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Strong multihomotopy and Steenrod loop spaces

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0. Introduction.

The strong shape category of metric compacta was introduced in 1973 by J. B. Quigley [**19**], although some notions related to strong shape were already considered by D. Christie [**6**] and T. Porter [**18**]. In particular, Christie defined the strong shape groups. In 1976 D. A. Edwards and H. M. Hastings [**11**], motivated by work of T. A. Chapman [**5**], obtained a category isomorphism between the strong shape category of compacta K in the pseudo-interior of the Hilbert cube, Q, and the proper homotopy category of their complements $Q-K$. Strong shape was extended to arbitrary topological spaces by F. W. Bauer [**1**] and Edwards and Hastings [**11**]. General information about the strong shape category of compacta is contained in the papers [**9**] by J. Dydak and J. Segal and [**3**] by F. W. Cathey. The first of them presents a geometric study of strong shape based on the notion of contractible telescope. The second one gives an account of several different approaches. We shall use in this paper the approach to strong shape given by J. B. Quigley [**19**] or, in a more general form, that given by Y. Kodama and J. Ono [**14**], [**15**] under the name of fine shape.

All the existing descriptions of the strong shape category of compacta use external elements to introduce the basic notion of strong shape. Compacta are generally assumed to lie in the Hilbert cube or in a convenient ambient space, like a manifold or a polyhedron, and maps take values in neighborhoods of the compacta in the ambient space. In other descriptions, compacta are presented as inverse limits of ANR systems and maps are defined between the systems and not directly between the compacta themselves.

We present in this paper a new description of strong shape. We eliminate all the external elements in our approach and obtain an intrinsic description of the strong shape category of compacta, completing in this way the program that was started in [**20**] and [**21**] for standard shape.

We use in our approach the theory of multivalued maps. Strong shape morphisms are characterized as homotopy classes of fine multivalued maps and

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a complete description of the category is given in terms of this notion. A topology is introduced on the set $M(X, Y)$ of fine multivalued maps between two compacta X and Y. This allows us to identify the shape morphisms from X to Y with the connected components of $M(X, Y)$ and the strong shape morphisms with the path components of $M(X, Y)$. By using this representation, we prove that every shape morphism is the closure of a strong shape morphism. If (X, x_0) is a pointed compactum, we define the Steenrod loop space $\Omega^s(X, x_0)$ as a useful tool to study the strong shape groups $\Pi_n^s(X, x_0)$. We have adopted this terminology since Steenrod's name has often been associated with strong shape (see [**12**]). We prove that $\Pi_n^s(X, x_0)=\Pi_{n-1}(\Omega^s(X, x_0), *)$ and, therefore, the calculation of strong shape groups can be reduced to that of the standard homotopy groups of the Steenrod loop space.

For information about shape theory we recommend the books [**2**], [**7**], [**8**] and [**17**] by K. Borsuk, J. M. Cordier and T. Porter, J. Dydak and J. Segal and S. Mardešić and J. Segal respectively. We also recommend the collection of open problems [**10**] by J. Dydak and J. Segal. For earlier results about the relationship between shape and multivalued maps see the papers [**4**], [**13**] and [**16**] by Z. Čerin and T. Watanabe, Y. Kodama and A. Koyama respectively.

1. Fine multivalued maps and strong shape morphisms.

Let X and Y be metric spaces. An upper semicontinuous multivalued function $F: X\to Y$ is a correspondence such that for every $x\in X$, $F(x)\neq\emptyset$ is a closed subset of Y and for every neighborhood V of $F(x)$ in Y there is a neighborhood U of x such that $F(U)=\bigcup_{y\in U} F(y)$ is contained in V. In the sequel, upper semicontinuous multivalued functions will be called multivalued maps for short. F is said to be ε-small if diameter $(F(x))<\varepsilon$ for every $x\in X$. Two multivalued maps $F, G: X\to Y$ are ε-homotopic if there exists an ε-small multivalued map $H: X\times I\to Y$ such that $H(x, 0)=F(x)$ and $H(x, 1)=G(x)$ for every $x\in X$.

In the sequel X and Y will always be compact metric spaces. A fine multivalued map from X to Y is a multivalued map $F: X\times \boldsymbol{R}_+\to Y$ such that for every $\varepsilon>0$ there is a $t_0\in\boldsymbol{R}_+=[0, \infty)$ such that diameter$(F(x, t))<\varepsilon$ for every $x\in X$ and every $t\geqq t_0$. Two fine multivalued maps $F, G: X\times\boldsymbol{R}_+\to Y$ are said to be homotopic if there exists a fine multivalued map $H: X\times[0, 1]\times\boldsymbol{R}_+\to Y$ such that $H(x, 0, t)=F(x, t)$ and $H(x, 1, t)=G(x, t)$ for every $(x, t)\in X\times\boldsymbol{R}_+$. F and G are said to be weakly homotopic if for every $\varepsilon>0$ there is a $t_0\in\boldsymbol{R}_+$ such that $F|_{X\times[t_0,\infty)}$ and $G|_{X\times[t_0,\infty)}$ are ε-homotopic. Homotopy and weak homotopy of fine multivalued maps are equivalence relations. The corresponding equivalence classes of F will be denoted by $[F]$ and $[F]_w$ respectively. Obviously $[F]\subset[F]_w$.

DEFINITION 1. Let $F: X\times \boldsymbol{R}_{+}\rightarrow Y$ and $G: Y\times \boldsymbol{R}_{+}\rightarrow Z$ be fine multivalued maps. A function $\alpha: \boldsymbol{R}_{+}\rightarrow \boldsymbol{R}_{+}$ is said to be a stretching map associated to the pair (F, G) if it is an increasing continuous function and there exist null sequences $\{\varepsilon_n\}$, $\{\eta_n\}$ such that a) $\operatorname{diam}(G(K\times\{t\}))<\varepsilon_n$ for every $K\subset Y$ with $\operatorname{diam}(K)<\eta_n$ and every $t\in[n, n+1]$ and b) $\operatorname{diam}(F(x, t))<\eta_n$ for every $x\in X$ and every $t>\alpha(n)$.

The next proposition shows that stretching maps always exist.

PROPOSITION 1. *Let $F: X\times \boldsymbol{R}_{+}\rightarrow Y$ and $G: Y\times \boldsymbol{R}_{+}\rightarrow Z$ be fine multivalued maps. Then, there exists a stretching map $\alpha: \boldsymbol{R}_{+}\rightarrow \boldsymbol{R}_{+}$ associated to (F, G).*

PROOF. Consider a null sequence $\{\varepsilon_n\}$ such that $\operatorname{diam}(G(y, t))<\varepsilon_n$ for every $y\in Y$ and every $t\geqq n$ and define by induction a null sequence $\eta_1>\eta_2>\cdots>\eta_n>\cdots$ such that $\operatorname{diam}(G(K\times\{t\}))<\varepsilon_n$ for every $K\subset Y$ with $\operatorname{diam}(K)<\eta_n$ and every $t\in[n, n+1]$. Since F is a fine multivalued map there is an unbounded sequence $0=t_0<t_1<t_2<\cdots<t_n<\cdots$ such that $\operatorname{diam}(F(x, t))<\eta_n$ for every $x\in X$ and every $t>t_n$ with $n\geqq 1$. Consider for every n the increasing linear homeomorphism $\alpha_n: [n, n+1]\rightarrow[t_n, t_{n+1}]$. Then, the obvious piecewise linear homeomorphism $\alpha: \boldsymbol{R}_{+}\rightarrow \boldsymbol{R}_{+}$ defined by means of the family $\{\alpha_n\}$ is a stretching map for (F, G).

As we see in the next proposition, stretching maps can be used to define a notion of composition of the homotopy classes $[F]$ and $[G]$.

PROPOSITION 2. *Let $[F]: X\times \boldsymbol{R}_{+}\rightarrow Y$ and $[G]: Y\times \boldsymbol{R}_{+}\rightarrow Z$ be homotopy classes of fine multivalued maps and suppose that $\alpha: \boldsymbol{R}_{+}\rightarrow \boldsymbol{R}_{+}$ is a stretching map for (F, G). Then the function $H: X\times \boldsymbol{R}_{+}\rightarrow Z$ defined by $H(x, t)=G(F(x, \alpha(t)), t)$ is a fine multivalued map and its homotopy class $[H]$ does not depend on the representatives of the classes $[F]$ and $[G]$ or on the particular stretching map α.*

PROOF. The first assertion is obvious.

To prove the second one we must show that if we have fine multivalued maps F' and G' homotopic to F and G respectively and if α' is a stretching map for (F', G'), then the map $H': X\times \boldsymbol{R}_{+}\rightarrow Z$ defined by

$$H'(x, t)=G'(F'(x, \alpha'(t)), t)$$

is homotopic to H.

First observe that if $\beta: \boldsymbol{R}_{+}\rightarrow \boldsymbol{R}_{+}$ is an increasing map such that $\beta(t)\geqq\alpha(t)$ for every $t\in \boldsymbol{R}_{+}$, then β is also a stretching map for (F, G). Moreover the fine multivalued map J given by the expression

$$J(x, t)=G(F(x, \beta(t)), t)$$

is homotopic to H by means of the fine homotopy $\phi: X\times \boldsymbol{R}_{+}\times[0, 1]\rightarrow Z$ defined by the expression

$$\phi(x, t, s) = G(F(x, \alpha(t)(1-s)+\beta(t)s), t).$$

Consider homotopies $F^*: X\times \boldsymbol{R}_+ \times [0, 1] \to Y$ and $G^*: Y\times \boldsymbol{R}_+\times[0, 1]\to Z$ connecting F with F' and G with G' respectively. We denote by

$$\tilde{F}: X\times\boldsymbol{R}_+\times[0, 1] \longrightarrow Y\times[0, 1]$$

the map $\tilde{F}(x, t, s)=(F^*(x, t, s), s)$ and select a stretching map $\alpha'': \boldsymbol{R}_+\to\boldsymbol{R}_+$ for the pair $(\tilde{F}, G^*)$ such that $\alpha''(t)\geqq\max\{\alpha(t), \alpha'(t)\}$ for every $t\in\boldsymbol{R}_+$. Then the expression $H^*(x, t, s)=G^*(F^*(x, \alpha''(t), s), t, s)$ defines a homotopy $H^*: X\times\boldsymbol{R}_+\times[0, 1]\to Z$ whose 0-level H_0^* is homotopic to H and whose 1-level H_1^* is homotopic to H'. This completes the proof of the proposition.

We are now in a position to state and prove the following result, which gives a new description of the strong shape category of compacta.

THEOREM 1. *If we consider the class of compact metric spaces and the homotopy classes of fine multivalued maps with the notion of composition previously defined we get a category, MSh, which is isomorphic to the strong shape category of compacta.*

PROOF. The identity morphism in $MSh(X, X)$ is the homotopy class of the map $I_X: X\times\boldsymbol{R}_+\to X$ defined by $I_X(x, t)=x$, then in order to show that MSh is a category it is only necessary to prove that if $F: X\times\boldsymbol{R}_+\to Y$, $G: Y\times\boldsymbol{R}_+\to Z$, and $H: Z\times\boldsymbol{R}_+\to W$ are fine multivalued maps then $[H]([G][F])=([H][G])[F]$. But $[H]([G][F])=[R]$ with

$$R(x, t) = H(G(F(x, \alpha_1(\alpha_2(t))), \alpha_2(t)), t)$$

and $([H][G])[F]=[S]$ with

$$S(x, t) = H(G(F(x, \beta_2(t)), \beta_1(t)), t),$$

where α_1, α_2, β_1, β_2 are suitable stretching maps, and it is easy to see that R and S are both homotopic to a common fine multivalued map of the kind

$$H(G(F(x, \gamma_1(t)), \gamma_2(t)), t),$$

where γ_1 and γ_2 are large enough stretching maps.

To prove that MSh and SSh are isomorphic we shall use the approach to strong shape given by Quigley [**19**] or Kodama and Ono [**14**] where compacta are assumed to lie in the Hilbert cube Q and strong shape morphisms from X to Y are homotopy classes of approaching maps (i.e., single-valued maps $f: Q\times\boldsymbol{R}_+\to Q$ such that for every neighborhood V of Y in Q there is a neighborhood U of X in Q and a $t_0\in\boldsymbol{R}_+$ such that $f(U\times[t_0, \infty))\subset V$).

Suppose that $f: Q\times\boldsymbol{R}_+\to Q$ is an approaching map from X to Y. We shall prove that there exists a fine multivalued map $F: X\times\boldsymbol{R}_+\to Y$ such that f is asymptotic to F, i.e., for every $\varepsilon>0$ there is a $t_0\in\boldsymbol{R}_+$ such that $d(f(x, t), F(x, t))<\varepsilon$, for every $x\in X$ and every $t\geqq t_0$. F is constructed in the following way: consider a null sequence $\varepsilon_1>\cdots>\varepsilon_n>\cdots$ such that $f(x, t)\in\bar{B}_{\varepsilon_n}(Y)$ (the closed ball in Q) for every $x\in X$ and every $t\geqq n$, then define $F(x, t)=\bar{B}_{\varepsilon_n}(f(x, t))\cap Y$ if $t\in(n, n+1]$. It is easy to see that in this way we get a fine multivalued map F such that f is asymptotic to F.

Suppose now that $g: Q\times\boldsymbol{R}_+\to Q$ is an approaching map homotopic to f and that $G: X\times\boldsymbol{R}_+\to Y$ is a fine multivalued map with g asymptotic to G. Consider an approaching homotopy $h: Q\times\boldsymbol{R}_+\times[0, 1]\to Q$ connecting f and g. If $H: X\times\boldsymbol{R}_+\times[0, 1]\to Y$ is a fine multivalued map asymptotic to h we have

$$d(F(x, t), H_0(x, t))\leqq d(F(x, t), f(x, t))+d(f(x, t), H_0(x, t))$$

and from this it follows that F and H_0 are asymptotic. We can then construct a fine homotopy $\phi: X\times\boldsymbol{R}_+\times[0, 1]\to Y$ connecting F and H_0 in the following way

$$\phi(x, t, s)=\begin{cases} F(x, t) & \text{if } 0\leqq s<\dfrac{1}{2} \\ F(x, t)\cup H_0(x, t) & \text{if } s=\dfrac{1}{2} \\ H_0(x, t) & \text{if } \dfrac{1}{2}<s\leqq 1\,. \end{cases}$$

It can be analogously proved that G is homotopic to H_1. Hence F and G are homotopic.

We have proved that there exists a well-defined correspondence

$$\Omega_{(X,Y)}: SSh(X, Y)\longrightarrow MSh(X, Y)$$

such that $\Omega_{(X,Y)}([f])=[F]$ where f is asymptotic to F.

In order to see that $\Omega_{(X,Y)}$ is surjective consider a fine multivalued map $F: X\times\boldsymbol{R}_+\to Y$ and select a null sequence $\varepsilon_1>\cdots>\varepsilon_n>\cdots$ such that $\operatorname{diam}(F(x, t))<\varepsilon_n$ for every $x\in X$ and every $t\geqq n$. Then for every $x\in X$ and every $t\in[n, n+1)$, there exists an open neighborhood $U^{(x,t)}$ of (x, t) contained in $X\times(n-1, n+1)$ such that

$$F(U^{(x,t)})\subset B_{\delta_{(x,t)}}(F(x, t)) \qquad \text{where } \delta_{(x,t)}=\frac{\varepsilon_n-\operatorname{diam}(F(x, t))}{2}.$$

Hence $\operatorname{diam}(F(U^{(x,t)}))<\varepsilon_n$. Now, by using the compactness of $X\times[1, n+1]$, we can define a sequence of open sets $U_1, U_2, U_3, \cdots$ and an increasing sequence of integers $k_1, k_2, k_3, \cdots$ such that for every n

$$X\times[1, n]\subset U_1\cup U_2\cup\cdots\cup U_{k_n}\subset X\times(0, n+1)$$

and for every k with $k_n<k\leqq k_{n+1}$ we have $U_k\subset X\times(n-1, n+2)$ and $\operatorname{diam}(F(U_k))$

$<\varepsilon_n$.

We consider $U_0=X\times[0, 1)$ and define for every $n\geqq 0$ a function $\delta_n: X\times\boldsymbol{R}_+\to\boldsymbol{R}$ such that

$$\delta_n(x, t)=\frac{d((x, t), (X\times\boldsymbol{R}_+)-U_n)}{\sum_k d((x, t), (X\times\boldsymbol{R}_+)-U_k)}.$$

The sum in the denominator is finite since $d((x, t), (X\times\boldsymbol{R}_+)-U_k)\neq 0$ if and only if $(x, t)\in U_k$.

We now choose for every n a point $y_n\in F(U_n)$ and define $f_0: X\times\boldsymbol{R}_+\to Q$ by the expression

$$f_0(x, t)=\sum\delta_n(x, t)y_n .$$

This is again a finite sum and, since $\sum\delta_n(x, t)=1$ and Q is convex, f_0 is a well-defined continuous function. Moreover, for every $(x, t)\in X\times\boldsymbol{R}_+$ consider the open sets $U_{i_1}, \cdots, U_{i_r}$ to which (x, t) belongs. Then if $y\in F(x, t)$ and $t\in[n, n+1)$ with $n\geqq 2$ we have

$$\begin{aligned} d(f_0(x, t), y) &= \|\sum\delta_{i_k}(x, t)y_{i_k}-\sum\delta_{i_k}(x, t)y\| \\ &= \|\sum\delta_{i_k}(x, t)(y_{i_k}-y)\| \\ &\leqq \sum\delta_{i_k}(x, t)\|y_{i_k}-y\| \\ &\leqq \max\{\|y_{i_k}-y\|\} \\ &\leqq \max\{\operatorname{diam}(F(U_{i_k}))\}<\varepsilon_n . \end{aligned}$$

In the above expressions we have used the norm $\| \ \|$ of the Hilbert space l_2 where Q is supposed to lie.

We have proved that for every $\varepsilon>0$ there exists $n\in\boldsymbol{N}$ verifying that $d(f_0(x, t), F(x, t))<\varepsilon$ for every $x\in X$ and every $t\geqq n$. An obvious consequence of this is that for every neighborhood V of Y in Q there exists a $t_0\in\boldsymbol{R}_+$ such that $f_0(X\times[t_0, \infty))\subset V$. It is now easy to see, by repeatedly applying the homotopy extension theorem, that f_0 can be extended to an approaching map $f: Q\times\boldsymbol{R}_+\to Q$. Since f is asymptotic to F, we have that $\Omega_{(X,Y)}$ is surjective.

Suppose now that $F, G: X\times\boldsymbol{R}_+\to Y$ are homotopic fine multivalued maps and that $f, g: Q\times\boldsymbol{R}_+\to Q$ are approaching maps asymptotic to F and G respectively. Consider a fine multivalued homotopy

$$H: X\times\boldsymbol{R}_+\times[0, 1]\longrightarrow Y$$

connecting F and G. It can be seen, by using arguments similar to those used before, that there is an approaching homotopy $h: Q\times\boldsymbol{R}_+\times[0, 1]\to Q$ asymptotic to H. Obviously h_0 is asymptotic to f and h_1 is asymptotic to g and this implies that f and g are homotopic. This proves that $\Omega_{(X,Y)}$ is injective.

In order to prove that SSh and MSh are isomorphic categories, consider fine multivalued maps $F: X\times\boldsymbol{R}_+\to Y$ and $G: Y\times\boldsymbol{R}_+\to Z$. Let $g: Q\times\boldsymbol{R}_+\to Q$ be an approaching map asymptotic to G. We are going to find an adequate representative of $\Omega_{(X,Y)}^{-1}([F])$. Let $\varepsilon_1>\cdots>\varepsilon_n>\cdots$ be a null sequence such that $\mathrm{diam}(G(y, t))<\varepsilon_n$ and $d(g(y, t), G(y, t))<\varepsilon_n$ for every $y\in Y$ and every $t\geqq n$. Let $\eta_1>\cdots>\eta_n>\cdots$ be another sequence such that $\mathrm{diam}(G(K\times\{t\}))<\varepsilon_n$ for every $K\subset Y$ with $\mathrm{diam}(K)<\eta_n$ and every $t\in[n, n+1]$ and such that $d(g(y, t), g(y', t))<\varepsilon_n$ for every $y, y'\in Q$ with $d(y, y')<\eta_n$ and every $t\in[n, n+1]$. Let $\alpha: \boldsymbol{R}_+\to\boldsymbol{R}_+$ be a stretching map for the pair (F, G) satisfying the condition $\mathrm{diam}(F(x, \alpha(t)))<\eta_n$ for every $t\geqq n$. By the argument given at the beginning of this proof there is an approaching map f such that $d(f(x, t), F(x, \alpha(t)))<\eta_n$ for $t\geqq n$. Obviously $\Omega_{(X,Y)}([f])=[F]$. Let us see now that $\Omega_{(X,Z)}([g][f])=[G][F]$. Since for every $t\geqq n$ we have that $d(F(x, \alpha(t)), f(x, t))<\eta_n$, it follows that there exists $y\in F(x, \alpha(t))$ such that $d(y, f(x, t))<\eta_n$ and, hence, $d(g(y, t), g(f(x, t), t))<\varepsilon_n$. On the other hand $d(g(y, t), G(y, t))<\varepsilon_n$ and this implies that $d(g(y, t), G(F(x, \alpha(t)), t))<\varepsilon_n$. Hence

$$d(g(f(x, t), t), G(F(x, \alpha(t)), t)) < 2\varepsilon_n .$$

As a consequence

$$\Omega_{(X,Z)}([g][f]) = [G][F] = \Omega_{(Y,Z)}[g]\Omega_{(X,Y)}[f]$$

and SSh and MSh are isomorphic categories. This completes the proof of the theorem.

If X is a closed subset of a compactum X' and $f: X\to Y$ is a (single-valued) map, we say that f is ε-extendable to X' if there exists an ε-small multivalued map $F: X'\to Y$ such that $F|_X=f$. The following result gives a characterization of the inclusions that induce strong shape isomorphisms. The proof is an application of the techniques developed in this section and is left to the reader.

THEOREM 2. *Let X be a closed subset of the compactum X'. Then the inclusion $i: X\to X'$ induces a strong shape equivalence if and only if for any (single-valued) map $f: X\to Y$, where Y is an arbitrary compactum, there exists for every $\varepsilon>0$ an ε-extension $F: X'\to Y$ and for any map $g: X'\times\{0, 1\}\cup X\times[0, 1]\to Y$ there exists an ε-extension $G: X'\times[0, 1]\to Y$.*

2. A topology for the space of fine multivalued maps.

DEFINITION 2. Let X and Y be compact metric spaces and by $M(X, Y)$ denote the set of all fine multivalued maps from X to Y. If $F, G\in M(X, Y)$ and ε is a positive number we say that $G\in B_\varepsilon(F)$ if there exists a sequence

$\{\varepsilon_n\}$ such that $\sum \varepsilon^k 2^{-k} < \varepsilon$ and the following holds:

a) For every $k \in \boldsymbol{N}$ and for every $(x, t) \in X \times [0, k]$ there exists $(x', t') \in X \times [0, k]$ such that $d((x, t), (x', t')) < \varepsilon_k$ and $G(x, t) \subset B_{\varepsilon_k}(F(x', t'))$.

b) For every $(x, t) \in X \times \boldsymbol{R}_+$ there exists $(x', t') \in X \times \boldsymbol{R}_+$ such that $d((x, t), (x', t')) < \varepsilon$ and $\operatorname{diam}(G(x, t)) < \operatorname{diam}(F(x', t')) + \varepsilon$.

We remark that if $G \in B_\varepsilon(F)$ then for every k and for every $(x, t) \in X \times [0, k]$ there exists $(x', t') \in X \times [0, k]$ such that $d((x, t), (x', t')) < 2^k \varepsilon$ and $G(x, t) \subset B_{2^k \varepsilon}(F(x', t'))$.

In the next proposition we show how to define a topology on the set $M(X, Y)$. If $F \in M(X, Y)$ we introduce the notation

$$\mathscr{B}(F) = \{B_\varepsilon(F) \mid \varepsilon > 0\}.$$

PROPOSITION 3. *The family* $\{\mathscr{B}(F) \mid F \in M(X, Y)\}$ *is a neighborhood system for the set* $M(X, Y)$. *The corresponding topological space will also be denoted by* $M(X, Y)$ *and is a topological invariant of the pair* (X, Y).

PROOF. In order to prove the first assertion, the only nontrivial fact is the following: If $G \in B_\varepsilon(F)$ then there exists a $\delta > 0$ such that $B_\delta(G) \subset B_\varepsilon(F)$. To see this, consider a sequence $\{\varepsilon_n\}$ such that $\sum \varepsilon_k 2^{-k} < \varepsilon$ and such that properties a) and b) in the definition hold. Let n_0 be a number such that $\operatorname{diam}(G(x, t)) < \varepsilon/2$ for every $x \in X$ and every $t \geqq n_0$ and select a $\delta > 0$ such that $\delta < \min\{\varepsilon/2, \varepsilon - \sum \varepsilon_k 2^{-k}\}$ and such that for every $(x, t) \in X \times [0, n_0]$ there exists a $(x', t') \in X \times \boldsymbol{R}_+$ with $d((x, t), (x', t')) < \varepsilon - \delta$ and

$$\operatorname{diam}(G(x, t)) < \operatorname{diam}(F(x', t')) + \varepsilon - \delta .$$

Let $H \in B_\delta(G)$, then there exists a sequence $\{\delta_n\}$ such that $\sum \delta_k 2^{-k} < \delta$ satisfying conditions a) and b) in the definition (with the obvious changes of notation). Hence $\sum (\varepsilon_k + \delta_k) 2^{-k} < \varepsilon$ and for every k and every $(x, t) \in X \times [0, k]$ there exists $(x', t') \in X \times [0, k]$ such that $d((x, t), (x', t')) < \delta_k$ and

$$H(x, t) \subset B_{\delta_k}(G(x', t')) .$$

On the other hand, there exists $(x'', t'') \in X \times [0, k]$ with $d((x', t'), (x'', t'')) < \varepsilon_k$ and $G(x', t') \subset B_{\varepsilon_k}(F(x'', t''))$. As a consequence $d((x, t), (x'', t'')) < \varepsilon_k + \delta_k$ and $H(x, t) \subset B_{\varepsilon_k + \delta_k}(F(x'', t''))$.

Furthermore, if $(x, t) \in X \times \boldsymbol{R}_+$, there exists $(x', t') \in X \times \boldsymbol{R}_+$ such that $d((x, t), (x', t')) < \delta$ and $\operatorname{diam}(H(x, t)) < \operatorname{diam}(G(x', t')) + \delta$. Then, if $(x', t') \in [0, n_0]$, there exists $(x'', t'') \in X \times \boldsymbol{R}_+$ such that $d((x', t'), (x'', t'')) < \varepsilon - \delta$ and $\operatorname{diam}(G(x', t')) < \operatorname{diam}(F(x'', t'')) + \varepsilon - \delta$ and, as a consequence, $d((x, t), (x'', t'')) < \varepsilon$ and

$$\operatorname{diam}(H(x, t)) < \operatorname{diam}(G(x', t')) + \delta < \operatorname{diam}(F(x'', t'')) + \varepsilon .$$

If $(x', t') \in [n_0, \infty)$, then $\operatorname{diam}(H(x, t)) < \operatorname{diam}(G(x', t')) + \delta < \varepsilon/2 + \delta < \varepsilon$. This shows that $\{\mathcal{B}(F) \mid F \in M(X, Y)\}$ is a neighborhood system for $M(X, Y)$.

In order to prove the second assertion, suppose that $h: X \to X'$ is a homeomorphism between compacta. There is an induced correspondence

$$\gamma_h : M(X', Y) \longrightarrow M(X, Y) \qquad \text{defined by } \gamma_h(F)(x, t) = F(h(x), t).$$

We shall show that γ_h is continuous. Let $F \in M(X', Y)$ and $\varepsilon > 0$. Select a k_0 such that

$$\sum_{k=k_0+1}^{\infty} \frac{\Delta}{2^k} < \frac{\varepsilon}{2}, \qquad \text{where } \Delta > \operatorname{diam}(Y)$$

and, using the uniform continuity of h^{-1}, take a $\delta_1 > 0$ such that for every pair of points $x', y' \in X'$ with $d(x', y') < \delta_1$ we have that

$$d(h^{-1}(x'), h^{-1}(y')) < \frac{\varepsilon}{4}.$$

Consider now a $\delta > 0$ such that $2^{k_0}\delta < \min\{\varepsilon/4, \delta_1\}$. We define a sequence $\{\varepsilon_k\}$ by

$$\varepsilon_k = \begin{cases} \dfrac{\varepsilon}{2} & \text{if } 1 \leqq k \leqq k_0 \\ \Delta & \text{if } k_0 < k. \end{cases}$$

We obviously have that $\sum \varepsilon_k 2^{-k} < \varepsilon$. We shall prove that if $G \in M(X', Y)$ and $G \in B_\delta(F)$ then $\gamma_h(G) \in B_\varepsilon(\gamma_h(F))$.

Let $(x, t) \in X \times [0, k_0]$, then $(h(x), t) \in X' \times [0, k_0]$ and there exists $(x'', t') \in X \times [0, k_0]$ such that $d((h(x), t), (x'', t')) < 2^{k_0}\delta$ and

$$G(h(x), t) \subset B_{2^{k_0}\delta}(F(x'', t')).$$

Since $d(h(x), x'') < 2^{k_0}\delta < \delta_1$, we have that $d(x, x') < \varepsilon/4$ where $x' = h^{-1}(x'')$ (observe that $d((x, t), (x', t')) < d(x, x') + d(t, t') < \varepsilon/4 + 2^{k_0}\delta < \varepsilon/2$). Hence, if $(x, t) \in X \times [0, k_0]$ there exists a $(x', t') \in X \times [0, k_0]$ with $d((x, t), (x', t')) < \varepsilon/2$ and

$$\gamma_h G(x, t) = G(h(x), t) \subset B_{2^{k_0}\delta}(F(h(x'), t')) \subset B_{\varepsilon/2}(\gamma_h F(x', t')).$$

If $t > k_0$, then $\gamma_h G(x, t) \subset B_\Delta(\gamma_h F(x, t))$.

On the other hand, for every $(x, t) \in X \times \boldsymbol{R}_+$ there exists $(x'', t') \in X' \times \boldsymbol{R}_+$ such that $d((h(x), t), (x'', t')) < \delta$ and

$$\operatorname{diam}(G(h(x), t)) < \operatorname{diam}(F(x'', t')) + \delta.$$

Let $x' = h^{-1}(x'')$. Since $d(h(x), x'') < \delta < \delta_1$, we have that $d(x, x') < \varepsilon/4$. Hence $d((x, t), (x', t')) < \varepsilon$ and

$$\operatorname{diam}(\gamma_h G(x, t)) = \operatorname{diam}(G(h(x), t)) < \operatorname{diam}(F(h(x'), t')) + \delta$$
$$< \operatorname{diam}(\gamma_h F(x', t')) + \varepsilon .$$

We conclude from this that $\gamma_h(G) \in B_\varepsilon(\gamma_h(F))$ and γ_h is continuous.

Moreover $\gamma_h \gamma_{h^{-1}} = id_{M(X,Y)}$ and this implies that γ_h is a homeomorphism.

It can be analogously proved that a homeomorphism $g: Y \to Y'$ induces a homeomorphism $\gamma^g: M(X, Y) \to M(X, Y')$ and from this readily follows the proof of the proposition.

REMARK 1. Condition b) in the definition of the topology for $M(X, Y)$ reflects the fact that close fine multivalued maps have comparable diameters. Condition a) is a reminiscence of the compact-open topology on spaces of multivalued maps between compacta. This can be made precise in the following way:

If $\Gamma(X, Y)$ represents the set of all upper semicontinuous multivalued maps from X to Y and by $\phi' \in B_\varepsilon(\phi)$ ($\phi, \phi' \in \Gamma(X, Y)$) we mean that for every $x \in X$ there is a $x' \in X$ with $d(x, x') < \varepsilon$ and $\phi'(x) \subset B_\varepsilon(\phi(x'))$, then the family $\{B_\varepsilon(\phi) \mid \phi \in \Gamma(X, Y)$ and $\varepsilon > 0\}$ defines a neighborhood system which induces exactly the compact-open topology on $\Gamma(X, Y)$.

The next theorem is a key result in this section. It refers to properties of exponential type in the space $M(X, Y)$ and the main results in this section and the next one will be derived from it.

THEOREM 3. *Let X, Y and Z be compact metric spaces and suppose that $F: X \times Z \times \boldsymbol{R}_+ \to Y$ is a fine multivalued map. Then, the function $F': Z \to M(X, Y)$ defined by $F'(z)(x, t) = F(x, z, t)$ is continuous. Conversely, if $F': Z \to M(X, Y)$ is a map, then the associated function $F: X \times Z \times \boldsymbol{R}_+ \to Y$ defined by $F(x, z, t) = F'(z)(x, t)$ is a fine multivalued map. As a consequence, there exists a natural bijection between the sets $C(Z, M(X, Y))$ and $M(X \times Z, Y)$, where $C(Z, M(X, Y))$ represents the set of maps from Z to $M(X, Y)$.*

PROOF. Let $z_0 \in Z$. We shall prove that if $F: X \times Z \times \boldsymbol{R}_+ \to Y$ is a fine multivalued map then F' is continuous at z_0. For a given $\varepsilon > 0$, select k_0 such that

$$\sum_{k=k_0+1}^{\infty} \frac{\Delta}{2^k} < \frac{\varepsilon}{2}, \qquad \text{where } \Delta > \operatorname{diam}(Y),$$

and such that $\operatorname{diam}(F(x, z, t)) < \varepsilon$ for every $t \geqq k_0$. For every $(x, t) \in X \times [0, k_0]$, there exists $\delta_{(x,t)} < \varepsilon/2$ such that $F(B_{\delta_{(x,t)}}(x, z_0, t)) \subset B_{\varepsilon/2}(F(x, z_0, t))$, and using the compactness of $X \times \{z_0\} \times [0, k_0]$, we can find a finite family of points $(x_1, t_1), \cdots, (x_n, t_n)$ and a $\delta > 0$ such that

$$X\times B_{\delta}(z_0)\times[0, k_0]\subset\bigcup_{i=1}^{n} B_{\delta(x_i,t_i)}(x_i, z_0, t_i)$$

and such that for every $(x, z, t)\in X\times B_{\delta}(z_0)\times[0, k]$ with $k\leqq k_0$, there exists $(x_i, t_i)\in X\times[0, k]$ such that $(x, z, t)\in B_{\delta(x_i,t_i)}(x_i, z_0, t_i)$. Now we define a sequence $\{\varepsilon_k\}$, verifying $\sum\varepsilon_k 2^{-k}<\varepsilon$, in the following way

$$\varepsilon_k=\begin{cases}\dfrac{\varepsilon}{2} & \text{if } 1\leqq k\leqq k_0\\ \Delta & \text{if } k_0<k\,.\end{cases}$$

Suppose that $z\in Z$ and $d(z, z_0)<\delta$. Then if $k\leqq k_0$, for every $(x, t)\in X\times[0, k]$ there exists $(x', t')\in B_{\varepsilon_k}(x, t)$ such that

$$F'(z)(x, t)\subset B_{\varepsilon_k}(F'(z_0)(x', t'))\,.$$

If $k>k_0$ then for every $(x, t)\in X\times[0, k]$, $F'(z)(x, t)\subset B_{\Delta}(F'(z_0)(x, t))$. It can also be readily shown that for every $(x, t)\in X\times\boldsymbol{R}_+$ there exists $(x', t')\in X\times\boldsymbol{R}_+$ such that $d((x, t), (x', t'))<\varepsilon$ and

$$\operatorname{diam}(F'(z)(x, t))<\operatorname{diam}(F'(z_0)(x', t'))+\varepsilon\,.$$

As a consequence $F'(z)\in B_{\varepsilon}(F'(z_0))$ and this proves the continuity of F' at z_0.

In order to prove the converse statement, consider a map $F'\colon Z\to M(X, Y)$ and let $F\colon X\times Z\times\boldsymbol{R}_+\to Y$ be its associated function. First we shall see that F is upper semicontinuous. Let $(x_0, z_0, t_0)\in X\times Z\times\boldsymbol{R}_+$ and let $\varepsilon>0$. Since $F'(z_0)$ is upper semicontinuous at (x_0, t_0), there exists $\delta_1>0$ such that

$$F'(z_0)(B_{\delta_1}(x_0, t_0))\subset B_{\varepsilon/2}(F(x_0, z_0, t_0))\,.$$

Select now a k_0 verifying that $(t_0-\delta_1, t_0+\delta_1)\subset[0, k_0]$. Then there exists $\delta_2<\min\{\delta_1/2, \varepsilon\}$ such that for every $z\in Z$ with $d(z, z_0)<\delta_2$ we have $F'(z)\in B_{\delta_1/2^{k_0+1}}(F'(z_0))$. Hence for every $(x, t)\in X\times[0, k_0]$ there exists $(x', t')\in B_{\delta_1/2}(x, t)$ such that $F(x, z, t)\subset B_{\delta_1/2}(F(x', z_0, t'))$. Consequently, for every $(x, z, t)\in X\times Z\times\boldsymbol{R}_+$ with $d((x, z, t), (x_0, z_0, t_0))<\delta_2$ there exists $(x', t')\in B_{\delta_1/2}(x, t)$ such that $F(x, z, t)\subset B_{\delta_1/2}(F(x', z_0, t'))$ and, since $d((x', t'), (x_0, t_0))<\delta_1/2+\delta_2<\delta_1$, we have that $F(x', z_0, t')\subset B_{\varepsilon/2}(F(x_0, z_0, t_0))$. Hence

$$F(x, z, t)\subset B_{\delta_1/2}(F(x', z_0, t'))\subset B_{\delta_1/2+\varepsilon/2}(F(x_0, z_0, t_0))\subset B_{\varepsilon}(F(x_0, z_0, t_0))$$

and F is upper semicontinuous at (x_0, z_0, t_0).

Moreover, if $\varepsilon>0$ then for every $z\in Z$ there exists a $\delta>0$ such that for every $z'\in Z$ with $d(z, z')<\delta$ we have that $F'(z')\in B_{\varepsilon/2}(F'(z))$. By the compactness of Z there exists a finite family of points $z_1, z_2, \cdots, z_n\in Z$ such that for every $z\in Z$ there exists $i\in\{1, \cdots, n\}$ with $F'(z)\in B_{\varepsilon/2}(F'(z_i))$ and from this it follows that for every $(x, t)\in X\times\boldsymbol{R}_+$ there exists $(x', t')\in X\times\boldsymbol{R}_+$ such that

$d((x, t), (x', t')) < \varepsilon/2$ and

$$\operatorname{diam}(F'(z)(x, t)) < \operatorname{diam}(F'(z_i)(x', t')) + \frac{\varepsilon}{2}.$$

Since $F'(z_i)$ is a fine multivalued map there exists $t_0 \in \boldsymbol{R}_+$ verifying that $\operatorname{diam}(F'(z_i)(x, t)) < \varepsilon/2$ for every $x \in X$, $t \geqq t_0$ and $i \in \{1, \cdots, n\}$. Hence for every $(x, z, t) \in X \times Z \times [t_0 + \varepsilon/2, \infty)$ there exists $(x', z_i, t') \in X \times Z \times [t_0, \infty)$ such that

$$\operatorname{diam}(F(x, z, t)) = \operatorname{diam}(F'(z)(x, t)) < \operatorname{diam}(F'(z_i)(x', t')) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore F is a fine multivalued map and this completes the proof of the theorem.

As a consequence of Theorem 3 we obtain the following result, which gives a representation of a strong shape morphisms as a certain subset of $M(X, Y)$.

COROLLARY 1. *Two fine multivalued maps $F, G : X \times \boldsymbol{R}_+ \to Y$ are homotopic if and only if they lie in the same path-component of $M(X, Y)$. As a consequence, the strong shape morphisms from X to Y can be identified with the path-components of $M(X, Y)$.*

PROOF. F and G are homotopic if and only if there exists a fine multivalued map $H: X \times I \times \boldsymbol{R}_+ \to Y$ such that $H(x, 0, t) = F(x, t)$ and $H(x, 1, t) = G(x, t)$ for every $(x, t) \in X \times \boldsymbol{R}_+$ but according to Theorem 3 this is equivalent to the existence of a map $h : I \to M(X, Y)$ with $h(0) = F$ and $h(1) = G$.

Two fine multivalued maps $F, G : X \times \boldsymbol{R}_+ \to Y$ are said to be weakly homotopic if for every $\varepsilon > 0$ there is a $t_0 \in \boldsymbol{R}_+$ such that $F|_{X \times [t_0, \infty)}$ is homotopic to $G|_{X \times [t_0, \infty)}$. In [20], the shape morphisms from X to Y are characterized as weak homotopy classes of fine multivalued maps $F: X \times \boldsymbol{R}_+ \to Y$. We denote by $[F]_w$ the weak homotopy class of F. Obviously $[F] \subset [F]_w$.

Our next result gives a new representation of shape morphisms and establishes a relationship between shape and strong shape morphisms.

THEOREM 4. *Two fine multivalued maps $F, G : X \times \boldsymbol{R}_+ \to Y$ are weakly homotopic if and only if they lie in the same connected component of $M(X, Y)$. As a consequence, the shape morphisms from X to Y can be identified with the connected components of $M(X, Y)$. Moreover $[F]_w = cl[F]$, i.e., every shape morphism is the closure of a strong shape morphism.*

PROOF. The proof will proceed in several steps. We first claim that if F is weakly homotopic to G then for every $\varepsilon > 0$ there is $F' \in B_\varepsilon(F)$ such that F' is homotopic to G. An obvious consequence of this is that $[F]_w \subset cl[F]$ and $[F]_w$ is connected.

In order to prove our claim consider for a given $\varepsilon>0$ a k_0 such that

$$\sum_{k=k_0+1}^{\infty}\frac{\Delta}{2^k}<\varepsilon\,, \qquad \text{where } \Delta>\operatorname{diam}(Y)\,,$$

and $F|_{X\times[k_0,\infty)}$ is ε-homotopic to $G|_{X\times[k_0,\infty)}$. Consider an ε-small multivalued map $\phi: X\times[0, 1]\to Y$ such that $\phi_0=F|_{X\times\{k_0\}}$ and $\phi_1=G|_{X\times\{k_0+1\}}$. We define $F': X\times \boldsymbol{R}_+\to Y$ by:

$$F'(x, t)=\begin{cases} F(x, t) & \text{if } 0\leqq t\leqq k_0 \\ \phi(x, t-k_0) & \text{if } k_0\leqq t\leqq k_0+1 \\ G(x, t) & \text{if } k_0+1\leqq t\,. \end{cases}$$

It is easy to see, and we leave it to the reader, that $F'\in B_\varepsilon(F)$ and $F'\simeq G$.

It can also be readily seen that if $F\in M(X, Y)$ and $\varepsilon>0$ then there exists a $\delta>0$ and a k_0 such that for every $F'\in B_\delta(F)$, we have that $F|_{X\times[k_0,\infty)}$ is ε-homotopic to $F'|_{X\times[k_0,\infty)}$. A consequence of this is that the shape morphism $[F]_w$ is closed in $M(X, Y)$ and, since $[F]\subset[F]_w\subset cl[F]$, we deduce that $cl[F]=[F]_w$.

Finally, we must show that if $A\subset M(X, Y)$ is connected and $F, G\in A$ then F and G are weakly homotopic. Consider for a given $\varepsilon>0$ the set

$$K_\varepsilon=\{H\in A\,|\,F \text{ is } \varepsilon\text{-homotopic to } H\}.$$

Suppose that $H'\in\overline{K}_\varepsilon$ (the closure of K_ε in A). Select a $\delta>0$ such that if $H''\in B_\delta(H')$ then H'' is ε-homotopic to H'. Since $B_\delta(H')\cap K_\varepsilon\neq\emptyset$, we deduce that H' is ε-homotopic to F. This shows that K_ε is closed in A and it can be easily shown that K_ε is also open. As a consequence $A=K_\varepsilon$ and $G\in K_\varepsilon$ for every $\varepsilon>0$. Hence F is weakly homotopic to G. This completes the proof of the theorem.

3. Strong shape groups and Steenrod loop spaces.

If (X, x_0) is a pointed compact metric space, the nth strong shape group $\Pi_n^s(X, x_0)$ can be viewed as the set of homotopy classes of fine multivalued maps $F: (I^n, \partial I^n)\times\boldsymbol{R}_+\to(X, x_0)$ with the group structure given by $[F]*[G]=[H]$, where

$$H(t_1, \cdots, t_n, t)=\begin{cases} F(2t_1, \cdots, t_n, t) & \text{if } 0\leqq t_1\leqq \dfrac{1}{2} \\ G(2t_1-1, \cdots, t_n, t) & \text{if } \dfrac{1}{2}\leqq t_1\leqq 1\,. \end{cases}$$

The proof of this fact is an easy consequence of the techniques developed in this paper, since all the approximation results admit relative versions for fine multivalued maps of the kind $F: (I^n, \partial I^n)\times\boldsymbol{R}_+\to(X, x_0)$.

We define the Steenrod loop space of (X, x_0) as the set $\Omega^s(X, x_0)$ of all the fine multivalued maps $F: (I, \partial I)\times \boldsymbol{R}_+ \to (X, x_0)$ endowed with the subspace topology of $M(I, X)$. We denote by $C_0: (I, \partial I)\times \boldsymbol{R}_+ \to (X, x_0)$ the constant loop $C_0(s, t)=x_0$. We also define $\Omega_2^s(X, x_0)=\Omega(\Omega^s(X, x_0), C_0)$ and, inductively, $\Omega_n^s(X, x_0)=\Omega(\Omega_{n-1}^s(X, x_0), *)$ (where $\Omega(\ )$ denotes the classical loop space).

We close our paper with a result that allows us to reduce the calculus of strong shape groups to that of standard homotopy groups.

THEOREM 5. *$\Pi_1^s(X, x_0)$ can be identified in a natural manner with the path components of the Steenrod loop space $\Omega^s(X, x_0)$. If $n\geqq 2$, $\Pi_n^s(X, x_0)$ is isomorphic to $\Pi_{n-1}(\Omega^s(X, x_0), C_0)$ and, hence, to $\Pi(\Omega_{n-1}^s(X, x_0), *)$.*

PROOF. We shall only prove the second half of the statement, the first being easier. If $[F]\in \Pi_n^s(X, x_0)$, then $F: (I^n=I^{n-1}\times I, \partial I^n)\times \boldsymbol{R}_+ \to (X, x_0)$ induces by Theorem 3 a map $F': I^{n-1}\to M(I, X)$ defined by

$$F'(t_1, \cdots, t_{n-1})(t, r) = F(t_1, \cdots, t_{n-1}, t, r).$$

Obviously $F'(\partial I^{n-1})=\{C_0\}$ and, hence, $[F']\in \Pi_{n-1}(\Omega^s(X, x_0), C_0)$.

If $G: (I^n, \partial I^n)\times \boldsymbol{R}_+ \to (X, x_0)$ is homotopic to F and $H: (I^n, \partial I^n)\times \boldsymbol{R}_+\times I\to (X, x_0)$ is a homotopy connecting F and G then the associated function $H': I^{n-1}\times I\to M(I, X)$ defined by

$$H'(t_1, \cdots, t_{n-1}, s)(t, r) = H(t_1, \cdots, t_{n-1}, t, r, s)$$

is continuous and $\operatorname{Im} H'\subset \Omega^s(X, x_0)$ and $H'(\partial I^{n-1}\times I)=\{C_0\}$. Hence H' connects in $\{\Omega^s(X, x_0), C_0\}$ F' and the map G' associated to G. This shows that the homotopy class $[F']$ does not depend on the representative of the homotopy class $[F]$ and we have defined a function

$$\alpha: \Pi_n^s(X, x_0) \longrightarrow \Pi_{n-1}(\Omega^s(X, x_0), C_0)$$

that is clearly a group homomorphism.

We shall see that α is surjective, the injectivity is left to the reader. If $[F']\in \Pi_{n-1}(\Omega(X, x_0), C_0)$ then by Theorem 3, the function $F: I^n\times \boldsymbol{R}_+\to X$ defined by $F(t_1, \cdots, t_n, r)=F'(t_1, \cdots, t_{n-1})(t_n, r)$ is a fine multivalued map. Moreover, if $(t_1, \cdots, t_n)\in \partial I^n$ then $(t_1, \cdots, t_{n-1})\in \partial I^{n-1}$ or $t_n\in \partial I$ and, in both cases $F'(t_1, \cdots, t_{n-1})(t_n, r)=x_0$. Hence $F: (I^n, \partial I^n)\times \boldsymbol{R}_+ \to (X, x_0)$ defines an element $[F]\in \Pi_n^s(X, x_0)$ such that $\alpha([F])=[F']$. This completes the proof of the theorem.

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