J. Math. Soc. Japan
Vol. 48, No. 3, 1996

Gradient estimates for a quasilinear parabolic equation of the mean curvature type

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(Received Sept. 7, 1994)

1. Introduction.

In this paper we are concerned with the gradient estimates of solutions to the initial boundary value problem of the quasilinear parabolic equation

$$u_t - div\{\sigma(|\nabla u|^2)\nabla u\} = 0 \quad \text{in } \Omega\times[0, \infty), \tag{1.1}$$

$$u(x, 0) = u_0(x) \text{ and } u(x, t)|_{\partial\Omega} = 0 \quad \text{for } t \geqq 0, \tag{1.2}$$

where Ω is a bounded domain in R^N with a smooth, say C^3 class, boundary $\partial\Omega$ and $\sigma(v)$ is a function like $\sigma(v)=1/\sqrt{1+v}$.

When $\sigma(v)=|v|^{(p-2)/2}$, $p\geqq 2$, Alikakos and Rostamian [**1**] derived an estimate for $\|\nabla u(t)\|_\infty$ for the solutions of the equation with Neumann boundary condition, which includes a smoothing effect and decay properties. The argument can be applied to the case of Dirichlet problem. In [**1**], a strong coerciveness condition on $-div\{\sigma(|\nabla u|^2)\nabla u\}$ is used essentially and the mean curvature type nonlinearity $\sigma(v)=1/\sqrt{1+v}$ is excluded.

Recently, Engler, Kawohl and Luckhaus [**2**] have treated the problem (1.1)-(1.2) for a class of $\sigma(v)$ including $\sigma(v)=|v|^{(p-2)/2}$ and $1/\sqrt{1+v}$ and derived estimates for $\|\nabla u(t)\|_q$, in particular if $\sigma'(v)\geqq\varepsilon_0>0$, the decay estimate

$$\|\nabla u(t)\|_q \leqq \|\nabla u_0\|_q e^{-\lambda t}, \quad \lambda > 0, \tag{1.3}$$

for any $q\geqq 2$. In [**2**], however, no result concerning smoothing effect nor decay estimate for $\|\nabla u(t)\|_\infty$ is given.

The object of this paper is to derive an estimate for $\|\nabla u(t)\|_\infty$ to the problem (1.1)-(1.2) with $\sigma(v)$ like $1/\sqrt{1+v}$. Our result includes both of smoothing effect and exponential decay. More precisely, we prove

$$\|\nabla u(t)\|_\infty \leqq C\|\nabla u_0\|_{p_0} t^{-\mu} e^{-\lambda t} \tag{1.4}$$

for $p_0>3N/2$ ($p_0\geqq 3$ if $N=1$), where λ is a positive constant and $\mu=N/(2p_0-3N)$.

As in [**1**] and [**2**] (see Serrin [**9**]) we make a certain geometric condition on $\partial\Omega$, which is essential for our argument. Such a condition is useful even for some type of quasilinear wave equations ([**6**]).

The equation (1.1) with $\sigma(v)=1/\sqrt{1+v}$ was treated by Lichnewsky and Temam [**4**], and there a decay property for $\|u(t)\|_1$ as well as the existence and uniqueness was discussed under a general boundary condition.

Quite recently, another type of mean curvature flow

$$u_t-\sqrt{1+|\nabla u|^2}\,div\left\{\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right\}=0$$

has been investigated by Oliker and Uraltseva [8]. In [8], a precise exponential decay estimate for $\|\nabla u(t)\|_\infty$ is established by Nash-Giorgi type argument combined with classical maximum principle. But, initial data are assumed to be C_0^2 class and smoothing effect near $t=0$ is not known at all.

For the proof of our result (1.4) we employ Moser's technique as in Alikakos and Rostamian [**1**] and make some device as in Nakao [**6**] to overcome the lack of coercivity of the nonlinear term $-div\{\sigma(|\nabla u|^2)\nabla u\}$. A delicate estimate near $t=0$ will be derived by use of a result for a singular differential inequality proved in Ohara [**7**]. Véron [**10**] is a pioneering work proving smoothing effect and decay for nonlinear parabolic equations by use of Moser's technique.

2. Preliminaries and result.

The function spaces we use are all standard and the definition of them are omitted. But, we note that $\|\cdot\|_p$, $1\leqq p\leqq\infty$, denotes L^p norm on Ω.

We make the following assumption on $\sigma(v)$.

Hyp. 1. $\sigma(v)$ belongs to $C^2(R^+)$, $R^+\equiv[0,\infty)$, and satisfies the conditions:

$$(1)\quad k_0(1+v)^{-1/2}\leqq\sigma(v)\leqq k_1,$$

$$(2)\quad \sigma(v)+2\sigma'(v)v\geqq k_0(1+v)^{-3/2},$$

and

$$(3)\quad |\sigma'(v)v|\leqq k_1\sigma(v),$$

with some positive constants k_0, k_1.

As a definition of solution for (1.1)-(1.2) we employ a standard one.

DEFINITION. We say a measurable function $u(x,t)$ on $\Omega\times R^+$ to be a solution of the problem (1.1)-(1.2) if

$$u(t)\in L_{loc}^2([0,\infty);W_0^{1,2}(\Omega))$$

and the variational equality

$$\int_0^\infty\int_\Omega\{-u\phi_t+\sigma(|\nabla u|^2)\nabla u\cdot\nabla\phi\}\,dx\,dt=\int_\Omega u_0\phi(0)dx \tag{2.1}$$

is valid for any $\phi\in C_0^1([0,\infty);C_0^1(\Omega))$.

Our result reads as follows.

THEOREM 1. *Suppose that the mean curvature $H(x)$ of $\partial\Omega$ at x with respect to the outward normal is nonnegative. Let $u_0 \in W_0^{1,p_0}(\Omega)$ with $p_0 > 3N/2$ if $N \geqq 2$ and $p_0 \geqq 3$ if $N=1$. Then, the problem* (1.1)-(1.2) *admits a unique solution $u(t)$ in the class*

$$L^\infty(R^+ ; L^\infty(\Omega)) \cap L^\infty(R^+ ; W_0^{1,p}(\Omega)) \cap L_{loc}^\infty(R^+ ; W_0^{1,\infty}(\Omega)) \cap W_0^{1,2}(R^+ ; L^2(\Omega)) \tag{2.2}$$

and the estimate

$$\|\nabla u(t)\|_\infty \leqq C \|\nabla u_0\|_{p_0} t^{-N/(2p_0-3N)} e^{-\lambda t}, \quad t > 0, \tag{2.3}$$

holds for some $\lambda > 0$, where C is a constant independent of u_0 and p_0.

For the proof of Theorem we use the following lemmas.

LEMMA 1 (Gagriardo-Nirenberg). *Let $1 \leqq r \leqq q \leqq Np/(N-p)$ ($1 \leqq r \leqq q \leqq \infty$ if $N<p$ and $1 \leqq r \leqq q < \infty$ if $N=p$). Then, for $u \in W^{1,p}(\Omega)$, $p \geqq 1$, we have*

$$\|u\|_q \leqq C \|u\|_r^{1-\theta} \|u\|_{W^{1,p}}^\theta \tag{2.4}$$

with

$$\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{N} - \frac{1}{p} + \frac{1}{r}\right)^{-1},$$

where C is a constant independent of p, q, r.

In fact we use Lemma 1 in the following form.

LEMMA 2. *If $|u|^\beta u \in W^{1,p}(\Omega)$, $p \geqq 1$, $\beta > 0$, we have*

$$\|u\|_q \leqq C^{1/(\beta+1)} \|u\|_r^{1-\theta} \| |u|^\beta u\|_{W^{1,p}}^{\theta/(\beta+1)} \tag{2.5}$$

with

$$\theta = (\beta+1)\left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{N} - \frac{1}{p} + \frac{\beta+1}{r}\right)^{-1}$$

where we assume $\beta+1 \leqq q$ and $1 \leqq r \leqq q \leqq (\beta+1)Np/(N-p)$ ($1 \leqq r \leqq q < \infty$ if $p=N$ and $1 \leqq r \leqq q \leqq \infty$ if $N<p$).

(Cf. Véron [**10**], Nakao [**5**], Ohara [**7**].)

LEMMA 3. *Let $y(t)$ be a nonnegative differentiable function on* $(0, T]$, $T>0$, *satisfying the inequality*

$$y'(t) + At^{\lambda\theta-1} y(t)^{1+\theta} \leqq By(t) + Ct^{-1-\delta}, \quad 0 < t < T, \tag{2.6}$$

with $A>0$, $B \geqq 0$, $C \geqq 0$, $\lambda > 0$, $\theta > 0$ and $-\infty < \delta < \infty$ such that $\lambda\theta \geqq 1$ and $\lambda > \delta$. Then, we have

$$y(t) \leqq \left\{\left(\frac{2\lambda+2BT}{A}\right)^{1/\theta}+\frac{2Ct^{\lambda-\delta}}{\lambda+BT}\right\}t^{-\lambda} \tag{2.7}$$

for $0<t\leqq T$.

For a proof of Lemma 3 see Ohara [**7**].

3. Some differential inequalities for $\|\nabla u(t)\|_p$.

In this section we want to derive some differential inequalities and a priori estimates concerning $\|\nabla u(t)\|_p$, $p\geqq 2$. For construction of the solutions, however, we treat in fact approximate solutions $u_\varepsilon(t)$.

Let $u_{0,\varepsilon}\in C_0^\infty(\Omega)$ and consider the approximate equations

$$u_t-div\{\sigma_\varepsilon(|\nabla u|^2)\nabla u\}=0 \quad \text{in } \Omega\times[0,\infty), \tag{3.1}$$

$$u(x,0)=u_{0,\varepsilon}(x) \text{ and } u(x,t)|_{\partial\Omega}=0, \tag{3.2}$$

where we set

$$\sigma_\varepsilon(v)=\sigma(v)+\varepsilon \tag{3.3}$$

and $u_{0,\varepsilon}$ should be chosen so that $u_{0,\varepsilon}\to u_0$ in $W_0^{1,p}$ as $\varepsilon\to 0$.

When $\varepsilon>0$ the nonlinear term in (3.1) is uniformly elliptic, and hence the problem (3.1)-(3.2) admits a unique smooth solution $u_\varepsilon(t)$ for each $u_{0,\varepsilon}$ (Ladyzhenskaya, Solonnikov and Uraltseva [**3**]).

We write u for u_ε for simplicity of notation.

The following is the basic differential inequality for our argument.

PROPOSITION 1. *For approximate solution* $u=u_\varepsilon$ *we have, for* $p\geqq 2$,

$$\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p+\frac{(p-1)}{4}\int_\Omega\{\varepsilon+k_0(1+|\nabla u|^2)^{-3/2}\}|\nabla u|^{p-4}|\nabla(|\nabla u|^2)|^2dx$$

$$\leqq -(N-1)\int_{\partial\Omega}\sigma_\varepsilon(|\nabla u|^2)|\nabla u|^pH(x)d\Gamma \tag{3.4}$$

where $H(x)$ *denotes the mean curvature of* $\partial\Omega$ *at* x.

RROOF. We write u_i for $\partial u/\partial x_i$ and employ the notation of summation convention.

Multiplying the equation (3.1) by $-(|\nabla u|^{p-2}u_j)_j$ and integrating over Ω we have, by integration by parts, (cf. [**1**] and [**2**]).

$$\int_\Omega|\nabla u|^{p-2}u_ju_{jt}dx$$

$$=\int_\Omega\{\sigma_\varepsilon(|\nabla u|^2)u_i\}_{ij}|\nabla u|^{p-2}u_jdx-\int_\partial\{\sigma_\varepsilon(|\nabla u|^2)u_i\}_i|\nabla u|^{p-2}u_jn_jd\Gamma$$

$$= -\int_{\Omega}\{\sigma_{\varepsilon}(|\nabla u|^{2})u_{i}\}_{j}(|\nabla u|^{p-2}u_{j})_{i}dx$$

$$+\int_{\partial\Omega}\{\{\sigma_{\varepsilon}(|\nabla u|^{2})u_{i}\}_{j}|\nabla u|^{p-2}u_{j}n_{i}-\{\sigma_{\varepsilon}(|\nabla u|^{2})u_{i}\}_{i}|\nabla u|^{p-2}u_{j}n_{j}\}\,d\Gamma$$

$$= -\int_{\Omega}\{\sigma_{\varepsilon}(|\nabla u|^{2})u_{i}\}_{j}\{|\nabla u|^{p-2}u_{j}\}_{i}dx-(N-1)\int_{\partial\Omega}\sigma_{\varepsilon}(|\nabla u|^{2})|\nabla u|^{p}H(x)d\Gamma \tag{3.5}$$

where $n=(n_{1}, \cdots, n_{N})$ denotes the exterior normal vector at the boundary. Here, we see

$$\begin{aligned}
&\{\sigma_{\varepsilon}(|\nabla u|^{2})u_{i}\}_{j}\{|\nabla u|^{p-2}u_{j}\}_{i}\\
&= \{\sigma_{\varepsilon}u_{ij}+2\sigma' u_{i}u_{k}u_{kj}\}\{|\nabla u|^{p-2}u_{ij}+(p-2)|\nabla u|^{p-4}u_{j}u_{l}u_{li}\}\\
&= \{\sigma_{\varepsilon}u_{ij}^{2}+2\sigma' u_{i}u_{ij}\cdot u_{k}u_{kj}\}|\nabla u|^{p-2}\\
&\quad+(p-2)|\nabla u|^{p-4}\{\sigma_{\varepsilon}u_{i}u_{ji}\cdot u_{l}u_{li}+2\sigma' u_{k}u_{j}u_{kj}\cdot u_{i}u_{l}u_{li}\}\\
&= \{\sigma_{\varepsilon}|\nabla^{2}u|^{2}+2\sigma'\sum_{j}|\nabla u\cdot\nabla u_{j}|^{2}\}|\nabla u|^{p-2}\\
&\quad+\frac{(p-2)}{4}|\nabla u|^{p-4}\{\sigma_{\varepsilon}|\nabla(|\nabla u|^{2})|^{2}+2\sigma'|\nabla u\cdot\nabla(|\nabla u|^{2})|^{2}\}\\
&\geqq \{\varepsilon+k_{0}(1+|\nabla u|^{2})^{-3/2}\}\{|\nabla u|^{p-2}|\nabla^{2}u|^{2}+\frac{(p-2)}{4}|\nabla u|^{p-4}|\nabla(|\nabla u|^{2})|^{2}\}\\
&\geqq \frac{(p-1)}{4}\{\varepsilon+k_{0}(1+|\nabla u|^{2})^{-3/2}\}|\nabla u|^{p-4}|\nabla(|\nabla u|^{2})|^{2}.
\end{aligned} \tag{3.6}$$

(Note that the term $|\nabla u|^{p-4}|\nabla(|\nabla u|^{2})|^{2}$ contains no singularity if $p\geqq 2$.)
(3.4) follows from (3.5) and (3.6).

From Proposition 1 we have further the following inequality by which we can overcome the difficulty of the noncoerciveness of $-div\{\sigma(|\nabla u|^{2})\nabla u\}$ and apply Moser's technique.

PROPOSITION 2. *Let $p_{0}>3N/2$ ($p_{0}\geqq 3$ if $N=1$) and assume that $H(x)\geqq 0$ on $\partial\Omega$. Then, for $p\geqq p_{0}$, we have*

$$\frac{d}{dt}\|\nabla u(t)\|_{p}^{p}+C_{0}\|\nabla(|\nabla u|^{p/2})\|_{1+\kappa}^{2}\leqq 0 \tag{3.7}$$

with $C_{0}=C\cdot(|\Omega|^{1/p_{0}}+\|\nabla u_{0}\|_{p_{0}})^{-3}$ and $\kappa=(p_{0}-3)/(p_{0}+3)$, where C is a positive constant independent of u, p and p_{0}.

PROOF. Since $H(x)\geqq 0$ we have from (3.4)

$$\frac{d}{dt}\|\nabla u(t)\|_{p}^{p}\leqq 0$$

and, in particular,

$$\|\nabla u(t)\|_{p_{0}}\leqq\|\nabla u_{0,\varepsilon}\|_{p_{0}},\quad t\geqq 0. \tag{3.8}$$

Now, noting that

$$|\nabla(|\nabla u|^{p/2})|^2=\frac{p^2}{4}|\nabla u|^{p-4}|\nabla(|\nabla u|^2)|^2 \tag{3.9}$$

we have

$$\begin{aligned}
&\|\nabla(|\nabla u|^{p/2})\|_{1+\kappa}^2\\
&=\frac{p^2}{4}\left\{\int_\Omega(|\nabla u|^{(p-4)/2}|\nabla(|\nabla u|^2)|)^{1+\kappa}dx\right\}^{2/(1+\kappa)}\\
&=\frac{p^2}{4}\left\{\int_\Omega\left[\frac{|\nabla u|^{p-4}|\nabla(|\nabla u|^2)|^2}{(1+|\nabla u|^2)^{3/2}}\right]^{(1+\kappa)/2}(1+|\nabla u|^2)^{3(1+\kappa)/4}dx\right\}^{2/(1+\kappa)}\\
&\leqq\frac{p^2}{4}\left\{\int_\Omega\frac{|\nabla u|^{p-4}|\nabla(|\nabla u|^2)|^2}{(1+|\nabla u|^2)^{3/2}}dx\right\}\left\{\int_\Omega(1+|\nabla u|)^{3(1+\kappa)/(1-\kappa)}dx\right\}^{(1-\kappa)/(1+\kappa)}\\
&\leqq\frac{p^2}{4}(|\Omega|^{1/p_0}+\|\nabla u_{0,\varepsilon}\|_{p_0})^3\int_\Omega\frac{|\nabla u|^{p-4}|\nabla(|\nabla u|^2)|^2}{(1+|\nabla u|^2)^{3/2}}dx,
\end{aligned} \tag{3.10}$$

where we have used (3.8) at the last step (note that $3(1+\kappa)/(1-\kappa)=p_0$). The inequalities (3.4) and (3.10) imply (3.7) immediately.

To derive the exponential decay of $\|\nabla u(t)\|_\infty$ as $t\to\infty$ we prepare:

PROPOSITION 3. *Assume that $H(x)\geqq 0$ on $\partial\Omega$ and there exists $t_0\geqq 0$ such that $M_0\equiv\|\nabla u(t_0)\|_\infty<\infty$. Then, for any $2\leqq p<\infty$, we have*

$$\|\nabla u(t)\|_p\leqq\|\nabla u(t)\|_p e^{-\lambda(t-t_0)} \quad \textit{for } t\geqq t_0, \tag{3.11}$$

where λ is a positive constant depending on M_0 and p.

PROOF. From (3.5) or (3.7) we have

$$\|\nabla u(t)\|_p\leqq\|\nabla u(t_0)\|_p<\infty \tag{3.12}$$

and hence, taking the limit as $p\to\infty$,

$$\|\nabla u(t)\|_\infty\leqq\|\nabla u(t_0)\|_\infty<\infty \tag{3.13}$$

for $t\geqq t_0$.

Once the boundedness of $\|\nabla u(t)\|_\infty$ is known the exponential decay (3.11) follows from an argument as in [2]. Indeed, setting $w=\sqrt{\sigma_\varepsilon(|\nabla u|)^2}|\nabla u|^p$ we see, by the assumption on σ, that

$$\begin{aligned}
|\nabla w|^2&=\frac{1}{16}\sigma_\varepsilon^{-1}|\nabla u|^{p-4}(p\sigma_\varepsilon+2\sigma'|\nabla u|^2)^2|\nabla(|\nabla u|^2)|^2\\
&\leqq\frac{1}{16}(p+2k_1)^2\sigma_\varepsilon|\nabla u|^{p-4}|\nabla(|\nabla u|^2)|^2\\
&\leqq C_p^{-1}\frac{(p-1)}{4}\{\varepsilon+k_0(1+|\nabla u|^2)^{-3/2}\}|\nabla u|^{p-4}|\nabla(|\nabla u|^2)|^2,
\end{aligned} \tag{3.14}$$

where we put

$$C_p^{-1} = \frac{k_1(p+2k_1)^2}{4k_0(p-1)}(1+M_0^2)^{3/2}.$$

Hence, by the inequality (3.4) we have

$$\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + C_p\|\nabla w(t)\|_2^2 + (N-1)\int_{\partial\Omega} H(x)w^2(x)d\Gamma \leqq 0. \tag{3.15}$$

Here, by an argument of elliptic eigenvalue problem there exists $\lambda_p>0$ such that

$$C_p\|\nabla w\|_2^2 + (N-1)\int_{\partial\Omega} H(x)w^2 d\Gamma \geqq \lambda_p\|w\|_2^2 \tag{3.16}$$

(cf. [2]).

Since

$$\|w(t)\|_2^2 = \int_\Omega \sigma_\varepsilon|\nabla u|^p dx \geqq k_0/\sqrt{1+M_0^2}\|\nabla u\|_p^p$$

we obtain from (3.15) and (3.16) that

$$\frac{d}{dt}\|\nabla u(t)\|_p^p + p\lambda\|\nabla u(t)\|_p^p \leqq 0 \tag{3.17}$$

with $\lambda \equiv \lambda_p k_0/\sqrt{1+M_0^2}$, which implies (3.11).

4. Estimate near $t=0$.

In this section we shall derive an estimate for $\|\nabla u_\varepsilon(t)\|_\infty$ near $t=0$, which will yield (2.3) near $t=0$, by taking the limit as $\varepsilon \to 0$.

Let $u=u_\varepsilon(t)$ be the approximate solution as in the previous section and set $v(t) \equiv |\nabla u(t)|$. First, we note that

$$\|v(t)\|_{p_0} \leqq \|\nabla u_{0,\varepsilon}\|_{p_0} \quad \text{for } t \geqq 0. \tag{4.1}$$

For a sequence $\{p_n\}$ defined by $p_n=2^n p_0$, $n=1, 2, \cdots$, we shall show that there exist sequences $\{\mu_n\}$ and $\{\xi_n\}$ of nonnegative numbers such that

$$\|v(t)\|_{p_n} \leqq \xi_n t^{-\mu_n} \quad \text{for } t \in (0, T], \tag{4.2}$$

where $T>0$ is an arbitrarily fixed number.

We prove (4.2) by induction. It holds certainly for $n=0$ by taking $\xi_0 = \|\nabla u_{0,\varepsilon}\|_{p_0}$ and $\mu_0=0$. Assume that it is valid for $n=k-1$. To show (4.2) for $n=k$ we utilize the inequality

$$\|v\|_{p_k} \leqq C^{2/p_k}\|v\|_{p_{k-1}}^{1-\theta}\{\|\nabla(v^{p_k/2})\|_{1+\kappa}^2 + \|v^{p_k/2}\|_{1+\kappa}^2\}^{\theta/p_k} \tag{4.3}$$

with $\kappa=(p_0-3)/(p_0+3)$ and $\theta=N(1+\kappa)/2(N\kappa+\kappa+1)$, which follows easily by

Lemma 2.

In what follows we denote by C general positive constants independent of k and ε. Now, by (3.7), (4.2) with $n=k-1$ and (4.3) we have

$$\frac{d}{dt}\|v(t)\|_{p_k}^{p_k}+C_0C^{-2/\theta}(\xi_{k-1}t^{-\mu_{k-1}})^{-p_k(1-\theta)/\theta}\|v(t)\|_{p_k}^{p_k/\theta}\leqq C_0\|v(t)^{p_k/2}\|_{1+\kappa}^2. \tag{4.5}$$

Since

$$\|v(t)^{p_k/2}\|_{1+\kappa}^2\leqq C\|v(t)\|_{p_k}^{p_k}$$

we have from (4.5)

$$\frac{d}{dt}\|v(t)\|_{p_k}+C_0C^{-2/\theta}p_k^{-1}(\xi_{k-1}t^{-\mu_{k-1}})^{-p_k(1-\theta)/\theta}\|v\|_{p_k}^{1-p_k+p_k/\theta}\leqq Cp_k^{-1}\|v(t)\|_{p_k}, \tag{4.6}$$

which is rewritten as

$$y'(t)+C_0C^{-2/\theta}p_k^{-1}\xi_{k-1}^{-\theta_k}t^{\mu_k\theta_k-1}y^{1+\theta_k}\leqq Cp_k^{-1}y(t) \tag{4.7}$$

where we set

$$y(t)=\|v(t)\|_{p_k},\quad \theta_k=p_k(1-\theta)/\theta \quad\text{and}\quad \mu_k=\mu_{k-1}+1/\theta_k.$$

Thus, applying Lemma 3 to (4.7) we obtain

$$\|v(t)\|_{p_k}\leqq\{C_0^{-1}C^{2/\theta}p_k\xi_{k-1}^{\theta_k}(2\mu_k+2Cp_k^{-1}T)\}^{1/\theta_k}t^{-\nu_k} \tag{4.8}$$

for $t\in(0,T]$, $T>0$. This inequality means that (4.2) is valid for $n=k$ if we define

$$\xi_k=\xi_{k-1}\{C_0^{-1}C^{2/\theta}p_k(2\mu_k+2Cp_k^{-1}T)\}^{1/\theta_k}. \tag{4.9}$$

To take the limit in (4.2) as $n\to\infty$ we must check the behaviour of $\{\mu_n\}$ and $\{\xi_n\}$. First, from the definition

$$\mu_n=\mu_{n-1}+\theta/2^np_0(1-\theta)\quad\text{and}\quad\mu_0=0$$

we see that

$$\mu_\infty\equiv\lim_{n\to\infty}\mu_n=\sum_{k=1}^{\infty}\frac{\theta}{2^kp_0(1-\theta)}=\frac{\theta}{p_0(1-\theta)}=\frac{N}{2p_0-3N}>0. \tag{4.10}$$

Next, we show that $\{\xi_n\}$ is bounded. Indeed, by the definition (4.9) we have

$$\begin{aligned}\log\xi_n&\leqq\log\xi_{n-1}+\frac{\theta}{p_n(1-\theta)}\{C+\log p_n\}\\&\leqq\log\xi_{n-1}+C(1+n)2^{-n}\end{aligned}$$

for some $C=C(T)>0$. Hence,

$$\log \xi_n \leqq \log \xi_0 + C \sum_{k=1}^{n} k/2^k$$
$$\leqq \log \xi_0 + C \equiv \log(\xi_0 \tilde{C}) \tag{4.11}$$

for some $\tilde{C}=\tilde{C}(T)$, that is,

$$\xi_n \leqq \tilde{C}\xi_0 \equiv \tilde{C}\|u_{0,\varepsilon}\|_{p_0}. \tag{4.12}$$

From (4.2), (4.10) and (4.12) we conclude that

$$\|\nabla u(t)\|_\infty \equiv \|v(t)\|_\infty \leqq \tilde{C}\|u_{0,\varepsilon}\|_{p_0} t^{-\nu_\infty} \tag{4.13}$$

for $t\in(0, T]$ with $\mu_\infty = N/(2p_0-3N)$.

5. Estimate for large t and completion of the proof of Theorem.

Let us proceed to the estimation of $\|\nabla u(t)\|_\infty$ for large t, where $u=u_\varepsilon(t)$ is the approximate solution of the problem (3.1)-(3.2). We take $T=1$ in (4.13) and fix this. Then,

$$\|\nabla u(1)\|_\infty \leqq \tilde{C}\|\nabla u_{0,\varepsilon}\|_{p_0} \tag{5.1}$$

and hence, by Proposition 3 and (3.13),

$$\|\nabla u(t)\|_\infty \leqq \tilde{C}\|\nabla u_{0,\varepsilon}\|_{p_0} \tag{5.2}$$

and

$$\|\nabla u(t)\|_{p_0} \leqq \|\nabla u(1)\|_{p_0} e^{-\lambda_0(t-1)} \leqq \|\nabla u_{0,\varepsilon}\|_{p_0} e^{-\lambda_0(t-1)} \tag{5.3}$$

for $t\geqq 1$ with some $\lambda_0>0$ independent of ε.

From (3.4) and (5.2) we have

$$\frac{d}{dt}\|v(t)\|_{p_k}^{p_k} + C_1\|\nabla(v^{p_k/2})\|_2^2 \leqq 0 \tag{5.4}$$

for some constant $C_1=C_1(\|\nabla u_{0,\varepsilon}\|)$ independent of k.

By Lemma 2 we see (cf. (4.3))

$$\|v\|_{p_k}^{p_k/\theta} \leqq C^{2/\theta}\|v\|_{p_{k-1}}^{p_k(1-\theta)/\theta}\{\|\nabla(v^{p_k/2})\|_2^2 + \|v\|_{p_k}^{p_k}\} \tag{5.5}$$

with $\theta=2/(N+2)$.

Since, generally, the inequality $a^{1/\beta}\leqq b(c+a)$, $0<\beta<1$, implies

$$a \leqq \max\{[(p+1)b]^{\beta/(1-\beta)}, p^{-1}c\} \tag{5.6}$$

for any $p>0$, we have from (5.5) that

$$\|v\|_{p_k}^{p_k} \leqq \max\{(p_k+1)^{\theta/(1-\theta)} C^{2/(1-\theta)}\|v\|_{p_{k-1}}^{p_k}, p_k^{-1}\|\nabla(v^{p_k/2})\|_2^2\}. \tag{5.7}$$

We shall derive exponential decay for $\|\nabla u(t)\|_\infty$ from (5.4) and (5.7). For this we shall prove

$$\|v(t)\|_{p_n} \leqq \eta_n e^{-\lambda(t-1)} \quad \text{for } t \geqq 1 \tag{5.8}$$

with a certain $\{\eta_n\}$ and $\lambda=\min\{\lambda_0, C_1\}$. (5.8) is valid for $n=0$ if we take $\eta_0=\tilde{C}\|\nabla u_{0,\varepsilon}\|_{p_0}$. Suppose that (5.8) is valid for $n=k-1$ and define

$$\eta_k = \{(p_k+1)^\theta C^2\}^{1/(1-\theta)p_k}\eta_{k-1}. \tag{5.9}$$

Then, by (5.7),

$$\|v(t)\|_{p_k}^{p_k} \leqq \max\{\eta_k^{p_k} e^{-\lambda p_k(t-1)},\ p_k^{-1}\|\nabla(v^{p_k/2})\|_2^2\}. \tag{5.10}$$

Here, we see

$$\eta_k \geqq \eta_{k-1} \geqq C^{1/p_k}\eta_0 = C^{1/p_k}\tilde{C}\|\nabla u_{0,\varepsilon}\|_{p_0} \geqq C^{1/p_k}\|v(1)\|_\infty \quad (C>1)$$

and hence, we may assume

$$\|v(1)\|_{p_k} \leqq \|v(1)\|_\infty |\Omega|^{1/p_k} < \eta_k$$

by taking $C>\max(1, |\Omega|)$. This means that (5.8) is valid on some interval $[1, 1+\delta]$, $\delta>0$. If (5.8) was false, then there would exist $t_*>1$ such that

$$\|v(t_*)\|_{p_k} = \eta_k e^{-\lambda(t_*-1)} \tag{5.11}$$

and

$$\|v(t)\|_{p_k} > \eta_k e^{-\lambda(t-1)} \tag{5.12}$$

for $t_*<t<t_*+\delta$ with some $\delta>0$.

But then, by (5.10) we have

$$\|v(t)\|_{p_k}^{p_k} \leqq p_k^{-1}\|\nabla(v(t))^{p_k/2}\|_2^2 \quad \text{on } [t_*, t_*+\delta]$$

and, by the differential inequality (5.4),

$$\frac{d}{dt}\|v(t)\|_{p_k}^{p_k}+\lambda p_k\|v(t)\|_{p_k}^{p_k} \leqq 0, \quad t_* \leqq t \leqq t_*+\delta, \tag{5.13}$$

where we note that $\lambda\leqq C_1$. This together with (5.11) implies

$$\begin{aligned}\|v(t)\|_{p_k}^{p_k} &\leqq \|v(t_*)\|_{p_k}^{p_k} e^{-\lambda p_k(t-t_*)}\\ &= \eta_k^{p_k} e^{-\lambda p_k(t-1)}\end{aligned}$$

for $t_*\leqq t\leqq t_*+\delta$, which contradicts to (5.12).

Thus, we conclude that (5.8) is valid for $n=k$ and consequently for all n.

Finally we shall check the boundedness of $\{\eta_n\}$ in (5.8). By the definition (5.9) we see

$$\log\eta_k-\log\eta_{k-1} = \frac{1}{(1-\theta)p_k}(\theta\log(1+p_k)+C)$$

and hence,

$$\log\frac{\eta_n}{\eta_0} \leqq \frac{\theta}{1-\theta}\left\{\sum_{k=1}^{\infty}\frac{\log(1+p_k)}{p_k}+C\sum_{k=1}^{\infty}\frac{1}{p_k}\right\}$$

$$\leqq \frac{C\theta}{1-\theta} \equiv \log C_2 < \infty .$$

Thus, we have

$$\eta_n \leqq C_2\eta_0 \tag{5.14}$$

and we conclude

$$\|\nabla u(t)\|_\infty \leqq \tilde{C}C_2\|\nabla u_{0,\varepsilon}\| e^{-\lambda(t-1)} \tag{5.15}$$

for $t \geqq 1$.

Combining (4.13) and (5.15) we obtain the desired estimate

$$\|\nabla u_\varepsilon(t)\|_\infty \leqq C\|\nabla u_{0,\varepsilon}\|_{p_0} t^{-N/(2p_0-3N)} e^{-\lambda t} \tag{5.16}$$

with some constant C independent of u_0 and p_0.

To show the convergence of u_ε as $\varepsilon \to 0$ we need further estimate:

$$\int_0^t \|u_{\varepsilon t}(s)\|_2^2 ds + F(\nabla u_\varepsilon(t)) = F(\nabla u_\varepsilon(0)) \leqq C\|\nabla u_{0,\varepsilon}\|_2^2 \tag{5.17}$$

for any $t>0$, where we set

$$F(\nabla u) \equiv \frac{1}{2}\int_\Omega \int_0^{|\nabla u|^2} \sigma(v)dvdx .$$

(5.17) follows easily if we multiply the equation (3.1) by $u_{\varepsilon t}$ and integrate.

Now, by a standard compactness argument we have, along a subsequence,

$$u_\varepsilon(t) \longrightarrow u(t) \text{ weakly* in } L^\infty_{loc}([0, \infty); W_0^{1,p_0}) \cap L^\infty_{loc}([0, \infty); W_0^{1,\infty})$$
$$\text{and } a.e. \text{ in } [0, \infty)\times\Omega,$$

$$u_{\varepsilon t}(t) \longrightarrow u_t(t) \text{ weakly in } L^2_{loc}([0, \infty); L^2(\Omega)),$$

and

$$A_\varepsilon u_\varepsilon \equiv -div\{\sigma_\varepsilon(|\nabla u_\varepsilon(t)|^2)\nabla u_\varepsilon(t)\} \longrightarrow \chi \text{ weakly in } L^2_{loc}([0, \infty); W^{-1,2})$$

for a measurable function $u(t)=u(t, x)$.

Since A_ε is monotone operator from $L^2_{loc}([0, \infty); W_0^{1,2})$ to $L^2_{loc}([0, \infty); W_0^{-1,2})$ we see $\chi=\sigma(|\nabla u|^2)\nabla u$ by Minty's trick. The limit function $u(t)$ satisfies (2.1) and the estimates (5.15) and (5.16) remain valid for $u(t)$ with $u_{0,\varepsilon}$ replaced by u_0. Uniqueness is trivial. The proof of Theorem is now complete.

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