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The group ring of $GL_n(q)$ and the q-Schur algebra

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Introduction.

Dipper-James [**5**] have introduced the q-Schur algebra $S_q(n)$ to study representations of $GL_n(q)$ in non-describing characteristic. The q-Schur algebra is a q-analogue of the usual Schur algebra, and its representations are equivalent to polynomial representations of quantum general linear group [**3**]. Dipper-James [**5**] have established an interesting relationship between representations of $GL_n(q)$ and the q-Schur algebra $S_q(n)$. They deal with not only unipotent representations but also cuspidal representations. In this paper, we restrict to unipotent representations and show there is a shorter realization of the Dipper-James correspondence in this case.

Let KG be the group algebra of $G=GL_n(q)$ over the field K whose characteristic does not divide q. Let B be the upper-triangular matrices and let $M=KG[B]$ the left ideal generated by $[B]$, the sum of all elements in B. Let I_M be the annihilator of M in KG. By *unipotent representations* of G, we mean left KG/I_M modules. Let **mod** KG/I_M be the category of all left KG/I_M modules.

Let λ be a partition of n. James [**9**] defines the Specht module S_λ and its irreducible quotient D_λ. Both are left KG/I_M modules, and the set of D_λ for all partitions λ of n exhausts all irreducible unipotent representations of G. On the other hand, Dipper-James [**6**] define the q-Weyl module W_λ and its irreducible quotient F_λ, which are left $S_q(n)$ modules. The purpose of this paper is to prove:

THEOREM. *Assume K has a primitive p-th root of* 1. *There is an idempotent E in KG/I_M satisfying the following properties*:

(a) *The algebra $E(KG/I_M)E$ is isomorphic to the q-Schur algebra $S_q(n)$.*

(b) *The functor $V\mapsto EV$ gives a category equivalence from* **mod** KG/I_M *to* **mod** $S_q(n)$.

(c) *Let λ be a partition of n, and let λ' be its dual partition. Under the category equivalence of* (b), *the KG/I_M module S_λ (resp. D_λ) corresponds to the $S_q(n)$ module $W_{\lambda'}$ (resp. $F_{\lambda'}$).*

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As a corollary, if λ and μ are two partitions of n, one sees the composition multiplicity of D_μ in S_λ equals the composition multiplicity of $F_{\mu'}$ in $W_{\lambda'}$ (cf. [**5**, Theorem 4.7]).

I hope the theorem above yields a better understanding of some part of the Dipper-James theory. It is proved as follows.

The endomorphism algebra $\mathrm{End}_{KG}(M)$ is isomorphic to the Hecke algebra $\mathscr{H}$ associated with the pair (G, B) [**8**][**1**]. We let $\mathscr{H}$ act on M on the right. We define some idempotent e in KG explicitly (1.1) and prove the following properties:

1) The algebra KG/I_M is Morita equivalent to $e(KG/I_M)e$ via the functor $V \mapsto eV$ (1.11).
2) The canonical algebra map $eKGe \to \mathrm{End}_{\mathscr{H}}(eM)$ is surjective (3.1).
3) The algebra $e(KG/I_M)e$ is isomorphic to the generalized q-Schur algebra S_Λ associated with some full labelling Λ (4.1).
4) In general, the generalized q-Schur algebra S_Λ associated with a full labelling Λ has an idempotent ε such that $\varepsilon S_\Lambda \varepsilon$ is isomorphic to $S_q(n)$ and that S_Λ is Morita equivalent to $S_q(n)$ via the functor $V \mapsto \varepsilon V$ (4.3).

Statements (a) and (b) of the theorem follow from 1), 3) and 4) above. (Item 2) is used to yield 3) and it requires a long calculation). The final statement (c) follows from definition of the idempotents e and ε together with the definition of modules S_λ and W_λ.

1. Representations of $GL_n(q)$

Let $G = GL_n(q)$ with q a power of the prime p. Let K be a field containing a primitive p^{th} root of 1. This means $\mathrm{char}\, K \neq p$ in particular. We construct q^{n-1} orthogonal idempotents of the group algebra KG, by using the method of [**9**, §9].

Let B be the subgroup of all upper triangular matrices in G, and let $M = KG[B]$ the left ideal generated by $[B]$, the sum of all b in B. (We are using Green's notation [**7**, 5.3]). Let $U^{\pm}$ be the subgroup of all upper (resp. lower) triangular unipotent matrices in G. Let W be the subgroup of all permutation matrices in G. We identify W with the symmetric group on n letters. (We let a permutation act on letters on the *left*).

Let $\chi_1, \cdots, \chi_q$ be the distinct linear K-characters of $(\boldsymbol{F}_q, +)$, where χ_1 is the trivial character (see [**9**, 9.1]). Let $I(q, n-1)$ be the set of all sequences $c = (c(1), \cdots, c(n-1))$ with $1 \leqq c(i) \leqq q$.

1.1. DEFINITION. For c in $I(q, n-1)$, define a linear K-character

$$\chi_c : U^- \longrightarrow K$$

by setting

$$\chi_c(g) = \prod_{i=1}^{n-1} \chi_{c(i)}(g_{i+1,i}), \qquad g = (g_{ij}) \text{ in } U^-.$$

Define an idempotent in KG

$$E_c = |U^-|^{-1} \sum_{g \text{ in } U^-} \chi_c(g) g .$$

(Note that the order $|U^-|$ is a power of q.) The intersection of all kernels $\mathrm{Ker}(\chi_c)$ coincides with the commutator subgroup $(U^-)'$ of U^-. We define

$$e = |(U^-)'|^{-1} \sum_{g \text{ in } (U^-)'} g .$$

1.2. Proposition. *$\{E_c | c$ in $I(q, n-1)\}$ is a set of orthogonal idempotents whose sum is e.*

This is obvious, since $\{\chi_c | c \text{ in } I(q, n-1)\}$ exhausts all linear K-characters of U^- (cf. [**9**, line -2, p. 43]). It follows that

$$eM = \bigoplus_{c \text{ in } I(q, n-1)} E_c M .$$

By a *composition* of n, we mean a sequence

$$\lambda = (\lambda_1, \lambda_2, \cdots)$$

of non-negative integers whose sum is n. It is called a *partition* if $\lambda_1 \geqq \lambda_2 \geqq \cdots$. Let $C(n)$ (resp. $\mathscr{P}(n)$) be the set of all compositions (resp. partitions) of n.

Usually, we denote the composition λ by a finite sequence

$$(\lambda_1, \lambda_2, \cdots, \lambda_h)$$

which means that $\lambda_{h+1} = \lambda_{h+2} = \cdots = 0$. We say the composition λ is *tight* if $\lambda_a = 0$ implies $\lambda_{a+1} = 0$. All partitions are tight.

1.3. Definition. For c in $I(q, n-1)$, let

$$c^{-1}(1) = \{\Lambda_1, \Lambda_2, \cdots, \Lambda_{h-1}\}$$

with $1 \leqq \Lambda_1 < \Lambda_2 < \cdots < \Lambda_{h-1} < n$. We define

$$\lambda^c = (\Lambda_1, \Lambda_2 - \Lambda_1, \cdots, \Lambda_{h-1} - \Lambda_{h-2}, n - \Lambda_{h-1})$$

which is a tight composition of n.

1.4. Proposition. *Let c and d be sequences in $I(q, n-1)$. If $\lambda^c = \lambda^d$, then E_c and E_d are conjugate by a diagonal matrix in G.*

This follows by using [**9**, (9.6)].

1.5. Definition. Conversely, let λ be a tight composition of n. Define a sequence $c_\lambda = (c_\lambda(1), \cdots, c_\lambda(n-1))$ in $I(q, n-1)$ as follows:

$$c_\lambda(i) = \begin{cases} 1 & \text{if } i = \lambda_1+\lambda_2+\cdots+\lambda_a \quad \text{for some } a \geqq 1 \\ 2 & \text{otherwise.} \end{cases}$$

We put

$$\chi_\lambda = \chi_{c_\lambda} \quad \text{and} \quad E_\lambda = E_{c_\lambda}.$$

Obviously, the composition associated with c_λ is λ. Note that our notation is a bit different from James [9, 11.4], where E_λ stands for our $E_{\lambda'}$ with λ' the partition dual to λ.

1.6. COROLLARY. *If $\lambda=\lambda^c$, then E_c is conjugate to E_λ by a diagonal matrix in G.*

We construct a K-basis for E_cM with c in $I(q, n-1)$.

1.7. FACT [9, 7.11]. *M has a K-basis $\{u\pi[B] \mid \pi$ in W, u in $U^-\cap\pi U^-\pi^{-1}\}$.*

If λ is a composition of n, we can decompose $\{1, 2, \cdots, n\}$ as the disjoint union of *intervals* $\{1, 2, \cdots, \lambda_1\}$, $\{\lambda_1+1, \lambda_1+2, \cdots, \lambda_1+\lambda_2\}, \cdots$. These intervals are called associated with the composition λ.

1.8. DEFINITION. A permutation π of n letters is *distinguished* relative to the composition λ if π^{-1} is increasing on each interval associated with λ. Let $\mathcal{D}_\lambda$ be the set of all permutations distinguished relative to λ.

For π in W and u in $U^-\cap\pi U^-\pi^{-1}$, we have

$$E_c u\pi[B] = \chi_c(u)^{-1}E_c\pi[B]$$

and it follows from [9, 10.2] that $E_c\pi[B]\neq 0$ if and only if $c(i)>1$ implies $\pi^{-1}(i+1)>\pi^{-1}(i)$, i.e., π is distinguished relative to λ^c. Thus we have the following:

1.9. PROPOSITION. *Let c be in $I(q, n-1)$ and $\lambda=\lambda^c$.*

(a) *For π in W, $E_c\pi[B]\neq 0$ if and only if π is distinguished relative to λ.*

(b) *The set $\{E_c\pi[B] \mid \pi$ in $\mathcal{D}_\lambda\}$ forms a K-basis for E_cM.*

We end this section by recalling the definition and main properties of the KG modules M_λ and S_λ associated with compositions λ of n.

Let P_λ be the parabolic subgroup of G corresponding to the composition λ. It consists of all $g=(g_{ij})$ in G such that $g_{ij}=0$ if $i>j$ and i and j belong to distinct intervals relative to λ. Let $M_\lambda=KG[P_\lambda]$ a left ideal contained in M.

The K vector space $E_{\lambda'}M_\lambda$ is one-dimensional [9, 11.7] and we define $S_\lambda = KGE_{\lambda'}M_\lambda$ a submodule of M_λ (λ' the partition dual to λ). S_λ has a unique maximal submodule $S_\lambda^{\max}$ and the quotient $D_\lambda=S_\lambda/S_\lambda^{\max}$ is an absolutely irreducible KG module [9, (11.12)].

The following properties are proved in [**9**].

1.10. PROPERTIES OF M_λ, S_λ, AND D_λ.

(1) M_λ, S_λ, and D_λ are defined over the prime field.

(2) $S_\lambda=D_\lambda$ in characteristic zero [**9**, 11.16].

(3) $\dim S_\lambda$ is independent of the field K [**9**, 16.5].

(4) For a prime $l\neq p$, the module S_{λ,F_l} is identified with the l-modular reduction of $S_{\lambda,Q}$ [**9**, 16.6].

(5) Let μ be the partition of n obtained by rearranging the parts of λ. Then

$$M_\lambda \cong M_\mu, \qquad S_\lambda \cong S_\mu, \qquad D_\lambda \cong D_\mu \qquad [\mathbf{9},\ 16.1].$$

(6) Every composition factor of the KG module M is isomorphic to D_μ for a uniquely determined partition μ of n [**9**, 16.4].

Let I_M be the annihilator of M in KG. We are concerned with representations of the quotient algebra KG/I_M, i.e., *unipotent representations* of G.

1.11. THEOREM. *The functor*

$$\mathbf{mod}\, KG/I_M \longrightarrow \mathbf{mod}\, e(KG/I_M)e, \qquad V\mapsto eV$$

is a category equivalence.

PROOF. The set of D_λ for all partitions λ of n gives a complete set of representatives of all irreducible KG/I_M modules, by 1.10 (6). By construction, $eD_\lambda\neq 0$ for all λ. In fact, if $E_{\lambda'}M_\lambda$ is spanned by v, then $v=ev$ but $v\notin S_\lambda^{\max}$. It follows from the arguments in Chap. 6 of [**7**] that the above functor is a category equivalence. Q.E.D.

2. The Hecke algebra.

We use [**8**], [**1**], [**4**] as basic references on Hecke algebras. Let $\mathcal{H}$ be the Hecke algebra $H_K(G, B)$ which has a K-basis T_π, π in W such that if $s=(i, i+1)$ is a basic transposition, then we have

$$T_\pi T_s=\begin{cases} T_{\pi s} & \text{if } \pi(i)<\pi(i+1) \\ qT_{\pi s}+(q-1)T_\pi & \text{if } \pi(i)>\pi(i+1). \end{cases}$$

There is a right $\mathcal{H}$ module structure on M which commutes with the left KG action. It is defined by

$$[B]T_\pi=[B\pi B]=\textstyle\sum_{u \text{ in } U^+\cap\pi U^-\pi^{-1}} u\pi[B], \quad \pi \text{ in } W.$$

The right $\mathcal{H}$ action induces an opposite algebra isomorphism

$$\mathcal{H} \underset{\text{opp}}{\cong} \mathrm{End}_{KG}(M).$$

Let λ be a composition of n. We denote by Y_λ the Young subgroup of W associated with λ. It consists of all permutations which leave the intervals $\{1, 2, \cdots, \lambda_1\}$, $\{\lambda_1+1, \lambda_1+2, \cdots, \lambda_1+\lambda_2\}$, $\cdots$ invariant. We define elements in $\mathcal{H}$

$$x_\lambda = \textstyle\sum_{\pi \text{ in } Y_\lambda} T_\pi, \qquad y_\lambda = \sum_{\pi \text{ in } Y_\lambda} (-q)^{-l(\pi)} T_\pi$$

where $l(\pi)$ denotes the length of π [**4**, § 3].

2.1. LEMMA (cf. [**10**, p. 235]). *We have* $[P_\lambda]=[B]x_\lambda$. *Hence* $M_\lambda=Mx_\lambda$.

PROOF. The set $\{u\pi \mid \pi \text{ in } Y_\lambda,\ u \text{ in } U^+\cap\pi U^-\pi^{-1}\}$ forms a system of left coset representatives for B in P_λ. Hence we have

$$[P_\lambda] = \textstyle\sum_{\pi, u} u\pi[B] = \sum_{\pi \text{ in } Y_\lambda} [B]T_\pi = [B]x_\lambda .$$

Q.E.D.

The multiplication of W induces a bijection

$$Y_\lambda \times \mathcal{D}_\lambda \cong W$$

and we have

$$x_\lambda T_\pi = q^{l(\pi)} x_\lambda, \quad y_\lambda T_\pi = (-)^{l(\pi)} y_{\lambda,\pi} \qquad \pi \text{ in } Y_\lambda .$$

The right ideal $x_\lambda\mathcal{H}$ (resp. $y_\lambda\mathcal{H}$) has a K-basis $x_\lambda T_d$ (resp. $y_\lambda T_d$) for d in $\mathcal{D}_\lambda$ [**4**, 3, 2].

If μ is a composition of n obtained by rearranging the parts of λ, there is a permutation d in $\mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$ such that $x_\lambda T_d = T_d x_\mu$ [**6**, (1.1)]. This implies $M_\lambda \cong M_\mu$ as KG modules (1.10 (5)).

There is an algebra automorphism

$$\# : \mathcal{H} \longrightarrow \mathcal{H}$$

such that $T_s{}^\# = q-1-T_s$ if $s=(i, i+1)$ is a basic transposition [**5**, § 2]. (Cf. the paragraph before [**5**, (2.1)].)

2.2. PROPOSITION. *If* $\lambda=(\lambda_1, \cdots, \lambda_h)$ *is a composition of* n,

$$x_\lambda{}^\# = q^{\lambda_1(\lambda_1-1)/2+\cdots+\lambda_h(\lambda_h-1)/2} y_\lambda .$$

PROOF. We have only to show

$$x_{(n)}{}^\# = q^{n(n-1)/2} y_{(n)} .$$

In fact, it follows from $l.3$–$l.5$, p. 25 [**5**] that $x_{(n)}{}^\# = r y_{(n)}$ with some scalar r. Comparison of the coefficients of T_{w_0} with the longest element w_0 yields $r=q^{n(n-1)/2}$. This shorter proof is due to the referee. Q.E.D.

We have a direct sum decomposition of the right $\mathcal{H}$ module

$$eM = \bigoplus_{c \text{ in } I(q, n-1)} E_c M .$$

We show that $E_c M$ is isomorphic to $y_\lambda \mathcal{H}$ with $\lambda = \lambda^c$.

For $i \neq j$ and α in $\boldsymbol{F}_q$, let $x_{ij}(\alpha) = I + \alpha E_{ij}$ with matrix units E_{ij} [**9**, §5].

2.3. LEMMA. *Let* $s = (i, i+1)$. *Then*

(i) $[B]T_s = s[B] + \sum_{\alpha \neq 0} x_{i+1, i}(\alpha)[B]$.

(ii) $[B]T_s^{\#} = -s[B] + \sum_{\alpha \text{ in } F_q} (1 - x_{i+1, i}(\alpha))[B]$.

PROOF. (ii) follows from (i). If $\alpha \neq 0$ in $\boldsymbol{F}_q$, then we have $x_{i, i+1}(\alpha)s[B] = x_{i+1, i}(\alpha^{-1})[B]$ by using

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha^{-1} & 1 \end{pmatrix}\begin{pmatrix} \alpha & 1 \\ 0 & -\alpha^{-1} \end{pmatrix}.$$

Hence,

$$[B]T_s = \sum_{\alpha \text{ in } F_q} x_{i, i+1}(\alpha)s[B] = s[B] + \sum_{\alpha \neq 0} x_{i+1, i}(\alpha^{-1})[B]$$

yielding (i). Q.E.D.

2.4. PROPOSITION. *Let c be in $I(q, n-1)$ and $\lambda = \lambda^c$. If π is a permutation in Y_λ, then we have*

$$E_c[B]T_\pi = (-)^{l(\pi)} E_c[B] .$$

PROOF. We may assume $\pi = s = (i, i+1)$, where $c(i) > 1$. By using 2.3 (i) above, we have

$$E_c[B]T_s = E_c s[B] + \sum_{\alpha \neq 0} E_c x_{i+1, i}(\alpha)[B] .$$

Let $\chi = \chi_{c(i)}{}^{-1}$ which is a nontrivial linear K-character of $(\boldsymbol{F}_q, +)$. Then we have

$$E_c x_{i+1, i}(\alpha) = \chi(\alpha) E_c \quad \text{and} \quad \sum_\alpha \chi(\alpha) = 0 .$$

On the other hand, $E_c s[B] = 0$, since s is not distinguished relative to λ (1.9 (a)). Hence we have

$$E_c[B]T_s = (\sum_{\alpha \neq 0} \chi(\alpha)) E_c[B] = -E_c[B] .$$

Q.E.D.

2.5. PROPOSITION. *Let c be in $I(q, n-1)$ and $\lambda = \lambda^c$. For d in $\mathcal{D}_\lambda$, we have*

$$E_c[B]T_d^{\#} = (-)^{l(d)} E_c d[B] .$$

PROOF. This is proved by induction on the length $l(d)$. Assume $l(d) > 0$. Take i with $1 \leqq i < n$ such that $d(i) > d(i+1)$ and write $d = \sigma s$ with $s = (i, i+1)$. Then $l(d) = l(\sigma) + 1$. Since d is distinguished relative to λ, $d(i)$ and $d(i+1)$ should belong to distinct intervals relative to λ. It follows that σ is distinguished relative to λ, too. We have

$$E_c[B]T_\sigma^{\#}=(-)^{l(\sigma)}E_c\sigma[B]$$

by the induction hypothesis. Using 2.3 (ii), we have

$$\begin{aligned}E_c[B]T_d^{\#}&=(-)^{l(\sigma)}E_c\sigma[B]T_s^{\#}\\&=(-)^{l(d)}E_cd[B]+(-)^{l(\sigma)}\textstyle\sum_{\alpha \text{ in } F_q}E_c\sigma(1-x_{i+1,i}(\alpha))[B]\,.\end{aligned}$$

We claim the second term vanishes. In fact,

$$\begin{aligned}E_c\sigma(1-x_{i+1,i}(\alpha))&=E_c(1-x_{\sigma(i+1),\sigma(i)}(\alpha))\sigma^{-1}\\&=\{1-\chi_c(x_{\sigma(i+1),\sigma(i)}(\alpha))\}E_c\sigma^{-1}=0\end{aligned}$$

since $\sigma(i)$ and $\sigma(i+1)$ belong to distinct intervals relative to λ. Q.E.D.

2.6. THEOREM. *Let c be a sequence in $I(q, n-1)$ and let $\lambda=\lambda^c$. There is an isomorphism of right $\mathcal{H}$ modules*

$$E_cM\cong y_\lambda\mathcal{H},\qquad E_c[B]h\longleftrightarrow y_\lambda h,\quad h\ \text{ in } \mathcal{H}.$$

PROOF. Proposition 2.4 implies the $\mathcal{H}$ homomorphism $y_\lambda\mathcal{H}\to E_cM$, $y_\lambda h\mapsto E_c[B]h$ is well-defined. Consider the composite

$$x_\lambda\mathcal{H}\xrightarrow{\#}y_\lambda\mathcal{H}\longrightarrow E_cM\,.$$

The basis element $x_\lambda T_d$ goes to $E_c[B]T_d^{\#}$ (up to a factor which is a power of q) for each d in $\mathcal{D}_\lambda$. Since $E_c[B]T_d^{\#}$, d in $\mathcal{D}_\lambda$, form a K-basis for E_cM by Propositions 1.9 and 2.5, the claim follows. Q.E.D.

3. Double centralizer theorem.

Since $V\mapsto eV$ is a category equivalence (1.11), the left KG module M and the left $eKGe$ module eM have isomorphic endomorphism algebras which are anti-isomorphic to $\mathcal{H}$. We prove the left $eKGe$ module eM has the following double centralizer property.

3.1. THEOREM. *The canonical algebra homomorphism*

$$eKGe\longrightarrow \mathrm{End}_{\mathcal{H}}(eM)$$

is surjective.

This property can also be thought of as an analogue of Schur's reciprocity theorem [7, (2.6c)]. The rest of this section is devoted to the proof of this theorem.

Let c, d be sequences in $I(q, n-1)$. We have only to show that every $\mathcal{H}$ homomorphism $E_dM \to E_cM$ is the left multiplication by some element in E_cKGE_d. If the idempotents E_c and E_d satisfy this property, then obviously the conjugates gE_cg^{-1} and hE_dh^{-1}, with g and h in G, also satisfy the same property. Hence we can assume

$$E_c = E_\lambda \quad \text{and} \quad E_d = E_\mu$$

with tight compositions λ and μ of n. There is a 1−1 correspondence

$$\mathrm{Hom}_{\mathcal{H}}(x_\mu \mathcal{H}, x_\lambda \mathcal{H}) \cong \mathrm{Hom}_{\mathcal{H}}(E_\mu M, E_\lambda M), \qquad \phi \leftrightarrow \psi \tag{3.2}$$

such that if $\phi(x_\mu) = x_\lambda h$ with h in $\mathcal{H}$, then $\psi(E_\mu[B]) = E_\lambda[B]h^{\#}$. This correspondence arises from #-isomorphisms

$$x_\lambda \mathcal{H} \underset{\#}{\cong} E_\lambda M \quad \text{and} \quad x_\mu \mathcal{H} \underset{\#}{\cong} E_\mu M$$

given by $x_\lambda h \leftrightarrow E_\lambda[B]h^{\#}$ and $x_\mu h \leftrightarrow E_\mu[B]h^{\#}$ for h in $\mathcal{H}$.

All $\mathcal{H}$ homomorphisms $x_\mu \mathcal{H} \to x_\lambda \mathcal{H}$ are described in [**4**] as follows. We set

$$\mathcal{D}_{\lambda,\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$$

which is a system of $Y_\lambda - Y_\mu$ double coset representatives in W. Let d be a permutation in $\mathcal{D}_{\lambda,\mu}$. There are tight compositions ν and γ of n such that

$$Y_\nu = d^{-1}Y_\lambda d \cap Y_\mu \quad \text{and} \quad Y_\gamma = dY_\mu d^{-1} \cap Y_\lambda .$$

A composition α is called a *refinement* of λ if $Y_\alpha \subset Y_\lambda$. Thus the composition ν (resp. γ) is a refinement of μ (resp. λ). We define elements in $\mathcal{H}$

$$\eta = \sum_{u \text{ in } Y_\lambda \cap \mathcal{D}_\gamma^{-1}} T_u \quad \text{and} \quad \eta' = \sum_{v \text{ in } Y_\mu \cap \mathcal{D}_\nu} T_v . \tag{3.3}$$

We have

$$\eta T_d x_\mu = \sum_{w \text{ in } Y_\lambda d Y_\mu} T_w = x_\lambda T_d \eta' .$$

We define an $\mathcal{H}$ homomorphism

$$\phi_d : x_\mu \mathcal{H} \longrightarrow x_\lambda \mathcal{H}$$

by setting

$$\phi_d(x_\mu h) = \eta T_d x_\mu h = x_\lambda T_d \eta' h, \qquad h \text{ in } \mathcal{H}.$$

3.4. Lemma [**4**, 3.4 Theorem]. *The $\mathcal{H}$ homomorphisms ϕ_d, d in $\mathcal{D}_{\lambda,\mu}$, form a K-basis for* $\mathrm{Hom}_{\mathcal{H}}(x_\mu \mathcal{H}, x_\lambda \mathcal{H})$.

For d in $\mathcal{D}_{\lambda,\mu}$, let

$$\psi_d : E_\mu M \longrightarrow E_\lambda M$$

be the $\mathcal{H}$ homomorphism corresponding to ϕ_d under the correspondence of (3.2).

Thus we have

$$\phi_d(E_\mu[B]) = E_\lambda[B]T_d{}^{\#}\eta'^{\#} = \sum_{v \text{ in } Y_{\mu} \cap \mathcal{D}_\nu} (-)^{l(d)+l(v)} E_\lambda dv[B]\,.$$

The double centralizer theorem will follow if we prove:

3.5. CLAIM. *Let λ and μ be tight compositions of n, and let d be a permutation in $\mathcal{D}_{\lambda,\mu}$. The $\mathcal{H}$ homomorphism ϕ_d is the left multiplication by some element in $E_\lambda KGE_\mu$.*

This claim is proved by induction on the fixed integer n. We will make some reductions.

3.6. REDUCTION. It is enough to show Claim 3.5 in the following three special cases.

(a) $d=1$ and λ is a refinement of μ. The corresponding ϕ_d is the inclusion $x_\mu\mathcal{H}\to x_\lambda\mathcal{H}$.

(b) $d=1$ and μ is a refinement of λ. The corresponding ϕ_d is the projection $x_\mu\mathcal{H}\to x_\lambda\mathcal{H}$, $x_\mu\mapsto x_\lambda$.

(c) $Y_\lambda d=dY_\mu$. In this case, we have $x_\lambda T_d=T_d x_\mu$, and

$$\phi_d(x_\mu h) = T_d x_\mu h = x_\lambda T_d h, \qquad h \text{ in } \mathcal{H}.$$

In fact, using the notation of (3.3), the $\mathcal{H}$ homomorphism ϕ_d factors as follows:

$$\phi_d : x_\mu\mathcal{H} \xrightarrow{\ i\ } x_\nu\mathcal{H} \xrightarrow{\ \zeta\ } x_\gamma\mathcal{H} \xrightarrow{\ p\ } x_\lambda\mathcal{H}$$

where i is the inclusion and ζ (resp. p) is the left multiplication by T_d (resp. η). Note that $p(x_\gamma)=x_\lambda$, since $\eta x_\gamma=x_\lambda$. These maps i, p, and ζ are of the form ϕ_d corresponding to cases (a), (b), and (c) respectively.

3.7. REDUCTION. We can further reduce the proof of 3.5 to the following cases.

(a) $d=1$, $\mu=(\mu_1, \cdots, \mu_{a-1}, m, \mu_{a+1}, \cdots, \mu_h)$ and $\lambda=(\mu_1, \cdots, \mu_{a-1}, i, m-i, \mu_{a+1}, \cdots, \mu_h)$.

(b) $d=1$, $\lambda=(\lambda_1, \cdots, \lambda_{a-1}, m, \lambda_{a+1}, \cdots, \lambda_h)$ and $\mu=(\lambda_1, \cdots, \lambda_{a-1}, i, m-i, \lambda_{a+1}, \cdots, \lambda_h)$.

(c) $\lambda=(\lambda_1, \cdots, \lambda_{a-1}, i, j, \lambda_{a+2}, \cdots, \lambda_h)$, $\mu=(\lambda_1, \cdots, \lambda_{a-1}, j, i, \lambda_{a+2}, \cdots, \lambda_h)$, and

$$d = \begin{pmatrix} k+1, & \cdots, & k+j, & k+1+j, & \cdots, & k+i+j \\ k+i+1, & \cdots, & k+i+j, & k+1, & \cdots, & k+i \end{pmatrix}$$

where $k=\lambda_1+\cdots+\lambda_{a-1}$.

PROOF. (a), (b) This is easy since any refinement of a composition is obtained by dividing a part into two parts successively.

(c) Assume that $Y_\lambda d=dY_\mu$. Then the composition λ is obtained by rearranging the parts of μ. There is a permutation π such that

$$\lambda=(\mu_{\pi(1)},\cdots,\mu_{\pi(h)}),$$
$$\mu=(\mu_1,\cdots,\mu_h),\qquad \text{with } \mu_a>0 \text{ if } 1\leqq a\leqq h.$$

The permutation d maps the a^{th} interval relative to μ onto the $\pi^{-1}(a)^{\text{th}}$ interval relative to λ as an order preserving isomorphism. Thus the pair (d,λ) is determined by the permutation π of the parts of μ. Let us write $d=d(\pi): \mu\to\lambda=\mu\pi$ symbolically in this case. We have

$$l(d)=\sum_{a<b,\ \pi^{-1}(a)>\pi^{-1}(b)}\mu_a\mu_b. \tag{3.7.1}$$

Take some a with $1\leqq a<h$ and $\pi^{-1}(a)>\pi^{-1}(a+1)$. Let $s=(a, a+1)$ and write $\pi=s\rho$. Then $d=d(\pi)=d(\rho)d(s)$, where $d(s): \mu\to\mu s$ and $d(\rho): \mu s\to\mu s\rho=\lambda$. Since $l(\pi)=l(\rho)+l(s)$, it follows from (3.7.1) that $l(d)=l(d(\rho))+l(d(s))$. This implies $T_d=T_{d(\rho)}T_{d(s)}$, i.e.,

$$\phi_d=\phi_{d(\rho)}\phi_{d(s)}: x_\mu\mathcal{H}\longrightarrow x_{\mu s}\mathcal{H}\longrightarrow x_\lambda\mathcal{H}.$$

Therefore 3.6 (c) reduces to the case $d(s): \mu\to\mu s$. Q.E.D.

3.8. OBSERVATION.

We identify a tight composition as a finite sequence of its nonzero parts. In each case of 3.7 above, the tight compositions λ and μ can be written in the form

$$\lambda=(\alpha,\tilde{\lambda},\beta)\quad\text{and}\quad\mu=(\alpha,\tilde{\mu},\beta)$$

where α (resp. β) is a tight composition of k (resp. m), and $\tilde{\lambda}$, $\tilde{\mu}$ are tight compositions of l, for some integers k, l, m whose sum is n. Let G_l be the subgroup of all $g=(g_{ij})$ in G such that $g_{ij}=\delta_{ij}$ unless $k+1\leqq i, j\leqq k+l$. Let $\mathcal{H}_l$ be the subalgebra of $\mathcal{H}$ spanned by all T_π with π in $W_l=W\cap G_l$. It is identified with the Hecke algebra $H_K(G_l, B_l)$ where $B_l=B\cap G_l$. In each case of 3.7, we may think d is a permutation in $\mathcal{D}_{\tilde{\lambda},\tilde{\mu}}$ $(=W_l\cap\mathcal{D}_{\lambda,\mu})$. In this case, the corresponding elements η, η' (3.3) belong to the subalgebra $\mathcal{H}_l$. Let $M_l=KG_l[B_l]$ which is a left ideal of KG_l. Let

$$\tilde{\phi}_d: x_{\tilde{\mu}}\mathcal{H}_l\longrightarrow x_{\tilde{\lambda}}\mathcal{H}_l\quad\text{and}\quad\tilde{\varphi}_d: E_{\tilde{\mu}}M_l\longrightarrow E_{\tilde{\lambda}}M_l$$

be the G_l-analogues of $\mathcal{H}$ homomorphisms ϕ_d and φ_d. It follows that there is some element ξ in KG_l (see above 3.5) such that

$$\varphi_d(E_\mu[B])=E_\lambda\xi[B]\quad\text{and}\quad\tilde{\varphi}_d(E_{\tilde{\mu}}[B_l])=E_{\tilde{\lambda}}\xi[B_l].$$

Assume there is an element ξ' in KG_l such that

(3.8.1) $$E_{\tilde{\lambda}}\xi[B_l] = \xi' E_{\tilde{\mu}}[B_l]\,.$$

We claim the same relation

(3.8.2) $$E_\lambda \xi[B] = \xi' E_\mu[B]$$

holds. If $l<n$, such an element ξ' exists by the induction hypothesis.

To prove the claim, let V be the subgroup of all $u=(u_{ij})$ in U^- such that $u_{ij}=\delta_{ij}$ if $k+1\leqq i,\ j\leqq k+l$. Note that the linear characters χ_λ and χ_μ coincide on V by (1.5). Let χ be the common restriction. The subgroup V is normalized by G_l and the character χ is G_l-invariant. Hence the idempotent

$$E = |V|^{-1}\textstyle\sum_{u \text{ in } V} \chi(u)u$$

commutes with the elements in G_l and we have by [**9**, 9.2]

$$E_\lambda = EE_{\tilde{\lambda}}, \qquad E_\mu = EE_{\tilde{\mu}}\,.$$

The equality (3.8.2) is obtained by applying $\sum_i E(-)b_i$ to (3.8.1), where $\{b_i\}$ is a system of right coset representatives for B_l in B.

Using this observation, we have arrived at the final step.

3.9. REDUCTION. It is enough to show Claim 3.5 in the following three special cases.

(a) $d=1$, $\lambda=(i,\ n-i)$, and $\mu=(n)$.

(b) $d=1$, $\lambda=(n)$, and $\mu=(i,\ n-i)$.

(c) $\lambda=(i,\ n-i)$, $\mu=(n-i,\ i)$, and $d=\begin{pmatrix} 1, & \cdots, & n-i, & n-i+1, & \cdots, & n \\ i+1, & \cdots, & n, & 1, & \cdots, & i \end{pmatrix}$.

3.10. PROPOSITION. *Claim* 3.5 *is true in each case of* 3.9.

PROOF. (a) $\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H},\ x_\lambda\mathcal{H})$ is one-dimensional, since $\mathcal{D}_{\lambda,\mu}=\{1\}$. We have only to show $E_\lambda KGE_\mu M\neq 0$. Let d be the permutation in (c) above. This d is in $\mathcal{D}_\lambda$. We claim $E_\lambda dE_\mu[B]\neq 0$. It is a linear combination of $E_\lambda\sigma[B]$ with σ in $\mathcal{D}_\lambda$ (1.9). We will compute the coefficient of $E_\lambda d[B]$. Let u be an element in U^-. It is easy to see du is in U^-dB if and only if u is of the form

$$u = \begin{pmatrix} u'' & 0 \\ 0 & u' \end{pmatrix}$$

where u' (resp. u'') is a lower unitriangular matrix of size i (resp. $n-i$). If this is the case, we have $du=\tilde{u}d$ and $\chi_\mu(u)=\chi_\lambda(\tilde{u})$ with

$$\tilde{u} = \begin{pmatrix} u' & 0 \\ 0 & u'' \end{pmatrix}.$$

Since all such matrices u form a subgroup of U^- of index $q^{i(n-i)}$, if we write $E_\lambda dE_\mu[B]$ as a linear combination of $E_\lambda\sigma[B]$ with σ in $\mathcal{D}_\lambda$, then it follows that the coefficient of $E_\lambda d[B]$ is $q^{-i(n-i)}$. Hence $E_\lambda dE_\mu[B]\neq 0$.

(b) $\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H}, x_\lambda\mathcal{H})$ is one-dimensional, too. We have shown above $E_\mu dE_\lambda[B]\neq 0$ (with λ, μ interchanged). There is a symmetric G-invariant non-degenerate bilinear form $\langle , \rangle$ on M defined as follows [**9**, 11.1]:

$$\langle g[B], h[B]\rangle = \begin{cases} 1 & \text{if } gB=hB \\ 0 & \text{otherwise.} \end{cases}$$

Let E'_λ and E'_μ be the images of E_λ and E_μ by the opposite automorphism $g\mapsto g^{-1}$. They are conjugate to E_λ and E_μ by diagonal matrices. Since $E_\mu dE_\lambda M \neq 0$, it follows that

$$\langle M, E'_\lambda d^{-1}E'_\mu M\rangle = \langle E_\mu dE_\lambda M, M\rangle \neq 0\,.$$

This implies $E'_\lambda KGE'_\mu M\neq 0$, or $E_\lambda KGE_\mu M\neq 0$.

(c) The set $\mathcal{D}_{\lambda,\mu}$ consists of permutations $\pi_0, \pi_1, \cdots, \pi_t$ with $t=\mathrm{Min}\,(i, n-i)$, where

$$\pi_a = \begin{pmatrix} a+1, & \cdots, & n-i, & n-i+1, & \cdots, & n-a \\ i+1, & \cdots, & n-a, & a+1, & \cdots, & i \end{pmatrix}, \quad 0\leqq a\leqq t.$$

Thus $\pi_0=d$. A computation similar as in (a) yields that if we write $E_\lambda dE_\mu[B]$ as a linear combination of $E_\lambda\sigma[B]$ with σ in $\mathcal{D}_\lambda$, then the coefficient of $E_\lambda d[B]$ is $q^{-i(n-i)}$. It follows that the left multiplication by $E_\lambda dE_\mu$ is of the form

$$(-q)^{-i(n-i)}\phi_{\pi_0}+c_1\phi_{\pi_1}+\cdots+c_t\phi_{\pi_t}, \qquad c_t \text{ in } K.$$

To finish the proof, it is enough to verify the homomorphism ϕ_{π_a} is the left multiplication by some element in $E_\lambda KGE_\mu$ if $a>0$. Indeed, decompose ϕ_{π_a} as the composition

$$\phi_{\pi_a}: E_\mu M \xrightarrow{\ i\ } E_\nu M \xrightarrow{\ \zeta\ } E_\gamma M \xrightarrow{\ p\ } E_\lambda M$$

as in 3.6. Here $\nu=(a, n-i-a, i-a, a)$ and $\gamma=(a, i-a, n-i-a, a)$. It follows from (a), (b) above that i and p are left multiplications by elements in KG and from 3.8 that ζ is also the left multiplication by some element in KG. Hence we are done. Q.E.D.

4. Generalized q-Schur algebras.

Let Λ be a finite set of compositions of n admitting some redundancy. Strictly speaking we are considering a pair (Λ, π) of a finite set Λ and a map $\pi: \Lambda\to C(n)$. Such a pair is called a *labelling* of compositions. Let M_Λ (resp.

$M_\Lambda^\#$) be the direct sum of right $\mathcal{H}$ modules $x_\lambda\mathcal{H}$ (resp. $y_\lambda\mathcal{H}$) for λ in Λ (λ meaning $\pi(\lambda)$ by abuse of notation). These right $\mathcal{H}$ modules have isomorphic endomorphism algebras. We put

$$S_\Lambda = \mathrm{End}_{\mathcal{H}}(M_\Lambda) \cong \mathrm{End}_{\mathcal{H}}(M_\Lambda^\#)$$

and call it the *generalized q-Schur algebra* associated with the labelling Λ.

4.1. EXAMPLES. (1) $\Lambda=\mathcal{P}(n)$. The corresponding q-Schur algebra is $\mathcal{S}(q, n)$ [**5**]. We prefer to denote it by $S_q(n)$.

(2) $\Lambda=\Lambda(d, n)$ the set of all sequences $(\lambda_1, \cdots, \lambda_d)$ of nonnegative integers whose sum is n. The corresponding q-Schur algebra is $S_q(d, n)$ [**6**][**3**] the q-analogue of the usual Schur algebra $S(d, n)$ [**7**].

(3) $\Lambda=I(q, n-1)$ with π the map $c \mapsto \lambda^c$ (1.3). The right $\mathcal{H}$ module eM is isomorphic to $M_\Lambda^\#$ by 2.6. It follows from 3.1 that the corresponding q-Schur algebra is identified with $e(KG/I_M)e$ where I_M denotes the annihilator of M in KG.

The representation theory of q-Schur algebras as developed in [**5, 6**] can be generalized to our algebras S_Λ by Morita theory. The main results will be reviewed in the following.

Let Λ be a labelling of compositions of n. For each λ in Λ let $\xi_\lambda : M_\Lambda \to x_\lambda\mathcal{H}$ be the projection onto the λ component. We get orthogonal idempotents ξ_λ (λ in Λ) in S_Λ whose sum is 1. If V is a left S_Λ module, it is the direct sum of K subspaces $V^\lambda=\xi_\lambda V$, the *λ-weight space*. If two compositions λ, μ are obtained from each other by rearranging the parts (in which case we write $\lambda\sim\mu$), then, we have $x_\lambda\mathcal{H}\cong x_\mu\mathcal{H}$, hence there are f, g in S_Λ such that $\xi_\lambda=fg$ and $\xi_\mu=gf$. This implies $\dim_K V^\lambda=\dim_K V^\mu$.

Let Λ^+ be the set of partitions α of n such that $\alpha\sim\lambda$ for some λ in Λ. We say Λ is a *full* labelling if $\Lambda^+=\mathcal{P}(n)$. Examples 4.1 (1) and (3) correspond to full labellings. The labelling $\Lambda(d, n)$ of (2) is full if $d\geq n$.

Let Λ be a full labelling. For each partition α of n, choose an element $\lambda(\alpha)$ in Λ such that $\alpha\sim\lambda(\alpha)$. Let ε be the sum of idempotents $\xi_{\lambda(\alpha)}$ for all partitions α of n. Since $\varepsilon M_\Lambda\cong M_{\mathcal{P}(n)}$ as right $\mathcal{H}$ modules, we have $\varepsilon S_\Lambda\varepsilon\cong S_q(n)$ and the following functor of Schur type:

$$\mathbf{mod}\, S_\Lambda \longrightarrow \mathbf{mod}\, S_q(n)\,, \qquad V \mapsto \varepsilon V\,. \tag{4.2}$$

The following proposition follows directly from Morita theory.

4.3. PROPOSITION. *Let Λ be a full labelling of compositions of n. With the idempotent ε defined above, we have*

(i) *$\varepsilon S_\Lambda\varepsilon\cong S_q(n)$ as algebras,*

(ii) *The functor* (4.2) *is a category equivalence.*

It follows that the following algebras are Morita equivalent with one another (with n fixed).

$$(1)\quad S_q(n),\qquad (2)\quad S_q(d,\ n),\ d\geqq n,\qquad (3)\quad KG/I_M.$$

The Morita equivalence of (3) and (1) is realized as the composite of category equivalences 1.11 and 4.2 with $\Lambda=I(q,\ n-1)$.

q-Weyl modules play a crucial role in the representation theory of S_Λ. The main results of [**6**] can be translated to S_Λ via the category equivalence 4.2 if Λ is full. Some of them hold even if Λ is not full.

Let Λ be a labelling of compositions of n. For a composition λ of n, let λ' be the partition dual to λ. It is known that $x_\lambda \mathcal{H} y_{\lambda'}$ is one-dimensional [**4**, 4.1]. If λ is in Λ, this space is identified with $(M_\Lambda)^\lambda y_{\lambda'}$ where $(M_\Lambda)^\lambda=x_\lambda \mathcal{H}$ the λ-weight space. Let W_λ be the S_Λ submodule of M_Λ generated by the subspace $(M_\Lambda)^\lambda y_{\lambda'}$. It is called the q-Weyl module associated with λ. If two weights λ, μ are equivalent under $\sim$, then $W_\lambda=W_\mu$. Therefore we can well-define the q-Weyl module W_α for each partition α in Λ^+.

The q-Weyl modules W_λ (for various labellings) correspond with one another under the Morita equivalence arising in 4.3. If $\Lambda=\Lambda(d,\ n)$ the q-Weyl modules coincide with those in [**6**].

If we use $M_\Lambda^\#$ instead of M_Λ, the q-Weyl module is defined by $W_\lambda= S_\Lambda(M_\Lambda^\#)^\lambda x_{\lambda'}$, λ in Λ.

The q-Weyl module W_λ is a highest weight module and has a unique maximal submodule $W_\lambda^{\max}$. The quotient S_Λ module $F_\lambda=W_\lambda/W_\lambda^{\max}$ is absolutely irreducible self-dual. If Λ is full, the modules F_α for all partitions α of n, give a complete set of non-isomorphic irreducible S_Λ modules, as a consequence of [**6**, 8.8] and the equivalence (4.2).

The following proposition yields (c) of Theorem in the Introduction.

4.4. PROPOSITION. *Under the category equivalence of* 1.11:

$$\mathbf{mod}\ KG/I_M \longrightarrow \mathbf{mod}\ e(KG/I_M)e=\mathbf{mod}\ S_\Lambda$$

with $\Lambda=I(q,\ n-1)$. *we have*

$$eS_\lambda \cong W_{\lambda'} \quad\text{and}\quad eD_\lambda \cong F_{\lambda'}$$

for all compositions λ *of* n.

PROOF. We have $M_\lambda=Mx_\lambda$ (2.1) and $M_\lambda\cong M_{\lambda''}$ since $\lambda\sim\lambda''$ (1.10) (5). We may identify $E_\lambda M$ with the λ-weight space $(M_\Lambda^\#)^\lambda$ by Theorem 2.6. It follows that

$$eS_\lambda = eKGE_{\lambda'}M_\lambda \cong eKGE_{\lambda'}M_{\lambda''} = S_\Lambda(M_\Lambda^\#)^{\lambda'}x_{\lambda''} = W_{\lambda'}\,.$$

Obviously, this induces $eD_\lambda\cong F_{\lambda'}$. Q.E.D.

Some of the properties 1.10 on KG modules S_λ and D_λ correspond to the analogous properties on S_A modules W_λ and F_λ via the category equivalence. The fact on decomposition numbers we mentioned after Theorem in the Introduction follows directly.

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