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Boundary distance functions and q-convexity of pseudoconvex domains of general order in Kähler manifolds

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Introduction.

Let M be an n-dimensional Kähler manifold with C^{∞} Kähler metric G, let D be an open subset of M, and let $d_{\partial D}$ be the boundary distance function of D induced by the metric G.

When D is pseudoconvex (in the usual sense) in M, the plurisubharmonicity of the function $-\log d_{\partial D}$ is closely related to the holomorphic bisectional curvature of M. Takeuchi [**26**] first showed that, if D is a pseudoconvex open subset of the complex projective space $P^{n}(\boldsymbol{C})$ and if $d_{\partial D}$ is the boundary distance function of D with respect to the Fubini-Study metric on $P^{n}(\boldsymbol{C})$, the function $-\log d_{\partial D}$ is strongly plurisubharmonic on D. After the works of Takeuchi [**27**], Elencwajg [**6**], Suzuki [**24**] and others, Greene-Wu [**11**] differential-geometrically gave an estimate from below for 'the modulus of plurisubharmonicity' of the function $-\log d_{\partial D}$, and showed that a relatively compact, pseudoconvex open subset D of M is 1-complete (and hence Stein) if M has positive holomorphic bisectional curvature.

In this paper, we shall extend the result to the case where D is pseudoconvex of order $n-q$ in M and show that D is q-convex or q-complete (with corners) in several cases.

An open subset D of M is said to be pseudoconvex of order $n-q$, $1\leqq q\leqq n$, in M if, roughly speaking, the complement $M\backslash D$ has the same continuity as an analytic set of pure dimension $n-q$. Pseudoconvex open subsets in the usual sense are pseudoconvex of order $n-1$. If $D\subset M$ is weakly q-convex, then D is pseudoconvex of order $n-q$ in M. However, when $2\leqq q\leqq n-1$, the converse is not valid even if $D\subset\boldsymbol{C}^{n}$ (see Diederich-Fornaess [**4**] and Matsumoto [**13**]). By Fujita [**8**], an open subset D of $\boldsymbol{C}^{n}$ is pseudoconvex of order $n-q$ in $\boldsymbol{C}^{n}$, if and only if D has an exhaustion function which is pseudoconvex of order $n-q$ on D. Therefore, by the approximation theorem of Bungart [**3**], an open subset D of M is pseudoconvex of order $n-q$ in M, if and only if D is locally q-complete with corners in M in the sense of Peternell [**16**] (for the precise, see §§ 1 and 2).

The main results of this paper are as follows.

At first, let M be an n-dimensional Kähler manifold with positive holomorphic bisectional curvature and let D be a relatively compact, pseudoconvex open subset of order $n-q$ in M. Then the function $-\log d_{\partial D}$ is strongly pseudoconvex of order $n-q$ whole on D and particularly q-convex on the open subset of D (if it exists) where $d_{\partial D}$ is of class C^2 (see Corollary 6.5). Therefore, by the approximation theorems of Bungart and Diederich-Fornaess, the set D is q-complete with corners and hence $\tilde{q}$-complete, where $\tilde{q}=n-[n/q]+1$ and [ ] denotes the Gauss symbol (see Theorem 6.6). Moreover, if the boundary ∂D is also a real submanifold of class C^2 in M, then D is q-convex (see Theorem 6.2).

Secondly, let M be an n-dimensional Stein manifold and let D be a pseudoconvex open subset of order $n-q$ in M. Let $d_{\partial D}$ be a boundary distance function of D induced by a complete Kähler metric on M. Then there exists a 1-convex function h on M such that the function $-\log d_{\partial D}+h$ is strongly pseudoconvex of order $n-q$ on D (see Proposition 7.2). Therefore, the set D is q-complete with corners and hence $\tilde{q}$-complete (see Theorem 7.3). Moreover, if the boundary ∂D is also a real submanifold of class C^2 in M, then D is q-complete (see Theorem 7.6).

The above results are extensions (and different proofs) of that of Barth [**2**] and that of Suria [**23**] (or Eastwood-Suria [**5**]), respectively.

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1. Pseudoconvex functions of general order and q-convex functions with corners.

Throughout this paper, let D be a paracompact complex manifold of pure dimension n and q an integer with $1 \leqq q \leqq n$. After §4 we consider only the case where D is an open subset of another connected Kähler manifold M, but we do not require D to be Kählerian in the first three sections.

A function $\varphi: D \to \boldsymbol{R}$ is said to be *q-convex* (resp. *weakly q-convex*), if φ is of class C^2 on D and if its Levi form $\partial\bar{\partial}\varphi$ has at least $n-q+1$ positive (resp. non-negative) eigenvalues on the holomorphic tangent space $T_P(D)$ for each $P \in D$ (see Andreotti-Grauert [**1**]). As extensions of the notion of weakly q-convex functions or (upper semi-continuous) plurisubharmonic functions, Hunt-Murray [**12**] and Fujita [**8**] introduced that of $(q-1)$-plurisubharmonic functions and that of pseudoconvex functions of order $n-q$, respectively. Further, Fujita [**9**] proved that they are equivalent. For the original definitions and fundamental properties of them, see Fujita [**8**], Hunt-Murray [**12**] and Slodkowski [**21**], [**22**].

In this paper we shall give the definition as follows.

DEFINITION 1.1. An upper semi-continuous function $\varphi: D \to \boldsymbol{R} \cup \{-\infty\}$ is said to be *pseudoconvex of order* $n-q$ *at* $P \in D$ if, for each weakly $(n-q+1)$-convex function f defined near P, one can find a neighborhood $U(f)$ of P, so that

$$(\varphi+f)(P) \leqq \max\{(\varphi+f)(Q) : Q \in \partial\Delta\}$$

for every domain Δ with $P \in \Delta$ and $\Delta \Subset U(f)$. A function $\varphi: D \to \boldsymbol{R} \cup \{-\infty\}$ is said to be *pseudoconvex of order* $n-q$ *on* D, if φ is upper semi-continuous on D and if φ is pseudoconvex of order $n-q$ at each $P \in D$.

Using the criterion of $(q-1)$-plurisubharmonicity due to Slodkowski ([**21**], Proposition 1.1, (iii)), we can immediately prove that φ is pseudoconvex of order $n-q$ on D in the sense of Definition 1.1, if and only if φ is $(q-1)$-plurisubharmonic on D in the sense of Hunt-Murray [**12**]. Therefore, φ is pseudoconvex of order $n-q$ on D in the sense of Definition 1.1, if and only if so is φ in the sense of Fujita [**8**].

Plurisubharmonic functions in the usual sense are pseudoconvex functions of order $n-1$.

If f is weakly $(n-q+1)$-convex and if h is weakly 1-convex, then $f+h$ is weakly $(n-q+1)$-convex. Using this fact, we can easily verify that if φ is pseudoconvex of order $n-q$ at P and if h is weakly 1-convex near P, then $\varphi+h$ is pseudoconvex of order $n-q$ at P.

LEMMA 1.2. *An upper semi-continuous function* $\varphi: D \to \boldsymbol{R} \cup \{-\infty\}$ *is pseudoconvex of order* $n-q$ *at* $P \in D$, *if there exists an* $(n-q+1)$*-dimensional complex submanifold* L *defined near* P *and containing* P *such that the restriction* $\varphi|_L$ *is pseudoconvex of order* $n-q$ *at* $P \in L$ (*and particularly plurisubharmonic near* $P \in L$).

PROOF. Let f be a weakly $(n-q+1)$-convex function defined near $P \in D$. Then $f|_L$ is also weakly $(n-q+1)$-convex near $P \in L$. If $\varphi|_L$ is pseudoconvex of order $n-q$ at $P \in L$, we can by definition find a neighborhood $U' = U'(f|_L)$ $(\subset L)$ of $P \in L$, so that

$$(\varphi|_L + f|_L)(P) \leqq \max\{(\varphi|_L + f|_L)(Q) : Q \in \partial\Delta'\}$$

for every domain Δ' with $P \in \Delta'$ and $\Delta' \Subset U'$. Choose a neighborhood $U = U(f)$ $(\subset D)$ of $P \in D$ so that $U \cap L \subset U'$. Let Δ be a domain with $P \in \Delta$ and $\Delta \Subset U$, and denote by Δ' the connected component of $\Delta \cap L$ containing P. Then $P \in \Delta'$ and $\Delta' \Subset U'$. Moreover, we have

$$\begin{aligned}(\varphi+f)(P) &\leqq \max\{(\varphi|_L + f|_L)(Q) : Q \in \partial\Delta'\} \\ &\leqq \max\{(\varphi+f)(Q) : Q \in \partial\Delta\}.\end{aligned}$$

This implies that φ is pseudoconvex of order $n-q$ at $P\in D$. □

A C^2 function φ is pseudoconvex of order $n-q$ on D, if and only if φ is weakly q-convex on D (see Fujita [8], Proposition 8). It is well-known that every (upper semi-continuous) plurisubharmonic function defined on an open subset of $\boldsymbol{C}^n$ can be approximated by 1-convex functions. However, pseudoconvex functions of order $n-q$ cannot be approximated by q-convex functions in general. We shall next recall the approximation theorems of Diederich-Fornaess and Bungart.

DEFINITION 1.3 (Diederich-Fornaess [4]). A function $\varphi: D\to\boldsymbol{R}$ is said to be *q-convex with corners* on D if, for each $P\in D$, there exist a neighborhood U of P and (strongly) q-convex functions $\varphi_1, \varphi_2, \cdots, \varphi_{t(P)}$ on U such that $\varphi|_U=\max\{\varphi_1, \varphi_2, \cdots, \varphi_{t(P)}\}$.

DEFINITION 1.4 (cf. Bungart [3]). A function $\varphi: D\to\boldsymbol{R}\cup\{-\infty\}$ is said to be *strongly pseudoconvex of order $n-q$* on D (or strictly $(q-1)$-plurisubharmonic on D in the sense of Bungart [3]) if, for each $P\in D$, there exist a neighborhood U of P and a (strongly) 1-convex function h on U such that $\varphi-h$ is pseudoconvex of order $n-q$ on U.

It is clear that every q-convex function with corners is strongly pseudoconvex of order $n-q$. Conversely, if φ is strongly pseudoconvex of order $n-q$ and if φ is piecewise C^2, that is, φ is locally a maximum of a finite number of C^2 functions, then φ is q-convex with corners (see Matsumoto [13], p. 73).

Diederich-Fornaess showed the following approximation theorem.

THEOREM 1.5 ([4], Theorem 1). *Let D be an n-dimensional paracompact complex manifold and φ a q-convex function with corners on D. Then, for any continuous function $\varepsilon>0$ on D, there exists a $\tilde{q}$-convex function ψ on D such that $|\varphi-\psi|<\varepsilon$ on D, where $\tilde{q}=n-[n/q]+1$ and $[\ ]$ denotes the Gauss symbol.*

Diederich-Fornaess ([4], Theorem 2) further showed that the number $\tilde{q}$ in Theorem 1.5 is best possible for any pair (n, q). Note that $\tilde{q}>q$ when $2\leqq q\leqq n-1$.

On the other hand, Bungart showed the following approximation theorem.

THEOREM 1.6 ([3], Theorem 5.3). *Let D be an n-dimensional paracompact complex manifold and φ a continuous strongly pseudoconvex function of order $n-q$ on D. Then, for any continuous function $\varepsilon>0$ on D, there exists a q-convex function ψ with corners on D such that $|\varphi-\psi|<\varepsilon$ on D.*

REMARK 1.7. Bungart [3] asserted Theorem 1.6 only when $D\subset\boldsymbol{C}^n$. In view of his proof, the theorem remains valid when D is a paracompact complex manifold.

REMARK 1.8. By the definition in this paper, a q-convex function with corners is piecewise C^2. Since every C^2 function can be locally approximated by C^∞ functions with respect to (Whitney) C^2 topology, every q-convex function with corners defined on a paracompact complex manifold can be globally approximated by such piecewise C^∞ functions. Therefore, we can choose the q-convex function ϕ with corners in Theorem 1.6 so that it is also piecewise C^∞.

2. Pseudoconvex domains of general order and q-convex domains with corners.

Let D be a complex manifold and $\varphi : D \to \boldsymbol{R} \cup \{-\infty\}$ an upper semi-continuous function. Then φ is said to be an *exhaustion function* of D if $\{P \in D : \varphi(P) < A\} \Subset D$ for every $A \in \boldsymbol{R}$.

A complex manifold D is said to be *q-convex* (resp. *q-convex with corners*) if D has a continuous exhaustion function which is q-convex (resp. q-convex with corners) outside some compact subset of D. Further, D is said to be *q-complete* (resp. *q-complete with corners*) if D has an exhaustion function which is q-convex (resp. q-convex with corners) whole on D (see Andreotti-Grauert [**1**] and Diederich-Fornaess [**4**]).

It is clear that D is q-convex (resp. q-complete) with corners if D is q-convex (resp. q-complete). When $2 \leqq q \leqq n-1$, the converse is not valid even if $D \subset \boldsymbol{C}^n$ (see Diederich-Fornaess [**4**] and Matsumoto [**13**]). By the Diederich-Fornaess approximation theorem (Theorem 1.5), an n-dimensional complex manifold D is $\tilde{q}$-convex (resp. $\tilde{q}$-complete) if D is q-convex (resp. q-complete) with corners, where $\tilde{q} = n - [n/q] + 1$. Moreover, by the Bungart approximation theorem (Theorem 1.6), D is q-complete with corners, if and only if D has an exhaustion function which is strongly pseudoconvex of order $n-q$ on D.

In what follows, let M be a connected, paracompact complex manifold of dimension n.

An open subset D of M is said to be *pseudoconvex of order $n-q$* in M, if the complement $M \setminus D$ satisfies 'the Hartogs continuity principle of dimension $n-q$' (see Tadokoro [**25**] for the precise definition; and see also Riemenschneider [**18**] and Fujita [**8**]).

The pseudoconvexity of order $n-q$ of an open subset D in M is a local property of the boundary ∂D ($\subset M$) of D. More precisely, D is pseudoconvex of order $n-q$ in M if, for each $Q \in \partial D$, there exists a neighborhood V ($\subset M$) of Q such that $D \cap V$ is pseudoconvex of order $n-q$ in V.

When $M = \boldsymbol{C}^n$, Fujita showed the following.

THEOREM 2.1 ([**8**], Théorème 2). *For an open subset D of $\boldsymbol{C}^n$, the following conditions are equivalent:*

(a) *D is pseudoconvex of order $n-q$ in $\boldsymbol{C}^n$.*

(b) *D has an exhaustion function which is pseudoconvex of order $n-q$ on D.*

(c) *$-\log d_{\partial D}(z)$ is pseudoconvex of order $n-q$ on D, where $d_{\partial D}(z)=\inf\{\|z-w\|: w\in\partial D\}$ is the Euclidean boundary distance of D at $z\in D$.*

Using Theorem 2.1 and the Bungart approximation theorem (Theorem 1.6), we can easily prove the following.

PROPOSITION 2.2. *An open subset D of $\boldsymbol{C}^n$ is pseudoconvex of order $n-q$ in $\boldsymbol{C}^n$, if and only if D is q-complete with corners. Therefore, an open subset D of an n-dimensional complex manifold M is pseudoconvex of order $n-q$ in M, if and only if D is locally q-complete with corners in M in the sense of Peternell* [**16**].

Now we shall give some examples of pseudoconvex open subsets of order $n-q$.

EXAMPLE 2.3. Let D be an open subset of an n-dimensional complex manifold M and suppose that the boundary ∂D is a real hypersurface of class C^2 in M, that is, there exist, for each $Q\in\partial D$, a neighborhood V of Q and a C^2 function $\rho: V\to\boldsymbol{R}$ such that $d\rho(Q)\neq 0$ and $D\cap V=\{P\in V: \rho(P)<0\}$. Then D is pseudoconvex of order $n-q$ in M, if and only if the Levi form $\partial\bar{\partial}\rho$ has at least $n-q$ non-negative eigenvalues on $T'_Q(\partial D)$ for each $Q\in\partial D$ and for each defining function ρ of D near Q, where $T'_Q(\partial D)$ $(\subset T_Q(\partial D))$ is the holomorphic tangent space of the real hypersurface ∂D at Q. (Eastwood-Suria [**5**] and Suria [**23**] called such a subset D a $(q-1)$-pseudoconvex open subset with C^2 boundary.)

EXAMPLE 2.4. Let S be an analytic subset of an n-dimensional complex manifold M and denote by k the minimum of dimensions of irreducible components of S. Then the complement $M\backslash S$ is pseudoconvex of order $n-q$ in M if and only if $k\geqq n-q$. Moreover, an open subset D of M is pseudoconvex of order $n-q$ in M if, for each $Q\in\partial D$, there exists a purely $(n-q)$-dimensional analytic subset S defined near Q such that $Q\in S$ and $S\subset M\backslash D$.

In this paper, we introduce the following condition (C_q).

DEFINITION 2.5. We say that an open subset D of an n-dimensional complex manifold M satisfies the condition (C_q) in M, if

(C_q) For each $Q\in\partial D$, there exists an $(n-q)$-dimensional complex submanifold defined near Q such that $Q\in S$ and $S\subset M\backslash D$.
For the sake of simplicity, we agree that M itself and the empty set satisfy the condition (C_q) in M.

Every open subset with the condition (C_q) in M is pseudoconvex of order $n-q$ in M. If S is a complex submanifold of M and if each connected component of S has at least dimension $n-q$, the complement $M\backslash S$ obviously satisfies the condition (C_q) in M.

LEMMA 2.6. *Let φ be a q-convex function with corners defined on a complex manifold D and suppose that φ is also piecewise C^∞. Then there exists a subset Λ of Lebesgue measure zero in $\boldsymbol{R}$ such that the set $\{P\in D: \varphi(P)<A\}$ satisfies the condition (C_q) in D for every $A\in\boldsymbol{R}\setminus\Lambda$.*

PROOF. Let U be an open subset of D and $\psi: U\to\boldsymbol{R}$ a q-convex function of class C^∞. For each $A\in\boldsymbol{R}$, define the set U_A by $U_A=\{P\in U: \psi(P)<A\}$. If the value A of ψ is not critical and if the boundary ∂U_A $(\subset U)$ of U_A is not empty, then ∂U_A is a real hypersurface of class C^∞ in U and so U_A satisfies the condition (C_q) in U. On the other hand, the Sard theorem asserts that the set of the critical values of ψ is of Lebesgue measure zero in $\boldsymbol{R}$, if $\psi: U\to\boldsymbol{R}$ is of class C^∞ (at least of class C^{2n}). The lemma follows from the two facts. □

Using Lemma 2.6 we can easily prove the following.

LEMMA 2.7. *If a complex manifold D is q-convex with corners, there exists a sequence $\{D_\nu\}_{\nu\in N}$ of open subsets with the condition (C_q) in D such that $D_\nu\Subset D_{\nu+1}\Subset D$ for each $\nu\in\boldsymbol{N}$ and $\bigcup_{\nu=1}^\infty D_\nu=D$.*

3. The definition and some properties of the operator W_q.

Throughout § 3, let M be a connected, paracompact complex manifold of dimension n and G a (fixed) Hermitian metric on M. Let D be an open subset of M and q an integer with $1\leqq q\leqq n$.

Given a continuous function $\varphi: D\to\boldsymbol{R}$ and a point $P\in D$, the quantity $W[\varphi](P)$ introduced by Takeuchi [**26**], [**27**] is very useful to study plurisubharmonic functions defined on Kähler manifolds (see also Elencwajg [**6**], Suzuki [**24**] and Greene-Wu [**11**]). Roughly speaking, the quantity $W[\varphi](P)$ means 'the modulus of plurisubharmonicity' of φ at P. In this section, we shall introduce the quantity $W_q[\varphi](P)$ meaning 'the modulus of pseudoconvexity of order $n-q$' of φ at P and give some properties of the operator W_q (see Remark 3.5 below for the relation between the operators W and W_q).

DEFINITION 3.1. A local coordinate system $(z_1, \cdots, z_n)$ around $P\in M$ is said to be *normal* at P (with respect to G), if

$$z_i(P)=0, \quad G\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\Big)(P)=\delta_{ij} \qquad \text{for } 1\leqq i,\ j\leqq n.$$

Every point P of M has a normal coordinate system at P. If local coordinate systems $(z_1, \cdots, z_n)$ and $(w_1, \cdots, w_n)$ are both normal at P, the transformation matrix $(\partial z_i/\partial w_j)$ is unitary at P. Therefore, if a function φ defined near P is of class C^2, all the eigenvalues of the Hermitian matrix $(\partial^2\varphi/\partial z_i\partial\bar{z}_j)(P)$ coincide those of $(\partial^2\varphi/\partial w_i\partial\bar{w}_j)(P)$. We shall only call them eigenvalues of the Levi form $\partial\bar{\partial}\varphi$ at P.

DEFINITION 3.2. Let $\varphi: D\to\boldsymbol{R}\cup\{-\infty\}$ be upper semi-continuous and $P\in D$. Let $z=(z_1, \cdots, z_n)$ be a normal coordinate system at P. We define the quantity $W_q[\varphi](P)$ as the supremum of $\alpha\in\boldsymbol{R}$ such that $\varphi-\alpha\|z\|^2$ is pseudoconvex of order $n-q$ at P, where $\|z\|^2=\sum_{i=1}^n|z_i|^2$. If no such $\alpha\in\boldsymbol{R}$ exists, we put $W_q[\varphi](P)=-\infty$.

The following lemma implies that the quantity $W_q[\varphi](P)$ is well-defined, that is, it is independent of the choice of a normal coordinate system at P.

LEMMA 3.3. *Let $\varphi: D\to\boldsymbol{R}\cup\{-\infty\}$ be upper semi-continuous and $P\in D$. Suppose that $z=(z_1, \cdots, z_n)$ and $w=(w_1, \cdots, w_n)$ are both normal coordinate systems at P. If $\varphi-\alpha\|z\|^2$ is pseudoconvex of order $n-q$ at P, so is $\varphi-\beta\|w\|^2$ for every $\beta<\alpha$.*

PROOF. We put $h=\alpha\|z\|^2-\beta\|w\|^2$. Then h is 1-convex near P because all the eigenvalues of $\partial\bar{\partial}h$ are equal to $\alpha-\beta$ (>0) at P. Therefore, $\varphi-\beta\|w\|^2=\varphi-\alpha\|z\|^2+h$ is pseudoconvex of order $n-q$ at P if so is $\varphi-\alpha\|z\|^2$. □

In particular, Lemma 3.3 implies that φ is pseudoconvex of order $n-q$ at P if $W_q[\varphi](P)>0$.

Using Lemma 3.3, we can immediately prove the following.

LEMMA 3.4. *Let $\varphi: D\to\boldsymbol{R}\cup\{-\infty\}$ be upper semi-continuous, $P\in D$, and $\alpha\in\boldsymbol{R}$. Then the following conditions are equivalent*:

(a) $W_q[\varphi](P)\geqq\alpha$.

(b) *There exists a normal coordinate system $z=(z_1, \cdots, z_n)$ at P such that $\varphi-\beta\|z\|^2$ is pseudoconvex of order $n-q$ at P for every $\beta<\alpha$.*

(c) *$\varphi-\beta\|z\|^2$ is pseudoconvex of order $n-q$ at P for every normal coordinate system $z=(z_1, \cdots, z_n)$ at P and for every $\beta<\alpha$.*

Let $\varphi: D\to\boldsymbol{R}$ be of class C^2 and $P\in D$. Denote all the eigenvalues of $\partial\bar{\partial}\varphi$ at P by $\alpha_1, \alpha_2, \cdots, \alpha_n$, where $\alpha_1\geqq\alpha_2\geqq\cdots\geqq\alpha_n$. Then we have $W_q[\varphi](P)=\alpha_{n-q+1}$. Moreover, $W_q[\varphi]: D\to\boldsymbol{R}$ is continuous if φ is of class C^2. When φ is not of class C^2, the function $W_q[\varphi]$ is not continuous in general.

REMARK 3.5. If W denotes the operator introduced by Takeuchi, then $W[\varphi]=4W_1[\varphi]$ for every C^2 function φ (see Takeuchi [**27**], p. 335). The author does not know whether the operators W and $4W_1$ exactly coincide or not.

A C^2 function $\varphi: D\to\boldsymbol{R}$ is q-convex (resp. weakly q-convex) on D if and only if $W_q[\varphi]>0$ (resp. $W_q[\varphi]\geqq 0$) on D. Moreover, we obtain the following.

PROPOSITION 3.6. *Let* $\varphi: D\to \boldsymbol{R}\cup\{-\infty\}$ *be an upper semi-continuous function. Then*

(a) φ *is pseudoconvex of order* $n-q$ *on* D *if and only if* $W_q[\varphi]\geqq 0$ *on* D.

(b) φ *is strongly pseudoconvex of order* $n-q$ *on* D *if and only if, for each* $P\in D$, *there exist a neighborhood* U *of* P *and a constant* $\varepsilon>0$ *such that* $W_q[\varphi]\geqq\varepsilon$ *on* U.

PROOF. The proof of (b) is easy. The necessity of (a) is obvious. To prove the sufficiency of (a), suppose that $W_q[\varphi]\geqq 0$ on D and (U, z), $z=(z_1, \cdots, z_n)$, is any coordinate neighborhood of D. For each $\nu\in\boldsymbol{N}$, define the function φ_ν on U by $\varphi_\nu=\varphi+(1/\nu)\|z\|^2$. Then $W_q[\varphi_\nu]>0$ on U. This implies that each φ_ν is pseudoconvex of order $n-q$ at each point of U and hence on U. Therefore, by Fujita ([8], Proposition 7), the limit φ of the decreasing sequence $\{\varphi_\nu\}_{\nu\in\boldsymbol{N}}$ is pseudoconvex of order $n-q$ on U, which proves the sufficiency of (a). □

PROPOSITION 3.7. *Let* $\varphi_\nu: D\to\boldsymbol{R}\cup\{-\infty\}$, $\nu\in\boldsymbol{N}$, *be upper semi-continuous and let* $\alpha: D\to\boldsymbol{R}$ *be continuous. Suppose that* $W_q[\varphi_\nu]\geqq\alpha$ *on* D *for all* $\nu\in\boldsymbol{N}$. *If the sequence* $\{\varphi_\nu\}_{\nu\in\boldsymbol{N}}$ *is decreasing or uniformly convergent on* D, *then* $W_q[\varphi]\geqq\alpha$ *on* D, *where* $\varphi=\lim_{\nu\to\infty}\varphi_\nu$.

PROOF. Let P be a point of D and β a real number with $\beta<\alpha(P)$. Let (U, z), $z=(z_1, \cdots, z_n)$, be a normal coordinate neighborhood at P. Choose a neighborhood V ($\subset U$) of P so that $W_1[\beta\|z\|^2]<\alpha$ on V. Then, for each $\nu\in\boldsymbol{N}$, we have $W_q[\varphi_\nu-\beta\|z\|^2]>0$ on V and so $\varphi_\nu-\beta\|z\|^2$ is pseudoconvex of order $n-q$ on V. Since the sequence $\{\varphi_\nu-\beta\|z\|^2\}_{\nu\in\boldsymbol{N}}$ is decreasing or uniformly convergent on V, it follows by Fujita ([8], Proposition 7) that the limit $\varphi-\beta\|z\|^2$ is also pseudoconvex of order $n-q$ on V. Therefore, we have $W_q[\varphi](P)\geqq\alpha(P)$ for every $P\in D$. □

The following criterion will be used frequently in this paper.

LEMMA 3.8. *Let* φ *and* ψ *be upper semi-continuous functions from* D *to* $\boldsymbol{R}\cup\{-\infty\}$ *and* P *a point of* D. *If* $\varphi(P)=\psi(P)$ *and* $\varphi\geqq\psi$ *on* D, *then* $W_q[\varphi](P)\geqq W_q[\psi](P)$.

PROOF. Let $z=(z_1, \cdots, z_n)$ be a normal coordinate system at P and α a real number with $\alpha<W_q[\psi](P)$. Then $\psi-\alpha\|z\|^2$ is pseudoconvex of order $n-q$ at P. Hence, for each weakly $(n-q+1)$-convex function f defined near P, one can find a neighborhood $U(f)$ of P, so that

$$(\psi-\alpha\|z\|^2+f)(P)\leqq\max\{(\psi-\alpha\|z\|^2+f)(Q): Q\in\partial\Delta\}$$

for every domain Δ with $P\in\Delta$ and $\Delta\Subset U(f)$. If $\varphi(P)=\psi(P)$ and $\varphi\geqq\psi$ on D, the above inequality replaced ψ with φ remains valid. Therefore, $\varphi-\alpha\|z\|^2$ is also pseudoconvex of order $n-q$ at P for every $\alpha<W_q[\psi](P)$ and hence we obtain $W_q[\varphi](P)\geqq W_q[\psi](P)$. □

Next, let L be a t-dimensional complex submanifold of D ($\subset M$), $1\leqq t\leqq n$. Then L has the C^∞ Hermitian metric $G|_L$ induced by the metric G on M. In exactly the same way as the definition of the operators W_q, $1\leqq q\leqq n$, on M with respect to the metric G on M, we can define the operators on L with respect to the metric $G|_L$ on L. We shall denote them by $W_q^{(L)}$, $1\leqq q\leqq t$. The results about $W_q=W_q^{(M)}$ are naturally valid for $W_q^{(L)}$.

LEMMA 3.9. *Let L be a t-dimensional complex submanifold of D. Let $\varphi: L\rightarrow \boldsymbol{R}\cup\{-\infty\}$ be upper semi-continuous, $P\in L$, and $\alpha\in\boldsymbol{R}$. Then the following conditions are equivalent*:

(a) $W_q^{(L)}[\varphi](P)\geqq\alpha$.

(b) *There exists a normal coordinate system $z=(z_1, \cdots, z_n)$ at $P\in D$ such that $\varphi-\beta(\|z\|^2)|_L$ is pseudoconvex of order $t-q$ at $P\in L$ for every $\beta<\alpha$.*

(c) *$\varphi-\beta(\|z\|^2)|_L$ is pseudoconvex of order $t-q$ at $P\in L$ for every normal coordinate system $z=(z_1, \cdots, z_n)$ at $P\in D$ and for every $\beta<\alpha$.*

PROOF. If $w=(w_1, \cdots, w_n)$ is a normal coordinate system of D at $P\in D$ with respect to the metric G on D and if L is written by $w_{t+1}=w_{t+2}=\cdots=w_n=0$ near P, then $w'=(w_1, \cdots, w_t)$ is a normal coordinate system of L at $P\in L$ with respect to the metric $G|_L$ on L. Hence it follows from Lemma 3.4 that $W_q^{(L)}[\varphi](P)\geqq\alpha$ if and only if $\varphi-\beta(\|w\|^2)|_L=\varphi-\beta\|w'\|^2$ is pseudoconvex of order $t-q$ at $P\in L$ for every $\beta<\alpha$. This implies that (c)$\Rightarrow$(a)$\Rightarrow$(b).

To prove (b)$\Rightarrow$(c), suppose that $z=(z_1, \cdots, z_n)$ and $w=(w_1, \cdots, w_n)$ are both normal coordinate systems of D at $P\in D$. Let β and γ be real numbers with $\beta<\gamma<\alpha$. Then the function $h:=\gamma\|w\|^2-\beta\|z\|^2$ is 1-convex near $P\in D$ and so the restriction $h|_L$ is also 1-convex near $P\in L$. Therefore, if $\varphi-\gamma(\|w\|^2)|_L$ is pseudoconvex of order $t-q$ at $P\in L$, so is $\varphi-\beta(\|z\|^2)|_L=\varphi-\gamma(\|w\|^2)|_L+h|_L$. This implies that (b)$\Rightarrow$(c). □

LEMMA 3.10. *Let $\varphi: D\rightarrow\boldsymbol{R}\cup\{-\infty\}$ be upper semi-continuous and $P\in D$. Then $W_q[\varphi](P)\geqq\alpha$, if there exists an $(n-q+1)$-dimensional complex submanifold L defined near P such that $P\in L$ and $W_1^{(L)}[\varphi|_L](P)\geqq\alpha$.*

PROOF. Let $z=(z_1, \cdots, z_n)$ be a normal coordinate system of D at $P\in D$ and β a real number with $\beta<\alpha$. Since $W_1^{(L)}[\varphi|_L](P)\geqq\alpha$, it follows from Lemma 3.9 that $(\varphi-\beta\|z\|^2)|_L=\varphi|_L-\beta(\|z\|^2)|_L$ is pseudoconvex of order $n-q$ $(=(n-q+1)-1)$ at $P\in L$. Hence, by Lemma 1.2, $\varphi-\beta\|z\|^2$ is pseudoconvex of order $n-q$ at $P\in D$ for every $\beta<\alpha$. This means that $W_q[\varphi](P)\geqq\alpha$. □

LEMMA 3.11. *Let $\varphi: D\rightarrow\boldsymbol{R}\cup\{-\infty\}$ be upper semi-continuous and $P\in D$. Then $W_1[\varphi](P)\geqq\alpha$ if, for every 1-dimensional $\boldsymbol{C}$-linear subspace E_P of $T_P(D)$, there* *exists a 1-dimensional complex submanifold E of D defined near P such that $P\in E$, $T_P(E)=E_P$ and $W_1^{(E)}[\varphi|_E](P)\geqq\alpha$.*

PROOF. Let $z=(z_1, \cdots, z_n)$ be a normal coordinate system of D at $P\in D$ and β a real number with $\beta<\alpha$. To prove the pseudoconvexity of order $n-1$ of $\varphi-\beta\|z\|^2$ at $P\in D$, let f be a weakly n-convex function defined near $P\in D$ and γ a real number with $\beta<\gamma<\alpha$. Since the function $h:=f+(\gamma-\beta)\|z\|^2$ is strongly n-convex near $P\in D$, there exists a 1-dimensional $\boldsymbol{C}$-linear subspace E_P of $T_P(D)$ such that $\partial\bar{\partial}h$ has a positive eigenvalue on E_P. By the assumption of the lemma, choose a 1-dimensional complex submanifold E of D defined near P such that $P\in E$, $T_P(E)=E_P$ and $W_1^{(E)}[\varphi|_E](P)\geqq\alpha$. Then $h|_E$ is 1-convex near $P\in E$ and $\varphi|_E-\gamma(\|z\|^2)|_E$ is pseudoconvex of order 0 $(=1-1)$ at $P\in E$. Hence we can find a neighborhood $U'=U'(h|_E)$ $(\subset E)$ of $P\in E$, so that

$$(\varphi-\gamma\|z\|^2+h)(P)\leqq\max\{(\varphi-\gamma\|z\|^2+h)|_E(Q): Q\in\partial\Delta'\}$$

for every domain Δ' with $P\in\Delta'$ and $\Delta'\Subset U'$. Choose a neighborhood $U=U(f)$ $(\subset D)$ of $P\in D$ so that $U\cap E\subset U'$. Let Δ be a domain with $P\in\Delta$ and $\Delta\Subset U$, and denote by Δ' the connected component of $\Delta\cap E$ containing P. Then $P\in\Delta'$ and $\Delta'\Subset U'$. Moreover, we have

$$\begin{aligned}(\varphi-\beta\|z\|^2+f)(P)&\leqq\max\{(\varphi-\beta\|z\|^2+f)|_E(Q): Q\in\partial\Delta'\}\\&\leqq\max\{(\varphi-\beta\|z\|^2+f)(Q): Q\in\partial\Delta\}.\end{aligned}$$

Therefore, $\varphi-\beta\|z\|^2$ is pseudoconvex of order $n-1$ at $P\in D$ for every $\beta<\alpha$ and hence $W_1[\varphi](P)\geqq\alpha$. □

4. Distance functions to complex submanifolds.

After §4, let M be an n-dimensional connected Kähler manifold with C^∞ Kähler metric G. Then M can be also regarded as a $2n$-dimensional Riemannian manifold with the C^∞ Hermitian metric $g=\mathrm{Re}\,G$. We denote by J the complex structure tensor field of M, and denote by ∇ and R the covariant derivation and the curvature tensor field (of covariant degree 4) with respect to the Riemannian connection of M, respectively.

If σ and τ are holomorphic planes, i.e., J-invariant planes in the (real) tangent space $T_P(M)$ at $P\in M$, the holomorphic bisectional curvature $H(\sigma, \tau)$ of them is defined by

$$\begin{aligned}H(\sigma, \tau)&:=R(X, JX, Y, JY)\\&=R(X, Y, X, Y)+R(JX, Y, JX, Y),\end{aligned}$$

where X and Y are unit vectors in σ and τ, respectively (see Goldberg-Kobayashi [**10**]).

For two points P and Q of M, denote by $d(P, Q)$ the distance between P and Q induced by the metric g ($=\operatorname{Re} G$). Given a subset E of M, we define the distance function $d_E: M \to \boldsymbol{R}$ to E by

$$d_E(P) = d(P, E) = \inf\{d(P, Q): Q \in E\} \qquad \text{for } P \in M.$$

When D is a pseudoconvex open subset (in the usual sense) in M, the plurisubharmonicity of the function $-\log d_{M\setminus D}$ was differential-geometrically studied by Takeuchi [**27**], Elencwajg [**6**], Suzuki [**24**] and Greene-Wu [**11**]. In this section we shall prove the following fundamental lemma. The proof is based on that of Greene-Wu ([**11**], Theorem 1).

LEMMA 4.1. *Let M be an n-dimensional Kähler manifold, D an open subset of M, and P a point of D. Suppose that there exists (at least one) $Q \in \partial D$ such that*

(i) $d_{\partial D}(P) = d(P, Q)$,

(ii) *The points P and Q can be joined by a geodesic ξ in M,*

(iii) *There exists an $(n-q)$-dimensional complex submanifold defined near Q such that $Q \in S$ and $S \subset M \setminus D$.*

Then we have the estimate

$$W_q[-\log d_{\partial D}](P) \geqq \frac{1}{4}\min\left\{\frac{\Theta}{3}, \Theta\right\},$$

where Θ is the minimum of the holomorphic bisectional curvatures of M on the geodesic ξ in (ii).

PROOF. If S is an $(n-q)$-dimensional complex submanifold defined near $Q \in \partial D$, and if $Q \in S$ and $S \subset M \setminus D$, we have $d_S \geqq d_{M\setminus D} = d_{\partial D}$ on D and hence $-\log d_{\partial D} \geqq -\log d_S$ on D. Moreover, since $d_S(P) = d(P, Q) = d_{\partial D}(P)$, we have $-\log d_{\partial D}(P) = -\log d_S(P)$. Hence, by Lemma 3.8, we first see

$$W_q[-\log d_{\partial D}](P) \geqq W_q[-\log d_S](P).$$

Let $\xi = \xi(t)$, $t \in [0, l]$, be a geodesic in M from $P \in D$ to $Q \in \partial D$, where $\xi(0) = P$, $\xi(l) = Q$, $l = d_{\partial D}(P) = d(P, Q)$, and the parameter t is canonical. Let N_t, $t \in [0, l]$, be the unit tangent vector field of $\xi = \xi(t)$. Then the vector N_l is orthogonal to the (real) tangent space $T_Q(S)$ at $Q = \xi(l) \in S$. Let F_P be the parallel translate of $T_Q(S)$ along ξ back to $P = \xi(0)$. Since $T_Q(S)$ is J-invariant and of real dimension $2(n-q)$, so is F_P. Moreover, F_P is orthogonal to both N_0 and JN_0. We denote by L_P the J-invariant $\boldsymbol{R}$-linear subspace of real dimension $2(n-q+1)$ in $T_P(M)$ which is generated by N_0, JN_0 and the elements of F_P.

Since the metric G on the complex manifold M is now Kählerian, we can choose a local coordinate system $(z_1, \cdots, z_n)$ around P, so that $(z_1, \cdots, z_n)$ is normal at P (in the sense of Definition 3.1) and moreover satisfies $(\partial G_{ij}/\partial z_k)(P)=0$ for $1\leqq i, j, k\leqq n$, where $G_{ij}=G(\partial/\partial z_i, \partial/\partial z_j)$. Let L be the $(n-q+1)$-dimensional complex submanifold defined near P such that $P\in L$, $T_P(L)=L_P$ and L is linear with respect to $(z_1, \cdots, z_n)$. Making a unitary transformation of $(z_1, \cdots, z_n)$ if necessary, we may assume that L is given by $z_{n-q+2}=z_{n-q+3}=\cdots=z_n=0$ near P.

We put $\alpha=\min\{\Theta/3, \Theta\}/4$. To prove $W_q[-\log d_S](P)\geqq\alpha$, it is sufficient by Lemma 3.10 to show that $W_1^{(L)}[(-\log d_S)|_L](P)\geqq\alpha$ for the L chosen above. Moreover, it is sufficient by Lemma 3.11 to show that $W_1^{(E)}[(-\log d_S)|_E](P)\geqq\alpha$ for every 1-dimensional complex submanifold E of L defined near P such that $P\in E$ and E is linear with respect to $(z_1, \cdots, z_n)$.

Making a unitary transformation of $(z_1, \cdots, z_{n-q+1})$ if necessary, we may without loss of generality assume that E is given by $z_2=z_3=\cdots=z_n=0$ near P. For the sake of simplicity, we write z instead of z_1, and put $z=x+\sqrt{-1}y$, $x, y\in\boldsymbol{R}$. Since the vector $(\partial/\partial z)_P$ is unit with respect to the metric G, the vectors $V_0=(\partial/\partial x)_P$ and $JV_0=(\partial/\partial y)_P$ are unit with respect to the metric g ($=\mathrm{Re}\,G$). Since $T_P(E)$ is a J-invariant $\boldsymbol{R}$-linear subspace of $L_P=T_P(L)$, we can, by making a rotation of z_1-plane if necessary, write $V_0=\alpha N_0+\beta X_0$ for some α, β and X_0, where $X_0\in F_P$ is unit and $\alpha^2+\beta^2=1$.

Let X_t, $t\in[0, l]$, be the parallel translate of X_0 along ξ to $\xi(t)$. Then the unit vectors X_t, JX_t, N_t and JN_t are mutually orthogonal at $\xi(t)$ for each $t\in[0, l]$. We now define the vector field V along ξ by

$$V_t=\left(\frac{l-t}{l}\right)\alpha N_t+\beta X_t \qquad \text{for } t\in[0, l],$$

and put $U_\varepsilon=\{(x, y)\in E : |x|<\varepsilon, |y|<\varepsilon\}$ for $\varepsilon>0$. Then, for sufficiently small $\varepsilon>0$, we can take a C^∞ mapping $k:[0, l]\times U_\varepsilon\to M$ such that

(i) $k(t;0, 0)=\xi(t)$,

(ii) $k_*\left(\frac{\partial}{\partial x}\right)_{(t;0,0)}=V(t)$, $k_*\left(\frac{\partial}{\partial y}\right)_{(t;0,0)}=JV(t)$,

(iii) $k(0;x, y)=x+\sqrt{-1}y\in E$, $k(l;x, y)\in S'$,

for $t\in[0, l]$ and $(x, y)\in U_\varepsilon$, where S' is some 1-dimensional complex submanifold of S defined near Q and containing Q, and k_* denotes the differential of the mapping k.

For $(x, y)\in U_\varepsilon$, we define the function $h: U_\varepsilon\to\boldsymbol{R}$ by

$$h(x, y)=\int_0^l\sqrt{g\left(k_*\left(\frac{\partial}{\partial t}\right), k_*\left(\frac{\partial}{\partial t}\right)\right)_{(t;x,y)}}\,dt,$$

i.e., the length of the curve $k_{(x,y)}=k_{(x,y)}(t):=k(t;x, y)\in M$, $t\in[0, l]$. Since

$h(P)=h(0,0)=l=d_S(P)$, we have $(-\log h)(P)=(-\log d_S)|_E(P)$. Moreover, it follows from the condition (iii) of the mapping k that $h \geqq d_S|_E$ on U_ε and hence $-\log h \leqq (-\log d_S)|_E$ on U_ε. Therefore, by Lemma 3.8, we have

$$W_1^{(E)}[(-\log d_S)|_E](P) \geqq W_1^{(E)}[-\log h](P).$$

Since the function $-\log h$ is of class C^∞ on U_ε $(\subset E)$ and the local coordinate $z=x+\sqrt{-1}\,y$ of E is normal at $P \in E$, we have

$$\begin{aligned}(1)\qquad W_1^{(E)}[-\log h](P) &= \frac{\partial^2}{\partial z \partial \bar{z}}(-\log h)(P)\\ &= \frac{1}{l^2}\left|\frac{\partial h}{\partial z}(P)\right|^2 - \frac{1}{l}\left(\frac{\partial^2 h}{\partial z \partial \bar{z}}\right)(P)\\ &= \frac{1}{4l^2}\left\{\left(\frac{\partial h}{\partial x}(P)\right)^2+\left(\frac{\partial h}{\partial y}(P)\right)^2\right\} - \frac{1}{4l}\left\{\frac{\partial^2 h}{\partial x^2}(P)+\frac{\partial^2 h}{\partial y^2}(P)\right\}.\end{aligned}$$

We shall now apply to (1) the variation formulas in Riemannian geometry. The first variation formula gives

$$\frac{\partial h}{\partial x}(P) = g(V_t, N_t)\Big|_{t=0}^{t=l} = -\alpha, \quad \frac{\partial h}{\partial y}(P) = g(JV_t, N_t)\Big|_{t=0}^{t=l} = 0,$$

and hence we first obtain

$$(2)\qquad \left(\frac{\partial h}{\partial x}(P)\right)^2+\left(\frac{\partial h}{\partial y}(P)\right)^2 = \alpha^2.$$

Next, the second variation formula gives

$$\begin{aligned}\frac{\partial^2 h}{\partial x^2}(P) &= g((\nabla_V V)_{(t;0,0)}, N_t)\Big|_{t=0}^{t=l}\\ &\quad +\int_0^l\left[-R(V_t, N_t, V_t, N_t)+g((\nabla_N V)_t, (\nabla_N V)_t)-\left\{\frac{d}{dt}g(V_t, N_t)\right\}^2\right]dt,\\ \frac{\partial^2 h}{\partial y^2}(P) &= g((\nabla_{JV} JV)_{(t;0,0)}, N_t)\Big|_{t=0}^{t=l}\\ &\quad +\int_0^l\left[-R(JV_t, N_t, JV_t, N_t)+g((\nabla_N JV)_t, (\nabla_N JV)_t)-\left\{\frac{d}{dt}g(JV_t, N_t)\right\}^2\right]dt,\end{aligned}$$

where we have put

$$V_{(t;x,y)} = k_*\left(\frac{\partial}{\partial x}\right)_{(t;x,y)}, \quad JV_{(t;x,y)} = k_*\left(\frac{\partial}{\partial y}\right)_{(t;x,y)}.$$

Now, by the condition (iii) of the mapping k, the vector fields

$$V_{(0;x,y)} \equiv \left(\frac{\partial}{\partial x}\right)_{(x,y)}, \quad JV_{(0;x,y)} \equiv \left(\frac{\partial}{\partial y}\right)_{(x,y)}$$

are restrictions to U_ε $(\subset E)$ of the coordinate vector fields with respect to the normal coordinate system $(z_1, \cdots, z_n)$ of M at $P \in M$. Hence we have

$$(\nabla_V V)_{(0;\,0,0)} = (\nabla_{JV} JV)_{(0;\,0,0)} = 0\,.$$

Moreover, since $V_{(l;\,x,y)}$ and $JV_{(l;\,x,y)}$ are vector fields on the complex submanifold S', and since the vector N_l is orthogonal to S' ($\subset S$) at Q, we have

$$g((\nabla_V V)_{(l;\,0,0)},\, N_l) + g((\nabla_{JV} JV)_{(l;\,0,0)},\, N_l) = g(J[JV,\, V]_{(l;\,0,0)},\, N_l) = 0$$

(see Frankel [**7**], p. 171). Therefore, we have

$$\frac{\partial^2 h}{\partial x^2}(P) + \frac{\partial^2 h}{\partial y^2}(P) = \frac{\alpha^2}{l} - \int_0^l R(V_t,\, JV_t,\, N_t,\, JN_t)dt, \tag{3}$$

exactly as in the proof of Greene-Wu ([**11**], pp. 177-178). Substituting (2) and (3) for (1), we obtain

$$W_1^{(E)}[-\log h](P) = \frac{1}{4l}\int_0^l R(V_t,\, JV_t,\, N_t,\, JN_t)dt. \tag{4}$$

If Θ is the minimum of the holomorphic bisectional curvatures of M on the geodesic $\xi = \xi(t)$, $t \in [0, l]$, then

$$R(V_t,\, JV_t,\, N_t,\, JN_t) \geqq \Theta\left\{\left(\frac{l-t}{l}\right)^2 \alpha^2 + \beta^2\right\} \qquad \text{for } t \in [0, l]\,.$$

Hence, by (4), we have

$$\begin{aligned} W_1^{(E)}[-\log h](P) &\geqq \frac{\Theta}{4l}\int_0^l \left\{\left(\frac{l-t}{l}\right)^2 \alpha^2 + \beta^2\right\}dt \\ &= \frac{\Theta}{4}\left(\frac{\alpha^2}{3} + \beta^2\right). \end{aligned}$$

Noting that $\alpha^2 + \beta^2 = 1$ and hence $1/3 \leqq (\alpha^2/3) + \beta^2 \leqq 1$, we finally obtain

$$W_q[-\log d_{\partial D}](P) \geqq W_1^{(E)}[-\log h](P) \geqq \frac{1}{4}\min\left\{\frac{\Theta}{3},\, \Theta\right\},$$

which completes the proof of the lemma. □

5. Boundary distance functions of pseudoconvex domains of general order.

Let M be a Kähler manifold and D an open subset of M. For $P \in M$ and $r > 0$, we use the notation

$$B(P,\, r) = \{Q \in M\colon d(P,\, Q) < r\}\,.$$

Then $B(P,\, d_{\partial D}(P)) \subset D$ for every $P \in D$. We further denote by $\Theta(P)$, $P \in D$, the infimum of the holomorphic bisectional curvatures on $B(P,\, d_{\partial D}(P))$. It is easy to see that the function $\Theta: D \to \boldsymbol{R}$ is continuous, if $D \cap B(P,\, r) \Subset M$ for every $P \in D$ and for every $r > 0$. Note that the condition is satisfied, either if M is complete or if $D \Subset M$.

As an application of Lemma 4.1, we shall first prove the following local result on boundary distance functions of pseudoconvex open subsets of general order.

PROPOSITION 5.1. *Let M be an n-dimensional Kähler manifold and let D be a pseudoconvex open subset of order $n-q$ in M. Then there exists an open subset Δ of M such that $\partial D \subset \Delta$ and*

$$W_q[-\log d_{\partial D}] \geqq \frac{1}{4}\min\left\{\frac{\Theta}{3},\, \Theta\right\} \qquad \text{on } D\cap\Delta,$$

where $\Theta=\Theta(P)$, $P\in D$, is the infimum of the holomorphic bisectional curvatures on $B(P, d_{\partial D}(P))$.

PROOF. We put $\alpha=\min\{\Theta/3, \Theta\}/4$. To prove the proposition, it is sufficient to show that each $Q\in\partial D$ has a neighborhood V such that $W_q[-\log d_{\partial D}] \geqq\alpha$ on $D\cap V$.

Let V^* be a Stein neighborhood of $Q\in\partial D$ which is relatively compact in some coordinate neighborhood of M. Then the set $D^*:=D\cap V^*$ is biholomorphic to a pseudoconvex open subset of order $n-q$ in $\boldsymbol{C}^n$. Hence, by Proposition 2.2 and Lemma 2.7, we can take a sequence $\{D_\nu^*\}_{\nu\in\boldsymbol{N}}$ of open subsets with the condition (C_q) in D^* such that $D_\nu^*\Subset D_{\nu+1}^*\Subset D^*$ for each $\nu\in\boldsymbol{N}$ and $\bigcup_{\nu=1}^{\infty} D_\nu^*=D^*$. Then, for each $P\in D_\nu^*$, there exists (at least one) $Q\in\partial D_\nu^*$ which satisfies the conditions (i), (ii) and (iii) of Lemma 4.1. Hence, by Lemma 4.1, we have

$$W_q[-\log d_{\partial D_\nu^*}] \geqq \frac{1}{4}\min\left\{\frac{\Theta^*}{3},\, \Theta^*\right\} \qquad \text{on } D_\nu^*$$

for each $\nu\in\boldsymbol{N}$, where $\Theta^*=\Theta^*(P)$, $P\in D^*$, is the infimum of the holomorphic bisectional curvatures on $B(P, d_{\partial D^*}(P))$. Note here that, because $D^*\Subset M$, Θ^* and hence $\alpha^*:=\min\{\Theta^*/3, \Theta^*\}/4$ are continuous functions from D^* to $\boldsymbol{R}$. On the other hand, for each $\nu\in\boldsymbol{N}$, the sequence $\{-\log d_{\partial D_\mu^*}\}_{\mu\geqq\nu}$ decreases on D_ν^* and converges to $-\log d_{\partial D^*}$. Therefore, it follows from Proposition 3.7 that $W_q[-\log d_{\partial D^*}]\geqq\alpha^*$ on D_ν^* for each $\nu\in\boldsymbol{N}$ and hence $W_q[-\log d_{\partial D^*}]\geqq\alpha^*\geqq\alpha$ on D^*.

Now choose $r>0$ so that $B(Q, 2r)\Subset V^*$, and put $V=B(Q, r)$. Then we have $d_{\partial D}=d_{\partial D^*}$ on $D\cap V$ $(\subset D^*)$, which implies that $W_q[-\log d_{\partial D}]\geqq\alpha$ on $D\cap V$ for this V. □

We shall later show that the estimate in Proposition 5.1 holds not only near ∂D but also whole on D in some cases (see Proposition 6.4 and Proposition 7.1). In this section we give the following global estimate for $W_q[-\log d_{\partial D}]$ under the assumption stated below.

LEMMA 5.2. *Let M be an n-dimensional Kähler manifold and let D be a pseudoconvex open subset of order $n-q$ in M such that $D\cap B(P, r)\Subset M$ for every*

$P\in D$ and for every $r>0$. Suppose that there exists an open subset Δ of M with $\partial D\subset\Delta$, and that one can for each $r>0$ find a positive number $C^{(r)}$ and a q-convex function $\phi^{(r)}$ with corners on $D^{(r)}\cap\Delta$ satisfying $|-\log d_{\partial D}-\phi^{(r)}|<C^{(r)}$ on $D^{(r)}\cap\Delta$, where $D^{(r)}=D\cap B(O, r)$ and $O\in\partial D$ is fixed. Then we have the estimate

$$W_q[-\log d_{\partial D}]\geqq\frac{1}{4}\min\left\{\frac{\Theta}{3},\ \Theta\right\}\qquad \textit{whole on } D,$$

where $\Theta=\Theta(P)$, $P\in D$, is the infimum of the holomorphic bisectional curvatures on $B(P, d_{\partial D}(P))$.

PROOF. We may assume that each $\phi^{(r)}$ is piecewise C^∞. Then, by Lemma 2.6, there exists a subset $\Lambda^{(r)}$ of Lebesgue measure zero in $\boldsymbol{R}$ such that the set $\{P\in D^{(r)}\cap\Delta : \phi^{(r)}(P)<A\}$ satisfies the condition (C_q) in $D^{(r)}\cap\Delta$ for every $A\in\boldsymbol{R}\setminus\Lambda^{(r)}$. On the other hand, by assumption, $D^{(r)}\Subset M$ and hence $D^{(r)}\setminus\Delta\Subset D$ for each $r>0$. We can thus choose $A_0^{(r)}>0$, so that

$$D^{(r)}\setminus\Delta\subset\{P\in D^{(r)} : -\log d_{\partial D}(P)+C^{(r)}<A_0^{(r)}\}.$$

For $A>0$ and $r>0$, we define the set $D_A^{(r)}$ by

$$D_A^{(r)}=(D^{(r)}\setminus\Delta)\cup\{P\in D^{(r)}\cap\Delta : \phi^{(r)}(P)<A\}.$$

Since $\phi^{(r)}>-\log d_{\partial D}-C^{(r)}$ on $D^{(r)}\cap\Delta$, we have $D_A^{(r)}\Subset D$ for every $A>0$. Moreover, since $\phi^{(r)}<-\log d_{\partial D}+C^{(r)}$ on $D^{(r)}\cap\Delta$, the set $D_A^{(r)}$ satisfies the condition (C_q) in $D^{(r)}$ if $A>A_0^{(r)}$ and $A\in\boldsymbol{R}\setminus\Lambda^{(r)}$.

For each $P\in D_A^{(r)}$, let $Q\in\partial D_A^{(2r)}$ be a point such that $d_{\partial D_A^{(2r)}}(P)=d(P, Q)$. Then the point Q is necessarily an interior point of $D^{(2r)}$ because $d_{\partial D_A^{(2r)}}(P)<d(O, P)<r$. Hence, if $A>A_0^{(2r)}$ and $A\in\boldsymbol{R}\setminus\Lambda^{(2r)}$, the point Q belongs to $D^{(2r)}\cap\Delta$ and satisfies $\phi^{(2r)}(Q)=A$, and fulfills the conditions (i), (ii) and (iii) of Lemma 4.1 with respect to the set $D_A^{(2r)}$. Therefore, it follows from Lemma 4.1 that

$$W_q[-\log d_{\partial D_A^{(2r)}}]\geqq\frac{1}{4}\min\left\{\frac{\Theta^{(2r)}}{3},\ \Theta^{(2r)}\right\}\qquad\text{on } D_A^{(r)}$$

for every A with $A>A_0^{(2r)}$ and $A\in\boldsymbol{R}\setminus\Lambda^{(2r)}$, where $\Theta^{(r)}=\Theta^{(r)}(P)$, $P\in D^{(r)}$, is the infimum of the holomorphic bisectional curvatures on $B(P, d_{\partial D^{(r)}}(P))$. Note here that $\Theta^{(r)} : D^{(r)}\to\boldsymbol{R}$ is continuous because $D^{(r)}\Subset M$. Furthermore, $-\log d_{\partial D_B^{(2r)}}$, where $B>A$, decreases on $D_A^{(r)}$ and converges to $-\log d_{\partial D}$ as $B\to\infty$. Therefore, using Proposition 3.7, we can conclude that

$$W_q[-\log d_{\partial D}]\geqq\frac{1}{4}\min\left\{\frac{\Theta^{(2r)}}{3},\ \Theta^{(2r)}\right\}\geqq\frac{1}{4}\min\left\{\frac{\Theta}{3},\ \Theta\right\}$$

on $D^{(r)}$ for every $r>0$, which proves the lemma. □

6. Pseudoconvex domains of general order in Kähler manifolds of positive holomorphic bisectional curvature.

In § 6, we consider the case where a Kähler manifold M has positive or non-negative holomorphic bisectional curvature.

The following is the direct result of Proposition 5.1 and Proposition 3.6.

COROLLARY 6.1. *Let M be an n-dimensional Kähler manifold with non-negative* (*resp. positive*) *holomorphic bisectional curvature and let D be a pseudoconvex open subset of order $n-q$ in M. Then there exists an open subset Δ of M such that $\partial D \subset \Delta$ and the function $-\log d_{\partial D}$ is pseudoconvex* (*resp. strongly pseudoconvex*) *of order $n-q$ on $D \cap \Delta$.*

If the boundary ∂D of an open subset D of M is a real submanifold of class C^2 in M (whose irreducible components may have different dimensions from each other), there exists an open subset Γ of M such that $\partial D \subset \Gamma$ and the boundary distance function $d_{\partial D}$ is of class C^2 on $D \cap \Gamma$ (see Matsumoto [**14**]). Using this fact and Proposition 6.1, we first obtain the following result on the q-convexity of domains.

THEOREM 6.2. *Let M be an n-dimensional Kähler manifold with non-negative* (*resp. positive*) *holomorphic bisectional curvature and let D be a pseudoconvex open subset of order $n-q$ in M. Moreover, suppose that $D \Subset M$ and the boundary ∂D is a real submanifold of class C^2 in M. Then D is weakly* (*resp. strongly*) *q-convex.*

REMARK 6.3. The n-dimensional complex projective space $P^n(\boldsymbol{C})$ has positive holomorphic bisectional curvature with respect to the Fubini-Study metric on $P^n(\boldsymbol{C})$. Theorem 6.2 is an extension of the Barth theorem ([**2**], Satz 3) asserting that the complement $P^n(\boldsymbol{C}) \setminus S$ is strongly q-convex, if S is a complex submanifold (and hence an algebraic submanifold) of $P^n(\boldsymbol{C})$ and if each connected component of S has at least dimension $n-q$ (cf. Example 2.4). When $M = P^n(\boldsymbol{C})$, Theorem 6.2 is the result of Schwarz ([**20**], Theorem 6.4) and Matsumoto ([**15**], Corollary of Theorem 2). As another extension of the Barth theorem, Schneider [**19**] has also showed the q-convexity of $M \setminus S$ under the assumption that M and S are compact and S has positive normal bundle in M.

In what follows, we consider only the case where M has positive holomorphic bisectional curvature. Then we can extend Proposition 5.1 to the following global result.

PROPOSITION 6.4. *Let M be an n-dimensional Kähler manifold with positive holomorphic bisectional curvature and let D be a pseudoconvex open subset of order $n-q$ in M. Moreover, suppose either that M is complete or that $D \Subset M$.*

Then we have the estimate

$$W_q[-\log d_{\partial D}] \geqq \frac{\Theta}{12} \qquad \textit{whole on } D,$$

where $\Theta=\Theta(P)$, $P\in D$, *is the infimum of the holomorphic bisectional curvatures on* $B(P, d_{\partial D}(P))$.

PROOF. By Corollary 6.1, there exists an open subset $\varDelta$ of M such that $\partial D\subset\varDelta$ and $-\log d_{\partial D}$ is strongly pseudoconvex of order $n-q$ on $D\cap\varDelta$. Hence, by the Bungart approximation theorem (Theorem 1.6), we can find a q-convex function ψ with corners on $D\cap\varDelta$ such that $|-\log d_{\partial D}-\psi|<1$ on $D\cap\varDelta$. The proposition thus follows from Lemma 5.2. □

COROLLARY 6.5. *Uuder the same assumption as in Proposition* 6.4, *the function* $-\log d_{\partial D}$ *is strongly pseudoconvex of order* $n-q$ *whole on* D.

Using the approximation theorems of Bungart and Diederich-Fornaess, we obtain from Corollary 6.5 the following theorem and its corollary on the q-completeness (with corners) of domains.

THEOREM 6.6. *Let* M *be an* n*-dimensional Kähler manifold with positive holomorphic bisectional curvature and let* D *be a relatively compact, pseudoconvex open subset of order* $n-q$ *in* M. *Then* D *is* q*-complete with corners.*

COROLLARY 6.7. *Under the same assumption as in Theorem* 6.6, D *is* $\tilde{q}$*-complete, where* $\tilde{q}=n-[n/q]+1$.

When $M=P^n(\boldsymbol{C})$, Theorem 6.6 is particularly stated as follows (see Proposition 2.2).

COROLLARY 6.8. *Let* D *be an open subset of* $P^n(\boldsymbol{C})$. *If* D *is locally* q*-complete with corners in* $P^n(\boldsymbol{C})$ *(in the sense of Peternell* [**16**]*), then* D *is globally* q*-complete with corners and hence globally* $\tilde{q}$*-complete, where* $\tilde{q}=n-[n/q]+1$. *In particular, if* S *is an algebraic subset of* $P^n(\boldsymbol{C})$ *and if each irreducible component of* S *has at least dimension* $n-q$, *then* $P^n(\boldsymbol{C})\backslash S$ *is globally* q*-complete with corners and hence globally* $\tilde{q}$*-complete.*

REMARK 6.9. In Corollary 6.8, the case where S is non-singular has been showed by Schwarz ([**20**], Theorem 6.5). When S is non-singular, the set $P^n(\boldsymbol{C})\backslash S$ is further $\min\{2q-1, \tilde{q}\}$-complete (see Peternell [**17**]).

7. Pseudoconvex domains of general order in Stein manifolds.

Finally in §7, we consider the case where a Kähler manifold M admits a (strongly) 1-convex function. Then we can extend Proposition 5.1 to the following global result.

Proposition 7.1. *Let M be an n-dimensional Kähler manifold and let D be a pseudoconvex open subset of order $n-q$ in M. Suppose that there exists an open subset Δ of M such that $\partial D \subset \Delta$ and Δ admits a 1-convex function. Moreover, suppose either that M is complete or that $D \Subset M$. Then we have the estimate*

$$W_q[-\log d_{\partial D}] \geqq \frac{1}{4}\min\left\{\frac{\Theta}{3}, \Theta\right\} \qquad \text{whole on } D,$$

where $\Theta=\Theta(P)$, $P \in D$, is the infimum of the holomorphic bisectional curvatures on $B(P, d_{\partial D}(P))$.

Proof. Shrinking Δ if necessary, we may assume that there exists a 1-convex function h which is defined on an open subset including $\bar{\Delta}$. Moreover, we may by Proposition 5.1 assume that the estimate in Proposition 7.1 holds on $D \cap \Delta$.

Let O be a fixed point of ∂D and put $D^{(r)}=D \cap B(O, r)$ for $r>0$. Then, by the assumption of the proposition, $D^{(r)} \Subset M$ for each $r>0$. We put

$$\alpha^{(r)} = \frac{1}{4}\inf\left\{\frac{\Theta}{3}(P), \Theta(P) : P \in D^{(r)} \cap \Delta\right\},$$

$$\beta^{(r)} = \inf\{W_1[h](P) : P \in D^{(r)} \cap \Delta\}.$$

Then $\alpha^{(r)} \in \boldsymbol{R}$ and $\beta^{(r)}>0$. If we choose $A^{(r)}>0$ so that $\alpha^{(r)}+A^{(r)}\beta^{(r)}>1$, we have $W_q[-\log d_{\partial D}+A^{(r)}h]>1$ on $D^{(r)} \cap \Delta$. By Proposition 3.6, the function $-\log d_{\partial D}+A^{(r)}h$ is strongly pseudoconvex of order $n-q$ on $D^{(r)} \cap \Delta$. Hence, by the Bungart approximation theorem (Theorem 1.6), we can find a q-convex function $\psi^{(r)}$ with corners on $D^{(r)} \cap \Delta$ such that

$$|-\log d_{\partial D}+A^{(r)}h-\psi^{(r)}| < 1 \qquad \text{on } D^{(r)} \cap \Delta.$$

If we choose $C^{(r)}>0$ so that $C^{(r)}>1+A^{(r)}|h|$ on $D^{(r)} \cap \Delta$, then $|-\log d_{\partial D}-\psi^{(r)}| < C^{(r)}$ on $D^{(r)} \cap \Delta$. The proposition thus follows from Lemma 5.2. □

In what follows, let M be a Stein manifold. Then M admits a complete Kähler metric.

Proposition 7.2. *Let M be an n-dimensional Stein manifold and let D be a pseudoconvex open subset of order $n-q$ in M. Let $d_{\partial D}$ be a boundary distance function of D induced by a complete Kähler metric on M. Then there exists a 1-convex function h on M such that the function $-\log d_{\partial D}+h$ is strongly pseudoconvex of order $n-q$ on D.*

Proof. Let f be a 1-convex exhaustion function of M. For each $\nu \in \boldsymbol{N}$, define the set D_ν by $D_\nu=\{P \in D : f(P)<\nu\}$ and denote by α_ν the infimum of the function $\{\Theta/3, \Theta\}/4$ on D_ν, where $\Theta=\Theta(P)$, $P \in D$, is the infimum of the holomorphic bisectional curvatures on $B(P, d_{\partial D}(P))$. Then, by Proposition 7.1, we have $W_q[-\log d_{\partial D}] \geqq \alpha_\nu$ on D_ν. Let β_ν be the infimum of the function $W_1[f]$ on D_ν. Then $\beta_\nu > 0$ because $D_\nu \Subset M$.

Take a sequence $\{C_\nu\}_{\nu \in \boldsymbol{N}}$ such that $0 < C_\nu < C_{\nu+1}$ and $\alpha_\nu + C_\nu \beta_\nu > 1$ for $\nu \in \boldsymbol{N}$. Choose a C^2 function $u : \boldsymbol{R} \to (1, +\infty)$ such that $u' > C_1 > 0$, $u'' > 0$ and $u'(\nu) \geqq C_{\nu+1}$ for $\nu \in \boldsymbol{N}$, and put $h = u \circ f$. Then h is 1-convex on M. On the other hand, since $W_1[h] \geqq C_\nu \beta_\nu$ on $D_\nu \backslash D_{\nu-1}$, we have $W_q[-\log d_{\partial D} + h] > 1$ on $D_\nu \backslash D_{\nu-1}$ for each $\nu \in \boldsymbol{N}$ and hence on D. Therefore, $-\log d_{\partial D} + h$ is strongly pseudoconvex of order $n-q$ on D. □

Using the approximation theorems of Bungart and Diederich-Fornaess, we obtain from Proposition 7.2 the following theorem and its corollary.

THEOREM 7.3. *Let M be an n-dimensional Stein manifold and let D be a pseudoconvex open subset of order $n-q$ in M. Then D is q-complete with corners.*

COROLLARY 7.4. *Under the same assumption as in Theorem 7.3, D is $\tilde{q}$-complete, where $\tilde{q} = n - [n/q] + 1$.*

REMARK 7.5. Using the Bungart approximation theorem, we can also obtain Theorem 7.3 directly from the result of Peternell ([**16**], Theorem 2) or that of Matsumoto ([**13**], Theorem 1).

If the boundary ∂D of an open subset D of M is a real submanifold of class C^2 in M (whose irreducible components may have different dimensions from each other), we further obtain the following.

THEOREM 7.6. *Let M be an n-dimensional Stein manifold and let D be a pseudoconvex open subset of order $n-q$ in M. Moreover, suppose that the boundary ∂D is a real submanifold of class C^2 in M. Then D is q-complete.*

PROOF. By Proposition 7.2, we can find a 1-convex function h on M such that $\varphi = -\log d_{\partial D} + h$ is strongly pseudoconvex of order $n-q$ on D, where $d_{\partial D}$ is a boundary distance function of D induced by a complete Kähler metric on M. Let Δ be an open subset of M such that $\partial D \subset \Delta$ and $d_{\partial D}$ is of class C^2 on $D \cap \Delta$. Then φ is (strongly) q-convex on $D \cap \Delta$.

Choose a 1-convex exhaustion function f of M so that $f > \varphi$ on $D \backslash \Delta$ and put $\Phi = \max\{\varphi, f\}$ on D. Since $\Phi \equiv f$ on $D \backslash \Delta$ and since φ is q-convex on $D \cap \Delta$, we can, by the Diederich-Fornaess approximation theorem (cf. [**4**], §5), find a q-convex function Ψ (without corners) on D such that $|\Phi - \Psi| < 1$ on D. Then the function Ψ is further an exhaustion function of D, which proves the theorem. □

REMARK 7.7. When ∂D is a real hypersurface of class C^2 in M, Theorem 7.6 has been showed by Suria [**23**] and Eastwood-Suria [**5**] (cf. Example 2.3).

Theorem 7.6 is an extension of the result. Schwarz ([**20**], Corollary 6.3) has also proved Theorem 7.6 in another way.

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