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# Periodic stability of solutions to some degenerate parabolic equations with dynamic boundary conditions

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## 0. Introduction.

This paper is concerned with a degenerate parabolic equation

$$u_t - \Delta\beta(u) = f \quad \text{in } Q := (t_0, \infty)\times\Omega \tag{0.1}$$

with dynamic boundary condition

$$\begin{cases} \dfrac{\partial\beta(u)}{\partial\nu} + \dfrac{\partial V}{\partial t} + h = 0 \\ V = \beta(u) \end{cases} \quad \text{on } \Sigma := (t_0, \infty)\times\Gamma, \tag{0.2}$$

where $t_0 \in \boldsymbol{R}$ or $t_0 = -\infty$; $\Omega$ is a bounded domain in $\boldsymbol{R}^N$ $(N \geqq 1)$ with smooth boundary $\Gamma := \partial\Omega$; $(\partial/\partial\nu)$ denotes the outward normal derivative on $\Gamma$; $\beta : \boldsymbol{R} \to \boldsymbol{R}$ is a given nondecreasing function; $f$ and $h$ are given functions on $Q$ and $\Sigma$, respectively. In this paper, we denote by "$SP$ on $(t_0, \infty)$" the system $\{(0.1), (0.2)\}$.

Equation (0.1) represents the enthalpy formulation of the Stefan problem, when

$$\beta(r) = \begin{cases} c_1(r-1) & \text{for } r \geqq 1, \\ 0 & \text{for } 0 < r < 1, \\ c_2 r & \text{for } r \leqq 0 \end{cases}$$

for some positive constants $c_1$, $c_2$. For the physical interpretation of boundary condition (0.2) we quote Langer [**11**] and Aiki [**1**]. As far as initial-boundary value problems for (0.1) with usual boundary conditions are concerned, there are some interesting results (e.g., [**16**, **14**, **13**]) dealing with existence and uniqueness of solutions. Recently, problems with similar boundary conditions were discussed by Mikelič-Primicerio [**12**] and Primicerio-Rodrigues [**15**].

In Aiki [**1**], the existence and uniqueness of a weak solution of

$$\begin{cases} u_t - \Delta\beta(u) = f \quad \text{in } (0,T)\times\Omega, \\ \dfrac{\partial\beta(u)}{\partial\nu} + \dfrac{\partial V}{\partial t} + g(t,x,V) = 0 \quad \text{on } (0,T)\times\Gamma, \\ V = \beta(u) \quad \text{on } (0,T)\times\Gamma, \\ u(0,\cdot) = u_0 \quad \text{in } \Omega, \\ V(0,\cdot) = V_0 \quad \text{on } \Gamma, \end{cases}$$

were proved, where $u_0$ and $V_0$ are given functions on $\Omega$ and $\Gamma$, respectively, and $g:(0,T)\times\Gamma\times\boldsymbol{R}\to\boldsymbol{R}$. One of the purposes of the present paper is to establish existence, uniqueness and comparison results on the initial-boundary value problem for $SP$. These results are not covered by Aiki [**1**], since in Aiki [**1**] it is necessary to assume that the weak solution belongs to the class $L^\infty((0,T)\times\Omega)$, and in this paper the boundedness of the weak solution is not required.

In this paper, we are mainly interested in the asymptotic stability of weak solutions to $SP$. This question is studied by reformulating $SP$ as a nonlinear evolution equation involving time-dependent subdifferential operators in a suitable Hilbert space. Such a technique was already employed in Damlamian [**5**], Damlamian-Kenmochi [**6**] and Haraux-Kenmochi [**7**]. We shall show that $SP$ can be reformulated as a nonlinear evolution equation of the form

$$v'(t) + \partial\varphi^t(v(t)) = f^* \qquad t \in (t_0, \infty), \tag{0.3}$$

in the dual space $X^*$ of the Hilbert space

$$X = \left\{ z \in H^1(\Omega);\ \int_\Omega z\,dx + \int_\Gamma z\,d\Gamma = 0 \right\}$$

with norm

$$|z|_X = \left\{ \int_\Omega |\nabla z(x)|^2 dx \right\}^{1/2},$$

where $d\Gamma$ is the surface element on $\Gamma$ and $\partial\varphi^t$ is the subdifferential of a convex function $\varphi^t$ on $X^*$.

Once the problem is represented in the form (0.3), we can apply some general results in Kenmochi-Ôtani [**9, 10**] on asymptotic to $SP$. Under periodicity conditions $h(t+T)=h(t)$ on $\boldsymbol{R}\times\Gamma$, $f(t+T)=f(t)$ on $\boldsymbol{R}\times\Omega$ and

$$\int_0^T\int_\Omega f\,dx\,dt - \int_0^T\int_\Gamma h\,d\Gamma\,dt = 0,$$

for some positive number $T$, we shall show that

(i) (existence of periodic solutions) $SP$ has at least one periodic solution on $\boldsymbol{R}$;

(ii) (order property of periodic solutions) if $u_1$, $u_2$ are periodic solutions of $SP$ on $\boldsymbol{R}$ such that

$$\int_\Omega u_1(0, x)dx+\int_\Gamma \beta(u_1(0, x))d\Gamma \geqq \int_\Omega u_2(0, x)dx+\int_\Gamma \beta(u_2(0, x))d\Gamma,$$

then $\beta(u_1)\geqq\beta(u_2)$ on $\boldsymbol{R}\times\Omega$;

(iii) (asymptotic stability of periodic solutions) if $u$ is a solution of $SP$ on $[t_0, \infty)$, then there is a periodic solution $w$ of $SP$ on $\boldsymbol{R}$ such that $\beta(u(nT+\cdot))\to\beta(w)$ in $L^2(0, T; H^1(\Omega))$.

Similar questions were discussed in Haraux-Kenmochi [7] and Aiki-Kenmochi-Shinoda [**2**].

Throughout this paper we use the following notations:

(1) For a real Banach space $W$ we denote by $W^*$ the topological dual of $W$ and $|\cdot|_W$ the norm in $W$. The duality pairing between $W^*$ and $W$ is written by $\langle\cdot, \cdot\rangle_W$. As a special case the inner product on a Hilbert space $W$ is denote by $(\cdot, \cdot)_W$.

(2) Let $W$ be a Hilbert space and $J$ be a compact subinterval of $\boldsymbol{R}$, $u$, $u_n\in C_w(J; W)$ for $n=1, 2, \cdots$. We denote by $u_n\to u$ in $C_w(J; W)$ if it satisfies $(u_n(t), z)_W\to(u(t), z)_W$ uniformly in $t\in J$ as $n\to\infty$ for each $z\in W$.

(3) We denote by $|\Omega|$ and $|\Gamma|$ the volume of $\Omega$ and the surface measure of $\Gamma$, respectively.

(4) For a proper lower semicontinuous (l. s. c.) convex function $\varphi$ on $W$, we denote by $D(\varphi)$ the effective domain $\{z\in W; \varphi(z)<+\infty\}$ and by $\partial\varphi$ the subdifferential of $\varphi$, i. e., $\partial\varphi$ is a (possibly multivalued) operator which assigns to each $z\in D(\varphi)$ the set $\partial\varphi(z)$ in $W$ defined by

$$\partial\varphi(z)=\{z^*\in W; (z^*, v-z)_W\leqq\varphi(v)-\varphi(z) \quad \text{for all } v\in W\}.$$

The domain of $\partial\varphi$ is the set $D(\partial\varphi)=\{z\in W; \partial\varphi(z)\neq\emptyset\}$. For general properties of subdifferential operators we refer to Brézis [**4**].

## 1. Main Results.

Throughout this paper we assume that the function $\beta: \boldsymbol{R}\to\boldsymbol{R}$ satisfies the following conditions ($\beta$1) and ($\beta$2):

($\beta$1) $\beta$ is non-decreasing and Lipschitz continuous on $\boldsymbol{R}$ with Lipschitz constant $C_\beta$ and $\beta(0)=0$.

($\beta$2) There are some positive constants $L_\beta$, $l_\beta$ such that

$$|\beta(r)|\geqq L_\beta|r|-l_\beta \qquad \text{for all } r\in\boldsymbol{R}.$$

For the sake of simplicity of notations we put

$$Y=H^1(\Omega), \quad W=L^2(\Omega)\times L^2(\Gamma),$$

$$A(u, v)=\int_\Omega \nabla u\cdot\nabla v dx \qquad \text{for } u, v\in Y,$$

$$(u, v)_Y = A(u, v)+\left(\int_\Omega u dx+\int_\Gamma u d\Gamma\right)\left(\int_\Omega v dx+\int_\Gamma v d\Gamma\right) \qquad \text{for } u, v \in Y,$$

$$(\vec{u}, \vec{v})_W = \int_\Omega uv dx+\int_\Gamma u_\Gamma v_\Gamma d\Gamma \qquad \text{for } \vec{u} = (u, u_\Gamma),\ \vec{v} = (v, v_\Gamma) \in W,$$

and $C_\Omega$ is a positive constant satisfying that

$$|v|_{L^2(\Omega)} \leqq C_\Omega |v|_Y, \quad |v|_{L^2(\Gamma)} \leqq C_\Omega |v|_Y \qquad \text{for any } v \in Y.$$

Also, we define an operator $E: Y \to W$ by putting

$$Ev = (v, v|_\Gamma) \qquad \text{for } v \in Y,$$

and denote by $F_Y$ the duality mapping from $Y$ to $Y^*$. Clearly, $Y$ and $W$ are Hilbert spaces with inner product $(\cdot, \cdot)_Y$ and $(\cdot, \cdot)_W$, respectively, and the range of $E$, $R(E)$, is a dense subspace of $W$ and $E$ is linear and compact. We identify $W$ with its dual $W^*$ and therefore, denoting by $E^*$ the dual operator of $E$ we have

$$\langle E^*(v, v_\Gamma), \eta \rangle_Y = \int_\Omega v\eta dx+\int_\Gamma v_\Gamma \eta d\Gamma \qquad \text{for any } (v, v_\Gamma) \in W \text{ and } \eta \in Y.$$

We introduce a notion of weak solution for $SP$.

DEFINITION 1.1. Let $J=[t_0, t_1]$ be a compact interval, $Q=(t_0, t_1)\times\Omega$, $\Sigma=(t_0, t_1)\times\Gamma$ and $f\in L^2(Q)$, $h\in L^2(\Sigma)$. Then a couple $\{u, V\}$ of functions $u: J\times\Omega\to \boldsymbol{R}$ and $V: J\times\Gamma\to\boldsymbol{R}$ is called a (weak) solution of $SP$ on $J$ if $u\in C_w(J; L^2(\Omega))$, $\beta(u)\in L^2(J; Y)$, $V\in C_w(J; L^2(\Gamma))$, $\beta(u)=V$ a.e. on $\Sigma$ and the following variational identity is satisfied:

$$(1.1) \quad -\int_Q u\eta_t dxdt-\int_\Sigma V\eta_t d\Gamma dt+\int_Q \nabla\beta(u)\cdot\nabla\eta dxdt+\int_\Sigma h\eta d\Gamma dt = \int_Q f\eta dxdt$$

$$\text{for any } \eta \in Z,$$

where $Z=\{\eta\in C^1(J; Y); \eta(t_0)=\eta(t_1)=0\}$.

DEFINITION 1.2. Let $J'$ be any interval in $\boldsymbol{R}$ and $f\in L^2_{loc}(J'; L^2(\Omega))$, $h\in L^2_{loc}(J'; L^2(\Gamma))$. Then a couple $\{u, V\}$ of functions $u: J'\times\Omega\to\boldsymbol{R}$ and $V: J'\times\Gamma\to\boldsymbol{R}$ is called a solution of $SP$ on $J'$ if for every compact subinterval $J=[t_0, t_1]$ of $J'$ the couple $\{u, V\}$ is a solution of $SP$ on $J$ in the sense of Definition 1.1.

Next, we formulate the Cauchy problem and the periodic problem in time for $SP$.

DEFINITION 1.3. (i) Let $J'=[t_0, t_1]$ or $[t_0, t_1)$, and let $u_0\in L^2(\Omega)$, $V_0\in L^2(\Gamma)$. Then a couple $\{u, V\}$ of functions $u: J'\to L^2(\Omega)$ and $V: J'\to L^2(\Gamma)$ is a solution of the Cauchy problem and initial conditions $u(t_0)=u_0$, $V(t_0)=V_0$, denoted by $CSP(u_0, V_0)$ on $J'$, for problem $SP$ on $J'$, if $\{u, V\}$ is a solution of $SP$ on $J'$

with $u(t_0)=u_0$, $V(t_0)=V_0$.

(ii) Let $T$ be a positive number, and let $\{u, V\}: \boldsymbol{R}\to V$ is a solution of $SP$ on $\boldsymbol{R}$ such that $u(t+T)=u(t)$, $V(t+T)=V(t)$ for all $t\in\boldsymbol{R}$. Then $\{u, V\}$ is called a $T$-periodic solution of $SP$ on $\boldsymbol{R}$.

For a compact interval $J=[t_0, t_1]$ let $\{u, V\}$ be a solution of $SP$ on $J$. Then it follows from (1.1) that for some constant $K\geqq 0$,

$$\left|\int_J \langle E^*(u, V), \eta_t\rangle_Y dt\right| \leqq K\left\{\int_Q |\nabla\beta(u)|^2 dxdt+\int_\Sigma h^2 d\Gamma dt+\int_Q f^2 dxdt\right\}^{1/2}$$
$$\times\left\{\int_Q (\eta^2+|\nabla\eta|^2)dxdt\right\}^{1/2} \quad \text{for any } \eta\in Z.$$

Hence, we infer that $E^*(u, V)\in W^{1,2}(J; Y^*)$, the function $t\to\langle E^*(u(t), V(t)), \eta(t)\rangle_Y$ is absolutely continuous on $J$ and

$$\int_s^t \left\langle \frac{d}{d\tau}E^*(u(\tau), V(\tau)), \eta(\tau)\right\rangle_Y d\tau+\int_s^t \langle E^*(u(\tau), V(\tau)), \eta_\tau(\tau)\rangle_Y d\tau$$
$$=\langle E^*(u(t), V(t)), \eta(t)\rangle_Y-\langle E^*(u(s), V(s)), \eta(s)\rangle_Y \quad \text{for any } \eta\in Z.$$

It follows that (1.1) can be written in the following form

$$\text{(1.2)}\quad \int_J\left\langle \frac{d}{dt}E^*(u, V), \eta\right\rangle_Y dt+\int_Q \nabla\beta(u)\cdot\nabla\eta dxdt+\int_\Sigma h\eta d\Gamma dt=\int_Q f\eta dxdt$$
$$\text{for any } \eta\in L^2(J; Y).$$

Besides, (1.2) is equivalent to

$$\text{(1.3)}\quad \left\langle \frac{d}{dt}E^*(u(t), V(t)), \eta\right\rangle_Y+A(\beta(u(t)), \eta)+(h(t), \eta)_{L^2(\Gamma)}=(f(t), \eta)_{L^2(\Omega)}$$
$$\text{for any } \eta\in Y \text{ and a.e. } t\in J.$$

It is then quite obvious to see the following proposition.

PROPOSITION 1.1. *Let $J$, $f$, $h$ be as in Definition 1.1 and consider $u: J\times\Omega\to\boldsymbol{R}$ and $V: J\times\Gamma\to\boldsymbol{R}$. Then $\{u, V\}$ is a solution of $SP$ on $J$ if and only if $E^*(u, V)\in W^{1,2}(J; Y^*)$, $u\in L^\infty(J; L^2(\Omega))$, $\beta(u)\in L^2(J; Y)$, $V\in L^\infty(J; L^2(\Gamma))$, $\beta(u)=V$ a.e. on $\Sigma$ and (1.3) is fulfilled.*

The first main result is concerned with existence and uniqueness of a solution to $SP$.

THEOREM 1.1. *Let $J'$, $f$, $h$ be as in Definition 1.3. Then for any $u_0\in L^2(\Omega)$, $V_0\in L^2(\Gamma)$ there exists one and only one solution $\{u, V\}$ of $CSP(u_0, V_0)$ on $J'$.*

The second one is concerned with a comparison result for $SP$.

THEOREM 1.2. *Let $\{u_1, V_1\}$, $\{u_2, V_2\}$ be solutions of $SP$ on $J=[t_0, t_1]$. Then, for any $s$, $t\in J$ with $s\leqq t$,*

$$|[u_1(t)-u_2(t)]^+|_{L^1(\Omega)}+|[V_1(t)-V_2(t)]^+|_{L^1(\Gamma)} \leqq |[u_1(s)-u_2(s)]^+|_{L^1(\Omega)}+|[V_1(s)-V_2(s)]^+|_{L^1(\Gamma)} \tag{1.4}$$

*and*

$$|u_1(t)-u_2(t)|_{L^1(\Omega)}+|V_1(t)-V_2(t)|_{L^1(\Gamma)} \leqq |u_1(s)-u_2(s)|_{L^1(\Omega)}+|V_1(s)-V_2(s)|_{L^1(\Gamma)}. \tag{1.5}$$

*In particular, if* $u_1(t_0, \cdot)\leqq u_2(t_0, \cdot)$ *a.e. on* $\Omega$ *and* $V_1(t_0, \cdot)\leqq V_2(t_0, \cdot)$ *a.e. on* $\Gamma$, *then*

$$u_1 \leqq u_2 \quad \text{a.e. on } J\times\Omega \quad \text{and} \quad V_1 \leqq V_2 \quad \text{a.e. on } J\times\Gamma. \tag{1.6}$$

Next we mention some results on $T$-periodic solutions of $SP$ on $\boldsymbol{R}$.

THEOREM 1.3. *We suppose that* $f\in L^2_{loc}(\boldsymbol{R}; L^2(\Omega))$, $h\in L^2_{loc}(\boldsymbol{R}; L^2(\Gamma))$. *Let* $T$ *be a positive number, and assume that*

$$\begin{aligned} &f(t+T, \cdot)=f(t, \cdot) \quad \text{a.e. on } \Omega \quad \text{and} \\ &h(t+T, \cdot)=h(t, \cdot) \quad \text{a.e. on } \Gamma \quad \text{for any } t\in\boldsymbol{R}, \end{aligned} \tag{1.7}$$

*and*

$$\int_0^T\int_\Omega f(t, x)dxdt-\int_0^T\int_\Gamma h(t, x)d\Gamma dt=0. \tag{1.8}$$

*Then the following statements* (i)~(iv) *hold.*

(i) *For each* $a_0\in\boldsymbol{R}$ *there exists a* $T$*-periodic solution* $\{u, V\}$ *of* $SP$ *on* $\boldsymbol{R}$ *such that*

$$\int_\Omega u(0, x)dx+\int_\Gamma V(0, x)d\Gamma=a_0.$$

(ii) *Let* $\{u, V\}$ *be a solution of* $SP$ *on* $\boldsymbol{R}$. *Then* $\{u, V\}$ *is* $T$*-periodic on* $\boldsymbol{R}$ *if and only if* $u\in L^\infty(\boldsymbol{R}; L^2(\Omega))$ *and* $V\in L^\infty(\boldsymbol{R}; L^2(\Gamma))$.

(iii) *Let* $\{u_1, V_1\}$, $\{u_2, V_2\}$ *be* $T$*-periodic solutions of* $SP$ *on* $\boldsymbol{R}$ *such that*

$$\int_\Omega u_1(0, x)dx+\int_\Gamma V_1(0, x)d\Gamma=\int_\Omega u_2(0, x)dx+\int_\Gamma V_2(0, x)d\Gamma.$$

*Then*

$$\beta(u_1)=\beta(u_2) \quad \text{a.e. on } \boldsymbol{R}\times\Omega,$$

*and there exist functions* $w\in L^2(\Omega)$, $w_\Gamma\in L^2(\Gamma)$ *with* $\int_\Omega w dx+\int_\Gamma w_\Gamma d\Gamma=0$ *such that*

$$\left.\begin{aligned} u_1(t, \cdot)-u_2(t, \cdot)&=w(\cdot) \quad \text{a.e. on } \Omega \\ V_1(t, \cdot)-V_2(t, \cdot)&=w_\Gamma(\cdot) \quad \text{a.e. on } \Gamma \end{aligned}\right\} \quad \text{for any } t\in\boldsymbol{R}.$$

(iv) *Let* $\{u_1, V_1\}$, $\{u_2, V_2\}$ *be two* $T$*-periodic solutions of* $SP$ *on* $\boldsymbol{R}$ *such that*

$$\int_\Omega u_1(0, x)dx+\int_\Gamma V_1(0, x)d\Gamma>\int_\Omega u_2(0, x)dx+\int_\Gamma V_2(0, x)d\Gamma.$$

*Then*

$$\beta(u_1) \geqq \beta(u_2) \quad a.e. \text{ on } \boldsymbol{R}\times\Omega. \tag{1.9}$$

We denote by $\mathcal{P}_T$ the set of all $T$-periodic solutions of $SP$ on $\boldsymbol{R}$. In Theorem 1.3, it is mentioned that $\{\beta(u); \{u, V\}\in\mathcal{P}_T\}$ is a totally ordered set with respect to the usual order of functions on $\boldsymbol{R}\times\Omega$.

Finally, as to the asymptotic stability of $T$-periodic solutions we prove the following.

THEOREM 1.4. *Suppose that all the conditions of Theorem* 1.3 *are satisfied. Let $t_0$ be any positive number and let $\{u, V\}$ be any solution of $SP$ on $[t_0, \infty)$. Then there exists a $T$-periodic solution $\{\bar{u}, \bar{V}\}$ of $SP$ on $\boldsymbol{R}$ such that*

$$\int_\Omega u(t, x)dx+\int_\Gamma V(t, x)d\Gamma=\int_\Omega \bar{u}(t, x)dx+\int_\Gamma \bar{V}(t, x)d\Gamma \qquad \text{for any } t\geqq t_0,$$

$$u(t)-\bar{u}(t)\to 0 \quad \text{weakly in } L^2(\Omega) \text{ as } t\to\infty, \tag{1.10}$$

$$V(t)-\bar{V}(t)\to 0 \quad \text{weakly in } L^2(\Gamma) \text{ as } t\to\infty, \tag{1.11}$$

*and*

$$\beta(u(nT+\cdot))\to\beta(\bar{u}) \quad \text{in } L^2(0, T; Y) \text{ as } n\to\infty. \tag{1.12}$$

## 2. Proof of Theorem 2.1.

Throughout this section we assume that $f\in L^2_{loc}(I; L^2(\Omega))$, $h\in L^2_{loc}(I; L^2(\Gamma))$ with $I=[t_0, +\infty)$, $t_0\in\boldsymbol{R}$, $u_0\in L^2(\Omega)$ and $V_0\in L^2(\Gamma)$.

First, we show a lemma about the solution of $CSP(u_0, V_0)$.

LEMMA 2.1. *Let $\{u, V\}$ be a solution of $CSP(u_0, V_0)$ on $I$. Then $\{u, V\}$ satisfies*

$$\begin{aligned}\int_\Omega u(t, x)dx+\int_\Gamma V(t, x)d\Gamma \\ =\int_\Omega u_0(x)dx+\int_\Gamma V_0(x)d\Gamma+\int_{t_0}^t\int_\Omega f(s, x)dxds-\int_{t_0}^t\int_\Gamma h(s, x)d\Gamma ds \\ \text{for any } t\in I.\end{aligned} \tag{2.1}$$

PROOF. Indeed, (2.1) is an immediate consequence of the integration of (1.3) with $\eta=1$ over $[t_0, t]$. Q.E.D.

We define a function $a$ on $I$ as follows:

$$a(t)=\frac{1}{|\Omega|+|\Gamma|}\{\langle E^*(u_0, V_0), 1\rangle_Y+\int_{t_0}^t\int_\Omega fdxds-\int_{t_0}^t\int_\Gamma hd\Gamma ds\}. \tag{2.2}$$

By (2.1) it is obvious that for a solution $\{u, V\}$ of $CSP(u_0, V_0)$ on $I$

$$\int_\Omega u(t, x)dx+\int_\Gamma V(t, x)d\Gamma=(|\Omega|+|\Gamma|)a(t) \qquad \text{for any } t\in I. \tag{2.3}$$

From now on we use the following function spaces and operators.

(i) $X=\{z\in Y; \int_\Omega zdx+\int_\Gamma zd\Gamma=0\}$ is a Banach space with norm $|z|_X=|\nabla z|_{L^2(\Omega)}$.

(ii) $H=\{(z, z_\Gamma)\in W; \int_\Omega zdx+\int_\Gamma z_\Gamma d\Gamma=0\}$ is a Hilbert space with the inner product $(\vec{u}, \vec{v})_H$ induced from the Hilbert space $W$, i. e.,

$$(\vec{u}, \vec{v})_H=(u, v)_{L^2(\Omega)}+(u_\Gamma, v_\Gamma)_{L^2(\Gamma)} \qquad \text{for } \vec{u}=(u, u_\Gamma),\ \vec{v}=(v, v_\Gamma)\in H.$$

We identify $H$ with its dual $H^*$.

(iii) $\hat{E}: X\to H$ is the natural injection from $X$ to $H$, that is, $\hat{E}z=(z, z|_\Gamma)$ for $z\in X$. Also, $\hat{E}^*: H\to X^*$ is the dual operator of $\hat{E}$ and $R(\hat{E}^*)$ is the range of $\hat{E}^*$; therefore

$$\langle\hat{E}^*(z, z_\Gamma), \eta\rangle_X=\int_\Omega z\eta dx+\int_\Gamma z_\Gamma\eta d\Gamma \qquad \text{for } (z, z_\Gamma)\in H \text{ and } \eta\in X.$$

(iv) $P_X: Y\to X$, $P_H: W\to H$ are operators defined as follows:

$$P_Xz=z-\frac{1}{|\Omega|+|\Gamma|}\left(\int_\Omega zdx+\int_\Gamma zd\Gamma\right) \qquad \text{for } z\in Y,$$

$$P_H(z, z_\Gamma)=(z-C(z, z_\Gamma), z_\Gamma-C(z, z_\Gamma)) \quad \text{for } (z, z_\Gamma)\in W,$$

where $C(z, z_\Gamma)=(|\Omega|+|\Gamma|)^{-1}(\int_\Omega zdx+\int_\Gamma z_\Gamma d\Gamma)$.

(v) $F_X: X\to X^*$ is the duality mapping from $X$ to $X^*$.

Obviously, we see that

$\hat{E}$ and $\hat{E}^*$ are linear and compact;

$X^*$ is a Hilbert space with inner product $(\cdot,\cdot)_{X^*}$ given by

$$(w, z)_{X^*}=\langle z, F_X^{-1}w\rangle_X(=\langle w, F_X^{-1}z\rangle_X) \qquad \text{for any } w, z\in X^*; \tag{2.4}$$

$$A(w, z)=\langle F_XP_Xw, P_Xz\rangle_X \qquad \text{for any } w, z\in Y; \tag{2.5}$$

$$|P_Xw|_X\leqq|w|_Y \qquad \text{for any } w\in Y. \tag{2.6}$$

Finally, we introduce

$$\hat{\beta}(r)=\int_0^r\beta(s)ds \qquad \text{for } r\in\boldsymbol{R}, \tag{2.7}$$

and for each $t\in I$ we define a function $\varphi^t: X^*\to(-\infty, +\infty]$ by the formula

$$(2.8)\qquad \varphi^t(z)=\begin{cases}\int_\Omega \beta(z+a(t))dx+\frac{1}{2}\int_\Gamma (z_\Gamma+a(t))^2 d\Gamma \\ \qquad \text{if } z^*\in R(\hat{E}^*) \text{ with } (z, z_\Gamma)=\hat{E}^{*-1}(z^*), \\ \infty \qquad\qquad \text{otherwise.}\end{cases}$$

Clearly, $\varphi^t$ is a proper l.s.c. convex function on $X^*$ and $D(\varphi^t)=R(\hat{E}^*)$ for each $t\in I$. Denoting by $\partial\varphi^t$ the subdifferential of $\varphi^t$ in $X^*$, we obtain the following lemma.

LEMMA 2.2. *For each $t\in I$, $\partial\varphi^t$ is singlevalued in $X^*$ with*

$$D(\partial\varphi^t)=\{z^*\in R(\hat{E}^*)\,;\,\beta(z+a(t))\in Y\},$$

*and for any $z^*=\hat{E}^*(z, z_\Gamma)\in D(\partial\varphi^t)$ with $(z, z_\Gamma)\in H$*

$$\partial\varphi^t(z^*)=F_X P_X \beta(z+a(t)) \quad \textit{in } X^* \textit{ and } \quad z_\Gamma+a(t)=\beta(z+a(t)) \quad \textit{a.e. on } \Gamma.$$

PROOF. Let $z', z^*\in X^*$. If $z'\in\partial\varphi^t(z^*)$, then there exists an element $(z, z_\Gamma)\in H$ such that $\hat{E}^*(z, z_\Gamma)=z^*$, and for any $(w, w_\Gamma)\in H$

$$(z', w^*-z^*)_{X^*}\leqq \varphi^t(w^*)-\varphi^t(z^*),$$

where $w^*=\hat{E}^*(w, w_\Gamma)$. By using (2.4) this can be written as

$$(2.9)\quad \langle w^*-z^*, F_X^{-1}z'\rangle_X$$
$$\leqq \int_\Omega \hat{\beta}(w+a(t))dx-\int_\Omega \beta(z+a(t))dx+\frac{1}{2}\int_\Gamma (u_\Gamma+a(t))^2 d\Gamma-\frac{1}{2}\int_\Gamma (z_\Gamma+a(t))^2 d\Gamma.$$

By definition of $\hat{E}^*$ we see that

$$(2.10)\qquad \langle w^*-z^*, F_X^{-1}z'\rangle_X=\int_\Omega (w-z)F_X^{-1}z'dx+\int_\Gamma (w_\Gamma-z_\Gamma)F_X^{-1}z'd\Gamma.$$

Choosing $w=\varepsilon v+z$, $w_\Gamma=\varepsilon v_\Gamma+z_\Gamma$, $\varepsilon>0$ in (2.9) and dividing by $\varepsilon$, we obtain by (2.10) that

$$\int_\Omega vF_X^{-1}z'dx+\int_\Gamma v_\Gamma F_X^{-1}z'd\Gamma$$
$$\leqq \frac{1}{\varepsilon}\int_\Omega \{\beta(z+a(t)+\varepsilon v)-\beta(z+a(t))\}dx$$
$$+\frac{1}{2\varepsilon}\int_\Gamma \{(z_\Gamma+a(t)+\varepsilon v_\Gamma)^2-(z_\Gamma+a(t))^2\}d\Gamma \qquad \text{for any } (v, v_\Gamma)\in H.$$

Then letting $\varepsilon\downarrow 0$ yields

$$\int_\Omega vF_X^{-1}z'dx+\int_\Gamma v_\Gamma F_X^{-1}z'd\Gamma\leqq \int_\Omega v\beta(z+a(t))dx+\int_\Gamma v_\Gamma(z_\Gamma+a(t))d\Gamma.$$

Hence,

$$(2.11)\qquad \int_\Omega vF_X^{-1}z'dx+\int_\Gamma v_\Gamma F_X^{-1}z'd\Gamma=\int_\Omega v\beta(z+a(t))dx+\int_\Gamma v_\Gamma(z_\Gamma+a(t))d\Gamma$$

for any $(v, v_\Gamma)\in H$.

Note here that

$$(2.12)\qquad \int_\Omega v\beta(z+a(t))dx+\int_\Gamma v_\Gamma(z_\Gamma+a(t))d\Gamma=((v, v_\Gamma), P_H(\beta(z+a(t)), z_\Gamma+a(t)))_H,$$

since $(v, v_\Gamma)\in H$. (2.11) and (2.12) imply in particular that $P_H(\beta(z+a(t)), z_\Gamma+a(t))=\hat{E}(F_X^{-1}z')$, and therefore $\beta(z+a(t))\in Y$, $z'=F_XP_X\beta(z+a(t))$ and $\beta(z+a(t))=z_\Gamma+a(t)$ a. e. on $\Gamma$.

Conversely, for $(z, z_\Gamma)\in H$, let $z^*=\hat{E}^*(z, z_\Gamma)$, $\beta(z+a(t))\in Y$, $z'=F_XP_X\beta(z+a(t))$ and $z_\Gamma=(\beta(z+a(t))-a(t))|_\Gamma$. Then for any $(v, v_\Gamma)\in H$ and $v^*=\hat{E}^*(v, v_\Gamma)$,

$$\begin{aligned}
&(z', v^*-z^*)_{X^*}\\
&=\langle v^*-z^*, F_X^{-1}z'\rangle_X\\
&=\int_\Omega(v-z)P_X\beta(z+a(t))dx+\int_\Gamma(v_\Gamma-z_\Gamma)P_X\beta(z+a(t))d\Gamma\\
&=\int_\Omega\{(v+a(t))-(z+a(t))\}\beta(z+a(t))dx+\int_\Gamma\{(v_\Gamma+a(t))-(z_\Gamma+a(t))\}(z_\Gamma+a(t))d\Gamma\\
&\leqq\int_\Omega\hat{\beta}(v+a(t))dx-\int_\Omega\hat{\beta}(z+a(t))dx+\frac{1}{2}\int_\Gamma(v_\Gamma+a(t))^2d\Gamma-\frac{1}{2}\int_\Gamma(z_\Gamma+a(t))^2d\Gamma\\
&=\varphi^t(v^*)-\varphi^t(z^*).
\end{aligned}$$

Therefore $z'\in\partial\varphi^t(z^*)$ and Lemma 2.2 has been completely proved. Q.E.D.

In order to apply the subdifferential theory to our problem we shall use the following lemma.

LEMMA 2.3. *Let* $\{\varphi^t\}_{t\in I}$ *be the family of proper l.s.c. convex functions on* $X^*$ *defined by* (2.8) *with* $a(t)$ *given by* (2.2). *Then for each compact interval* $J=[t_0, t_1]\subset I$ *there is a constant* $K>0$ *such that*

$$(2.13)\qquad |\varphi^s(z^*)-\varphi^t(z^*)|\leqq K|a(t)-a(s)|(1+|\varphi^s(z^*)|)$$

*for all* $s, t\in J$ *and* $z^*\in R(\hat{E}^*)$,

*where* $K$ *depends only on* $J$, $\beta$ *and the restriction of* $a(\cdot)$ *to* $J$.

PROOF. This is easily derived from (2.8) with the help of ($\beta$1), ($\beta$2) and (2.2). Q.E.D.

We now consider the evolution equation

$$(2.14)\qquad \frac{d}{dt}v^*(t)+\partial\varphi^t(v^*(t))=f^*(t),\qquad t\in J=[t_0, t_1],$$

where $f^*\in L^2(J; X^*)$. Under (2.13), the next result follows easily from Attouch-Damlamian [**3**] and Kenmochi [**8**].

LEMMA 2.4. *For any* $v_0^*\in R(\hat{E}^*)$ *there exists a unique function* $v^*: J\to R(\hat{E}^*)$ *such that* $v^*\in W^{1,2}(J; X^*)$, $\hat{E}^{*-1}(v^*)\in L^\infty(J; H)$ *and* $v^*$ *satisfies* (2.14) *in* $X^*$ *a.e. on* $J$ *and the initial condition* $v^*(t_0)=v_0^*$ *in* $X^*$.

The relationship between the original problem $CSP(u_0, V_0)$ and equation (2.14) is now clarified as follows.

PROPOSITION 2.1. *Let* $J=[t_0, t_1]$ *be a compact interval,* $Q=(t_0, t_1)\times\Omega$, $\Sigma=(t_0, t_1)\times\Gamma$, $f\in L^2(Q)$, $h\in L^2(\Sigma)$, $u_0\in L^2(\Omega)$ *and* $V_0\in L^2(\Gamma)$. *Then a couple* $\{u, V\}$ *of functions* $u: J\times\Omega\to\boldsymbol{R}$ *and* $V: J\times\Gamma\to\boldsymbol{R}$ *is a solution of* $CSP(u_0, V_0)$ *if and only if* $v^*:=\hat{E}^*(v, v_\Gamma)$, *with* $v:=u-a$ *and* $v_\Gamma:=V-a$, *is the solution of* (2.14) *on* $J$ *satisfying the initial condition* $v^*(t_0)=\hat{E}^*P_H(u_0, V_0)$ *in* $X^*$, *where* $f^*\in L^2(J; X^*)$ *is given by*

$$\langle f^*(t), \eta\rangle_X = (f(t), \eta)_{L^2(\Omega)}-(h(t), \eta)_{L^2(\Gamma)} \qquad \text{for } \eta\in X \text{ and a.e. } t\in J.$$

PROOF. First let $\{u, V\}$ be a solution of $CSP(u_0, V_0)$. Then it follows from (1.3) that for a.e. $t\in J$

$$\left\langle \frac{d}{dt}E^*(u(t), V(t)), \eta\right\rangle_Y + A(\beta(u(t)), \eta)+(h(t), \eta)_{L^2(\Gamma)} = (f(t), \eta)_{L^2(\Omega)} \tag{2.15}$$
$$\text{for any } \eta\in Y \text{ and a.e. } t\in J.$$

By (2.3) we see that $(v(t), v_\Gamma(t))=P_H(u(t), V(t))\in H$ for any $t\in J$, that is, $v^*(t)\in R(\hat{E}^*)$ for any $t\in J$. From (2.15) and (2.5) for any $\eta\in X$ and a.e. $t\in J$ we have

$$\begin{aligned} \left\langle \frac{d}{dt}v^*(t), \eta\right\rangle_X &= \left\langle \frac{d}{dt}E^*(u(t), V(t)), \eta\right\rangle_Y - a'(t)\left(\int_\Omega \eta\, dx+\int_\Gamma \eta\, d\Gamma\right) \\ &= \left\langle \frac{d}{dt}E^*(u(t), V(t)), \eta\right\rangle_Y \\ &= -A(\beta(u(t)), \eta)+(f(t), \eta)_{L^2(\Omega)}-(h(t), \eta)_{L^2(\Gamma)} \\ &= -\langle F_XP_X\beta(u(t)), \eta\rangle_X+\langle f^*(t), \eta\rangle_X \\ &= -\langle F_XP_X\beta(v(t)+a(t)), \eta\rangle_X+\langle f^*(t), \eta\rangle_X. \end{aligned} \tag{2.16}$$

It is clear that for a.e. $t\in J$

$$v_\Gamma(t)+a(t) = \beta(v(t)+a(t)) \qquad \text{a.e. on } \Gamma. \tag{2.17}$$

On account of Lemma 2.2, (2.16) and (2.17) we infer that

$$\frac{d}{dt}v^*(t)+\partial\varphi^t(v^*(t)) = f^*(t) \qquad \text{in } X^* \text{ for a.e. } t\in J.$$

Obviously, $v^*\in W^{1,2}(J;X^*)$ and $\hat{E}^{*-1}(v^*)=(u-a, V-a)\in L^\infty(J;H)$. Hence $v^*$ is the solution of (2.14) on $J$ satisfying $v^*(t_0)=\hat{E}^*(P_H(u_0, V_0))$.

Conversely, if $v^*$ is the solution of (2.14) on $J$ satisfying the initial condition $v^*(t_0)=\hat{E}^*(P_H(u_0, V_0))$, then $E^*(u(t_0), V(t_0))=E^*(u_0, V_0)$ and $u(t_0)=u_0$, $V(t_0)=V_0$. Furthermore, $u\in L^\infty(J;L^2(\Omega))$, $V\in L^\infty(J;L^2(\Gamma))$, $\beta(u)\in L^2(J;Y)$ and $\beta(u)=V$ a.e. on $\Sigma$, and by (2.6) for any $\eta\in Y$ and a.e. $t\in J$

$$\begin{aligned}\left\langle\frac{d}{dt}E^*(u(t), V(t)), \eta\right\rangle_Y &= \left\langle\frac{d}{dt}v^*(t), P_X\eta\right\rangle_X+a'(t)\left(\int_\Omega\eta dx+\int_\Gamma\eta d\Gamma\right)\\ &\leqq \left|\frac{d}{dt}v^*(t)\right|_{X^*}|P_X\eta|_X+|a'(t)||\eta|_Y\\ &\leqq \left(\left|\frac{d}{dt}v^*(t)\right|_{X^*}+|a'(t)|\right)|\eta|_Y.\end{aligned}$$

Therefore, $E^*(u, V)\in W^{1,2}(J;Y^*)$. Moreover, for any $\eta\in Y$ and a.e. $t\in J$ we have

$$\begin{aligned}&\left\langle\frac{d}{dt}E^*(u(t), V(t)), \eta\right\rangle_Y\\ &= \left\langle\frac{d}{dt}v^*(t), P_X\eta\right\rangle_X+a'(t)\left(\int_\Omega\eta dx+\int_\Gamma\eta d\Gamma\right)\\ &= -\langle\partial\varphi^t(v^*(t)), P_X\eta\rangle_X+\langle f^*(t), P_X\eta\rangle_X+a'(t)\left(\int_\Omega\eta dx+\int_\Gamma\eta d\Gamma\right)\\ &= -\langle F_XP_X\beta(v(t)+a(t)), P_X\eta\rangle_X+\langle f^*(t), P_X\eta\rangle_X+a'(t)\left(\int_\Omega\eta dx+\int_\Gamma\eta d\Gamma\right)\\ &= -A(\beta(v(t)+a(t)), \eta)+(f(t), \eta)_{L^2(\Omega)}-(h(t), \eta)_{L^2(\Gamma)}.\end{aligned}$$

Hence (1.3) holds. We conclude by Proposition 1.1 that $\{u, V\}$ is a solution of $CSP(u_0, V_0)$. Q.E.D.

PROOF OF THEOREM 1.1. As a consequence of Lemma 2.4, for any $v_0^*\in R(\hat{E}^*)$ and any compact interval $J=[t_0, t_1]\subset I$ there exists a function $v^*\in W^{1,2}(J;X^*)$, $\hat{E}^{*-1}(v^*)\in L^\infty(J;H)$ and $\beta(v(t)+a(t))\in Y$ for a.e. $t\in J$, which satisfies (2.14) in $X^*$ a.e. on $J$ and the initial condition $v^*(t_0)=v_0^*$. The conclusion of Theorem 1.1 then follows immediately from Proposition 2.1. Q.E.D.

### 3. Comparison result for $SP$.

The aim of this section is to prove Theorem 1.2. Throughout this section we suppose that $f\in L^2(J;L^2(\Omega))$, $h\in L^2(J;L^2(\Gamma))$ for $J=[t_0, t_1]$.

Let $\{u, V\}$ be a solution of $SP$ on $J$ and $\{\beta_\varepsilon\}$, $\{f_\varepsilon\}$, $\{h_\varepsilon\}$ and $\{z_\varepsilon\}$ be smooth approximations of $\beta$, $f$, $h$ and $u(t_0, \cdot)$, respectively, such that

$$0<\varepsilon \leqq \frac{d}{dr}\beta_\varepsilon(r) \leqq C_\beta+1 \qquad \text{for any } r\in \boldsymbol{R} \text{ and } \beta_\varepsilon(0)=0,$$

$$\beta_\varepsilon \to \beta \text{ uniformly on each compact interval of } \boldsymbol{R} \text{ as } \varepsilon\downarrow 0,$$

$$f_\varepsilon \to f \text{ in } L^2(J\,;L^2(\Omega)),\ h_\varepsilon\to h \text{ in } L^2(J\,;L^2(\Gamma)) \text{ as } \varepsilon\downarrow 0,$$

and

$$(z_\varepsilon,\ \beta_\varepsilon(z_\varepsilon)|_\Gamma)\to(u(t_0),\ V(t_0)) \text{ in } H \text{ as } \varepsilon\downarrow 0.$$

We use the following lemmas in our proof of Theorem 1.2.

LEMMA 3.1. *For each $\varepsilon>0$, there is a unique function $u_\varepsilon\colon J\to L^2(\Omega)$ such that*

(3.1) $u_\varepsilon\in W^{1,2}(J\,;L^2(\Omega))$, $\beta_\varepsilon(u_\varepsilon)\in L^\infty(J\,;Y)$ *and* $\beta_\varepsilon(u_\varepsilon)|_{J\times\Gamma}\in W^{1,2}(J\,;L^2(\Gamma))$,

(3.2) $(u_{\varepsilon t},\ \eta)_{L^2(\Omega)}+(\beta_\varepsilon(u_\varepsilon)_t,\ \eta)_{L^2(\Gamma)}+A(\beta_\varepsilon(u_\varepsilon),\ \eta)+(h_\varepsilon,\ \eta)_{L^2(\Gamma)}=(f_\varepsilon,\ \eta)_{L^2(\Omega)}$
*for any $\eta\in Y$ and a.e. $t\in J$,*

(3.3) $u_\varepsilon(t_0)=z_\varepsilon$.

We can prove this lemma in a way similar to that of [**1**; Section 2]. So we omit the proof.

LEMMA 3.2. *For each $\varepsilon>0$ let $u_\varepsilon$ be a solution of* (3.1)~(3.3). *Then*

$$\left\{\begin{array}{l} u_\varepsilon\to u \quad in\ C_w(J\,;L^2(\Omega)),\\ \beta_\varepsilon(u_\varepsilon)\to V \quad in\ C_w(J\,;L^2(\Gamma)),\\ E^*(u_\varepsilon,\ \beta_\varepsilon(u_\varepsilon))\to E^*(u,\ V) \quad weakly\ in\ W^{1,2}(J\,;Y^*),\\ (u_\varepsilon(t_0),\ \beta_\varepsilon(u_\varepsilon(t_0))|_\Gamma)\to(u(t_0),\ V(t_0)) \quad in\ H,\\ \beta_\varepsilon(u_\varepsilon)\to\beta(u) \quad weakly\ in\ L^2(J\,;W), \end{array}\right. \tag{3.4}$$

*as $\varepsilon\downarrow 0$.*

PROOF. Putting $\eta=\beta_\varepsilon(u_\varepsilon)$ in (3.2), for a.e. $t\in J$ we have

$$\frac{d}{dt}\int_\Omega\hat\beta_\varepsilon(u_\varepsilon(t))dx+\frac{1}{2}\frac{d}{dt}\int_\Gamma\beta_\varepsilon(u_\varepsilon(t))^2d\Gamma+|\nabla\beta_\varepsilon(u_\varepsilon(t))|^2_{L^2(\Omega)}+(h_\varepsilon(t),\ \beta_\varepsilon(u_\varepsilon))_{L^2(\Gamma)} \tag{3.5}$$
$$=(f_\varepsilon(t),\ \beta_\varepsilon(u_\varepsilon))_{L^2(\Omega)},$$

where $\hat\beta_\varepsilon(\xi)=\int_0^\xi\beta_\varepsilon(r)dr$ for any $\xi\in\boldsymbol{R}$. For $s\in J$, integrating (3.5) on $(t_0,\ s)$, we see that

$$\int_\Omega\hat\beta_\varepsilon(u_\varepsilon(s))dx+\frac{1}{2}\int_\Gamma\beta_\varepsilon(u_\varepsilon(s))^2d\Gamma+\int_{t_0}^s|\nabla\beta_\varepsilon(u_\varepsilon)|^2_{L^2(\Omega)}dt+\int_{t_0}^s(h_\varepsilon,\ \beta_\varepsilon(u_\varepsilon))_{L^2(\Gamma)}dt$$
$$=\int_\Omega\hat\beta_\varepsilon(z_\varepsilon)dx+\frac{1}{2}\int_\Gamma\beta_\varepsilon(z_\varepsilon)^2d\Gamma+\int_{t_0}^s(f_\varepsilon,\ \beta_\varepsilon(u_\varepsilon))_{L^2(\Omega)}dt,$$

which implies that

$$\{(u_\varepsilon,\ \beta_\varepsilon(u_\varepsilon)|_\Gamma)\} \text{ is bounded in } L^\infty(J\,;H), \tag{3.6}$$

and

(3.7) $\{\beta_\varepsilon(u_\varepsilon)\}$ is bounded in $L^2(J;Y)$.

In particular, from (3.2) and (3.7) it follows that

(3.8) $\{E^*(u_\varepsilon, \beta_\varepsilon(u_\varepsilon))\}$ is bounded in $W^{1,2}(J;Y^*)$.

By estimates (3.6)$\sim$(3.8) it is possible to extract a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ (as $n\to\infty$) such that

$$(3.9)\qquad \begin{cases} u_{\varepsilon_n} =: u_n \to \tilde{u} \quad \text{weakly* in } L^\infty(J;L^2(\Omega)), \\ \beta_{\varepsilon_n}(u_{\varepsilon_n})|_{J\times\Gamma} =: V_n \to \tilde{V} \quad \text{weakly* in } L^\infty(J;L^2(\Gamma)), \\ E^*(u_n, V_n) \to \tilde{E} \quad \text{weakly in } W^{1,2}(J;Y^*), \\ (u_n(t_0), V_n(t_0)) \to (u(t_0), V(t_0)) \quad \text{in } H, \\ \beta_{\varepsilon_n}(u_{\varepsilon_n}) =: U_n \to \tilde{U} \quad \text{weakly in } L^2(J;Y). \end{cases}$$

Since $E^*$ is compact as an operator from $W$ to $Y^*$, (3.6) and (3.9) show that

(3.10) $E^*(u_n, V_n) \to \tilde{E}$ in $C(J;Y^*)$.

Immediately, $\tilde{U}=\tilde{V}$ a.e. on $J\times\Gamma$, $\tilde{E}=E^*(\tilde{u}, \tilde{V})$, $u_n\to\tilde{u}$ in $C_w(J;L^2(\Omega))$ and $V_n\to\tilde{V}$ in $C_w(J;L^2(\Gamma))$. Now, we note from the monotonicity of $\beta_{\varepsilon_n}$ that

(3.11) $\int_J (u_n - w, U_n - \beta_{\varepsilon_n}(w))_{L^2(\Omega)}dt \geqq 0$ for any $w \in L^2(J;L^2(\Omega))$.

According to (3.9) and (3.10),

$$\begin{aligned} &\liminf_{n\to\infty}\int_J(\tilde{u}-u_n, U_n)_{L^2(\Omega)}dt \\ &= \liminf_{n\to\infty}\Big(\int_J\langle E^*(\tilde{u}, \tilde{V})-E^*(u_n, V_n), U_n\rangle_Y dt - \int_J(\tilde{V}-V_n, V_n)_{L^2(\Gamma)}dt\Big) \geqq 0. \end{aligned}$$

Hence, letting $n\to\infty$ in (3.11), we obtain that

$$\int_J(\tilde{u}-w, \tilde{U}-\beta(w))_{L^2(\Omega)}dt \geqq 0 \qquad \text{for any } w \in L^2(J;L^2(\Omega)).$$

It results from this inequality that $\tilde{U}=\beta(\tilde{u})$, because $\beta$ is maximal monotone as a mapping in $L^2(J;L^2(\Omega))$, and $\{\tilde{u}, \tilde{V}\}$ is the solution of $CSP(u(t_0), V(t_0))$ on $J$. Thus $\tilde{u}=u$ by uniqueness, and the convergences (3.4) hold.

PROOF OF THEOREM 1.2. For $J=[t_0, t_1]$ and $i=1, 2$ let $\{u_i, V_i\}$ be the solution of $SP$ on $J$ and let $u_{i\varepsilon}$ ($\varepsilon\in(0, 1]$) be the smooth approximate solutions of $CSP(u_i(t_0), V_i(t_0))$ as constructed in Lemma 3.1. Then, by the standard $L^1$-space technique, we have

$$\begin{aligned} &|[u_{1\varepsilon}(t)-u_{2\varepsilon}(t)]^+|_{L^1(\Omega)} + |[\beta_\varepsilon(u_{1\varepsilon}(t))-\beta_\varepsilon(u_{2\varepsilon}(t))]^+|_{L^1(\Gamma)} \\ &\leqq |[u_{1\varepsilon}(t_0)-u_{2\varepsilon}(t_0)]^+|_{L^1(\Omega)} + |[\beta_\varepsilon(u_{1\varepsilon}(t_0))-\beta_\varepsilon(u_{2\varepsilon}(t_0))]^+|_{L^1(\Gamma)} \quad \text{for any } t\in J. \end{aligned}$$

Therefore, on account of Lemma 3.2, letting $\varepsilon\downarrow 0$ gives (1.4) for any $t\geqq s=t_0$. By the same argument as above we see that (1.4) holds for general $t\geqq s$. Inequalities (1.5) and (1.6) follow immediately from (1.4). Q.E.D.

## 4. Boundedness of solutions to $SP$ on $[t_0, \infty)$.

In this section, we take $I=[t_0, \infty)$ and $T>0$, and we assume that $f$, $h$ satisfy the following conditions

$$(4.1)\qquad \begin{cases} f\in L^2_{loc}(I\,;\,L^2(\Omega)),\ h\in L^2_{loc}(I\,;\,L^2(\Gamma)),\\ f(t+T,\cdot)=f(t,\cdot)\quad \text{a.e. on } \Omega \text{ for } t\in I,\\ h(t+T,\cdot)=h(t,\cdot)\quad \text{a.e. on } \Gamma \text{ for } t\in I,\\ \int_{t_0}^{t_0+T}\int_\Omega f(\tau, x)dxd\tau-\int_{t_0}^{t_0+T}\int_\Gamma h(\tau, x)d\Gamma d\tau=0. \end{cases}$$

The purpose of this section is to prove the following proposition.

PROPOSITION 4.1. *Assume that $f$ and $h$ satisfy the conditions* (4.1). *Then any solution $\{u, V\}$ of $SP$ on $I$ satisfies that*

$$(4.2)\qquad u: I\to L^2(\Omega)\ \textit{and}\ V: I\to L^2(\Gamma)\ \textit{are bounded,}$$

*and*

$$(4.3)\qquad \left\{\int_{t_0+(n-1)T}^{t_0+nT}|\nabla\beta(u(\tau))|^2_{L^2(\Omega)}d\tau\right\}_{n=1}^{\infty}\ \textit{is bounded.}$$

For the proof of Proposition 4.1 we prepare the following lemmas.

LEMMA 4.1. *Under the same assumptions as in Proposition* 4.1, *there are positive constants $\mu_1$, $K_1$ depending only on $C_\beta$, $L_\beta$ and $\Omega$ such that for any $s$, $t\in I$ with $s\leqq t$*

$$(4.4)\qquad |E^*(u(t), V(t))|^2_{Y^*}$$
$$\leqq e^{-\mu_1(t-s)}|E^*(u(s), V(s))|^2_{Y^*}+K_1\int_s^t(1+|a(\tau)|^2+|h(\tau)|^2_{L^2(\Gamma)}+|f(\tau)|^2_{L^2(\Omega)})d\tau,$$

$$(4.5)\qquad \mu_1\left(\int_s^t|u(\tau)|^2_{L^2(\Omega)}d\tau+\int_s^t|V(\tau)|^2_{L^2(\Gamma)}d\tau\right)$$
$$\leqq\frac{1}{2}|E^*(u(s), V(s))|^2_{Y^*}+K_1\int_s^t(1+|a(\tau)|^2+|h(\tau)|^2_{L^2(\Gamma)}+|f(\tau)|^2_{L^2(\Omega)})d\tau,$$

*where $a$ is the function defined by* (2.2) *with $u_0=u(t_0)$ and $V_0=V(t_0)$.*

PROOF. By the definition of $F_Y$, we see that for any $t\in I$

$$(4.6)\qquad \langle E^*(u(t),\ V(t)),\ 1\rangle_Y$$
$$= (F_Y^{-1}E^*(u(t),\ V(t)),\ 1)_Y$$
$$= \left(\int_\Omega F_Y^{-1}E^*(u(t),\ V(t))dx+\int_\Gamma F_Y^{-1}E^*(u(t),\ V(t))d\Gamma\right)(|\Omega|+|\Gamma|).$$

On account of (4.6) and (2.3), we have

$$(4.7)\qquad \int_\Omega F_Y^{-1}E^*(u(t),\ V(t))dx+\int_\Gamma F_Y^{-1}E^*(u(t),\ V(t))d\Gamma = a(t) \qquad \text{for } t\in I.$$

Putting $\eta=F_Y^{-1}E^*(u(t),\ V(t))$ in (1.3), from definition of inner product $\langle\cdot,\cdot\rangle_Y$ and (4.7) we infer that for a. e. $t\in I$

$$(4.8)\qquad \frac{1}{2}\frac{d}{dt}|E^*(u(t),\ V(t))|_{Y^*}^2$$
$$= \left\langle \frac{d}{dt}E^*(u(t),\ V(t)),\ F_Y^{-1}E^*(u(t),\ V(t))\right\rangle_Y$$
$$= -A(\beta(u(t)),\ F_Y^{-1}E^*(u(t),\ V(t)))$$
$$+(f(t),\ F_Y^{-1}E^*(u(t),\ V(t)))_{L^2(\Omega)}-(h(t),\ F_Y^{-1}E^*(u(t),\ V(t)))_{L^2(\Gamma)}$$
$$= -\langle E^*(u(t),\ V(t)),\ \beta(u(t))\rangle_Y+a(t)\left(\int_\Omega\beta(u(t))dx+\int_\Gamma V(t)d\Gamma\right)$$
$$+(f(t),\ F_Y^{-1}E^*(u(t),\ V(t)))_{L^2(\Omega)}-(h(t),\ F_Y^{-1}E^*(u(t),\ V(t)))_{L^2(\Gamma)}.$$

Now, noting that for any $t\in I$

$$(4.9)\qquad \langle E^*(u(t),\ V(t)),\ \beta(u(t))\rangle_Y \geqq \mu_1\left(\int_\Omega|u(t)|^2dx+\int_\Gamma|V(t)|^2d\Gamma\right)-\frac{l_\beta^2}{2L_\beta}|\Omega|$$

where $\mu_1=\min\{L_\beta/2,\ 1\}$,

$$(4.10)\qquad |a(t)|\left(\int_\Omega\beta(u(t))dx+\int_\Gamma V(t)d\Gamma\right)$$
$$\leqq \left(\frac{\mu_1}{4}\int_\Omega|u(t)|^2dx+\int_\Gamma|V(t)|^2d\Gamma\right)+\frac{|a(t)|^2}{\mu_1}(C_\beta^2|\Omega|+|\Gamma|)$$

and

$$(4.11)\qquad |E^*(u(t),\ V(t))|_{Y^*}^2 \leqq \int_\Omega|u(t)|^2dx+\int_\Gamma|V(t)|^2d\Gamma.$$

From (4.8)~(4.11) it follows that for a. e. $t\in I$

$$\frac{1}{2}\frac{d}{dt}|E^*(u(t),\ V(t))|_{Y^*}^2+\frac{\mu_1}{2}|E^*(u(t),\ V(t))|_{Y^*}^2$$
$$\leqq \frac{2C_\Omega^2}{\mu_1}(|f(t)|_{L^2(\Omega)}^2+|h(t)|_{L^2(\Gamma)}^2)+\frac{|a(t)|^2}{\mu_1}(C_\beta^2|\Omega|+|\Gamma|)+\frac{l_\beta^2}{2L_\beta}|\Omega|.$$

We put

$$K_1 = \frac{4C_\Omega^2}{\mu_1} + \frac{2}{\mu_1}(C_\beta^2|\Omega| + |\Gamma|) + \frac{l_\beta^2}{2L_\beta}|\Omega|,$$

then for a.e. $t \in I$

$$\frac{d}{dt}\{e^{\mu_1 t}|E^*(u(t), V(t))|_{Y^*}^2\} \leqq K_1 e^{\mu_1 t}(|a(t)|^2 + |h(t)|_{L^2(\Gamma)}^2 + |f(t)|_{L^2(\Omega)}^2 + 1). \tag{4.12}$$

Hence, for $s, t \in J$ with $s \leqq t$, by integrating (4.12) on $(s, t)$, we obtain (4.4). Moreover, it follows from (4.8)$\sim$(4.11) that

$$\frac{1}{2}\frac{d}{dt}|E^*(u(t), V(t))|_{Y^*}^2 + \frac{\mu_1}{2}\left(\int_\Omega |u(t)|^2 dx + \int_\Gamma |V(t)|^2 d\Gamma\right)$$
$$\leqq K_1(|a(t)|^2 + |h(t)|_{L^2(\Gamma)}^2 + |f(t)|_{L^2(\Omega)}^2 + 1).$$

Clearly, we infer that (4.5) holds. Thus we prove this lemma. Q.E.D.

Next, in order to get some other estimates for the solution of $SP$ on $I$, introduce a function $j$ on $H$ given by

$$j(z, z_\Gamma) = \int_\Omega \hat{\beta}(z) dx + \frac{1}{2}\int_\Gamma z_\Gamma^2 d\Gamma \quad \text{for } (z, z_\Gamma) \in H,$$

where $\hat{\beta}$ is the function defined in (2.7). Then we have:

LEMMA 4.2. *We suppose that all the conditions of Proposition* 4.1 *are satisfied. Then the following statements* (1) *and* (2) *are valid.*

(1) *$j(u(t), V(t))$ is absolutely continuous on each compact subinterval $[t_0, t_1]$ of $I$ and*

$$\frac{d}{dt}j(u(t), V(t)) = \left\langle \frac{d}{dt}E^*(u(t), V(t)), \beta(u(t))\right\rangle_Y \quad \text{for a.e. } t \in I. \tag{4.13}$$

(2) *For any $s, t \in I$ with $s \leqq t$,*

$$\begin{aligned} &j(u(t), V(t)) + \int_s^t |\nabla\beta(u(\tau))|_{L^2(\Omega)}^2 d\tau \\ &\leqq j(u(s), V(s)) + \int_s^t \frac{1}{2}(|f(\tau)|_{L^2(\Omega)}^2 + |h(\tau)|_{L^2(\Gamma)}^2) d\tau \\ &\quad + K_2\int_s^t (|u(\tau)|_{L^2(\Omega)}^2 + |V(\tau)|_{L^2(\Gamma)}^2) d\tau, \end{aligned} \tag{4.14}$$

*and*

$$\begin{aligned} &(t-s)j(u(t), V(t)) + \int_s^t (\tau - s)|\nabla\beta(u(\tau))|_{L^2(\Omega)}^2 d\tau \\ &\leqq \int_s^t \frac{\tau - s}{2}(|f(\tau)|_{L^2(\Omega)}^2 + |h(\tau)|_{L^2(\Gamma)}^2) d\tau \\ &\quad + K_2\int_s^t \left(\frac{\tau - s}{2} + 1\right)(|u(\tau)|_{L^2(\Omega)}^2 + |V(\tau)|_{L^2(\Gamma)}^2) d\tau, \end{aligned} \tag{4.15}$$

*where* $K_2=1+C_\beta^2+C_\beta/2$.

PROOF. (1) is already shown in [**1**; Lemma 4.2], so we omit the proof. By virtue of (4.13) and (1.3) we see that for a.e. $\tau\in I$

$$
\begin{aligned}
(4.16)\qquad & \frac{d}{d\tau}j(u(\tau),\ V(\tau)) \\
&= \left\langle \frac{d}{dt}E^*(u(\tau),\ V(\tau)),\ \beta(u(\tau))\right\rangle_Y \\
&= -|\nabla\beta(u(\tau))|_{L^2(\Omega)}^2-(h(\tau),\ \beta(u(\tau)))_{L^2(\Gamma)}+(f(\tau),\ \beta(u(\tau)))_{L^2(\Omega)} \\
&\leqq -|\nabla\beta(u(\tau))|_{L^2(\Omega)}^2+\frac{1}{2}|h(\tau)|_{L^2(\Gamma)}^2+\frac{1}{2}|f(\tau)|_{L^2(\Omega)}^2 \\
&\quad +\frac{C_\beta^2}{2}|u(\tau)|_{L^2(\Omega)}^2+\frac{1}{2}|V(\tau)|_{L^2(\Gamma)}^2.
\end{aligned}
$$

From (4.16) it is clear that (4.14) holds. Furthermore, multiplying (4.16) by $(\tau-s)$ and integrating over $(s,\ t)$, we have

$$
\begin{aligned}
&(t-s)j(u(t),\ V(t))+\int_s^t(\tau-s)|\nabla\beta(u(\tau))|_{L^2(\Omega)}^2d\tau \\
&\leqq \int_s^t\frac{\tau-s}{2}(|f(\tau)|_{L^2(\Omega)}^2+|h(\tau)|_{L^2(\Gamma)}^2)d\tau+\int_s^t\frac{\tau-s}{2}(C_\beta^2|u(\tau)|_{L^2(\Omega)}^2+|V(\tau)|_{L^2(\Gamma)}^2)d\tau \\
&\quad +\int_s^t j(u(\tau),\ V(\tau))d\tau \\
&\leqq \int_s^t\frac{\tau-s}{2}(|f(\tau)|_{L^2(\Omega)}^2+|h(\tau)|_{L^2(\Gamma)}^2)d\tau \\
&\quad +K_2\int_s^t\left(\frac{\tau-s}{2}+1\right)(|u(\tau)|_{L^2(\Omega)}^2+|V(\tau)|_{L^2(\Gamma)}^2)d\tau.
\end{aligned}
$$

Lemma 4.2 has been proved. Q.E.D.

PROOF OF PROPOSITION 4.1. First, from (4.1) we observe that $a(t+T)=a(t)$ for $t\in I$. For simplicity, we put

$$k(t)=K_1(1+|a(t)|^2+|f(t)|_{L^2(\Omega)}^2+|h(t)|_{L^2(\Gamma)}^2)\qquad \text{for } t\in I.$$

By (4.4) for each $n=1, 2, \cdots$, we have

$$
\begin{aligned}
&|E^*(u(t_0+nT),\ V(t_0+nT))|_{Y^*}^2 \\
&\leqq e^{-\mu_1 T}|E^*(u(t_0+(n-1)T),\ V(t_0+(n-1)T))|_{Y^*}^2+\int_{(n-1)T+t_0}^{nT+t_0}k(\tau)d\tau.
\end{aligned}
$$

By an elementary calculation, we infer that

$$(4.17)\qquad \{|E^*(u(t_0+nT),\ V(t_0+nT))|_{Y^*}\}_{n=0}^\infty \text{ is bounded.}$$

On account of (4.15) and (4.5), for each $n=1, 2, \cdots$, putting $m=n-1$, we have

(4.18) $$\begin{aligned} &Tj(u(t_0+nT),\ V(t_0+nT)) \\ &\leqq \int_{t_0+mT}^{t_0+nT}\frac{\tau-s}{2}k(\tau)d\tau+K_2\int_{t_0+mT}^{t_0+nT}\Big(\frac{\tau-s}{2}+1\Big)(|u(\tau)|_{L^2(\Omega)}^2+|V(\tau)|_{L^2(\Gamma)}^2)d\tau \\ &\leqq \frac{T}{2}\int_{t_0+mT}^{t_0+nT}k(\tau)d\tau \\ &\quad+\Big(\frac{T}{2}+1\Big)\frac{K_2}{\mu_1}\Big(\frac{1}{2}|E^*(u(t_0+mT),\ V(t_0+mT))|_{Y^*}^2+\int_{t_0+mT}^{t_0+nT}k(\tau)d\tau\Big). \end{aligned}$$

It follows from (4.17), (4.18) and ($\beta 2$) that

(4.19) $\{|u(t_0+nT)|_{L^2(\Omega)}\}_{n=0}^{\infty}$ and $\{|V(t_0+nT)|_{L^2(\Gamma)}\}_{n=0}^{\infty}$ are bounded.

As a consequence of (4.5), (4.14), (4.17), (4.18), (4.19) and ($\beta 2$) we conclude that (4.2) holds. Immediately, (4.2) and (4.14) imply (4.3). Thus we have proved the proposition. Q. E. D.

## 5. Periodic solutions.

The assertions of Theorem 1.3 (i)~(iii) are obtained as direct applications of the abstract results Kenmochi-Ôtani [**9**, **10**] concerning asymptotics as $t\to\infty$ in the framework of problem (2.15).

Throughout this section we suppose that $f\in L_{loc}^2(\boldsymbol{R}; L^2(\Omega))$, $h\in L_{loc}^2(\boldsymbol{R}; L^2(\Gamma))$ and for some positive number $T$, $f$ and $h$ satisfy (1.7) and (1.8). Let $a_0$ and $t_0$ be two real numbers. Here we can choose functions $u_0\in L^2(\Omega)$ and $V_0\in L^2(\Gamma)$ so that $a_0=\langle E^*(u_0, V_0), 1\rangle_Y$. Let $a$ be a function defined by (2.2) and $\varphi^t$ be the function on $X^*$ defined by (2.8) for each $t\in\boldsymbol{R}$ and for each $t\in\boldsymbol{R}$, $f^*(t)\in X^*$ is given by

$$\langle f^*(t), \eta\rangle_X = (f(t), \eta)_{L^2(\Omega)}-(h(t), \eta)_{L^2(\Gamma)} \qquad \text{for } \eta\in X.$$

By assumptions $a$, $f^*$ and $\varphi^t$ are $T$-periodic on $\boldsymbol{R}$, that is, for all $t\in\boldsymbol{R}$

$$\begin{cases} a(t)=a(t+T), \\ f^*(t)=f^*(t+T) \quad \text{in } X^*, \\ \varphi^t=\varphi^{t+T} \quad \text{on } X^*. \end{cases}$$

PROOF OF THEOREM 1.3 (i) AND (ii). Let $\{u, V\}$ be a solution of $CSP(u_0, V_0)$ on $I:=[t_0, \infty)$. Then, from Proposition 4.1 we see that $u: I\to L^2(\Omega)$ and $V: I\to L^2(\Gamma)$ are bounded. Also, by Proposition 2.1, $v^*:=\hat{E}^*(u-a, V-a)$ is a solution of (2.15) on $I$ and $\{v^*(t); t\in I\}$ is precompact in $X^*$, because $\hat{E}^*$ is compact. Hence, by [**9**; Lemma 5],

(5.1) there is a solution $\hat{v}^*$ of (2.15) on $\boldsymbol{R}$ and $\{\hat{v}^*(t); t\in\boldsymbol{R}\}$ is precompact in $X^*$.

Besides, from Lemma 2.2 we see that $\partial\varphi^t$ is single-valued for each $t\in\boldsymbol{R}$. Hence, [**9**; Theorems 2 and 3] imply that $\hat{v}^*$ is $T$-periodic on $\boldsymbol{R}$. Therefore, by using Proposition 2.1 again, we get the assertion of Theorem 1.3 (i).

Moreover, on account of (5.1), Lemma 2.2, Proposition 2.1 and [**9**; Theorems 2 and 3], Theorem 1.3 (ii) holds.

PROOF OF THEOREM 1.3 (iii). For $i=1, 2$, let $\{u_i, V_i\}$ be $T$-periodic solutions of $SP$ on $\boldsymbol{R}$ such that

$$\int_\Omega u_1(0, x)dx+\int_\Gamma V_1(0, x)d\Gamma=\int_\Omega u_2(0, x)dx+\int_\Gamma V_2(0, x)d\Gamma.$$

From (2.1) we have

$$\int_\Omega u_1(t, x)dx+\int_\Gamma V_1(t, x)d\Gamma=\int_\Omega u_2(t, x)dx+\int_\Gamma V_2(t, x)d\Gamma \quad \text{for any } t\in\boldsymbol{R}.$$

Hence, for a. e. $t\in\boldsymbol{R}$

$$\begin{aligned}(5.2)\qquad &\frac{1}{2}\frac{d}{dt}|E^*(u_1, V_1)-E^*(u_2, V_2)|_{Y^*}^2\\ &=\left\langle\frac{d}{dt}E^*(u_1, V_1)-\frac{d}{dt}E^*(u_2, V_2), F_Y^{-1}E^*(u_1, V_1)-F_Y^{-1}E^*(u_2, V_2)\right\rangle_Y\\ &=-A(\beta(u_1)-\beta(u_2), F_Y^{-1}E^*(u_1, V_1)-F_Y^{-1}E^*(u_2, V_2))\\ &=-\langle E^*(u_1, V_1)-E^*(u_2, V_2), \beta(u_1)-\beta(u_2)\rangle_Y\\ &=-\int_\Omega(u_1-u_2)(\beta(u_1)-\beta(u_2))dx-\int_\Gamma(V_1-V_2)^2d\Gamma\leqq 0.\end{aligned}$$

By periodicity of solutions

$$(5.3)\qquad \frac{d}{dt}|E^*(u_1(t), V_1(t))-E^*(u_2(t), V_2(t))|_{Y^*}^2=0 \qquad \text{for a. e. } t\in\boldsymbol{R}.$$

According to (5.2) and (5.3) we see that for a. e. $t\in\boldsymbol{R}$

$$\int_\Omega(u_1-u_2)(\beta(u_1)-\beta(u_2))dx+\int_\Gamma(V_1-V_2)^2d\Gamma=0,$$

which shows that $\beta(u_1)=\beta(u_2)$ a. e. on $\boldsymbol{R}\times\Omega$ and $V_1=V_2$ a. e. on $\boldsymbol{R}\times\Gamma$, since $\beta$ is monotone increasing. Furthermore, (1.3) implies that

$$\frac{d}{dt}(E^*(u_1(t), V_1(t))-E^*(u_2(t), V_2(t)))=0 \quad \text{in } Y^* \text{ for a. e. } t\in\boldsymbol{R},$$

that is, $E^*(u_1(t), V_1(t))-E^*(u_2(t), V_2(t))$ is independent on $t\in\boldsymbol{R}$. Then Theorem 1.3 (iii) has been proved. Q. E. D.

Before proving Theorem 1.3 (iv) we show Theorem 1.4.

PROOF OF THEOREM 1.4. Let $\{u, V\}$ be any solution of $SP$ on $[t_0, \infty)$. We put $v^*:=\hat{E}^*(u-a, V-a)$, where $a$ is a function defined by (2.2) with $u_0=u(t_0)$,

$V_0=V(t_0)$. In a similar way to the proof of Theorem 1.2 (i) and (ii), we have (5.1). By [**9**; Theorems 2 and 3], there is a solution $w^*$ of (2.15) on $\boldsymbol{R}$ satisfying that $w^*$ is $T$-periodic and

$$v^*(t)-w^*(t)\to 0 \qquad \text{in } X^* \text{ as } n\to\infty. \tag{5.4}$$

By definition of $\varphi^t$ it is clear that $w^*(t)\in R(\hat{E}^*)$ for any $t\in\boldsymbol{R}$. Therefore, there are functions $\bar{v}:\boldsymbol{R}\to L^2(\Omega)$ and $\bar{v}_\Gamma:\boldsymbol{R}\to L^2(\Gamma)$ such that $w^*(t)=\hat{E}^*(\bar{v}(t), \bar{v}_\Gamma(t))$. Hence, the couple $\{\bar{u}, \bar{V}\}$ of functions $\bar{u}(t):=\bar{v}(t)+a(t)$, $\bar{V}(t):=\bar{v}_\Gamma(t)+a(t)$ is $T$-periodic solutions of $SP$ on $\boldsymbol{R}$. (5.4) implies (1.10) and (1.11).

By Proposition 4.1 and ($\beta$1) there is a number $\delta$ with $\delta\in[0, T]$ such that $\{\beta(u(t_0+\delta+nT))\}_{n=1}^\infty$ contains a bounded subsequence in $Y$. Therefore, we can choose a subsequence $\{n_k\}$ (depending on $\delta$) of $\{n\}$ such that

$$E(\beta(u(t_0+\delta+n_kT)))\to\bar{U} \qquad \text{in } W \text{ as } k\to\infty. \tag{5.5}$$

Since $u(t_0+\delta+nT)\to\bar{u}(t_0+\delta)$ weakly in $L^2(\Omega)$, $V(t_0+\delta+nT)\to\bar{V}(t_0+\delta)$ weakly in $L^2(\Gamma)$ and $\beta$ is maximal monotone on $L^2(\Omega)$, it follows from (5.5) that $\bar{U}=E(\beta(\bar{u}(t_0+\delta)))$ and

$$j(u(t_0+\delta+n_kT),\ V(t_0+\delta+n_kT))\to j(\bar{u}(t_0+\delta),\ \bar{V}(t_0+\delta)) \quad \text{as } k\to\infty. \tag{5.6}$$

We put $t_1:=t_0+\delta$, $u_k(t):=u(t_1+n_kT+t)$ and $V_k(t):=V(t_1+n_kT+t)$. Besides, by taking a subsequence of $\{n_k\}$ if necessary, we may assume by Proposition 4.1 and (1.3) that

$$\begin{aligned} &\beta(u_k)\to\tilde{U} && \text{weakly in } L^2(0, T; Y), \\ &E^*(u_k, V_k)\to\tilde{E} && \text{weakly in } W^{1,2}(0, T; Y^*). \end{aligned} \tag{5.7}$$

Just as in the proof of Lemma 3.2, we can prove that

$$\beta(\bar{u}(t_1+\cdot))=\tilde{U} \quad \text{and} \quad E^*(\bar{u}(t_1+\cdot), \bar{V}(t_1+\cdot))=\tilde{E}.$$

From Lemma 4.2 (i) we have

$$\frac{d}{dt}j(\bar{u}(t), \bar{V}(t))=\left\langle\frac{d}{dt}E^*(\bar{u}(t), \bar{V}(t)), \beta(\bar{u}(t))\right\rangle_Y \qquad \text{for a.e. } t\in\boldsymbol{R}. \tag{5.8}$$

It follows from (1.3) that

$$\begin{aligned} &\left\langle\frac{d}{dt}E^*(u_k(t), V_k(t)), \beta(u_k(t))-\beta(\bar{u}(t_1+t))\right\rangle_Y \\ &\quad +A(\beta(u_k(t)), \beta(u_k(t))-\beta(\bar{u}(t_1+t))) \\ &\quad +(h(t_1+t), \beta(u_k(t))-\beta(\bar{u}(t_1+t)))_{L^2(\Gamma)} \\ &= (f(t_1+t), \beta(u_k(t))-\beta(\bar{u}(t_1+t)))_{L^2(\Omega)} \qquad \text{for a.e. } t\in[0, \infty). \end{aligned} \tag{5.9}$$

Here, on account of (5.6) and (5.7) we have

$$\liminf_{k\to\infty}\int_0^T\left\langle\frac{d}{dt}E^*(u_k(t),\ V_k(t)),\ \beta(u_k(t))-\beta(\bar{u}(t_1+t))\right\rangle_Y dt$$

$$= \liminf_{k\to\infty}\Big\{j(u_k(T),\ V_k(T))-j(u_k(0),\ V_k(0))$$

$$-\int_0^T\left\langle\frac{d}{dt}E^*(u_k(t),\ V_k(t)),\ \beta(\bar{u}(t_1+t))\right\rangle_Y dt\Big\}$$

$$\geqq j(\bar{u}(T+t_1),\ \bar{V}(T+t_1))-j(\bar{u}(t_1),\ \bar{V}(t_1))$$

$$-\int_0^T\left\langle\frac{d}{dt}E^*(\bar{u}(t_1+t),\ \bar{V}(t_1+t)),\ \beta(\bar{u}(t_1+t))\right\rangle_Y dt$$

$$=0,$$

since $j$ is weakly l. s. c. on $W$ and the last inequality due to (5.8). Therefore, by integrating (5.9) over $[0, T]$ and letting $k\to\infty$, we have

$$\limsup_{k\to\infty}\int_0^T A(\beta(u_k(t)),\ \beta(u_k(t))-\nabla\beta(\bar{u}(t_1+t)))dt \leqq 0.$$

This implies that $\nabla\beta(u_k)\to\nabla\beta(\bar{u}(t_1+\cdot))$ in $L^2(0, T; L^2(\Omega))$, whence (1.12) holds without extracting any subsequence $\{n_k\}$ of $\{n\}$.

PROOF OF THEOREM 1.3 (iv). It is easy to choose pairs $\{z_i, z_{\Gamma,i}\}$ $(i=1, 2)$ of functions $(z_i, z_{\Gamma,i})\in W$ such that

$$z_1 \leqq z_2 \quad \text{a. e. on } \Omega, \qquad z_{\Gamma,1} \leqq z_{\Gamma,2} \quad \text{a. e. on } \Gamma,$$

$$\int_\Omega u_i(0,\ x)dx+\int_\Gamma V_i(0,\ x)d\Gamma = \int_\Omega z_i(x)dx+\int_\Gamma z_{\Gamma,i}(x)d\Gamma.$$

Denote by $\{\tilde{u}_i, \tilde{V}_i\}$ the solutions of $CSP(z_i, z_{\Gamma,i})$ on $J=[0, \infty)$. Then by Theorem 1.2,

$$\tilde{u}_1 \geqq \tilde{u}_2 \quad \text{a. e. on } J\times\Omega \quad \text{and} \quad \tilde{V}_1 \geqq \tilde{V}_2 \quad \text{a. e. on } J\times\Gamma. \tag{5.10}$$

Now, applying Theorem 1.4, we see that there is a $T$-periodic solution $\{w_i, w_{\Gamma,i}\}$, $i=1, 2$, such that

$$\begin{aligned} &\tilde{u}_i(t)-w_i(t)\to 0 \qquad \text{weakly in } L^2(\Omega) \text{ as } t\to\infty,\\ &\tilde{V}_i(t)-w_{\Gamma,i}(t)\to 0 \quad \text{weakly in } L^2(\Gamma) \text{ as } t\to\infty, \end{aligned} \tag{5.11}$$

$$\int_\Omega u_i(0,\ x)dx+\int_\Gamma V_i(0,\ x)d\Gamma = \int_\Omega w_i(x)dx+\int_\Gamma w_{\Gamma,i}(x)d\Gamma. \tag{5.12}$$

Moreover, by (5.10) and (5.11), we have

$$w_1 \geqq w_2, \quad \text{hence } \beta(w_1)\geqq\beta(w_2) \quad \text{a. e. on } \boldsymbol{R}\times\Omega.$$

By Theorem 1.2 (iii), (5.12) implies that

$$\beta(u_i)=\beta(w_i) \qquad \text{a. e. on } \boldsymbol{R}\times\Omega.$$

Accordingly, (1.9) holds. Q. E. D.

## References


[1] T. Aiki, Multi-dimensional Stefan problems with dynamic boundary conditions, Tech. Rep. Math. Sci., Chiba Univ., No. **18**, 1992.

[2] T. Aiki, N. Kenmochi and J. Shinoda, Periodic stability for some degenerate parabolic equations with nonlinear flux conditions, Nonlinear Anal. TMA., **17** (1991), 885-902.

[3] H. Attouch and A. Damlamian, Problèmes d'évolution dans les Hilbert et applications, J. Math. Pures Appl., **54** (1975), 53-74.

[4] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les spaces de Hilbert, North-Holland, Amsterdam, 1973.

[5] A. Damlamian, Some results on the multi-phase Stefan problem, Comm. Partial Differential Equations, **2** (1977), 1017-1044.

[6] A. Damlamian and N. Kenmochi, Periodicity and almost periodicity of solutions to a multi-phase Stefan problem in several space variables, Nonlinear Anal. TMA., **12** (1988), 921-934.

[7] A. Haraux and N. Kenmochi, Asymptotic behavior of solutions to some degenerate parabolic equations, Funkcial Ekvac., **34** (1991), 19-38.

[8] N. Kenmochi, Solvability of nonlinear evolution equations with time-dependent constraints and applications, Bull. Fac. Education, Chiba Univ., **30** (1981), 1-87.

[9] N. Kenmochi and M. Ôtani, Asymptotic behavior of periodic systems generated by time-dependent subdifferential operators, Funkcial. Ekvac., **29** (1986), 219-236.

[10] N. Kenmochi and M. Ôtani, Nonlinear evolution equations governed by subdifferential operators with almost periodic time-dependence, Rend. Accad. Naz. Sci. XL Mem. Mat., **104** (1986), 288-291.

[11] R. E. Langer, A problem in diffusion or in the flow of heat for a solid in contact with fluid, Tôhoku Math. J., Ser. 1, **35** (1932), 260-275.

[12] A. Mikelič and M. Primicerio, Homogenization of the heat equation for a domain with a network of pipes with a well-mixed fluid, to appear.

[13] M. Niezgodka and I. Pawlow, A generalized Stefan problem in several space variables, Appl. Math. Optim., **9** (1983), 193-224.

[14] M. Niezgodka, I. Pawlow and A. Visintin, Remarks on the paper by A. Visintin, Sur le problème de Stefan avec flux non linéaire, Boll. Un. Mat. Ital., Anal. Funz. Appl. Serie V, **18** (1981), 87-88.

[15] M. Primicerio and J. F. Rodrigues, The Hele-Shaw problem with nonlocal injection condition, Nonlinear Mathematical Problems in Industry, Gakuto, Tokyo, 1993.

[16] A. Visintin, Sur le problème de Stefan avec flux non linéaire, Boll. Un. Mat. Ital., Anal. Funz. Appl. Serie V, **18** (1981), 63-86.



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