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Linear differential equations with rational coefficients

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1. Introduction and results.

We consider the nth order linear differential equation

$$w^{(n)}+r_{n-2}(z)w^{(n-2)}+\cdots+r_0(z)w=0 \tag{1}$$

with rational coefficients. Unlike the case of polynomial coefficients, we know, Eq. (1) may have some solutions multivalued in the complex plane. In this paper, however, we always assume that each solution of Eq. (1) is single-valued, meromorphic in the complex plane. Then it is clear to see that each solution of Eq. (1) has only a finite number of poles, so that its Valiron deficient at ∞ is equal to 1.

Now let us introduce some notations. For a system of rays

$$D=\bigcup_{j=1}^{m}\{z\,|\arg z=\theta_j\},\ 0\leqq\theta_1<\cdots<\theta_m<\theta_{m+1}=\theta_1+2\pi, \tag{2}$$

we define

$$\omega(D)=\max\left\{\frac{\pi}{\theta_{j+1}-\theta_j}\,|1\leqq j\leqq m\right\}$$

and

$$G(D,\varepsilon)=\boldsymbol{C}\setminus\bigcup_{j=1}^{m}\{z\,|\,|\arg z-\theta_j|<\varepsilon\}.$$

Let $f(z)$ be a function meromorphic in the complex plane. We shall say that the zeros of $f(z)$ are attracted to D, provided that for any $\varepsilon>0$

$$n\left(r,\,G(D,\varepsilon),\,\frac{1}{f}\right)=o(T(r,f)), \tag{3}$$

as $r\to+\infty$, where $n(r, G(D,\varepsilon), 1/f)$ is the number of zeros of $f(z)$ lying in $G(D,\varepsilon)\cap\{|z|<r\}$. And we always denote the order and the lower order of $f(z)$ by $\lambda(f)$ and $\rho(f)$, respectively. We assume that the reader is familiar with Nevanlinna theory of meromorphic functions and standard notations.

THEOREM 1. *Assume that Eq.* (1) *is given with rational coefficients* $r_j(z)$. *Suppose that there exists a fundamental set* (*FS*) $\{w_1,\cdots,w_n\}$ *of Eq.* (1) *with the* *property that the zeros of each* w_j *are attracted to the rays* (2). *Then either*

$$\lambda := \max\{\lambda(w_j) \mid 1 \leqq j \leqq n\} \leqq \omega(D), \tag{4}$$

or else each w_j *has Borel Exceptional Value (BEV)* 0 *of order* λ, *namely, for some* $\varepsilon > 0$, $n(r, 1/f) < r^{\lambda-\varepsilon}$.

A conclusion immediately follows from Theorem 1.

COROLLARY. *Under the same assumption as in Theorem* 1, *in addition, suppose that* λ *is not an integer. Then* (4) *always holds.*

Moreover, for special case when the system of rays (2) is formed by the real axis, the following is also a further consequence of Theorem 1, which was listed in Brüggemann [**4**], but as pointed out in Zheng [**12**], Brüggemann's proof is incomplete.

THEOREM 2. *Let Eq.* (1) *with rational coefficients* $r_j(z)$ *be given. Assume that there exists a FS* $\{w_1, \cdots, w_n\}$ *such that each* w_j *has the zeros to be attracted to the real axis and at least one* w_j *with maximal order* λ *does not have BEV* 0. *Then*

$$r_j(z) = a_j + O\left(\frac{1}{z}\right), \quad (|z| \to +\infty, \ a_j \in \boldsymbol{C})$$

and either there is at least one j *such that* $a_j \neq 0$ *or for each* j,

$$\lambda(w_j) = \frac{1}{2}. \tag{5}$$

REMARK. 1) Brüggemann [**3**] and Steinmetz [**9**] independently proved Frank-Wittich conjecture that Eq. (1) with polynomial coefficients has no FS each element of which has BEV 0, unless its coefficients are constants. However, unfortunately, that is not the same story for the case of rational coefficients. Let $H(z)$ be a rational function and of the form P/Q where P and Q are polynomials. Define

$$di(H) = degree\ P - degree\ Q\,.$$

By a theorem of Bank and Laine [**1**, Theorem 1(b)], as Hellerstein and Rossi [**7**] did, we can find a rational function $H(z)$ with $di(H)$ being a positive integer such that

$$w'' + Hw = 0 \tag{6}$$

has two linearly independent solutions f_1 and f_2 meromorphic in $\boldsymbol{C}$ with only finitely many real zeros. And $f_1 f_2$ is rational. Therefore, the assumption that at least one w_j with maximal order λ does not have BEV 0 can not be removable, but I guess it may be exactly modified. Now here is one question.

Does that each of a FS $\{w_1, \cdots, w_n\}$ has BEV 0 of order λ imply that $E=w_1 \cdots w_n$ is rational?

As pointed out in the sequel, the question is positive for the second order linear differential equation (6). This is because any meromorphic solution of Eq. (6) with BEV 0 must have 0 as a Picard Exceptional Value (PEV 0) providing $di(H) \geqq 0$.

2) A simple calculation shows that the second order linear differential equation

$$w''+\frac{1}{z}w'+\frac{1}{4z^2}w=0 \tag{7}$$

possesses two linearly independent entire solutions $\cos\sqrt{z}$ and $\cos\sqrt{-z}$, which, respectively, only have positive real zeros and negative real zeros and are of order 1/2. Obviously, we can not transform Eq. (7) into the form of Eq. (6) with $di(H)=0$ by a transformation similar to ones listed in Hellerstein-Rossi conjecture (cf. [**2**], Problem 2.72). Hence the conjecture does not hold for the case of rational coefficients. But I can do nothing for asserting whether or not the conclusion (5) in Theorem 2 occurs. Entire functions $\cos\sqrt{z}$ and $\cos\sqrt{-z}$ solve the following equation

$$w'''-\frac{1}{z^2}w'-\frac{3}{4z^3}w=0.$$

The present author conjectures that the case (5) in Theorem 2 does not occur.

And by applying Theorem 2, we easily get the following result which is Theorem 1 of Hellerstein and Rossi [**7**].

THEOREM A. *Let $H(z)$ be a rational function. If the linear differential equation*

$$w''+Hw=0 \tag{6}$$

admits two linearly independent solutions w_1 and w_2 such that the zeros of w_1 and w_2 are attracted to the real axis and $E=w_1w_2$ is transcendental, then $di(H)=0$.

In fact, we consider Eq. (6), $di(H) \neq -1$, otherwise Eq. (6) can not possess two linearly independent solutions, which is found in Theorem 2 of [**7**]. And $\lambda(w_i) \neq 1/2$, $i=1, 2$. If $di(H) \geqq 0$, then it follows from Theorem 4 of [**7**] that any meromorphic solution of Eq. (6) having BEV 0 must have PEV 0. If $di(H) \leqq -2$, then Eq. (6) can not have any transcendental solutions.

In one word, if it has BEV 0, then any meromorphic solution of Eq. (6) has PEV 0. And we can easily see that both w_1 and w_2 have PEV 0 if and only if $E=w_1w_2$ is rational. Thus Theorem A follows.

One question is immediately raised as follows.

Must any meromorphic solution of Eq. (1) *with BEV* 0 *have* 0 *as a PEV provided* $di(r_j)\geqq 0$ $(1\leqq j\leqq n-2)$?

In general, the question is negative, even if all the r_j's are polynomials. This can be described from the example that the entire function $(e^z-1)\exp z^2$ solves the equation

$$w'''-(7+6z+12z^2)w'+2z(1+2z)(1+4z)w=0.$$

However, the question is positive for some special cases such as

$$w^{(n)}+Hw=0, \tag{8}$$

where $n\geqq 2$ and H is rational.

Therefore, we can make a generalization of Theorem A.

THEOREM 3. *Let $H(z)$ of Eq.* (8) *be given. Assume that Eq.* (8) *has a FS* $\{w_1, \cdots, w_n\}$ *each element of which has the zeros to be attracted to the real axis. Then either* $E=w_1w_2\cdots w_n$ *is rational or else* $di(H)=0$.

In section 2, we exhibit some auxiliary results for the proof of our theorems. In section 3, we state the proofs of our theorems.

2. Auxiliary results.

The following is well known and comes from the theory of asymptotic integration, please refer to Brüggemann [**4**] and Zheng [**12**] as well:

Eq. (1) has n linearly independent formal solutions

$$w_j^0=\exp(P_j(z))z^{\rho_j}[\log z^{1/p}]^{m_j}Q_j(z, \log z), \quad 1\leqq j\leqq n, \tag{9}$$

where $P_j(z)$ is a polynomial in $z^{1/p}$, $m_j\in \boldsymbol{N}_0$, $\rho_j\in \boldsymbol{C}$ and Q_j is a polynomial in $\log z$ over the field of formal series $\sum_{s\in \boldsymbol{N}_0} a_s z^{-s/p}$, and $Q_j(z, \log z)=1+O(1/\log z)$, as $|z|\to+\infty$.

Then given a ray $\arg z=\theta$, there exists a sufficiently small $h>0$ (which depends on θ) such that $\{w_j^0|1\leqq j\leqq n\}$ represents a fundamental set of Eq. (1) in the sector $S: |\arg z-\theta|<h$.

Let $w(z)$ be a nontrivial solution of Eq. (1) (admitting multiplicited values). Then in S

$$w=c_1w_{i_1}^0+c_2w_{i_2}^0+\cdots+c_mw_{i_m}^0, \quad c_j\neq 0,\ 1\leqq i_1<\cdots<i_m\leqq n. \tag{10}$$

DEFINITION. A ray $\arg z=\theta\in \boldsymbol{R}$ is called a Stokes ray of $w(z)$, provided that for some δ, $0<\delta<h$, there exist $P_{i_v}(z)$ and $P_{i_k}(z)$ with $P_{i_v}(z)\neq P_{i_k}(z)$ such that for $\theta<\varphi<\theta+\delta$ and every $P_{i_s}(z)\neq P_{i_v}(z)$, we have

$$Re(P_{i_v}(re^{i\varphi})-P_{i_s}(re^{i\varphi}))\to+\infty,$$

as $r \to \infty$ and $P_{i_k}(z)$ has the same property for $\theta-\delta<\varphi<\theta$. And further if $\lambda(w)=\deg(P_{i_v}-P_{i_k})$, then $\arg z=\theta$ is called a Stokes ray of w of order $\lambda(w)$.

It follows from the proof of Lemma 1 in [4] that a ray $\arg z=\theta$ is a Stokes ray of w of order $\lambda(w)$ if and only if for some $c>0$,

$$n(r, S, w) = cr^{\lambda(w)}(1+o(1)),$$

where S is an arbitrary small sector containing the ray $\arg z=\theta$. And Corollary 1 of Steinmetz [8] and Theorem 2 of Zheng [12] show that the zeros of any nontrivial solution w of Eq. (1) are attracted to a system of its finitely many Stokes rays of order $\lambda(w)$.

Now we define the indicator function $h_w(\theta)$ of w with $0<\lambda(w)<+\infty$ by

$$h_w(\theta) := \limsup_{r\to\infty} \frac{\log|w(re^{i\theta})|}{r^{\lambda(w)}}, \quad (\theta \in \boldsymbol{R}), \tag{11}$$

and the formal indicator function by

$$I_j(\theta) := \lim_{r\to\infty} \frac{Re P_j(re^{i\theta})}{r^{\lambda_j}}, \quad (\theta \in \boldsymbol{R}),$$

where $\lambda_j=\deg P_j(z)$. Obviously, $h_w(\theta)$ is periodic with period 2π. The basic relationship between the indicator and formal indicator functions are clearly listed in Brüggeman [4].

3. Proofs of theorems.

Proof of Theorem 1. Set $E=w_1w_2\cdots w_n$. If $\lambda(E)=0$, then a well-known result of Wittich [11] concerning that $\lambda(w_j)>0$ $(1\leqq j\leqq n)$ or w_j is rational implies that each w_j has BEV 0. Now we assume that $\lambda(E)>0$. By Theorems 1 and 2 of Steinmetz [8], we have for some $d>0$,

$$T(r, E) = dr^{\lambda(E)}+o(r^{\lambda(E)}), \quad r \to +\infty.$$

Since each w_j has only finitely many poles, Valiron deficient $\Delta(\infty, E)=1$. Also

$$m\left(r, \frac{1}{E}\right) = m\left(r, \frac{W}{E}\right)+O(1) = O(\log r),$$

where $W=W(w_1, \cdots, w_n)$ is Wronskian of $w_1, w_2, \cdots, w_n$ and therefore $\Delta(0, E)=0$. It is clear that the zeros of E are attracted to the rays (2). Then from Theorem 1 of Gol'dberg [6], we conclude

$$\lambda(E) \leqq \omega(D).$$

If $\lambda(E)=\lambda=\max\{\lambda(w_j)\mid 1\leqq j\leqq n\}$, then the inequality (4) follows.

If $\lambda(E)<\lambda$, then each w_j obviously has BEV 0 of order λ.

REMARK. The above proof is in essence due to Steinmetz [**10**] in which he proved the Hellerstein-Rossi conjecture by an excellent method.

PROOF OF THEOREM 2. The application of Theorem 1 concludes that $\lambda(w_j)\leqq 1$, $1\leqq j\leqq n$. Each w_j of the FS has at most two Stokes rays of order $\lambda(w_j)$ at $\arg z=0, \pi$. Thus from a result of Dietrich [**5**] (cf. Brüggemann [**4**, Theorem C]), we need to treat two cases on the form of indicator function $h_{w_j}(\theta)$ of w_j as follows:

1) When w_j has exactly two Stokes rays of order $\lambda_j=\lambda(w_j)$ at $\arg z=0$ and π, $h_{w_j}(\theta)$ has the form

$$h_{w_j}(\theta)=\begin{cases}|b_j|\cos(\arg b_j+\lambda_j\theta), & 0\leqq\theta\leqq\pi,\\ |c_j|\cos(\arg c_j+\lambda_j\theta), & \pi<\theta<2\pi,\end{cases}\tag{12}$$

where b_j and c_j are constants and $b_j\neq c_j$.

2) When w_j has at most one Stokes ray of order $\lambda_j=\lambda(w_j)$ at $\arg z=0$ or π, the form of $h_{w_j}(\theta)$ is

$$h_{w_j}(\theta)=|d_j|\cos(\arg d_j+\lambda_j\theta),\quad 0\leqq\theta<2\pi \text{ or } -\pi\leqq\theta<\pi,$$

where d_j is a constant and $d_j\neq 0$. Analysing the definition of the Stokes rays of order λ, we easily see that at the Stokes ray $\arg z=\varphi$ of order λ,

$$Re(P_{i_v}(re^{i\varphi})-P_{i_k}(re^{i\varphi}))=o(r^{\lambda}),$$

as $r\to\infty$, that is,

$$I_{i_v}(\varphi)=I_{i_k}(\varphi).$$

Therefore for Case 1), we have

$$|b_j|\cos(\arg b_j)=|c_j|\cos(\arg c_j),$$

and

$$|b_j|\cos(\arg b_j+\lambda_j\pi)=|c_j|\cos(\arg c_j+\lambda_j\pi).$$

And a simple calculation concludes that λ_j is an integer.

For Case 2), since $h_{w_j}(\theta+2\pi)=h_{w_j}(\theta)$, we have

$$|d_j|\cos(\arg d_j)=|d_j|\cos(\arg d_j+2\lambda_j\pi),$$

or

$$|d_j|\cos(\arg d_j-\lambda_j\pi)=|d_j|\cos(\arg d_j+\lambda_j\pi).$$

And consequently, λ_j is a half of an integer.

In one word, $\lambda(w_j)=1$ or $1/2$, $1\leqq j\leqq n$.

The algebraic equation corresponding to Eq. (1)

$$H(z, y) = y^n + r_{n-2}(z)y^{n-2} + \cdots + r_1(z)y + r_0(z) = 0$$

has the solutions y_j $(1 \leqq j \leqq n)$ with the form

$$y_j = \alpha_j z^{\lambda_j - 1} + \cdots, \quad 1 \leqq j \leqq n,$$

near $z = \infty$.

If $\lambda = \max\{\lambda_j \mid 1 \leqq j \leqq n\} = 1$, there exists at least one j_0 such that $\lambda_{j_0} = 1$ and $\alpha_{j_0} \neq 0$. By the relation formula of the roots and coefficients of polynomials, we have by a simple calculation

$$r_j(z) = a_j + O\left(\frac{1}{z}\right), \quad (|z| \to \infty, \ a_j \in \boldsymbol{C}),$$

and at least one $a_j \neq 0$.

If $\lambda = 1/2$, then $\lambda(w_j) = 1/2$, $1 \leqq j \leqq n$.

Now Theorem 2 follows.

PROOF OF THEOREM 3. Set

$$H = -\alpha z^m + \cdots, \quad \text{near } z = \infty,$$

where $\alpha \in \boldsymbol{C} \setminus \{0\}$ and m is an integer. Then the algebraic equation corresponding to Eq. (8)

$$F(z, y) = y^n + H = 0$$

has the solutions y_j $(1 \leqq j \leqq n)$ with the forms

$$y_j = \omega^j \sqrt[n]{\alpha} z^{m/n} + \cdots, \quad \text{near } z = \infty,$$

where ω is a non-real nth root of unity and $\sqrt[n]{\alpha}$ is a defined complex number with $0 \leqq \arg \sqrt[n]{\alpha} < 2\pi/n$

Then Eq. (8) has a formal FS with the forms like (9) in a sufficiently small sector S around a given ray $\arg z = \theta$, where

$$P_j(z) = d_j z^{(m+n)/n} + \cdots, \quad \text{near } z = \infty, \tag{13}$$

where $d_j = \omega^j n \sqrt[n]{\alpha}/(m+n)$. Obviously, $d_i \neq d_j$ $(i \neq j)$, so P_j's have distinct leading terms.

It is clear to see that when $(m+n)/n \leqq 0$, i.e., $m \leqq -n$, Eq. (8) has no transcendental meromorphic solutions. Now we assume that $m > -n$ and set $\lambda = (m+n)/n > 0$. Then any meromorphic solution of Eq. (8) is of order λ.

Let w be a meromorphic solution of Eq. (8) and have BEV 0. We can write

$$w = ve^u, \tag{14}$$

where v is a meromorphic function with a finite number of poles and $\lambda(v) < \lambda$, and u is a polynomial with degree λ and $u = d_s z^\lambda + \cdots$ for some s. This is because $d_j \neq d_i$ $(i \neq j)$ in (13), and further there exists one and only one j such that $c_j \neq 0$ in the formula (10). Then by the definition of the Stokes rays, we know that w has no Stokes rays. Corollary 1 of Steinmetz [8] asserts that

$$n\left(r, \frac{1}{w}\right) = O(\log r),$$

that is, $\lambda(v)=0$. Substitution of (14) into Eq. (8) concludes that v solves an nth order linear differential equation with rational coefficients. And hence either $\lambda(v)>0$ or v is rational. Thus w has only finitely many zeros.

Suppose that each element of the FS $\{w_1, \cdots, w_n\}$ has BEV 0, that is, PEV 0. Then for $0 \leqq k \leqq n-1$ and $1 \leqq j \leqq n$, we have $w_j^{(k)}/w_j$ is rational so that

$$\frac{c}{E} = \frac{W(w_1, w_2, \cdots, w_n)}{E}, \quad (c \in \boldsymbol{C} \setminus \{0\})$$

is also rational, and E is rational.

Now, we can assume that at least one of the FS has no BEV 0. Applying Theorem 2 to the FS, we can conclude that either

$$di(H) = 0, \quad \text{or} \quad \lambda = 1/2.$$

Below we prove that $\lambda \neq 1/2$. Suppose that $\lambda = 1/2$. Then each w_j $(1 \leqq j \leqq n)$ has and only has one Stokes ray of order 1/2 at $\arg z=0$ or π. Since $d_j \neq d_i$ $(i \neq j)$ and d_j is the leading term of $P_j(z)$ in the formal solution $w_j^0(z)$ of (9), we have the following two cases.

Case 1). When w_j has one Stokes ray of order 1/2 at $\arg z=0$, then we can write

$$w_j = c_j w_{k_j}^0(z)(1+o(1)), \quad 0 \leqq \arg z < 2\pi,$$

near $z=\infty$.

Case 2). When w_j has one Stokes ray of order 1/2 at $\arg z=\pi$, then we have

$$w_j = \begin{cases} c_j w_{k_j}^0(z)(1+o(1)), & 0 \leqq \arg z \leqq \pi, \\ c_j' w_{k_j'}^0(z)(1+o(1)), & \pi \leqq \arg z < 2\pi, \end{cases}$$

near $z=\infty$, where $c_j, c_j' \in \boldsymbol{C} \setminus \{0\}$ and $1 \leqq k_j, k_j' \leqq n$, $k_j \neq k_j'$, and when $i \neq j$, $k_j \neq k_i$ and $k_j' \neq k_i'$.

Thus near $z=\infty$,

$$E = \prod_{j=1}^{n} w_j = \begin{cases} \prod_{j=1}^{n} c_j w_{k_j}^0(1+o(1)), & 0 \leqq \arg z \leqq \pi, \\ \prod' c_j w_{k_j}^0 \prod'' c_j' w_{k_j'}^0(1+o(1)), & \pi \leqq \arg z < 2\pi, \end{cases}$$

where $\prod'$ is a product taking over j such that w_j has the Stokes ray of order 1/2 at $\arg z=0$ and so is $\prod''$ at $\arg z=\pi$.

Since $\sum_{j=1}^{n} d_j=0$,

$$M(r,\ E)=\max\{|E(re^{i\theta})|\,|0\leqq\theta<2\pi\}<\exp(dr^{\lambda-\varepsilon}),$$

for sufficiently large r, some $d>0$ and $\varepsilon>0$. And

$$T(r,\ E)<\log M(r,\ E)+O(\log r)\leqq(d+o(1))r^{\lambda-\varepsilon},$$

that is, $\lambda(E)<\lambda=1/2$. It follows immediately that each w_j has BEV 0. This is a contradiction.

Thus Theorem 3 follows.

The following is an immediate conclusion from the proof of Theorem 3.

THEOREM 4. *Let Eq.* (8) *be given with rational coefficient* $H(z)=\alpha z^m+\cdots$, *near* $z=\infty$. *Then any nontrivial solution of Eq.* (8) *has only the Stokes rays of order* λ *with the formula*

$$\arg z=\frac{1}{m+n}[t\pi-\arg\alpha],$$

where $0\leqq t\leqq 2(m+n)-1$.

In addition, suppose that $\arg z=0$ *or* π *is a Stokes ray of order* λ, *then* α *is a real number.*

In fact, for $z=re^{i\theta}$ and $j>i$,

$$\begin{aligned}\arg(d_jz^\lambda-d_iz^\lambda)&=\lambda\theta+\arg\frac{1}{\lambda}\alpha(\omega^j-\omega^i)\\&=\lambda\theta+\frac{1}{n}(\arg\alpha+\pi)+\arg(\omega^j-\omega^i)\\&=\lambda\theta+\frac{\arg\alpha+\pi}{n}+\frac{\pi}{2}+\frac{(i+j)\pi}{n}.\end{aligned}$$

If $\arg z=\theta_0$ is a Stokes ray of order λ, then for some $1\leqq i,\ j\leqq n$,

$$\lambda\theta_0+\frac{\arg\alpha+\pi}{n}+\frac{\pi}{2}+\frac{(i+j)\pi}{n}=\frac{\pi}{2}+k\pi,$$

where k is a positive integer,

$$\theta_0=\frac{(nk-i-j-1)\pi-\arg\alpha}{n+m}. \tag{15}$$

Set $t=nk-i-j-1$, $0\leqq t<2(m+n)-1$.

If $\arg z=0$ or π is a Stokes ray of order λ, then it follows from (15) that $\arg\alpha=s\pi$, s is an integer, and α is a real number.

Theorem 4 follows.

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