On the filtration of topological and pro-l mapping class groups of punctured Riemann surfaces

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Introduction.

Let $R_{g,n}$ be a Riemann surface of genus $g \geq 0$ with $n \geq 0$ punctures, and $\Gamma_{g,n}$ be the pure mapping class group of $R_{g,n}$. As is well known, $\Gamma_{g,n}$ is isomorphic to a certain subgroup of the outer automorphism group of the topological fundamental group $\pi_{g,n}$ of $R_{g,n}$. The group $\pi_{g,n}$ has a natural filtration $\{\pi_{g,n}(m)\}_{m=1}^{\infty}$ called the weight filtration, which is introduced by T. Oda. This filtration naturally induces a filtration $\{\Gamma_{g,n}[m]\}_{m=0}^{\infty}$ of $\Gamma_{g,n}$. The aim of this paper is to serve some basic results about this filtration. In this paper, we assume that $2-2g-n < 0$, which is equivalent to that the group $\pi_{g,n}$ is non-abelian.

To explain our first result, let us recall that there exists a canonical exact sequence

$$ 1 \longrightarrow \pi_{g,n-1} \overset{d_*}{\longrightarrow} \Gamma_{g,n} \overset{p_*}{\longrightarrow} \Gamma_{g,n-1} \longrightarrow 1 . $$

The homomorphism p_* is induced from "forgetting" the n-th puncture and the homomorphism d_* can be explicitly described by a result of Birman. Then, it can be shown that the homomorphisms d_* and p_* preserve the filtrations, hence we have a complex

(*) $$ 0 \longrightarrow \mathrm{gr}(\pi_{g,n-1}) \longrightarrow \mathrm{gr}(\Gamma_{g,n}[1]) \longrightarrow \mathrm{gr}(\Gamma_{g,n-1}[1]) \longrightarrow 0 . $$

Here, each associated graded module $\mathrm{gr}(\ )$ has a structure of a graded Lie algebra. Moreover, if $g \geq 1$, the Siegel modular group $\mathrm{Sp}(2g; Z)$ naturally acts on them.

THEOREM A. *If $n \geq 2$, the complex (*) is an exact sequence of graded Lie algebras with $\mathrm{Sp}(2g; Z)$-action.*

Our second result is about the comparison of $\Gamma_{g,n}$ with the pro-l mapping class group, l being a fixed prime number. Using the pro-l completion $\pi_{g,n}^{\mathrm{pro}\text{-}l}$ of $\pi_{g,n}$ instead of $\pi_{g,n}$, we can define the pro-l mapping class group purely algebraically, which is denoted by $\Gamma_{g,n}^{(l)}$. (For the definition, see § 4-1.) The group $\Gamma_{g,n}^{(l)}$ also has a filtration induced from the weight filtration of $\pi_{\sigma,n}^{\mathrm{pro}\text{-}l}$.

Then we have the canonical homomorphism

$$\varphi_l : \Gamma_{g,n} \longrightarrow \Gamma_{g,n}^{(l)}$$

and φ_l preserves the filtration of $\Gamma_{g,n}$ and $\Gamma_{g,n}^{(l)}$. Hence φ_l induces the homomorphism

$$\mathrm{gr}^m(\varphi_l) : \mathrm{gr}^m(\Gamma_{g,n}) \longrightarrow \mathrm{gr}^m(\Gamma_{g,n}^{(l)}) \qquad (m \geq 1).$$

If $g \geq 1$, the group $\mathrm{Sp}(2g ; \boldsymbol{Z}_l)$ naturally acts on $\mathrm{gr}^m(\Gamma_{g,n}^{(l)})$ and $\mathrm{gr}^m(\varphi_l)$ is $\mathrm{Sp}(2g)$-equivariant.

Let $\bar{\Gamma}_{g,n}$ denote the topological closure of $\varphi_l(\Gamma_{g,n})$ in $\Gamma_{g,n}^{(l)}$. Then $\bar{\Gamma}_{g,n}$ is a profinite group and has two filtrations. The first one is $\{\overline{\Gamma_{g,n}[m]}\}$ induced from $\{\Gamma_{g,n}[m]\}$ and the second one is $\{\bar{\Gamma}_{g,n}[m]\}$ induced from $\{\Gamma_{g,n}^{(l)}[m]\}$. We shall study the homomorphism $\mathrm{gr}^m(\varphi_l)$. Our results are summarized as the following

THEOREM B.   (i) *For each* $m \geq 1$, *the* $\boldsymbol{Z}$-*module* $\mathrm{gr}^m(\Gamma_{g,n})$ *and the* $\boldsymbol{Z}_l$-*module* $\mathrm{gr}^m(\bar{\Gamma}_{g,n})$ *are free of finite rank and* $\mathrm{gr}^m(\varphi_l)$ *induces an injective homomorphism*

$$(*) \qquad\qquad \mathrm{gr}^m(\Gamma_{g,n}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Z}_l \longrightarrow \mathrm{gr}^m(\bar{\Gamma}_{g,n}).$$

If $g \geq 1$, *this homomorphism is* $\mathrm{Sp}(2g)$-*equivariant.*

   (ii) *Assume that* $g \neq 1$. *Then we have*

$$\overline{\Gamma_{g,n}[m]} = \bar{\Gamma}_{g,n}[m] \qquad (m \geq 0),$$

hence the homomorphism (*) *is an isomorphism.*

There are two crucial points for the proof of our results. One of them is that the graded Lie algebras $\mathrm{gr}(\pi_{g,n})$ and $\mathrm{gr}(\pi_{g,n}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{F}_p$ (p : prime) have trivial center. Another one, which is for the proof of Theorem B (ii), is some properties of the symplectic group, especially the congruence subgroup property of $\mathrm{Sp}(2g ; \boldsymbol{Z})$ ($g \geq 2$). In fact, in the case that $g = 1$ (and $n = 1$), analogous result does not hold.

The motivation of the present work is some observations by T. Oda (cf. e. g., [O₁]), which we shall briefly explain. Let us consider a pair (C, S), where C is a complete non-singular curve defined over a finite algebraic number field $k \subset C$ and S be a finite set of k-rational points on C with its cardinality n. Then the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ acts naturally on the pro-l fundamental group $\pi_1^{\mathrm{pro}\text{-}l}(C \underset{k}{\otimes} \bar{k} \backslash S)$ and we have a Galois representation

$$\rho_l : \mathrm{Gal}(\bar{k}/k) \longrightarrow \mathrm{Out}(\pi_1^{\mathrm{pro}\text{-}l}(C \underset{k}{\otimes} \bar{k} \backslash S)).$$

The image of this representation is contained in $\Gamma_{g,n}^{(l)}$, provided that the generators of $\pi_1^{\mathrm{pro}\text{-}l}(C \underset{k}{\otimes} \bar{k} \backslash S)$ are suitably chosen. Moreover Oda has observed

that (i) the image is contained in the normalizer $N(\bar{\Gamma}_{g,n})$ of $\bar{\Gamma}_{g,n}$ in $\Gamma^{(l)}_{g,n}$ and that (ii) the composite of ρ_l with the canonical projection $N(\bar{\Gamma}_{g,n}) \rightarrow N(\bar{\Gamma}_{g,n})/\bar{\Gamma}_{g,n}$ does not depend on (C, S) and is determined only by g, n and l. And he has proposed to investigate the groups $N(\bar{\Gamma}_{g,n})$ and $N(\bar{\Gamma}_{g,n})/\bar{\Gamma}_{g,n}$. Since we have little knowledge about these groups, to establish some basic properties of $\bar{\Gamma}_{g,n}$ seems to be useful in further investigations.

The organization of this paper is as follows. In § 1, we shall give some preparations for the proof of main results. The aim of § 2 is to establish some basic properties of the homomorphisms d_* and p_*. § 3 is devoted to the proof of Theorem A. In § 4, we first summarize some known facts about the pro-l fundamental groups and the pro-l mapping class groups. Then the proof of Theorem B is given. In the last section, we shall summarize some open problems and discuss them.

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§ 1. Preliminaries for the proof of results.

1-1. Weight filtration of the fundamental group and its associated graded Lie algebra.

In this subsection, we shall recall the weight filtration of the topological fundamental group of a punctured Riemann surface introduced by T. Oda and summarize some basic properties of its associated graded Lie algebra. (Cf. Asada-Matsumoto-Oda [**AMO**], Kaneko [**K**].)

Let $\pi_{g,n}$ denote the group defined by the following generators and a defining relation:

generators: $x_1, \cdots, x_{2g}, z_1, \cdots, z_n$

relation : $[x_1, x_{g+1}] \cdots [x_g, x_{2g}]z_1 \cdots z_n = 1$.

ere the bracket $[\ ,\ ]$ denotes the commutator; $[a, b] = aba^{-1}b^{-1}$.

The weight filtration $\{\pi_{g,n}(m)\}_{m=1}^{\infty}$ of $\pi_{g,n}$ is defined as follows:

$$\pi_{g,n}(1) = \pi_{g,n}$$

$$\pi_{g,n}(2) = \langle [\pi_{g,n}, \pi_{g,n}], z_1, \cdots, z_n \rangle_{\text{normal}}$$

$$\pi_{g,n}(m) = \langle [\pi_{g,n}(m_1), \pi_{g,n}(m_2)] \mid m_1 + m_2 = m \rangle_{\text{normal}} \quad (m \geq 3).$$

Here $\langle\ \rangle_{\text{normal}}$ denotes the smallest normal subgroup of $\pi_{g,n}$ containing all elements inside.

Then, $\{\pi_{g,n}(m)\}_{m=1}^{\infty}$ is a decreasing sequence of normal subgroups of $\pi_{g,n}$.

Moreover, it is a central filtration, i.e.,

$$[\pi_{g,n}(m), \pi_{g,n}(m')] \subset \pi_{g,n}(m+m')$$

holds for all m, $m' \geq 1$.

REMARK. We note that in the case of $n=0$ and 1, the weight filtration coincides with the (usual) lower central filtration. But in the case of $n \geq 2$, they are different and should be distinguished.

As is well known, the graded module

$$\mathrm{gr}(\pi_{g,n}) = \bigoplus_{m=1}^{\infty} \mathrm{gr}^m(\pi_{g,n}) \quad (\mathrm{gr}^m(\pi_{g,n}) = \pi_{g,n}(m)/\pi_{g,n}(m+1))$$

has a structure of a graded Lie algebra, the Lie bracket being naturally induced by the commutator (cf. e. g., Bourbaki [**Bo**, Ch. 2]). By theorems of Witt [**W**] and Labute [**La**], the generators and a defining relation of the Lie algebra $\mathrm{gr}(\pi_{g,n})$ are given as follows:

generators: $X_1, \cdots, X_{2g}, Z_1, \cdots, Z_n$

relation   : $\displaystyle\sum_{i=1}^{g} [X_i, X_{g+i}] + \sum_{j=1}^{n} Z_j = 0,$

where $X_i = x_i \bmod \pi_{g,n}(2)$ $(1 \leq i \leq 2g)$ and $Z_j = z_j \bmod \pi_{g,n}(3)$ $(1 \leq j \leq n)$.

The basic facts we shall use throughout this paper about the Lie algebra $\mathrm{gr}(\pi_{g,n})$ is summarized as the following

THEOREM 1 (Witt [**W**], Labute [**La**]). (i) *As a **Z**-module, $\mathrm{gr}(\pi_{g,n})$ is free.*
(ii) *The Lie algebra $\mathrm{gr}(\pi_{g,n})$ has trivial center.*
(iii) *The Lie algebra $\mathrm{gr}(\pi_{g,n}) \underset{Z}{\otimes} F_p$ has trivial center for all prime number p.*

For the proof of (ii) and (iii) in the case of $n=0$, cf. Asada [**As**]. See also Bass-Lubotzky [**BL**].

1-2. Induced filtration of the mapping class group.

In this subsection, we shall start with an algebraic definition of the pure mapping class group and recall its filtration induced from the weight filtration of the fundamental group.

Let us denote the automorphism group of $\pi_{g,n}$ by $\mathrm{Aut}(\pi_{g,n})$ and put

$$\mathrm{Aut}_{\{z_i\}_i}(\pi_{g,n}) = \{\sigma \in \mathrm{Aut}(\pi_{g,n}) \mid \sigma(z_j) \sim z_j \; {}^{\forall}j\},$$

where $\sim$ denotes the conjugacy in $\pi_{g,n}$. Then, $\mathrm{Aut}_{\{z_i\}_i}(\pi_{g,n})$ is a subgroup of $\mathrm{Aut}(\pi_{g,n})$, stabilizes the filtration $\{\pi_{g,n}(m)\}_{m \geq 1}$, and contains the inner automorphism group $\mathrm{Int}(\pi_{g,n})$ of $\pi_{g,n}$.

Assume that $g \geq 1$. Then, the group $\mathrm{Aut}_{\{z_i\}_i}(\pi_{g,n})$ acts naturally on $\mathrm{gr}^1(\pi_{g,n})$, which is a free **Z**-module of rank $2g$ with a basis $\{x_i \bmod \pi_{g,n}(2)\}_{1 \leq i \leq 2g}$. This

gives a representation

$$\rho : \mathrm{Aut}_{\{z_i\}_i}(\pi_{g,n}) \longrightarrow \mathrm{GL}(2g ; \boldsymbol{Z}),$$
$$\sigma \longrightarrow (\rho_{ij})$$

where $\sigma(x_i) = \prod_{k=1}^{2g} x_k^{\rho_{ki}} \bmod \pi_{g,n}(2)$. Then, it can be verified that the image of ρ is contained in the group

$$\mathrm{GSp}(2g ; \boldsymbol{Z}) = \{A \in \mathrm{GL}(2g ; \boldsymbol{Z}) \mid {}^t A J_g A = \chi(A) J_g, \ \chi(A) = \pm 1\},$$

with $J_g = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}$.

We define the group $\tilde{\Gamma}_{g,n}$ by

$$\tilde{\Gamma}_{g,n} = \{\sigma \in \mathrm{Aut}_{\{z_i\}_i}(\pi_{g,n}) \mid \chi(\rho(\sigma)) = 1\}$$
$$= \{\sigma \in \mathrm{Aut}(\pi_{g,n}) \mid \sigma(z_j) \sim z_j \ {}^\forall j, \ \chi(\rho(\sigma)) = 1\}.$$

(If $g=0$, the condition that $\chi(\rho(\sigma))=1$ is empty.) The quotient group of $\tilde{\Gamma}_{g,n}$ by $\mathrm{Int}(\pi_{g,n})$ is denoted by $\Gamma_{g,n}$;

$$\Gamma_{g,n} = \tilde{\Gamma}_{g,n} / \mathrm{Int}(\pi_{g,n}),$$

and is called the pure mapping class group of $R_{g,n}$.

Now, for each non-negative integer m, set

$$\tilde{\Gamma}_{g,n}[m] = \{\sigma \in \tilde{\Gamma}_{g,n} \mid x_i^\sigma x_i^{-1} \in \pi_{g,n}(m+1) \ {}^\forall i, \ z_j^\sigma \underset{m}{\sim} z_j \ {}^\forall j\}.$$

Here, $\underset{m}{\sim}$ denotes conjugacy by an element of $\pi_{g,n}(m)$. Then, $\{\tilde{\Gamma}_{g,n}[m]\}_{m \geq 0}$ gives a decreasing sequence of normal subgroups of $\tilde{\Gamma}_{g,n}$;

$$\tilde{\Gamma}_{g,n} = \tilde{\Gamma}_{g,n}[0] \supset \tilde{\Gamma}_{g,n}[1] \supset \cdots \supset \tilde{\Gamma}_{g,n}[m] \supset \tilde{\Gamma}_{g,n}[m+1] \supset \cdots.$$

If $g=0$, $\tilde{\Gamma}_{g,n} = \tilde{\Gamma}_{g,n}[1]$. If $g \geq 1$, $\tilde{\Gamma}_{g,n}[1]$ is nothing but the kernel of ρ. According to a classical result of Nielsen (cf. e.g., Magnus-Karrass-Solitar [**MKS**, § 3.7]), we have the following

THEOREM 2. *The representation ρ induces an isomorphism*

$$\tilde{\Gamma}_{g,n}[0] / \tilde{\Gamma}_{g,n}[1] \cong \mathrm{Sp}(2g ; \boldsymbol{Z}).$$

REMARK. The original Nielsen's result is the cases that $n=0$ and $n=1$. The case that $n \geq 2$ is easily reduced to the case that $n=1$ by using the forgetful homomorphism $\tilde{\Gamma}_{g,n} \to \tilde{\Gamma}_{g,1}$, which is surjective (cf. § 3-1).

The following proposition is basic.

PROPOSITION 1. *We have*

$$[\tilde{\Gamma}_{g,n}[m], \tilde{\Gamma}_{g,n}[m']] \subset \tilde{\Gamma}_{g,n}[m+m'] \qquad \textit{for all } m, m' \geq 0.$$

In particular, $\tilde{\Gamma}_{g,n}[m]$ *is a normal subgroup of* $\tilde{\Gamma}_{g,n}$ $(m \geq 0)$ *and* $\tilde{\Gamma}_{g,n}[m]/\tilde{\Gamma}_{g,n}[m+1]$ *is an abelian group if* $m \geq 1$.

For the proof, see e. g., [K] (in the context of pro-l groups) or [Bo, Ch. 2, Exercises §4.3] in a more general setting.

The filtration $\{\tilde{\Gamma}_{g,n}[m]\}_{m \geq 0}$ of $\tilde{\Gamma}_{g,n}$ naturally induces a filtration of $\Gamma_{g,n}$, namely, we put

$$\Gamma_{g,n}[m] = \tilde{\Gamma}_{g,n}[m]\,\mathrm{Int}(\pi_{g,n})/\mathrm{Int}(\pi_{g,n}) \qquad (m \geq 0).$$

By Proposition 1, $\{\Gamma_{g,n}[m]\}_{m \geq 0}$ is a decreasing sequence of subgroups of $\Gamma_{g,n}$ and satisfies the same properties stated in Proposition 1 for $\{\tilde{\Gamma}_{g,n}[m]\}_{m \geq 0}$.

We note that, since $\{\tilde{\Gamma}_{g,n}[m]\}_{m \geq 1}$ (resp. $\{\Gamma_{g,n}[m]\}_{m \geq 1}$) is a central filtration of $\tilde{\Gamma}_{g,n}[1]$ (resp. $\Gamma_{g,n}[1]$), the associated graded modules

$$\mathrm{gr}(\tilde{\Gamma}_{g,n}[1]) = \bigoplus_{m=1}^{\infty} \mathrm{gr}^m(\tilde{\Gamma}_{g,n}) \quad (\mathrm{gr}^m(\tilde{\Gamma}_{g,n}) = \tilde{\Gamma}_{g,n}[m]/\tilde{\Gamma}_{g,n}[m+1])$$

$$\mathrm{gr}(\Gamma_{g,n}[1]) = \bigoplus_{m=1}^{\infty} \mathrm{gr}^m(\Gamma_{g,n}) \quad (\mathrm{gr}^m(\Gamma_{g,n}) = \Gamma_{g,n}[m]/\Gamma_{g,n}[m+1])$$

have a natural structure of a graded Lie algebra. Moreover, in the case that $g \geq 1$, the conjugate action of $\tilde{\Gamma}_{g,n}$ (resp. $\Gamma_{g,n}$) on $\mathrm{gr}^m(\tilde{\Gamma}_{g,n})$ (resp. $\mathrm{gr}^m(\Gamma_{g,n})$) factors through $\tilde{\Gamma}_{g,n}/\tilde{\Gamma}_{g,n}[1]$ (resp. $\Gamma_{g,n}/\Gamma_{g,n}[1]$). Hence $\mathrm{gr}^m(\tilde{\Gamma}_{g,n})$ and $\mathrm{gr}^m(\Gamma_{g,n})$ are naturally $\mathrm{Sp}(2g\,;\,Z)$-modules. For the structure of modules $\mathrm{gr}^m(\Gamma_{g,n}) \bigotimes_Z Q$ $(1 \leq m \leq 3)$, see Asada-Nakamura [AN].

For the intersection of the filtration $\{\Gamma_{g,n}[m]\}_{m \geq 0}$, we have the following

PROPOSITION 2. $\bigcap\limits_{m=1}^{\infty} \Gamma_{g,n}[m] = \{1\}$ *if* $(g, n) \neq (2, 0)$.

For the proof, cf. [BL, §11]. Whether Proposition 2 holds also in the case of $(g, n) = (2, 0)$ seems to be unknown.

We shall see in the next section that $\mathrm{gr}^m(\Gamma_{g,n})$ is a free Z-module of finite rank (Proposition B1). Combining this with Proposition 2, we conclude that, if $(g, n) \neq (2, 0)$, the group $\Gamma_{g,n}[1]$ is torsion-free. But this is well known for all (g, n) (cf. Serre [Se₁]).

1-3. Coordinate module and its submodules.

To describe the module $\mathrm{gr}^m(\Gamma_{g,n})$ $(m \geq 1)$, let us recall the coordinate module and its submodules (cf. Nakamura-Tsunogai [NT]).

First, the coordinate module $C_m(2g, n)$ $(m \geq 1)$ is defined by

$$C_m(2g,\ n) = \begin{cases} (\mathrm{gr}^{m+1}(\pi_{g,n}))^{\oplus 2g} \oplus (\mathrm{gr}^m(\pi_{g,n}))^{\oplus n} & m \neq 2 \\ (\mathrm{gr}^{m+1}(\pi_{g,n}))^{\oplus 2g} \oplus \bigoplus_{j=1}^{n} (\mathrm{gr}^m(\pi_{g,n})/Z\bar{z}_j) & m = 2. \end{cases}$$

Here $^-$ denotes the image in the quotient.

Secondly, let $\tilde{f}_m$ denote the following Z-linear homomorphism

$$\tilde{f}_m : C_m(2g, n) \longrightarrow \mathrm{gr}^{m+2}(\pi_{g, n})$$

$$(S_i) \times (T_j) \longrightarrow \sum_{i=1}^{g} ([x_i, S_{g+i}] + [S_i, x_{g+i}]) + \sum_{j=1}^{n} [T_j, z_j]$$

and denote the kernel of $\tilde{f}_m$ by $M_m(2g, n)$.

Thirdly, we define the mapping

$$c_m(2g, n) : \tilde{\Gamma}_{g, n}[m] \longrightarrow C_m(2g, n)$$

as follows. For $\sigma \in \tilde{\Gamma}_{g, n}[m]$, put $s_i(\sigma) = \sigma(x_i)x_i^{-1}$ ($1 \le i \le g$), and let t_j be an element of $\pi_{g, n}(m)$ such that $\sigma(z_j) = t_j z_j t_j^{-1}$ ($1 \le j \le n$). Since the centralizer of z_j in $\pi_{g, n}$ is the infinite cyclic group generated by z_j, t_j is uniquely determined if $m \ne 2$ and $\bar{t}_j \bmod Z\bar{z}_j$ is uniquely determined if $m = 2$. We define

$$c_m(2g, n)(\sigma) = \overline{(s_i(\sigma))}_{1 \le i \le 2g} \times (\bar{t}_j)_{1 \le j \le n}.$$

Since we have $s_i(\sigma\tau) = \sigma(s_i(\tau))s_i(\sigma)$ ($\sigma, \tau \in \tilde{\Gamma}_{g, n}[m]$) and $\tilde{\Gamma}_{g, n}[m]$ acts trivially on $\mathrm{gr}^{m+1}(\pi_{g, n})$, $c_m(2g, n)$ induces an injective homomorphism

$$\tilde{\iota}_m(2g, n) : \mathrm{gr}^m(\tilde{\Gamma}_{g, n}) \longrightarrow C_m(2g, n).$$

We denote the image of $\tilde{\iota}_m(2g, n)$ by $N_m(2g, n)$.

The following lemma is basic. See e.g., [K] (in the context of pro-l groups), Morita [M₁] (in the case of $n=1$).

LEMMA 1.　$M_m(2g, n) \supset N_m(2g, n)$.

Let $g_m(2g, n)$ denote the following Z-linear homomorphism

$$g_m(2g, n) : \mathrm{gr}^m(\pi_{g, n}) \longrightarrow C_m(2g, n)$$

$$\bar{t} \longrightarrow (\overline{[t, x_i]})_{1 \le i \le 2g} \times (\bar{t}_j)_{1 \le j \le n-1}$$

and denote the image of $g_m(2g, n)$ by $I_m(2g, n)$.

The module $I_m(2g, n)$ is contained in $N_m(2g, n)$. In fact, for each $t \in \pi_{g, n}(m)$, we have

$$\tilde{\iota}_m(2g, n)(\mathrm{Int}(t) \bmod \tilde{\Gamma}_{g, n}[m+1]) = g_m(2g, n)(\bar{t}).$$

Here, $\mathrm{Int}(t)$ is the inner automorphism of $\pi_{g, n}$ induced from conjugation by t; $\mathrm{Int}(t)(x) = txt^{-1}$ ($x \in \pi_{g, n}$), which belongs to $\tilde{\Gamma}_{g, n}[m]$.

LEMMA 2.　*The Z-module $C_m(2g, n)/I_m(2g, n)$ is free.*

PROOF.　By Theorem 1 (ii), $g_m(2g, n) \underset{Z}{\otimes} F_p$ is injective for all prime number p. Since $\mathrm{gr}^m(\pi_{g, n})$ and $C_m(2g, n)$ are both free Z-modules of finite rank, by the theory of elementary divisors, it follows that $C_m(2g, n)/I_m(2g, n)$ is free.

PROPOSITION B1. *The group* $\mathrm{gr}^m(\Gamma_{g,n})$ *is a free* **Z**-*module of finite rank for* $m \geq 1$.

In the case of $n=0$, this is proved in [**As**] (cf. also [**BL**]). In the case of $n \geq 1$, in the pro-l context, corresponding result is proved in [**K**] (cf. also [**NT**]). Although the discrete case can be treated almost in the same way, we shall give a proof for the sake of completeness.

PROOF. For each positive integer m, set

$$\mathrm{Int}_{\pi_{g,n}}(\pi_{g,n}(m)) = \{\sigma \in \mathrm{Int}(\pi_{g,n}) \mid \sigma = \mathrm{Int}(t) \text{ with } t \in \pi_{g,n}(m)\}.$$

Let us consider the following homomorphism;

$$\mathrm{Int}_m : \mathrm{gr}^m(\pi_{g,n}) \longrightarrow \mathrm{Int}_{\pi_{g,n}}(\pi_{g,n}(m))/\mathrm{Int}_{\pi_{g,n}}(\pi_{g,n}(m+1))$$
$$\bar{t} \longrightarrow \text{the class of } \mathrm{Int}(t).$$

Then, by Theorem 1 (ii), it follows that Int_m is an isomorphism and

$$(1.3.1) \qquad \tilde{\Gamma}_{g,n}[m] \cap \mathrm{Int}(\pi_{g,n}) = \mathrm{Int}_{\pi_{g,n}}(\pi_{g,n}(m)) \qquad \text{for all } m \geq 1.$$

(See [**As**, Lemma 4].) Hence, we have the following commutative diagram;

$$0 \longrightarrow \mathrm{Int}_{\pi_{g,n}}(\pi_{g,n}(m))/\mathrm{Int}_{\pi_{g,n}}(\pi_{g,n}(m+1)) \longrightarrow \mathrm{gr}^m(\tilde{\Gamma}_{g,n}) \longrightarrow \mathrm{gr}^m(\Gamma_{g,n}) \longrightarrow 0$$
$$\mathrm{Int}_m \uparrow \qquad\qquad g_m(2g,n) \qquad\qquad \downarrow \tilde{\iota}_m(2g,n) \qquad \text{(exact)}$$
$$\mathrm{gr}^m(\pi_{g,n}) \xrightarrow{\qquad\qquad\qquad} C_m(2g,n).$$

Thus, $\tilde{\iota}_m(2g,n)$ induces an injective homomorphism

$$\iota_m(2g,n) : \mathrm{gr}^m(\Gamma_{g,n}) \longrightarrow C_m(2g,n)/I_m(2g,n).$$

The proposition follows from Lemma 2.

REMARK 1. By (1.3.1), we have an exact sequence

$$(1.3.2) \qquad 1 \longrightarrow \mathrm{Int}_{\pi_{g,n}}(\pi_{g,n}(m)) \longrightarrow \tilde{\Gamma}_{g,n}[m] \longrightarrow \Gamma_{g,n}[m] \longrightarrow 1.$$

REMARK 2. It seems to be a difficult problem to determine the image of the homomorphism $\iota_m(2g,n)$. In the case of $m=1$ (and $n=0$ or 1), it is determined by Johnson [**J**]. (Using this, we shall determine the image of $\iota_m(2g,n)$ for all $n \geq 0$ in § 3-1.) The case of $m=2$ has been treated by Morita [**M₁**] and he gives a new restriction on the image of $\iota_m(2g,1)$ for odd m in Morita [**M₂**]. In particular, it gives an upper bound for the rank of $\mathrm{gr}^m(\Gamma_{g,1})$. By using his result, the case of $m=3$ has been treated in [**AN**]. Recently, a lower bound for the rank of $\mathrm{gr}^m(\Gamma_{g,0})$ has also been obtained by Oda [**O₂**].

§ 2. Forgetful homomorphism.

2-1. Forgetful homomorphism.

Let $\pi_{g,n}$ and $\pi_{g,n-1}$ be the group defined by the following generators and relations:

$$\pi_{g,n} \quad \begin{cases} \text{generators}: & x_1, \cdots, x_{2g}, z_1, \cdots, z_n \\ \text{relation} & : \quad [x_1, x_{g+1}] \cdots [x_g, x_{2g}]z_1 \cdots z_n = 1 \end{cases}$$

$$\pi_{g,n-1} \quad \begin{cases} \text{generators}: & \xi_1, \cdots, \xi_{2g}, \zeta_1, \cdots, \zeta_{n-1} \\ \text{relation} & : \quad [\xi_1, \xi_{g+1}] \cdots [\xi_g, \xi_{2g}]\zeta_1 \cdots \zeta_{n-1} = 1. \end{cases}$$

Then, there exists uniquely a homomorphism

$$p : \pi_{g,n} \longrightarrow \pi_{g,n-1}$$

satisfying

$$p(x_i) = \xi_i \qquad (1 \leq i \leq 2g),$$

$$p(z_j) = \zeta_j \qquad (1 \leq j \leq n-1),$$

$$p(z_n) = 1.$$

Obviously, p is surjective and it follows easily that p preserves the weight filtrations, i.e., $p(\pi_{g,n}(m)) \subset \pi_{g,n-1}(m)$ $(m \geq 1)$. Since an element of $\tilde{\Gamma}_{g,n}$ preserves the conjugacy class determined by z_n, p induces homomorphisms

$$\tilde{p}_* : \tilde{\Gamma}_{g,n} \longrightarrow \tilde{\Gamma}_{g,n-1},$$

$$p_* : \Gamma_{g,n} \longrightarrow \Gamma_{g,n-1}.$$

It is well known that p_* is surjective. In the case of $n \geq 2$, this can be proved rather easily. In fact, we shall prove a more precise result in § 3-1. In the case of $n=1$, this is obtained by Nielsen. (Cf. e.g., [**MKS**, § 3.7 Theorem N10].)

Now, by a result of Birman [**Bi**$_1$, § 3], the kernel of p_* is isomorphic to $\pi_{g,n-1}$;

$$1 \longrightarrow \pi_{g,n-1} \xrightarrow{\ d_*\ } \Gamma_{g,n} \xrightarrow{\ p_*\ } \Gamma_{g,n-1} \longrightarrow 1 \quad \text{(exact)},$$

and the homomorphism d_* is described explicitly as we shall explain below.

Let us choose a base point of $R_{g,n}$ sufficiently close to the n-th puncture and let the generators $x_1, \cdots, x_{2g}, z_1, \cdots, z_n$ be represented by the simple closed curves in Figure 1. Let U_i be a cylindrical neighborhood of x_i containing the n-th puncture. The boundary of U_i consists of two simple closed curves $c_0^{(i)}$ and $c_1^{(i)}$. Then, $d_*(\xi_i)$ is the product of a pair of Dehn twists, in opposite directions, about these curves. The case of ζ_j is described in the same way, although one of the Dehn twists is trivial.
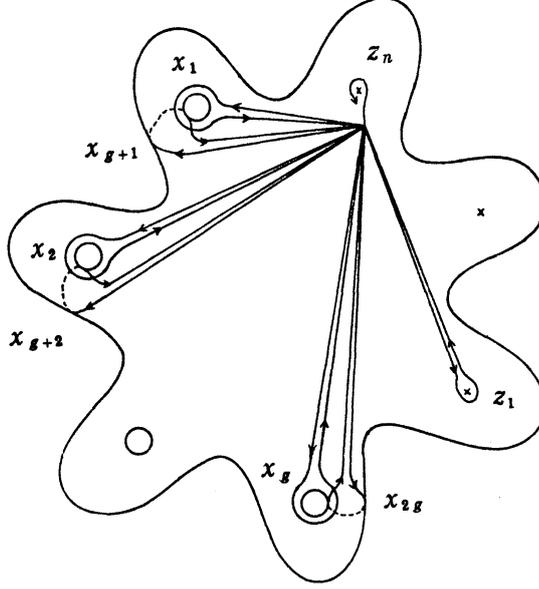
Figure 1.

Easy calculations show that these Dehn twists are represented by the following automorphisms of $\pi_{g,n}$:

$\sigma_0^{(i)}$ $(1 \leqq i \leqq g)$

$$x_{g+i} \longrightarrow x_{g+i} x_i^{-1}$$

$\sigma_1^{(i)}$ $(1 \leqq i \leqq g)$

$$x_i \longrightarrow (z_n x_i) x_i (z_n x_i)^{-1}$$

$$x_{g+i} \longrightarrow x_{g+i} (z_n x_i)^{-1}$$

$$z_n \longrightarrow (z_n x_i) z_n (z_n x_i)^{-1}$$

$$x_j \longrightarrow [z_n,\ x_i] x_j [z_n,\ x_i]^{-1} \quad (1 \leqq j \leqq i-1,\ g+1 \leqq j \leqq g+i-1)$$

$\sigma_0^{(i)}$ $(g+1 \leqq i \leqq 2g)$

$$x_i \longrightarrow x_i x_{g+i}$$

$\sigma_1^{(i)}$ $(g+1 \leqq i \leqq 2g)$

$$x_i \longrightarrow [z_n,\ x_{g+i}^{-1}] x_i (z_n x_{g+i}^{-1})^{-1}$$

$$x_{g+i} \longrightarrow (z_n x_{g+i}^{-1}) x_{g+i} (z_n x_{g+i}^{-1})^{-1}$$

$$z_n \longrightarrow (z_n x_{g+i}^{-1}) z_n (z_n x_{g+i}^{-1})^{-1}$$

$$x_j \longrightarrow [z_n,\ x_{g+i}^{-1}] x_j [z_n,\ x_{g+i}^{-1}]^{-1} \quad (1 \leqq j \leqq i-1,\ g+1 \leqq j \leqq g+i-1)$$

$\tau^{(i)}$ $(1 \leqq i \leqq n-1)$

$$x_j \longrightarrow [z_n,\ z_i] x_j [z_n,\ z_i]^{-1} \quad (1 \leqq j \leqq 2g)$$

$$z_j \longrightarrow [z_n, z_i] z_j [z_n, z_i]^{-1} \quad (1 \leq j \leq i-1)$$

$$z_i \longrightarrow (z_n z_i) z_i (z_n z_i)^{-1}$$

$$z_n \longrightarrow (z_n z_i) z_n (z_n z_i)^{-1}$$

(The generators not written explicitly above should be regarded as being fixed.) Thus, we can summarize a result of Birman as the following

THEOREM 3 (Birman [**Bi**₁, § 3]). *For suitably chosen generators* $\xi_1, \cdots, \xi_{2g}$, $\zeta_1, \cdots, \zeta_{n-1}$ *of* $\pi_{g, n-1}$, *the homomorphism* d_* *satisfies*

$$d_*(\xi_i) = \sigma_1^{(i)} (\sigma_0^{(i)})^{-1} \bmod \mathrm{Int}(\pi_{g, n}) \quad (1 \leq i \leq 2g),$$

$$d_*(\zeta_j) = \tau^{(j)} \bmod \mathrm{Int}(\pi_{g, n}) \quad (1 \leq j \leq n-1).$$

COROLLARY. $d_*(\pi_{g, n-1}) \subset \Gamma_{g, n}[1]$.

PROPOSITION 3. *The homomorphisms* d_* *and* p_* *preserve the filtrations and the associated Lie algebra homomorphisms*

$$\mathrm{gr}(d_*) \colon \mathrm{gr}(\pi_{g, n-1}) \longrightarrow \mathrm{gr}(\Gamma_{g, n}[1])$$

$$\mathrm{gr}(p_*) \colon \mathrm{gr}(\Gamma_{g, n}[1]) \longrightarrow \mathrm{gr}(\Gamma_{g, n-1}[1])$$

are Sp(2g ; **Z**)-*equivariant* (*in the case that* $g \geq 1$).

PROOF. That the homomorphism p_* preserves the filtrations and $\mathrm{gr}(p_*)$ is Sp(2g ; **Z**)-equivariant follows immediately from the definition.

Let us show that d_* preserves the filtrations. By Corollary to Theorem 3, $d_*(\pi_{g, n-1}) \subset \Gamma_{g, n}[1]$ holds. If we can show that

(2.1.1) $$d_*(\pi_{g, n-1}(2)) \subset \Gamma_{g, n}[2],$$

then, by using Proposition 1, $d_*(\pi_{g, n-1}(m)) \subset \Gamma_{g, n}[m]$ follows by induction on m. To show (2.1.1), it suffices to verify that $d_*(\zeta_j) = \tau^{(j)}$ $(1 \leq j \leq n-1)$ belongs to $\Gamma_{g, n}[2]$. But this is obvious.

To see that $\mathrm{gr}(d_*)$ is Sp(2g ; **Z**)-equivariant, we first note that

$$\mathrm{gr}(d_*)(A\bar{\xi}_i) = A \, \mathrm{gr}(d_*)(\bar{\xi}_i) \quad (1 \leq i \leq 2g)$$

$$\mathrm{gr}(d_*)(A\bar{\zeta}_j) = A \, \mathrm{gr}(d_*)(\bar{\zeta}_j) \quad (1 \leq j \leq n-1)$$

holds for all $A \in \mathrm{Sp}(2g ; \boldsymbol{Z})$. Here, $\bar{\xi}_i = \xi_i \bmod \pi_{g, n-1}(2)$ and $\bar{\zeta}_j = \zeta_j \bmod \pi_{g, n-1}(2)$. This follows from Theorem 3 and the well known fact that if c is a simple closed curve on $R_{g, n}$ and D_c is the Dehn twist about c, $\rho D_c \rho^{-1} = D_{\rho(c)}$ holds for all $\rho \in \Gamma_{g, n}$. Now, set

$$L = \{W \in \mathrm{gr}(\pi_{g, n-1}) \mid \mathrm{gr}(d_*)(AW) = A \, \mathrm{gr}(d_*)(W) \ {}^{\forall}A \in \mathrm{Sp}(2g ; \boldsymbol{Z})\}.$$

Then L is a Lie subalgebra of $\mathrm{gr}(\pi_{g, n-1})$ and contains its generators $\bar{\xi}_i$ $(1 \leq i \leq 2g)$

and $\bar{\zeta}_j$ ($1 \leq j \leq n-1$). Hence $L = \mathrm{gr}(\pi_{g,n-1})$, which shows the $\mathrm{Sp}(2g; \boldsymbol{Z})$-equivariance of $\mathrm{gr}(d_*)$.

2-2. A lemma.

In this section, we assume that $g \geq 1$. We shall prove a lemma which determines the image of the homomorphism

$$\iota_1(2g, n) \circ \mathrm{gr}^1(d_*): \mathrm{gr}^1(\pi_{g,n-1}) \longrightarrow C_1(2g, n)/I_1(2g, n).$$

This will be used to give a mild generalization of a result of Johnson in the next section.

The homomorphism p induces naturally a homomorphism from the coordinate module $C_m(2g, n)$ to $C_m(2g, n-1)$ for each $m \geq 1$, the last component of $C_m(2g, n)$ being forgotten. This induces a homomorphism

$$p_m: M_m(2g, n) \longrightarrow M_m(2g, n-1).$$

Let $K_m(2g, n)$ denote the kernel of p_m and consider the case $m=1$. Then we have the following commutative diagram:

$$
\begin{array}{ccccc}
\mathrm{gr}^1(\pi_{g,n-1}) & \xrightarrow{\mathrm{gr}^1(d_*)} & \mathrm{gr}^1(\Gamma_{g,n}) & \xrightarrow{\mathrm{gr}^1(p_*)} & \mathrm{gr}^1(\Gamma_{g,n-1}) \\
& & \downarrow{\scriptstyle \iota_1(2g,n)} & & \downarrow{\scriptstyle \iota_1(2g,n-1)} \\
\overline{K_1(2g, n)} & \longrightarrow & \overline{M_1(2g, n)} & \xrightarrow{\bar{p}_1} & \overline{M_1(2g, n-1)}.
\end{array}
$$

Here, $\overline{}$ denotes the image in $C_1(2g, n)/I_1(2g, n)$ and the image of $\iota_1(2g, n) \circ \mathrm{gr}^1(d_*)$ is contained in $\overline{K_1(2g, n)}$.

LEMMA 3. *The image of $\iota_1(2g, n) \circ \mathrm{gr}^1(d_*)$ coincides with $\overline{K_1(2g, n)}$.*

For the proof, we need the following sublemma whose proof is an easy exercise.

SUBLEMMA. *The $\boldsymbol{Z}$-module $K_1(2g, n)$ is free of rank $2g$ with a basis*

$$(-z_n, 0, \cdots\cdots\cdots\cdots\cdots, 0) \times (0, \cdots, 0, -x_{g+1}) = v_1$$
$$(0, -z_n, 0, \cdots\cdots\cdots\cdots, 0) \times (0, \cdots, 0, -x_{g+2}) = v_2$$
$$\vdots$$
$$(0, \cdots, 0, -z_n, 0, \cdots\cdots, 0) \times (0, \cdots, 0, -x_{2g}) = v_g$$
$$(0, \cdots\cdots, 0, -z_n, 0, \cdots, 0) \times (0, \cdots, 0, -x_1) = v_{g+1}$$
$$\vdots$$
$$(0, \cdots\cdots\cdots\cdots\cdots, 0, -z_n) \times (0, \cdots, 0, -x_g) = v_{2g}.$$

PROOF OF LEMMA 3. By Theorem 3, the image of $\iota_1(2g, n) \circ \mathrm{gr}^1(d_*)$ is generated by the classes of $c_1(2g, n)(\sigma_1^{(i)}\{\sigma_0^{(i)}\}^{-1})$ ($1 \leq i \leq 2g$). Easy calculations show that

$$c_1(2g,\ n)(\sigma_1^{(i)}\{\sigma_0^{(i)}\})^{-1} = \begin{cases} v_{g+i} & (1 \le i \le g) \\ v_i & (g+1 \le i \le 2g) \end{cases},$$

which proves the lemma.

COROLLARY. *The homomorphism* $\mathrm{gr}^1(d_*)$ *is injective.*

PROOF. By Sublemma, we have $K_1(2g,\ n) \cap I_1(2g,\ n) = \{0\}$. Hence the $\mathbb{Z}$-modules $\overline{K_1(2g,\ n)}$ and $\mathrm{gr}^1(\pi_{g,n-1})$ are both free of rank $2g$. Since $\iota_1(2g,\ n)$ is injective (Proof of Proposition B1), the corollary follows.

§3. Proof of Theorem A.

3-1. Surjectivity of p_*.

PROPOSITION A1. *For each $m \ge 0$, the homomorphism*

$$\tilde{p}_* | \tilde{\Gamma}_{g,n[m]} : \tilde{\Gamma}_{g,n}[m] \longrightarrow \tilde{\Gamma}_{g,n-1}[m]$$

is surjective if $n \ge 2$.

PROOF. We first define a section s of the homomorphism p. Since $\pi_{g,n-1}$ is a free group on $\xi_1, \cdots, \xi_{2g}, \zeta_1, \cdots, \zeta_{n-2}$, there exists uniquely a homomorphism

$$s : \pi_{g,n-1} \longrightarrow \pi_{g,n}$$

satisfying

$$s(\xi_i) = x_i \qquad (1 \le i \le 2g)$$
$$s(\zeta_j) = z_j \qquad (1 \le j \le n-2).$$

Then, $s(\zeta_{n-1}) = z_{n-1}z_n$ and s is a section of p. Moreover, s preserves the filtrations, i.e., $s(\pi_{g,n-1}(m)) \subset \pi_{g,n}(m)$. We regard $\pi_{g,n-1} \subset \pi_{g,n}$ via s.

Let σ be an element of $\tilde{\Gamma}_{g,n-1}[m]$ such that

$$\sigma(\zeta_j) = t_j \zeta_j t_j^{-1} \qquad (1 \le j \le n-1).$$

Since $\pi_{g,n}$ is a free group on $x_1, \cdots, x_{2g}, z_1, \cdots, z_{n-1}$, there exists uniquely a homomorphism

$$\tilde{\sigma} : \pi_{g,n} \longrightarrow \pi_{g,n}$$

satisfying

$$\tilde{\sigma}(x_i) = \sigma(x_i) \qquad (1 \le i \le 2g)$$
$$\tilde{\sigma}(z_j) = t_j z_j t_j^{-1} \qquad (1 \le j \le n-1).$$

Then, $\tilde{\sigma}$ is an automorphism of $\pi_{g,n}$. In fact, since σ is surjective, the image of $\tilde{\sigma}$ contains $x_1, \cdots, x_{2g}, z_1, \cdots, z_{n-2}$. As $\tilde{\sigma}(\sigma^{-1}(t_{n-1}^{-1})z_{n-1}\sigma^{-1}(t_{n-1})) = z_{n-1}$, the image of $\tilde{\sigma}$ contains z_{n-1}. Hence, $\tilde{\sigma}$ is surjective. Since $\pi_{g,n}$ is a free group, $\tilde{\sigma}$ is bijective.

Easy calculations show that $\tilde{\sigma}(z_n)=t_{n-1}z_n t_{n-1}^{-1}$. Hence, $\tilde{\sigma}$ belongs to $\tilde{\Gamma}_{g,n}$. Since $\pi_{g,n-1}(m)\subset\pi_{g,n}(m)$, $\tilde{\sigma}$ belongs to $\tilde{\Gamma}_{g,n}[m]$. Obviously, $\tilde{p}_*(\tilde{\sigma})=\sigma$. Hence, the proof is completed.

COROLLARY. *For each $m\geq0$, the homomorphism*

$$p_*|_{\Gamma_{g,n}[m]}:\Gamma_{g,n}[m]\longrightarrow\Gamma_{g,n-1}[m]$$

is surjective if $n\geq2$.

Assume that $g\geq1$. By using Proposition A1 and a result in §2-2, we can give a mild generalization of a result of Johnson [J], that is, the determination of the image of $\iota_1(2g,n)$ for all $n\geq0$.

PROPOSITION 4. (i) *The complex of* Sp($2g$; Z)*-modules*

$$0\longrightarrow\mathrm{gr}^1(\pi_{g,n-1})\xrightarrow{\mathrm{gr}^1(d_*)}\mathrm{gr}^1(\Gamma_{g,n})\xrightarrow{\mathrm{gr}^1(p_*)}\mathrm{gr}^1(\Gamma_{g,n-1})\longrightarrow0$$

is exact for all $n\geq1$.

(ii) *The image of $\iota_1(2g,n)$ coincides with $\overline{M_1(2g,n)}$ for all $n\geq0$.*

PROOF. (i) By Lemma 3 and its corollary, it suffices to show that $\mathrm{gr}^1(p_*)$ is surjective. In the case that $n\geq2$, this follows from Proposition A1. In the case that $n=1$, this is a consequence of a result of Nielsen as follows.

It suffices to show that $\tilde{p}_*|_{\tilde{\Gamma}_{g,1}[1]}:\tilde{\Gamma}_{g,1}[1]\to\tilde{\Gamma}_{g,0}[1]$ is surjective. Let σ be an element of $\tilde{\Gamma}_{g,0}[1]$. By a result of Nielsen (cf. e. g., [MKS, §3.7 Theorem N10]), there exists $\tilde{\sigma}\in\tilde{\Gamma}_{g,1}$ such that $\tilde{p}_*(\tilde{\sigma})=\sigma$. Since the homomorphism p induces an isomorphism $\mathrm{gr}^1(\pi_{g,1})\cong\mathrm{gr}^1(\pi_{g,0})$ and σ acts trivially on $\mathrm{gr}^1(\pi_{g,0})$, it follows that $\tilde{\sigma}$ acts trivially on $\mathrm{gr}^1(\pi_{g,1})$, i. e., $\tilde{\sigma}\in\tilde{\Gamma}_{g,1}[1]$. This shows that $\tilde{p}_*|_{\tilde{\Gamma}_{g,1}[1]}$ is surjective.

(ii) We note that in the cases of $n=0$ and 1 this has been proved in [J]. Then this can be proved by induction on n, using (i) and the diagram (2.2.1).

3-2. Injectivity of $\mathrm{gr}(d_*)$.

PROPOSITION A2. *The Lie algebra homomorphism*

$$\mathrm{gr}(d_*):\mathrm{gr}(\pi_{g,n-1})\longrightarrow\mathrm{gr}(\Gamma_{g,n}[1])\qquad(n\geq1)$$

induced from d_* is injective.

For the proof, we need the following

LEMMA 4 (Ihara). *Let G be a group and N be a normal subgroup of G. Let $\{G_m\}_{m=1}^\infty$ (resp. $\{N_m\}_{m=1}^\infty$) be a central filtration of G (resp. N) satisfying $N_m\subset G_m$ for all $m\geq1$. Assume that the following hold:*

(i)   *The centralizer of* $\mathrm{gr}^1(N)$ *in* $\mathrm{gr}(N)$ *reduces to* $\{0\}$.

(ii)   *The conjugate action of* G *on* N *stabilizes the filtration* $\{N_m\}_{m=1}^{\infty}$ *and induces the trivial action on* $\mathrm{gr}(N)$.

Then, the canonical Lie algebra homomorphism

$$\mathrm{gr}(N) \longrightarrow \mathrm{gr}(G)$$

is injective.

This is a slight modification of a lemma of Ihara [**I**, Lemma 3.1.1]. The above lemma can be proved completely in the same way.

PROOF OF PROPOSITION A2.   Assume that $g \geq 1$. Then this can be proved by applying Lemma 4 to the case of $G = \Gamma_{g,n}$ and $N = d_*(\pi_{g,n-1})$. The condition (i) is satisfied by Theorem 1 (ii).   Since the homomorphism $\Gamma_{g,n-1} \to \mathrm{Out}(\pi_{g,n-1})$ induced from the conjugate action of $\Gamma_{g,n}$ on $d_*(\pi_{g,n-1})$ is nothing but the inclusion, the condition (ii) is also satisfied.   In the case that $g = 0$, noting that the weight filtration is essentially the lower central filtration, this can be proved by applying original Ihara's lemma.

3-3.   Proof of Theorem A.

By Corollary to Proposition A1 and Proposition A2, we have an exact sequence

$$(3.3.1) \qquad 1 \longrightarrow \pi_{g,n-1}(m) \longrightarrow \Gamma_{g,n}[m] \longrightarrow \Gamma_{g,n-1}[m] \longrightarrow 1.$$

The theorem follows immediately from this.

§4.   Comparison with the pro-*l* case.

4-1.   Pro-*l* fundamental groups and pro-*l* mapping class groups.

Let l be a fixed prime number.   Let $\pi_{g,n}^{\mathrm{pro}\text{-}l}$ denote the pro-l completion of $\pi_{g,n}$, i.e., $\pi_{g,n}^{\mathrm{pro}\text{-}l} = \varprojlim \pi_{g,n}/N$, where N runs over all normal subgroups of $\pi_{g,n}$ with indices powers of l.   The weight filtration of $\pi_{g,n}^{\mathrm{pro}\text{-}l}$ is defined in the same way as that of $\pi_{g,n}$.   The associated graded module $\mathrm{gr}(\pi_{g,n}^{\mathrm{pro}\text{-}l})$ has a natural structure of a Lie algebra over Z_l.   The following proposition is more or less well known.   (Cf. Lubotzky [**Lu**, 2.6].)   We shall give a proof here for the convenience of the readers.

PROPOSITION 5.   *The canonical homomorphism* $\mathrm{gr}(\pi_{g,n}) \to \mathrm{gr}(\pi_{g,n}^{\mathrm{pro}\text{-}l})$ *induces an isomorphism*

$$\mathrm{gr}(\pi_{g,n}) \underset{Z}{\otimes} Z_l \cong \mathrm{gr}(\pi_{g,n}^{\mathrm{pro}\text{-}l}).$$

PROOF.   Fix a positive integer m and consider the nilpotent group $G_m = \pi_{g,n}/\pi_{g,n}(m+1)$.   By Theorem 1 (ii), it follows easily that the center of G_m

coincides with $\pi_{g,n}(m)/\pi_{g,n}(m+1)$. (Cf. e. g., [**As**, Lemma 4].) Hence, it follows that the lower central series of G_m coincides with the upper central series of G_m. Then, the assertion follows from a result of Warfield [**Wa**, Theorem 7.4].

We put

$$\tilde{\Gamma}^{(l)}_{g,n} = \{\tilde{\sigma} \in \mathrm{Aut}(\pi^{\mathrm{pro}\text{-}l}_{g,n}) \mid \tilde{\sigma}(z_j) \sim z_j^\alpha,\ \alpha \in \mathbf{Z}_l^* \ (1 \leq j \leq n)\},$$

$$\Gamma^{(l)}_{g,n} = \tilde{\Gamma}^{(l)}_{g,n}/\mathrm{Int}(\pi^{\mathrm{pro}\text{-}l}_{g,n}),$$

and call $\Gamma^{(l)}_{g,n}$ the pro-l mapping class group. The group $\tilde{\Gamma}^{(l)}_{g,n}$ is a closed subgroup of $\mathrm{Aut}(\pi^{\mathrm{pro}\text{-}l}_{g,n})$, which is naturally a profinite group. Hence $\tilde{\Gamma}^{(l)}_{g,n}$ and its quotient group $\Gamma^{(l)}_{g,n}$ are both profinite groups. The weight filtration of $\pi^{\mathrm{pro}\text{-}l}_{g,n}$ induces a filtration $\{\Gamma^{(l)}_{g,n}[m]\}_{m=1}^\infty$ of $\Gamma^{(l)}_{g,n}$ in the same way as the case of $\pi_{g,n}$ and $\Gamma_{g,n}$. Then, we have the following

THEOREM 4. (i) *Assume that* $g \geq 1$. *Then the natural action of* $\Gamma^{(l)}_{g,n}$ *on* $\mathrm{gr}^1(\pi^{\mathrm{pro}\text{-}l}_{g,n})$ *induces an isomorphism*

$$\Gamma^{(l)}_{g,n}/\Gamma^{(l)}_{g,n}[1] \cong \mathrm{GSp}(2g\ ;\ \mathbf{Z}_l).$$

Here, $\mathrm{GSp}(2g\ ;\ \mathbf{Z}_l) = \{A \in \mathrm{GL}(2g\ ;\ \mathbf{Z}_l) \mid {}^t A J_g A = \mu(A) J_g,\ \mu(A) \in \mathbf{Z}_l^*\}$.

(ii) *For each* $m \geq 1$, *the image of the homomorphism*

$$\iota^{(l)}_m(2g,\ n) : \mathrm{gr}^m(\Gamma^{(l)}_{g,n}) \longrightarrow C_m(2g,\ n) \underset{\mathbf{Z}}{\otimes} \mathbf{Z}_l/I_m(2g,\ n) \underset{\mathbf{Z}}{\otimes} \mathbf{Z}_l$$

coincides with $M_m(2g,\ n) \underset{\mathbf{Z}}{\otimes} \mathbf{Z}_l/I_m(2g,\ n) \underset{\mathbf{Z}}{\otimes} \mathbf{Z}_l$. *Here,* $\iota^{(l)}_m(2g,\ n)$ *is defined in the same way as* $\iota_m(2g,\ n)$ *is. In particular,* $\mathrm{gr}^m(\Gamma^{(l)}_{g,n})$ *is a free* $\mathbf{Z}_l$-*module of finite rank* r_m, *where* r_m *can be calculated explicitly by a formula of Labute.*

(iii) *We have* $\bigcap_{m=1}^\infty \Gamma^{(l)}_{g,n}[m] = \{0\}$.

For the proof cf. [**As**], Asada-Kanedo [**AK**], and [**K**]. See also [**NT**]. In [**As**], the proof of (iii) is given in the case of $n=0$. The proof in the case of $n \geq 1$ can be done completely in the same way.

4-2. The image of $\Gamma_{g,n}$ in $\Gamma^{(l)}_{g,n}$.

Since an element of $\tilde{\Gamma}_{g,n}$ induces naturally an element of $\tilde{\Gamma}^{(l)}_{g,n}$ and an inner automorphism of $\pi_{g,n}$ induces that of $\pi^{(l)}_{g,n}$, we have the canonical homomorphism

$$\varphi_l : \Gamma_{g,n} \longrightarrow \Gamma^{(l)}_{g,n}.$$

It is easily verified that φ_l preserves the filtrations.

Let $\bar{\Gamma}_{g,n}$ denote the closure of $\varphi_l(\Gamma_{g,n})$ in $\Gamma^{(l)}_{g,n}$. Then, $\bar{\Gamma}_{g,n}$ is a profinite group and we define its filtration $\{\bar{\Gamma}_{g,n}[m]\}_{m=0}^\infty$ by

$$\bar{\Gamma}_{g,n}[m] = \bar{\Gamma}_{g,n} \cap \Gamma^{(l)}_{g,n}[m] \qquad (m \geq 0).$$

Then, we have

$$\varphi_l(\Gamma_{g,n}[m]) \subset \bar{\Gamma}_{g,n}[m] \subset \Gamma^{(l)}_{g,n}[m] \qquad (m \geq 0),$$

and this induces a sequence

$$\mathrm{gr}^m(\Gamma_{g,n}) \xrightarrow{\ \mathrm{gr}^m(\varphi_l)\ } \mathrm{gr}^m(\bar{\Gamma}_{g,n}) \longrightarrow \mathrm{gr}^m(\Gamma^{(l)}_{g,n}) \qquad (m \geq 0).$$

Here, the right homomorphism is induced from the inclusion and is obviously injective.

We first prove the following

PROPOSITION B2. *For each* $m \geq 1$, $\mathrm{gr}^m(\varphi_l)$ *is injective and the closure of the image is isomorphic to* $\mathrm{gr}^m(\Gamma_{g,n}) \underset{\mathbf{Z}}{\otimes} \mathbf{Z}_l$.

PROOF. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{gr}^m(\Gamma_{g,n}) & \xrightarrow{\ \mathrm{gr}^m(\varphi_l)\ } & \mathrm{gr}^m(\Gamma^{(l)}_{g,n}) \\
\downarrow {\scriptstyle \iota_m(2g,\,n)} & \overset{\otimes \mathbf{Z}_l}{} & \downarrow {\scriptstyle \iota^{(l)}_m(2g,\,n)} \\
N_m(2g,\,n)/I_m(2g,\,n) & \longrightarrow & C_m(2g,\,n)\underset{\mathbf{Z}}{\otimes}\mathbf{Z}_l/I_m(2g,\,n)\underset{\mathbf{Z}}{\otimes}\mathbf{Z}_l\,.
\end{array}
$$

Here, $\iota_m(2g,\,n)$ and $\iota^{(l)}_m(2g,\,n)$ are both injective. (Cf. Proof of Proposition B1.) It follows from Lemma 2 that the canonical homomorphism $\otimes \mathbf{Z}_l$ is injective and the closure of the image is isomorphic to $(N_m(2g,\,n)/I_m(2g,\,n))\underset{\mathbf{Z}}{\otimes}\mathbf{Z}_l$. From this, the proposition follows immediately.

COROLLARY 1. *For each* $m \geq 1$, *the homomorphism*

$$\Gamma_{g,n}/\Gamma_{g,n}[m] \longrightarrow \Gamma^{(l)}_{g,n}/\Gamma^{(l)}_{g,n}[m]$$

induced from φ_l *is injective.*

By Proposition 2, we have the following

COROLLARY 2. *The homomorphism* φ_l *is injective if* $(g,\,n) \neq (2,\,0)$.

Let $\overline{\Gamma_{g,n}[m]}$ denote the closure of $\varphi_l(\Gamma_{g,n}[m])$ for each $m \geq 0$, so that we have

$$\overline{\Gamma_{g,n}[m]} \subset \bar{\Gamma}_{g,n}[m].$$

PROPOSITION B3. *Assume that* $g \neq 1$. *Then, for each* $m \geq 0$, *we have*

$$\overline{\Gamma_{g,n}[m]} = \bar{\Gamma}_{g,n}[m].$$

PROOF. We shall prove this by induction on m, the case $m=0$ being trivial. If $g=0$, the case $m=1$ is also trivial.

Assume that $m=1$ and $g \geq 2$. We have a sequence

$$\Gamma_{g,n}/\Gamma_{g,n}[1] \xrightarrow{\ \mathrm{gr}^0(\varphi_l)\ } \bar{\Gamma}_{g,n}/\overline{\Gamma_{g,n}[1]} \longrightarrow \bar{\Gamma}_{g,n}/\bar{\Gamma}_{g,n}[1] \subset \Gamma^{(l)}_{g,n}/\Gamma^{(l)}_{g,n}[1]\,.$$

Here, $\Gamma_{g,n}/\Gamma_{g,n}[1]$ is isomorphic to $\mathrm{Sp}(2g\,;Z)$ by Theorem 2 and it follows that $\bar{\Gamma}_{g,n}/\bar{\Gamma}_{g,n}[1]$ is isomorphic to $\mathrm{Sp}(2g\,;Z_l)$. Since $\bar{\Gamma}_{g,n}/\overline{\Gamma_{g,n}[1]}$ is a profinite group, $\mathrm{gr}^0(\varphi_l)$ factors through the profinite completion $(\Gamma_{g,n}/\Gamma_{g,n}[1])^{\mathrm{profinite}}$ of $\Gamma_{g,n}/\Gamma_{g,n}[1]$. As is well known, the group $\mathrm{Sp}(2g\,;Z)$ has the congruence subgroup property if $g\geq 2$ (cf. Serre [Se₂]) and the strong approximation theorem holds for it (Shimura [Sh]). Thus, we have an isomorphism

$$(\Gamma_{g,n}/\Gamma_{g,n}[1])^{\mathrm{profinite}} \cong \prod_p \mathrm{Sp}(2g\,;Z_p)$$

and the commutative diagram;

pr_l being the projection to the l-component. Therefore, we have a surjective homomorphism

$$\prod_{p\neq l}\mathrm{Sp}(2g\,;Z_p) \longrightarrow \bar{\Gamma}_{g,n}[1]/\overline{\Gamma_{g,n}[1]}.$$

By Theorem 4, $\bar{\Gamma}_{g,n}[1]/\overline{\Gamma_{g,n}[1]}$ is a pro-l group, hence the identity group by the following

LEMMA 5. *The group* $\mathrm{Sp}(2g\,;Z_p)$ *has no finite abelian quotient of order prime to* p.

PROOF. Let G be a finite abelian group of order prime to p and $f:\mathrm{Sp}(2g\,;Z_p)\to G$ be a homomorphism. Let N be the subgroup of $\mathrm{Sp}(2g\,;Z_p)$ consisting of all matrices in $\mathrm{Sp}(2g\,;Z_p)$ which are congruent to the unit matrix modulo p. Then N is a pro-p group and normal. Thus, f induces a homomorphism $\bar{f}:\mathrm{Sp}(2g\,;F_p)\to G$. It follows from a well known property of $\mathrm{Sp}(2g\,;F_p)$ that $\bar{f}(A)=1$ for all $A\in\mathrm{Sp}(2g\,;F_p)$ (cf. e.g., Artin [Ar, Th. 5.1]).

Assume that $m\geq 1$ and that we have $\overline{\Gamma_{g,n}[m]}=\bar{\Gamma}_{g,n}[m]$. Then we have a sequence

$$\mathrm{gr}^m(\Gamma_{g,n}) \xrightarrow{\mathrm{gr}^m(\varphi_l)} \bar{\Gamma}_{g,n}[m]/\overline{\Gamma_{g,n}[m+1]} \longrightarrow \bar{\Gamma}_{g,n}[m]/\bar{\Gamma}_{g,n}[m+1].$$

By the induction assumption, $\mathrm{gr}^m(\varphi_l)$ has dense image. Since $\bar{\Gamma}_{g,n}[m]/\overline{\Gamma_{g,n}[m+1]}$ is a pro-l group, $\mathrm{gr}^m(\varphi_l)$ factors through the pro-l completion of $\mathrm{gr}^m(\Gamma_{g,n})$, i.e.,

$$\mathrm{gr}^m(\Gamma_{g,n})\underset{Z}{\otimes}Z_l\,;$$

$$\mathrm{gr}^m(\Gamma_{g,n}) \xrightarrow{\mathrm{gr}^m(\varphi_l)} \bar{\Gamma}_{g,n}[m]/\bar{\Gamma}_{g,n}[m+1] \xrightarrow{\mathrm{pr}} \bar{\Gamma}_{g,n}[m]/\bar{\Gamma}_{g,n}[m+1].$$

with the maps through $\mathrm{gr}^m(\varphi_l)_l$ and $\mathrm{gr}^m(\Gamma_{g,n}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$

Since $\mathrm{pr} \circ \mathrm{gr}^m(\varphi_l)_l$ is an isomorphism by Proposition B2, pr is an isomorphism. Hence we have $\overline{\Gamma_{g,n}[m+1]} = \bar{\Gamma}_{g,n}[m+1]$ and the proof is completed.

4-3. Proof of Theorem B.

That $\bar{\Gamma}_{g,n}/\bar{\Gamma}_{g,n}[1] \cong \mathrm{Sp}(2g\,;\mathbb{Z}_l)$ follows from Theorem 2 and the $\mathrm{Sp}(2g)$-equivariance of (*) is obvious. The rest follows from Propositions B1, B2, B3 and Theorem 4 (iii).

4-4. Pro-*l* monodromy representation associated with the configuration space.

Let us consider the homomorphism

$$(4.4.1) \qquad \varphi_l \circ d_* : \pi_{g,n-1} \longrightarrow \Gamma^{(l)}_{g,n}[1].$$

Since $\Gamma^{(l)}_{g,n}[1]$ is a pro-*l* group, $\varphi_l \circ d_*$ factors through $\pi^{\mathrm{pro}\text{-}l}_{g,n-1}$;

$$\pi_{g,n-1} \xrightarrow{d_*} \Gamma_{g,n}[1] \xrightarrow{\varphi_l} \Gamma^{(l)}_{g,n}[1].$$

with the maps through $\pi^{\mathrm{pro}-l}_{g,n-1}$

Note that all homomorphisms in the above diagram preserve filtrations.

Then we have the following

PROPOSITION 6. *The induced homomorphism*

$$(4.4.2) \qquad \pi^{\mathrm{pro}\text{-}l}_{g,n-1} \longrightarrow \Gamma^{(l)}_{g,n}[1]$$

is injective. In other words, the closure of the image of the homomorphism (4.4.1) *is isomorphic to* $\pi^{\mathrm{pro}\text{-}l}_{g,n-1}$.

PROOF. It suffices to show that the induced homomorphism

$$(4.4.2)_m \qquad \mathrm{gr}^m(\pi^{\mathrm{pro}\text{-}l}_{g,n-1}) \longrightarrow \mathrm{gr}^m(\Gamma^{(l)}_{g,n})$$

is injective for all $m \geq 1$.

Let us consider the sequence

$$\mathrm{gr}^m(\pi_{g,n-1}) \xrightarrow{\mathrm{gr}^m(d_*)} \mathrm{gr}^m(\Gamma_{g,n}) \xrightarrow{\mathrm{gr}^m(\varphi_l)} \mathrm{gr}^m(\Gamma^{(l)}_{g,n}).$$

By Proposition A2, $\mathrm{gr}^m(d_*)$ is injective. Thus, by Proposition B2, the induced homomorphism

$$\mathrm{gr}^m(\pi_{g,\,n-1}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Z}_l \longrightarrow \mathrm{gr}^m(\varGamma^{(l)}_{g,\,n})$$

is injective. Therefore, by Proposition 5, the homomorphism $(4.4.2)_m$ is injective and the proof is completed.

We shall explain the implication of Proposition 6. As before, let $R_{g,\,n}$ denote a Riemann surface of genus g with n punctures. Let $\varDelta$ be the diagonal subset of $R_{g,\,n-1} \times R_{g,\,n-1}$ and consider the fibration

$$\mathrm{pr}_1 : R_{g,\,n-1} \times R_{g,\,n-1} \backslash \varDelta \longrightarrow R_{g,\,n-1},$$

pr_1 being the projection to the first component. The homotopy exact sequence of this fibration gives an exact sequence

$$(4.4.3) \qquad 1 \longrightarrow \pi_{g,\,n} \longrightarrow \pi_1(R_{g,\,n} \times R_{g,\,n} \backslash \varDelta) \longrightarrow \pi_{g,\,n-1} \longrightarrow 1$$

(Birman [$\mathbf{Bi}_2$, Prop. 1.3]). Then, the conjugate action of $\pi_1(R_{g,\,n-1} \times R_{g,\,n-1} \backslash \varDelta)$ on the normal subgroup $\pi_{g,\,n}$ induces a non-abelian monodromy representation of the group $\pi_{g,\,n-1}$;

$$\pi_{g,\,n-1} \longrightarrow \mathrm{Out}(\pi_{g,\,n}).$$

As is well known, this is nothing but the homomorphism d_* (Birman [$\mathbf{Bi}_1$, Corollary 1.4]). In particular, this representation is faithful.

Let us apply the pro-l completion functor to the exact sequence (4.4.3). Since $\pi_{g,\,n}^{\mathrm{pro}\text{-}l}$ has trivial center, and the image of $\varphi_l \circ d_*$ is contained in $\varGamma^{(l)}_{g,\,n}[1]$ which is a pro-l group (Theorem 4 (ii), (iii)), we obtain an exact sequence

$$1 \longrightarrow \pi_{g,\,n}^{\mathrm{pro}\text{-}l} \longrightarrow \pi_1(R_{g,\,n} \times R_{g,\,n} \backslash \varDelta)^{\mathrm{pro}\text{-}l} \longrightarrow \pi_{g,\,n-1}^{\mathrm{pro}\text{-}l} \longrightarrow 1$$

(Anderson [$\mathbf{An}$]; cf. also Ihara-Kaneko [$\mathbf{IK}$, (1.2.2)]). Then, the associated non-abelian monodromy representation

$$(4.4.4) \qquad\qquad \pi_{g,\,n-1}^{\mathrm{pro}\text{-}l} \longrightarrow \mathrm{Out}(\pi_{g,\,n}^{\mathrm{pro}\text{-}l})$$

is nothing but the homomorphism (4.4.2). Hence, Proposition 6 implies that the associated pro-l monodromy representation (4.4.4) is faithful.

§ 5. Problems and discussions.

5-1. P1. If $g \geq 2$, is the homomorphism

$$(5.1.1)_m \qquad\qquad p_* |_{\varGamma_{g,\,1}[m]} : \varGamma_{g,\,1}[m] \longrightarrow \varGamma_{g,\,0}[m] \qquad (m \geq 1)$$

surjective?

A closely related problem is

P2. If $g \geq 2$, is the sequence of Sp($2g$; $\boldsymbol{Z}$)-modules

$$(5.1.2)_m \qquad 0 \longrightarrow \mathrm{gr}^m(\pi_{g,0}) \xrightarrow{\mathrm{gr}^m(d_*)} \mathrm{gr}^m(\Gamma_{g,1}) \xrightarrow{\mathrm{gr}^m(p_*)} \mathrm{gr}^m(\Gamma_{g,0}) \longrightarrow 0$$

exact?

To investigate these problems, as before, let $K_m(2g, 1)$ denote the kernel of the homomorphism $p_m : M_m(2g, 1) \to M_m(2g, 0)$ (cf. § 2-2). Then we have the following commutative diagram:

$$(5.1.2)_m \qquad 0 \longrightarrow \mathrm{gr}^m(\pi_{g,0}) \xrightarrow{\mathrm{gr}^m(d_*)} \mathrm{gr}^m(\Gamma_{g,1}) \xrightarrow{\mathrm{gr}^m(p_*)} \mathrm{gr}^m(\Gamma_{g,0}) \longrightarrow 0$$

$$\qquad\qquad \downarrow \qquad\qquad \iota_m(2g,1) \downarrow \qquad\qquad \iota_m(2g,0) \downarrow$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \bar{p}_m$$

$$(5.1.3)_m \qquad 0 \longrightarrow \overline{K_m(2g, 1)} \longrightarrow \overline{M_m(2g, 1)} \longrightarrow \overline{M_m(2g, 0)} \longrightarrow 0.$$

Here, $^{-}$ denotes the image in $C_m(2g, *)/I_m(2g, *)$. Recall that $\iota_m(2g, *)$ and $\mathrm{gr}^m(d_*)$ are both injective (Proof of Proposition 3, Proposition A2). The surjectivity of $\bar{p}_m$ is proved in [**K**] (in the context of pro-l groups). We easily obtain the following

LEMMA 6. (i) *If $(5.1.1)_m$ and $(5.1.1)_{m+1}$ are both surjective, then $(5.1.2)_m$ is exact.*

(ii) *If $(5.1.1)_m$ is surjective and the homomorphism $\mathrm{gr}^m(\pi_{g,0}) \to \overline{K_m(2g, 1)}$ is surjective, then $(5.1.2)_m$ is exact.*

(iii) *If $(5.1.2)_m$ is exact, then $(5.1.1)_{m+1}$ is surjective.*

At present, we can show the following

PROPOSITION 7. (i) *If $m=1, 2$, $p_* |_{\Gamma_{g,1}[m]}$ is surjective. The complex $(5.1.2)_1$ is exact.*

(ii) *If $m=3, 4$, the image of $p_* |_{\Gamma_{g,1}[m]}$ is of finite index.*

(iii) *If $1 \leq m \leq 3$, we have an exact sequence of Sp($2g$; $\boldsymbol{Z}$)-modules*

$$0 \longrightarrow \mathrm{gr}^m(\pi_{g,0}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Q} \longrightarrow \mathrm{gr}^m(\Gamma_{g,1}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Q} \longrightarrow \mathrm{gr}^m(\Gamma_{g,0}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Q} \longrightarrow 0.$$

PROOF. (i) The surjectivity of $p_* |_{\Gamma_{g,1}[1]}$ and the exactness of $(5.1.2)_1$ are verified in Proposition 4. The surjectivity of $p_* |_{\Gamma_{g,1}[2]}$ follows from Lemma 3 and Lemma 6 (ii).

(ii) It suffices to show that

$$\mathrm{gr}^m(p_*) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Q} : \mathrm{gr}^m(\Gamma_{g,1}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Q} \longrightarrow \mathrm{gr}^m(\Gamma_{g,0}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{Q}$$

is surjective for $m=3, 4$. If we replace each $\boldsymbol{Z}$-module in the diagram $(5.1.2)_m$, $(5.1.3)_m$ with its tensor product with $\boldsymbol{Q}$, we still have the commutative diagram,

$\mathrm{gr}^m(p_*)\underset{Z}{\bigotimes}Q$ and $\iota_m(2g, *)\underset{Z}{\bigotimes}Q$ being injective. Since $\mathrm{gr}^2(p_*)\underset{Z}{\bigotimes}Q$ is surjective, it suffices to show that $\mathrm{gr}^m(\pi_{g,0})\underset{Z}{\bigotimes}Q\to\overline{K_m(2g, 1)}\underset{Z}{\bigotimes}Q$ is surjective for $m=2, 3$. By using Labute's formula ([L] cf. also [K]), we have

$$\mathrm{rank}\ \mathrm{gr}^m(\pi_{g,0}) = \mathrm{rank}\ \overline{M_m(2g, 1)} - \mathrm{rank}\ \overline{M_m(2g, 0)}$$

for $m=2, 3$. Hence the proof is completed.

(iii) This follows from Proposition A1 and (i), (ii).

5-2. As is indicated in [NT, §2], two exact sequences (1.3.2) and (3.3.1) can be related by a certain homomorphism

$$\delta_{g, n-1}: \Gamma_{g, n} \longrightarrow \tilde{\Gamma}_{g, n-1}.$$

Let us recall the definition of $\delta_{g, n-1}$. For $\sigma\in\Gamma_{g, n}$, let $\tilde{\sigma}\in\tilde{\Gamma}_{g, n}$ be a representative such that $\tilde{\sigma}(z_n)=z_n$. Since the centralizer of z_n in $\pi_{g, n}$ is the infinite cyclic group generated by z_n, $\tilde{\sigma}$ is unique up to right multiplications by $\mathrm{Int}(z_n^a)$ $(a\in Z)$. The homomorphism $\delta_{g, n-1}$ is defined by

$$\delta_{g, n-1}(\sigma) = \tilde{p}_*(\tilde{\sigma}).$$

It is easy to see that $\delta_{g, n-1}$ preserves the filtrations and that the diagram

$$
\begin{array}{ccccccc}
1 & \longrightarrow & \pi_{g, n-1}[m] & \longrightarrow & \Gamma_{g, n}[m] & \longrightarrow & \Gamma_{g, n-1}[m] & \longrightarrow 1 \\
 & & \downarrow \mathrm{Int} & & \downarrow \delta_{g, n-1} & & \downarrow id. & \\
1 & \longrightarrow & \mathrm{Int}_{\pi_{g, n-1}}(\pi_{g, n-1}[m]) & \longrightarrow & \tilde{\Gamma}_{g, n-1}[m] & \longrightarrow & \Gamma_{g, n-1}[m] & \longrightarrow 1
\end{array}
$$

is commutative. By Corollary to Proposition A1, we have the following

PROPOSITION 8. *If* $n\geq 2$, *the homomorphism* $\delta_{g, n-1}|_{\Gamma_{g, n}[m]}$ *is bijective.*

5-3. It seems to be an interesting problem to characterize the closure of $\varphi_l(\Gamma_{g, n}[1])$ (or that of $\varphi_l(\Gamma_{g, n})$, which seems to be much more difficult). Since $\Gamma_{g, n}^{(l)}[1]$ is a pro-l group, the closure of $\varphi_l(\Gamma_{g, n}[1])$ is a pro-l group. Then a natural problem is

P3. Is the closure of $\varphi_l(\Gamma_{g, n}[1])$ isomorphic to the pro-l completion of $\Gamma_{g, n}[1]$?

For the group $\bar{\Gamma}_{g, n}$, let us consider the case $g=n=1$. Then $\Gamma_{1,1}[1]=\{1\}$ and $\Gamma_{1,1}$ is isomorphic to $\mathrm{Sp}(2; Z)=\mathrm{SL}(2; Z)$. A result of Bloch [Bl] and a recent result of Nakamura [N] indicate that $\bar{\Gamma}_{1,1}[1]\neq\{1\}$, which means that Proposition B3 does not hold in this case. In other words, the topology of $\Gamma_{1,1}$ $(=\mathrm{SL}(2; Z))$ induced from $\Gamma_{1,1}^{(l)}$ (via φ_l) is "stronger" than the topology induced from the congruence subgroups with l-power levels. It seems to be an interesting problem to determine what this topology is.

References

[An] M. P. Anderson, Exactness properties of profinite completion functors, Topology, 13 (1974), 229-239.

[Ar] E. Artin, Geometric Algebra, Interscience, New York, 1957.

[As] M. Asada, Two properties of the filtration of the outer automorphism groups of certain groups, Math. Z., 218 (1995), 123-133.

[AK] M. Asada and M. Kaneko, On the automorphism group of some pro-l fundamental groups, In Galois Representations and Arithmetic Algebraic Geometry, (ed. Y. Ihara), Adv. Stud. Pure Math., 12 (1987), 137-159.

[AMO] M. Asada, M. Matsumoto and T. Oda, Local monodromy on the fundamental groups of algebraic curves along a degenerate stable curve, to appear in J. Pure Appl. Algebra.

[AN] M. Asada and H. Nakamura, On graded quotient modules of mapping class groups of surfaces, Israel J. Math., 90 (1995), 93-113.

[BL] H. Bass and A. Lubotzky, Linear-central filtrations on groups, Contemp. Math., 169 (1994), 45-98.

[Bi₁] J. S. Birman, Mapping class groups and their relationship to braid groups, Comm. Pure Appl. Math., 22 (1969), 213-238.

[Bi₂] J. S. Birman, Braids, Links and Mapping Class Groups, Ann. of Math. Stud., 82, Princeton Univ. Press, 1975.

[Bi₃] J. S. Birman, The algebraic structure of surface mapping class groups, In Discrete Groups and Automorphic Functions, (ed. W. Harvey), Academic Press, New York, 1977.

[Bl] S. Bloch, A letter to P. Deligne, 1984.

[Bo] N. Bourbaki, Groupes et algèbres de Lie, Chapitres 2 et 3, Hermann, Paris, 1972.

[I] Y. Ihara, Automorphisms of pure sphere braid groups and Galois representations, In The Grothendieck Festshrift, Vol. 2, Birkhäuser, Boston, 1990.

[IK] Y. Ihara and M. Kaneko, Pro-l pure braid groups of Riemann surfaces and Galois representations, Osaka J. Math., 29 (1992), 1-19.

[J] D. Johnson, An abelian quotient of the mapping class group I_g, Math. Ann., 249 (1980), 225-242.

[K] M. Kaneko, Certain automorphism groups of pro-l fundamental groups of punctured Riemann surfaces, J. Fac. Sci. Univ. Tokyo, 36 (1989), 363-372.

[La] J. Labute, On the descending central series of groups with a single defining relation, J. Algebra, 14 (1970), 16-23.

[Lu] A. Lubotzky, Combinatorial group theory for pro-p groups, J. Pure Appl. Algebra, 25 (1982), 311-325.

[MKS] W. Magnus, A. Karrass and D. Solitar, Combinatorial Group Theory, Interscience Publ. Wiley and Sons, 1966.

[M₁] S. Morita, Casson's invariant for homology 3-spheres and characteristic classes of surface bundles I, Topology, 28 (1989), 305-323.

[M₂] S. Morita, Abelian quotients of subgroups of the mapping class group of surfaces, Duke Math. J., 70 (1993), 699-726.

[N] H. Nakamura, On exterior Galois representations associated with open elliptic curves, J. Math. Sci. Univ. Tokyo, 2 (1995), 197-231.

[NT] H. Nakamura and H. Tsunogai, Some finiteness theorems on Galois centralizers in pro-l mapping class groups, J. Reine Angew. Math., 441 (1993), 115-144.

[O₁]   T. Oda, Galois action on the pro-nilpotent completion of the fundamental groups of algebraic curves, (in Japanese), In Algebraic Analysis and Number Theory, RIMS report, **810** (1992), 318-323.

[O₂]   T. Oda, A lower bound for the graded modules associated with the relative weight filtration on the Teichmüller group, preprint, 1992.

[Se₁]   J.-P. Serre, Rigidité du foncteur de Jacobi d'échelon $n \geq 3$, Appendice à l'exposé 17 du Séminaire Cartan, 1960-1961.

[Se₂]   J.-P. Serre, Groupes de congruence, (d'après H. Bass, H. Matsumoto, J. Mennicke, J. Milnor, C. Moore), Séminaire Bourbaki, 1966/67, **330**.

[Sh]   G. Shimura, Arithmetic of alternating forms and quaternion hermitian forms, J. Math. Soc. Japan, **15** (1963), 33-65.

[Wa]   R.B. Warfield, Nilpotent groups, Lecture Notes in Math., **513**, Springer, Berlin-Heidelberg-New York, 1976.

[Wi]   E. Witt, Treue Darstellung Liescher Ringe, J. Reine Angew. Math., **177** (1937), 152-160.

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