|+\|P_{\Delta'}GY\|_2^2)) \\
\leqq\ & C|\Delta'|,
\end{aligned}
$$

where we have used inequalities $|\det_2(I+A)|\leqq\exp(C\|A\|_2^2)$ (see [**10**, Theorem 9.2 (b)]) and $|e^a-e^b|\leqq|a-b|e^{2(|a|+|b|)}$ in the first step and Lemma 4.4 and Lemma 4.5 in the second and third steps. Therefore we have completed the proof of Proposition 4.1. □

5. Proof of Theorem 2.5: Fundamental solution.

In this section we prove that the limit

$$
\begin{aligned}
K(\hbar, t, s, x, y) &= \lim_{|\Delta|\to 0} K(\Delta\,; \hbar, t, s, x, y) \\
&= \Big(\frac{1}{2\pi i\hbar(t-s)}\Big)^{d/2}\exp(i\hbar^{-1}S(t, s, x, y))k(\hbar, t, s, x, y), \qquad (5.1)
\end{aligned}
$$

where $k(\hbar, t, s, x, y)$ is of the form given by (2.32), is the fundamental solution of the Cauchy problem of the Schrödinger equation (1.1). We begin with a proposition. Let δ be a positive constant such that Propositions 2.1, 2.2 and Theorem 2.3 hold if $|t-s|\leqq\delta$.

PROPOSITION 5.1. *Let* $|t-s|\leqq\delta$ *with* $t>s$. *Then we have*

$$
\Big[\frac{\hbar}{i}\partial_t+\frac{1}{2}\Big(\frac{\hbar}{i}\partial_x-A(t, x)\Big)^2+V(x)\Big]K(\hbar, t, s, x, y)=0, \qquad (5.2)
$$

$$
\Big[-\frac{\hbar}{i}\partial_s+\frac{1}{2}\Big(\frac{\hbar}{i}\partial_y-A(s, y)\Big)^2+V(y)\Big]K(\hbar, t, s, x, y)=0. \qquad (5.3)
$$

PROOF. We show only (5.2). Using the Hamilton-Jacobi equation

$$
\partial_t S(t, s, x, y)+\frac{1}{2}(\partial_x S(t, s, x, y)-A(t, x))^2+V(x)=0,
$$

we have

$$
\begin{aligned}
&\Big[\frac{\hbar}{i}\partial_t+\frac{1}{2}\Big(\frac{\hbar}{i}\partial_x-A(t, x)\Big)^2+V(x)\Big]K(\hbar, t, s, x, y) \\
&=\frac{\hbar}{i}\Big(\frac{1}{2\pi i\hbar(t-s)}\Big)^{d/2}\exp(i\hbar^{-1}S(t, s, x, y))h(\hbar, t, s, x, y),
\end{aligned}
$$

where

$$h(\hbar, t, s, x, y)$$
$$= \partial_t k(\hbar, t, s, x, y) + \frac{1}{2}\Big(\Delta_x S(t, s, x, y) - \frac{d}{t-s} - (\mathrm{div}\, A)(t, x)\Big) k(\hbar, t, s, x, y)$$
$$+ \partial_x k(\hbar, t, s, x, y)(\partial_x S(t, s, x, y) - A(t, x)) + \frac{\hbar}{2i} \Delta_x k(\hbar, t, s, x, y). \tag{5.4}$$

So we show that $h(\hbar, t, s, x, y)=0$. We take the limit of

$$\frac{k(\hbar, t+\varepsilon, s, x, y) - k(\hbar, t, s, x, y)}{\varepsilon}$$

as $\varepsilon \to 0$ to calculate $\partial_t k(\hbar, t, s, x, y)$. We may suppose $\varepsilon > 0$. The case $\varepsilon < 0$ is dealt with in the same way. We put $\tau = \varepsilon(t-s)/(\varepsilon+t-s)$ and

$$D(S(t+\varepsilon, t, x, z^*) + S(t, s, z^*, y))$$
$$= \tau^d \det \mathrm{Hess}_z(S(t+\varepsilon, t, x, z) + S(t, s, z, y))|_{z=z^*}.$$

By the stationary phase method [**6**, Lemma 3.5] or [**11**, Lemma 4.1], we have the equality:

$$\Big(\frac{1}{2\pi i \hbar \varepsilon}\Big)^{d/2} \Big(\frac{1}{2\pi i \hbar (t-s)}\Big)^{d/2} \int_{\boldsymbol{R}^d} \exp(i\hbar^{-1}(S(t+\varepsilon, t, x, z) + S(t, s, z, y)))$$
$$\times k(\hbar, t+\varepsilon, t, x, z) k(\hbar, t, s, z, y) dz$$
$$= \Big(\frac{1}{2\pi i \hbar (\varepsilon + t - s)}\Big)^{d/2} \exp(i\hbar^{-1} S(t+\varepsilon, s, x, y))$$
$$\times D(S(t+\varepsilon, t, x, z^*) + S(t, s, z^*, y))^{-1/2} b(x, y), \tag{5.5}$$

where

$$b(x, y) = k(\hbar, t+\varepsilon, t, x, z^*) k(\hbar, t, s, z^*, y)$$
$$- \frac{\hbar\tau}{2i} D(S(t+\varepsilon, t, x, z^*) + S(t, s, z^*, y))^{-1}$$
$$\times \Delta_z (k(\hbar, t+\varepsilon, t, x, z^*) k(\hbar, t, s, z^*, y)) + \tau^2 b'(x, y), \tag{5.6}$$

with some $b'(x, y)$ in $\mathcal{B}(\boldsymbol{R}^d \times \boldsymbol{R}^d)$. By the definition of $k(\hbar, t+\varepsilon, s, x, y)$, we have

$$k(\hbar, t+\varepsilon, s, x, y) = D(S(t+\varepsilon, t, x, z^*) + S(t, s, z^*, y))^{-1/2} b(x, y). \tag{5.7}$$

Since we have by (3.19),

$$D(S(t+\varepsilon, t, x, z^*) + S(t, s, z^*, y))^{-1/2}$$
$$= 1 - \frac{\tau}{2} \Delta_z (\omega(t+\varepsilon, t, x, z^*) + \omega(t, s, z^*, y)) + \tau^2 c'(x, y), \tag{5.8}$$

with some $c'(x, y)$ in $\mathcal{B}(\boldsymbol{R}^d \times \boldsymbol{R}^d)$, and by (2.23),

$$k(\hbar, t+\varepsilon, t, x, z) = 1-\frac{\varepsilon}{2}d(t+\varepsilon, t, x, z)+\varepsilon^2 r_1(x, z), \tag{5.9}$$

with some $r_1(x, z)$ in $\mathscr{B}(\boldsymbol{R}^d\times\boldsymbol{R}^d)$, it follows that

$$\begin{aligned}
&k(\hbar, t+\varepsilon, s, x, y)\\
&=\Big[1-\frac{\tau}{2}\Delta_z(\omega(t+\varepsilon, t, x, z^*)+\omega(t, s, z^*, y))+\tau^2 c'(x, y)\Big]\\
&\quad\times\Big[k(\hbar, t, s, z^*, y)-\frac{\hbar\tau}{2i}\Delta_z(k(\hbar, t+\varepsilon, t, x, z^*)k(\hbar, t, s, z^*, y))\\
&\qquad-\frac{\varepsilon}{2}d(t+\varepsilon, t, x, z^*)k(\hbar, t, s, z^*, y)+\tau^2 r_2(x, y)\Big]\\
&=k(\hbar, t, s, z^*, y)-\frac{\tau}{2}\Delta_z(\omega(t+\varepsilon, t, x, z^*)+\omega(t, s, z^*, y))k(\hbar, t, s, z^*, y)\\
&\quad-\frac{\hbar\tau}{2i}\Delta_z(k(\hbar, t+\varepsilon, t, x, z^*)k(\hbar, t, s, z^*, y))\\
&\quad-\frac{\varepsilon}{2}d(t+\varepsilon, t, x, z^*)k(\hbar, t, s, z^*, y)+\tau^2 r_3(x, y),
\end{aligned}\tag{5.10}$$

with some r_2, r_3 in $\mathscr{B}(\boldsymbol{R}^d\times\boldsymbol{R}^d)$. (5.10) yields that

$$\begin{aligned}
&\frac{k(\hbar, t+\varepsilon, s, x, y)-k(\hbar, t, s, x, y)}{\varepsilon}=\frac{k(\hbar, t, s, z^*, y)-k(\hbar, t, s, x, y)}{\varepsilon}\\
&-\frac{\tau}{2\varepsilon}\Delta_z(\omega(t+\varepsilon, t, x, z^*)+\omega(t, s, z^*, y))k(\hbar, t, s, z^*, y)\\
&-\frac{\hbar\tau}{2i\varepsilon}\Delta_z(k(\hbar, t+\varepsilon, t, x, z^*)k(\hbar, t, s, z^*, y))\\
&-\frac{1}{2}d(t+\varepsilon, t, x, z^*)k(\hbar, t, s, z^*, y)+\frac{\tau^2}{\varepsilon}r_3(x, y).
\end{aligned}\tag{5.11}$$

Let ε tend to zero in (5.11). The left-hand side of (5.11) converges to $\partial_t k(\hbar, t, s, x, y)$ as $\varepsilon\downarrow 0$. The first term on the right-hand side of (5.11)

$$\frac{k(\hbar, t, s, z^*, y)-k(\hbar, t, s, x, y)}{\varepsilon}=\frac{z^*-x}{\varepsilon}\int_0^1(\partial_x k)(\hbar, t, s, x+\theta(z^*-x), y)d\theta$$

converges to

$$-\partial_x k(\hbar, t, s, x, y)(\partial_x S(t, s, x, y)-A(t, x)) \tag{5.12}$$

since z^* tends to x and $(z^*-x)/\varepsilon$ tends to $-v(t)$ as $\varepsilon\downarrow 0$, and since

$$\partial_x S(t, s, x, y)=v(t)+A(t, x).$$

In the second term on the right-hand side of (5.11), the factor

$$\begin{aligned}&\Delta_z(\omega(t+\varepsilon, t, x, z^*)+\omega(t, s, z^*, y))\\&=(\Delta_z\omega)(t+\varepsilon, t, x, z^*)+(\operatorname{div} A)(t, z^*)+(\Delta_z\omega)(t, s, z^*, y)-(\operatorname{div} A)(t, z^*)\end{aligned}\tag{5.13}$$

converges to

$$\Delta_x S(t, s, x, y)-\frac{d}{t-s}-(\operatorname{div} A)(t, x), \quad \text{as } \varepsilon\downarrow 0, \tag{5.14}$$

since z^* tends to x, and we have

$$|(\Delta_z\omega)(t+\varepsilon, t, x, z^*)+(\operatorname{div} A)(t, z^*)| \leqq C(\varepsilon+|x-z^*|)$$

(see Lemma 3.3 (i)) and

$$\Delta_z S(t, s, z, y)=\frac{d}{t-s}+\Delta_z\omega(t, s, z, y).$$

Hence the second term on the right-hand side of (5.11) converges to

$$-\frac{1}{2}\Big(\Delta_x S(t, s, x, y)-\frac{d}{t-s}-(\operatorname{div} A)(t, x)\Big)k(\hbar, t, s, x, y). \tag{5.15}$$

The third term on the right-hand side of (5.11) converges to

$$-\frac{\hbar}{2i}\Delta_x k(\hbar, t, s, x, y), \tag{5.16}$$

by (5.9). For the forth term we have

$$d(t+\varepsilon, t, x, z^*)k(\hbar, t, s, z^*, y)\to 0 \quad \text{as } \varepsilon\downarrow 0, \tag{5.17}$$

since we have, by (2.21)

$$|d(t+\varepsilon, t, x, z^*)| \leqq C(\varepsilon+|x-z^*|).$$

The last term converges to zero, since τ^2/ε tends to 0.

Therefore (5.12, 15, 16) and (5.17) yield that

$$\begin{aligned}&\partial_t k(\hbar, t, s, x, y)\\&=-\partial_x k(\hbar, t, s, x, y)(\partial_x S(t, s, x, y)-A(t, x))\\&\quad-\frac{1}{2}\Big(\Delta_x S(t, s, x, y)-\frac{d}{t-s}-(\operatorname{div} A)(t, x)\Big)k(\hbar, t, s, x, y)\\&\quad-\frac{\hbar}{2i}\Delta_x k(\hbar, t, s, x, y).\end{aligned}\tag{5.18}$$

So we have proved $h(\hbar, t, s, x, y)=0$ and so (5.2). (5.3) can be proved similarly to the above. □

Let $J=[a, b]$ be an interval with $|a-b|\leqq\delta$. For $t, s\in J$ with $t>s$ we introduce the operators

$$K(\hbar, t, s)f(x)=\int_{\boldsymbol{R}^d}K(\hbar, t, s, x, y)f(y)dy \tag{5.19}$$

and

$$K(\hbar, s, t)f(x)=\int_{\boldsymbol{R}^d}\overline{K(\hbar, t, s, y, x)}f(y)dy, \quad f\in C_0^\infty(\boldsymbol{R}^d). \tag{5.20}$$

We apply [**1**, Theorem] to (5.19, 20) and see that there exists a constant $C>0$ independent of $t, s\in J$ such that

$$\|K(\hbar, t, s)f\|_{L^2(\boldsymbol{R}^d)}\leqq C\|f\|_{L^2(\boldsymbol{R}^d)}. \tag{5.21}$$

Let $s, r, t\in J$ with $s<r<t$. Then we have

$$K(\hbar, t, r)K(\hbar, r, s)=K(\hbar, t, s). \tag{5.22}$$

So we use (5.22) to extend the definition of $K(\hbar, t, s)$ for any $t, s\in \boldsymbol{R}$ by (5.22). The next theorem shows Theorem 2.5.

THEOREM 5.2. *Let $r, s, t\in\boldsymbol{R}$.*

(i)
$$\operatorname*{s-lim}_{t\to s}K(\hbar, t, s)=I, \quad \textit{in } L^2(\boldsymbol{R}^d). \tag{5.23}$$

(ii) *$K(\hbar, t, s)$ is unitary on $L^2(\boldsymbol{R}^d)$ and*

$$K(\hbar, t, r)K(\hbar, r, s)=K(\hbar, t, s). \tag{5.24}$$

(iii) *If $f\in C_0^\infty(\boldsymbol{R}^d)$, then $K(\hbar, t, s)f$ is strongly differentiable in t and s and we have*

$$i\hbar\partial_t K(\hbar, t, s)f=H(\hbar, t)K(\hbar, t, s)f, \quad \textit{in } L^2(\boldsymbol{R}^d). \tag{5.25}$$

PROOF. We write the $L^2(\boldsymbol{R}^d)$-norm as $\|\cdot\|$. (i). If $f\in C_0^\infty(\boldsymbol{R}^d)$, we have $\|K(\hbar, t, s)f-f\|\to 0$ as $t\to s$, by the stationary phase method. So (i) follows from (5.21).

(iii). Let $f\in C_0^\infty(\boldsymbol{R}^d)$. By Proposition 5.1, for any $x\in\boldsymbol{R}^d$ we have

$$i\hbar\partial_t K(\hbar, t, s)f(x)=\left[\frac{1}{2}\Big(\frac{\hbar}{i}\partial_x-A(t, x)\Big)^2+V(x)\right]K(\hbar, t, s)f(x).$$

Moreover, since the right-hand side of this is in $L^2(\boldsymbol{R}^d)$, it follows that

$$i\hbar(K(\hbar, t', s)f-K(\hbar, t, s)f)=\int_t^{t'}H(\hbar, r)K(\hbar, r, s)f\,dr$$

as a Bochner integral in $L^2(\boldsymbol{R}^d)$. Therefore we have proved (iii).

(ii). By (iii) and the symmetry property of $H(\hbar, t)$, we have $\|K(\hbar, t, s)f\| = \|f\|$. And similarly we have $\|K^*(\hbar, t, s)f\| = \|f\|$, where $K^*(\hbar, t, s)$ is the adjoint of $K(\hbar, t, s)$. These yield that Range$K(\hbar, t, s) = L^2(\boldsymbol{R}^d)$ and so $K(\hbar, t, s)$ is unitary. Hence we have

$$K(\hbar, t, s)^{-1} = K^*(\hbar, t, s) = K(\hbar, s, t). \tag{5.26}$$

By (5.22) and (5.26), we have (5.23) for any $t, r, s \in \boldsymbol{R}$. □

Appendix.

The next lemma is a simplified version of [8, Lemma 2.14].

LEMMA A. *For a positive integer L, we put $x=(x_1, \cdots, x_L) \in \boldsymbol{R}^L$. Let J, d and K be positive integers. Let $f_j(x_{j_1}, \cdots, x_{j_d})$, $j=1, \cdots, J$, be $C^\infty(\boldsymbol{R}^L)$-functions. Assume that there exist positive functions $\varepsilon_j(x)$, $j=1, \cdots, J$, such that for any $j=1, \cdots, J$,*

$$\Big| \prod_{u=1}^{d} \partial_{x_{j_u}}^{\alpha_{j_u}} f_j(x_{j_1}, \cdots, x_{j_d}) \Big| \leqq \varepsilon_j(x),$$

if $|\alpha_{j_u}| \leqq K$, $u=1, \cdots, d$. Then there exist positive constants C_K and C'_K such that if $|\alpha_l| \leqq K$, $l=1, \cdots, L$, then

$$\Big| \prod_{l=1}^{L} \partial_{x_l}^{\alpha_l} \Big[\prod_{j=1}^{J} (1+f_j(x_{j_1}, \cdots, x_{j_d}))-1 \Big] \Big| \leqq \prod_{j=1}^{J} (1+C_K \varepsilon_j(x))-1 \tag{A.1}$$

and

$$\Big| \prod_{l=1}^{L} \partial_{x_l}^{\alpha_l} \Big[\prod_{j=1}^{J} (1+f_j(x_{j_1}, \cdots, x_{j_d}))-1-\sum_{j=1}^{J} f_j(x_{j_1}, \cdots, x_{j_d}) \Big] \Big|$$
$$\leqq \Big(\sum_{j=1}^{J} \varepsilon_j(x) \Big)^2 \exp\Big[C'_K \Big(\sum_{j=1}^{J} \varepsilon_j(x)+1 \Big) \Big]. \tag{A.2}$$

The constants C_K and C'_K are independent of L and J but dependent on K and d.

PROOF. The proof can be done by induction on J. □

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