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On classification of non-Gorenstein Q-Fano 3-folds of Fano index 1

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1. Introduction.

First of all we recall some definitions.

DEFINITION 1.1. A d-dimensional normal complex projective variety X is called a *Q-Fano d-fold* if it has only terminal singularities and the anti-canonical Weil divisor $-K_X$ is ample (cf. [**KMM**]). The *index of singular point p* is defined to be the smallest positive integer i_p such that i_pK_X is a Cartier divisor near p. A singular point of singularity index one is called *Gorenstein singularity*. *Singularity index* $I(X)$ of X is defined to be the smallest positive integer such that IK_X is a Cartier divisor. Hence there is a positive integer r and a Cartier divisor H such that $-IK_X \sim rH$. Taking the largest number of such r, we call r/I the *Fano index* of X.

Q-Fano d-folds whose Fano indices are greater than $d-2$ are classified by [**Sa**] under the assumption that they are not Gorenstein, that is, their singular indices are greater than one. In this paper we shall consider Fano 3-folds of Fano index 1 and not Gorenstein. Classifying these Fano 3-folds also answers the next problem presented by G. Fano, A. Conte and J. P. Murre (cf. [**CM**]) in the case that they have only terminal singularities.

PROBLEM. *Classify the projective* 3-*folds having Enriques surfaces as hyperplane sections.*

In general case, this problem seems very hard to solve because their singularities may not be Q-Gorenstein, that is, $-mK$ is not Cartier for any positive integer m.

In this article we shall obtain next result.

THEOREM 1.1. *Let X be a Q-Fano 3-fold of Fano index* 1 *having only cyclic quotient singularities. We take a canonical cover*:

$$Y = Spec_X \bigoplus_{m=0}^{I-1} \mathcal{O}_X(m(K_X+H)) \xrightarrow{I:1} X .$$

Then I is 2 and Y is one of the following smooth Fano 3-folds.

No.	$(-K_Y)_t$	Y
1	4	$(2, 4) \subset \boldsymbol{P}(1, 1, 1, 1, 1, 2)$
2	8	$(2, 2, 2) \subset \boldsymbol{P}^6$
3	8	*the blowup of* $(4) \subset \boldsymbol{P}(1, 1, 1, 1, 2)$ *with center an elliptic curve which is an intersection of two member of* $\lvert -(1/2)K \rvert$
4	12	$\boldsymbol{P}^1 \times S_2$
5	12	*a double cover of* $\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$ *whose branch locus is a divisor of tridegree* $(2, 2, 2)$
6	12	*a double cover of* $(1, 1) \subset \boldsymbol{P}^2 \times \boldsymbol{P}^2$ *whose branch locus is a member of* $\lvert -K \rvert$
7	16	*the blow-up of* $(2, 2) \subset \boldsymbol{P}^5$ *with center an elliptic curve which is an intersection of two hyperplain sections*
8	16	$(4) \subset \boldsymbol{P}(1, 1, 1, 1, 2)$
9	20	*the complete intersection of three divisors of bi-degree* $(1, 1)$ *in* $\boldsymbol{P}^3 \times \boldsymbol{P}^3$
10	24	$\boldsymbol{P}^1 \times S_4$
11	24	$(1, 1, 1, 1) \subset \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$
12	32	$(2, 2) \subset \boldsymbol{P}^5$
13	36	$\boldsymbol{P}^1 \times S_6$
14	48	$\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$

There is a smooth Fano 3-fold Y among each deformation type, which has an involution θ such that Y/θ is a $\boldsymbol{Q}$-Fano 3-fold of Fano index 1.

REMARK 1.2. We can easily classify all involutions θ of each Y such that Y/θ is a $\boldsymbol{Q}$-Fano 3-fold of Fano index 1. But this is a very tiresome work, so we will constract only one example for each type.

Notation.

In this paper we always assume that the ground field is complex number field $\boldsymbol{C}$, and we will follow the notation and the terminology of [**KMM**]. The following symbols will be frequently used with no mention.

$\sim$: linear equivalence

$\sim_{\boldsymbol{Q}}$: $\boldsymbol{Q}$-linear equivalence

$\equiv$: numerical equivalence
K_X : canonical divisor of X
$\rho(X)$: the Picard number of X, i.e., rank Pic X
$h^i(D) := \dim_C H^i(D)$
$\chi(D) := \sum_i (-1)^i h^i(X, D)$
$c_i(X)$: i-th Chern class of X
$B_i(X)$: i-th Betti number of X.

2. Preliminaries.

Let X be a $\boldsymbol{Q}$-Fano 3-fold of Fano index 1 with only cyclic quotient singularities. We take the canonical cover:

$$Y = Spec_X \bigoplus_{m=0}^{I-1} \mathcal{O}_X(m(K+H)) .$$

In this section we will obtain bounds of $(-K_Y)^3$, I and the number of singular points. We recall here three fundamental theorems.

THEOREM 2.1 (Riemann-Roch Theorem for singular variety [**Ka**], [**Re**]). *Let V be a normal projective 3-fold with only terminal singularities, and D be a Weil divisor on V. If $\mathcal{O}_V(D) \cong \mathcal{O}_V(K_V)$ in a neighbourhood of each point of V, then*

$$\chi(\mathcal{O}_V(D)) = \chi(\mathcal{O}_V) + \frac{1}{12} D(D-K_V)(2D-K_V) + \frac{1}{12} D \cdot c_2(V) - \frac{1}{12} \sum_{i \mid I} \left(i - \frac{1}{i}\right) n_i$$

where n_i is the number of singular points of index i counted with multiplicites. If D is a Cartier divisor (not requiring $\mathcal{O}_V(D) \cong \mathcal{O}_V(K_V)$ in a neighbourhood of each point), then the same equality holds but the last term $(1/12)\sum_{i \mid I} (i-1/i) n_i$ does not appear.

THEOREM 2.2 (Vanishing Theorem [**KMM**]). *Let V be a normal projective variety with only $\boldsymbol{Q}$-factorial terminal singularities, and D be a Weil divisor on V. If $D-K_V$ is ample, then*

$$H^i(V, \mathcal{O}_V(D)) = 0 \qquad \forall i > 0 .$$

THEOREM 2.3 (Lefschetz fixed point formula. Cf. [**GH**]). *Let θ be an automorphism of smooth compact complex manifold M which fixes only finite points. Assume that θ is non-degenerate at each fixed point p, i.e.,* $\det(J_p(\theta) - I) \neq 0$. *Then the number of fixed points of θ is given by next formula.*

$$\sum (-1)^{p+q} \operatorname{trace} \theta^* |_{H^{p,q}(M)} .$$

LEMMA 2.4. *$n_2 = 8$ or $(n_2, n_4) = (3, 2)$, and the other $n_i = 0$. In particular $I(X) = 2$ or 4.*

PROOF. Put $D:=K_X+H$. Since D is a torsion divisor and $-K_X+D$ is ample, the Vanishing Theorem and Riemann-Roch Theorem gives $0=\chi(D)=1-(1/12)\sum(i-1/i)n_i$, hence

$$\sum\left(i-\frac{1}{i}\right)n_i=12.$$

The assertion can be obtained by solving this equality. ∎

LEMMA 2.5. *Let Y be a smooth Fano 3-fold. Assume that Y has an automorphism θ of index 2 or 4 which fixes just $2n$ points. Then the parities of $\rho(Y)$ and $B_3/2$ are same when n is odd, and the parities of $\rho(Y)$ and $B_3/2$ are distinct when n is even.*

PROOF. The following are easily verified.

$$\operatorname{Pic} Y\cong H^2(Y,\boldsymbol{Z}),\qquad H^2(Y,\boldsymbol{C})\cong H^{1,1},$$

$$H^3(Y,\boldsymbol{C})\cong H^{1,2}\oplus H^{2,1}.$$

Hence $h^{p,q}$ data are as follows.

$h^{p,q}$ q				
3	0	0	0	1
2	0	$\frac{1}{2}B_3$	$\rho(Y)$	0
1	0	$\rho(Y)$	$\frac{1}{2}B_3$	0
0	1	0	0	0
	0	1	2	3 p

By Lefschetz fixed point formula,

$$2+2\operatorname{trace}\theta^*|_{\operatorname{Pic}Y\otimes C}-2\operatorname{trace}\theta^*|_{H^{1,2}}=2n.$$

Hence the parities of $\operatorname{trace}\theta^*|_{\operatorname{Pic}Y\otimes C}$ and $\operatorname{trace}\theta^*|_{H^{1,2}}$ are same when n is odd, and are distinct when n is even. Note that the action of θ on $H^{p,q}$ is described by

$$\theta^*=\begin{pmatrix}\pm1 & & & & & 0\\ & \ddots & & & & \\ & & \pm1 & & & \\ & & & \pm\sqrt{-1} & & \\ & & & & \ddots & \\ 0 & & & & & \pm\sqrt{-1}\end{pmatrix}.$$

Hence the parities of $\rho(Y)$ and $B_3/2$ are same when n is odd, and are distinct when n is even.

COROLLARY 2.6. *The singularity index $I(X)$ is 2. The parities of $\rho(Y)$ and $B_3/2$ are distinct.*

LEMMA 2.7.

$$2\mid(-K_X)^3\,, \quad \textit{hence} \quad 4\mid(-K_Y)^3\,.$$

PROOF. Recall that there is a Cartier divisor H which is $\boldsymbol{Q}$ linearly equal to $-K_X$. Set $D=H$ and by applying Theorem 2.1, we obtain the assertion. ■

The way of classification.

We mention here the way of classification roughly. Smooth Fano 3-folds have been classified (cf. [**Is**], [**MM**]). We will investigate whether there is an involution which fixes just 8 points for each Fano 3-fold. First we use Corollary 2.6 and Lemma 2.7. Next we consider by its structure whether there exists the involution. If we cannot make decision easily, we take a chain of smooth Fano 3-folds and involutions:

$$(Y,\theta)\xrightarrow{f_1}(Y_1,\theta_1)\xrightarrow{f_2}\cdots\xrightarrow{f_{s-1}}(Y_{s-1},\theta_{s-1})\xrightarrow{f_s}(Y_s,\theta_s)\,,$$

where $f_i: Y_i \to Y_{i+1}$ is a contraction of the θ_i-invariant extremal face and θ_i is the lift of θ_{i+1}. We take a special assumption that the dimension of each contracted extremal face is one or two, and if it is 2, f_i is the inverse of a blowup with center two disjoint curves. This chain can be made by investigating the final column of the table in [**MM**].

DEFINITION 2.8. We call above Y_i "*a former*" associated to Y.

The structures of formers are simpler than that of Y, so we investigate formers instead of Y.

REMARK 2.9. If for some i, the dimension of the fixed locus of θ_i is not less than 1, then so is that of θ.

3. Proof of the Theorem.

We will carry out the classification along the way we mentioned in the last section.

1. Case $\rho(Y)=1$.

In this case $\operatorname{Pic} Y=\boldsymbol{Z}H$, where H is an ample divisor. Hence $\theta^*H=H$, so $\operatorname{trace}\theta^*|_{H^{1,1}}=\operatorname{trace}\theta^*|_{H^{2,2}}=1$. Then $\operatorname{trace}\theta^*|_{H^{1,2}}=\operatorname{trace}\theta^*|_{H^{2,1}}=-2$.

A smooth Fano 3-fold with $\rho(Y)=1$, $4|(-K_Y)^3$, $2|(B_3/2)$ and $B_3/2\geqq 2$ is one of the following (cf. [**Is**]).

No.	Y
1	$(4)\subset \boldsymbol{P}^4$
2	V_4, i.e., $(2,2)\subset \boldsymbol{P}^5$
3	$(2,2,2)\subset \boldsymbol{P}^6$
4	V_2, i.e., $(4)\subset \boldsymbol{P}(1,1,1,1,2)$
5	$(2,4)\subset \boldsymbol{P}(1,1,1,1,1,2)$

No. 1, 2, 3.

Each of these is embedded by $|H|$. Therefore θ is a restriction of a projective transformation to Y, so θ can be described by

$$[x_0:\cdots:x_l:x_{l+1}:\cdots:x_n]\longmapsto[x_0:\cdots:x_l:-x_{l+1}:\cdots:-x_n]$$

where $X_0,\cdots,X_n$ are homogeneous coordinates. The fixed locus of this involution consists of

$$V_+(X_0,\cdots,X_l)\quad\text{and}\quad V_+(X_{l+1},\cdots,X_n)\,.$$

No. 1.

In this case θ fixes infinitely many points, so this case never occur.

No. 2.

In this case θ should be

$$[x_0:x_1:x_2:x_3:x_4:x_5]\longmapsto[x_0:x_1:x_2:-x_3:-x_4:-x_5]\,.$$

Let $Y\subset\boldsymbol{P}^5$ be the complete intersection defined by 2 quadrics

$$Q_i(X_0,X_1,X_2)+Q_i'(X_3,X_4,X_5)\qquad(i=1,2)\,.$$

θ fixes just 8 points of Y. Hence Y/θ is a $\boldsymbol{Q}$-Fano 3-fold of Fano index is 1 or 1/2. Let S be $Y\cap Q_3$, where Q_3 is a third quadric of $\boldsymbol{P}^5$ defined by

$$Q_3(X_0,X_1,X_2)+Q_3'(X_3,X_4,X_5)\,.$$

Thus S is a member of $|-K_Y|$ and we can take S such that θ acts S without fixed points. So the quotient Y/θ is a $\boldsymbol{Q}$-Fano 3-fold of Fano index 1. To check the existence of such $S\in|-K_Y|$ is easy like this, so we omit the argument in what follows.

No. 3.

θ should be

$$[x_0:x_1:x_2:x_3:x_4:x_5:x_6]\longmapsto[x_0:x_1:x_2:x_3:-x_4:-x_5:-x_6]\,.$$

Let $Y \subset \boldsymbol{P}^6$ be complete intersection defined by the 3 quadrics

$$Q_i(X_0, X_1, X_2, X_3) + Q_i'(X_4, X_5, X_6) \qquad (i=1, 2, 3).$$

Generally, θ fixes just 8 points.

No. 4.

Let X_0, X_1, X_2, X_3, X_4 be homogeneous coordinates with $\deg X_i = 1$ $(0 \leqq i \leqq 3)$, $\deg X_4 = 2$. We define the involution

$$\theta : [X_0 : X_1 : X_2 : X_3 : X_4] \longmapsto [X_0 : X_1 : -X_2 : -X_3 : -X_4].$$

Then the fixed locus of this involution consists of

$$V_+(X_0, X_1, X_4), \qquad V_+(X_2, X_3, X_4), \quad [0:0:0:0:1].$$

Let Y be the hypersurface of $\boldsymbol{P}(1, 1, 1, 1, 2)$ defined by

$$X_0^4 + X_1^4 + X_2^4 + X_3^4 + X_4^2.$$

Then the fixed locus of θ on Y consists of 8 points.

No. 5.

Let $X_0, X_1, X_2, X_3, X_4, X_5$ be homogeneous with $\deg X_i = 1$ $(0 \leqq i \leqq 4)$, $\deg X_5 = 2$. We defined the involution

$$\theta : [x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \longmapsto [x_0 : x_1 : x_2 : -x_3 : -x_4 : -x_5].$$

Hence the fixed locus of this involution consists of

$$V_+(X_0, X_1, X_2, X_5), \qquad V_+(X_3, X_4, X_5), \quad [0:0:0:0:0:1].$$

Let Y be a weighted complete intersection $(2, 4) \subset \boldsymbol{P}(1, 1, 1, 1, 1, 2)$, defined by next two equations:

$$f_1(X_0, X_1, X_2) + f_1'(X_3, X_4) + X_5$$

$$f_2(X_0, X_1, X_2) + f_2'(X_3, X_4, X_5).$$

Generally, the fixed locus of θ on Y consists of 8 points.

2. Case $\rho(Y)=2$.

In this case trace $\theta^*|_{H^{1,1}} = $ trace $\theta^*|_{H^{2,2}} = 0$ or 2. Thus by Lefschetz fixed point formula trace $\theta^*|_{H^{1,2}} = $ trace $\theta^*|_{H^{2,1}} = -3$ or -1. A smooth Fano 3-fold with $\rho(Y)=2$, $4 \mid (-K_Y)^3$ and $2 \nmid (B_3/2)$, $B_3/2 > 0$ is one of the following (cf. [**MM**]).

No.	Y	one of the formers
1	$(2, 2) \subset \boldsymbol{P}^2 \times \boldsymbol{P}^2$	none
2	a double cover of $(1, 1) \subset \boldsymbol{P}^2 \times \boldsymbol{P}^2$ whose branch locus is a member of $\|-K\|$	none
3	the blowup of V_2 with center an elliptic curve which is V_2 an intersection of two member of $\|-(1/2)K\|$	V_2
4	the blowup of $V_4 = (2, 2) \subset \boldsymbol{P}^5$ with center an elliptic curve which is an intersection of two hyperplane sections	V_4
5	$(1, 1)^3 \subset \boldsymbol{P}^3 \times \boldsymbol{P}^3$	*
6	the blowup of $V_5 = Gr(1, 4) \cdot L_1 \cdot L_2 \cdot L_3 \subset \boldsymbol{P}^6$ with center an elliptic curve which is an intersection of two hyperplane sections	V_5
7	*	$\boldsymbol{P}^3$

Where Q is a quadric in $\boldsymbol{P}^4$, V_d is a Del Pezzo 3-fold of degree$=d$ and (*) means abbreviation because it is not necessary for the proof. The column "*the formers*" is the list of smooth Fano 3-folds which are obtained by contraction the each extremal ray.

No. 1.

If an involution of Y fixes only finite points, then they are less than 8 points.

No. 2.

There is an example. Let Z denote the manifold $(1, 1) \subset \boldsymbol{P}^2 \times \boldsymbol{P}^2$, π the morphism of Y to Z and $B \in |-K_Z|$ the branch locus. Let λ be the covering action. We define an involution τ of Z as

$$[x_0 : x_1 : x_2] \times [y_0 : y_1 : y_2] \longmapsto [y_0 : y_1 : y_2] \times [x_0 : x_1 : x_2] .$$

The fixed locus is just the diagonal set Δ. We define the involution μ of Y as extention of τ to Y:

$$\begin{tikzcd} Y \cong Z \times_Z Y \arrow[r, "\mu"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\pi"] \\ Z \arrow[r, "\tau"] & Z . \end{tikzcd}$$

And define θ to be the composition of λ and μ. There is natural one to one correspondence between the fixed locus of θ and $\Delta \cap B$. Thus the fixed locus of θ consists of just 8 points.

No. 3, 4.

There is an example. The curve C which does not through the fixed points and satisfy $\theta(C)=C$ can be taken as blowing up center. Indeed $C:=V_+(X_0, X_2)$ is enough for No. 3.

No. 5.

There is an example. Indeed the diagonal involution θ fixes just 8 points.

No. 6.

The former of Y is only V_5. Hence θ fixes the extremal ray and is the lift of an involution τ of V_5. If τ fixes finite points, they are 4 points by Lefschetz fixed point formula (use $\rho(V_5)=1$, $B_3(V_5)=0$). Since τ is a restriction of projective transform of $\boldsymbol{P}^6$, it fixes at least 5 points. This is a contradiction.

No. 7.

In this case θ is the lift of an involution of $\boldsymbol{P}^3$, so it fixes infinite points.

3. Case $\rho(Y)=3$.

The smooth Fano 3-fold with $\rho(Y)=3$, $4|(-K_Y)^3$ and $2|(B_3/2)$ is one of the following.

No.	$\frac{1}{2}B_3$	Y	one of the formers
1	*	$\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$	none
2	*	a double cover of $\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$ whose branch locus is a divisor of tridegree (2, 2, 2)	none
3	0	the blowup of the cone over a smooth quadric surface in $\boldsymbol{P}^3$ with center the vertex	none
4	*	a smooth divisor on $\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^2$ of tridegree (1, 1, 1)	none
5	*	*	$\boldsymbol{P}^1\times\boldsymbol{P}^2$
6	*	*	$\boldsymbol{P}^3$

No. 1.

There is an example. We define θ as

$$[X_0:X_1]\times[Y_0:Y_1]\times[Z_0:Z_1]\longmapsto[X_0:-X_1]\times[Y_0:-Y_1]\times[Z_0:-Z_1],$$

then θ fixes just 8 points.

No. 2.

There is an example. We define an involution τ on $\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$ as

$$[X_0:X_1]\times[Y_0:Y_1]\times[Z_0:Z_1]\longmapsto[X_0:-X_1]\times[Y_0:-Y_1]\times[Z_0:-Z_1]$$

and θ as the lift of τ. The fixed locus of θ fixes just 8 points.
No. 3.

The $h^{p,q}$ data are as follows.

0	0	0	1
0	0	3	0
0	3	0	0
1	0	0	0

Thus θ fixes $\operatorname{Pic} Y$, so it is the lift of an involution τ on the cone over a smooth quadric surface in $\boldsymbol{P}^3$. But τ fixes infinitely many points, so this case cannot occur.
No. 4.

θ fixes $\operatorname{Pic} Y$ by the same reason of the No. 3. Thus θ must be

$$[X_0:X_1]\times[Y_0:Y_1]\times[Z_0:Z_1:Z_2]\longmapsto[X_0:-X_1]\times[Y_0:-Y_1]\times[Z_0:Z_1:-Z_2].$$

But by considering the form of the defining polynomial, it is easy to check that this can not fix just 8 points.
No. 5, 6.

Note that any involution of $\boldsymbol{P}^1\times\boldsymbol{P}^2$ or $\boldsymbol{P}^3$ fixes infinitely many points.

4. Case $\rho(Y)=4$.

The smooth Fano 3-fold with $\rho(Y)=4$, $4|(-K_Y)^3$ and $2\nmid(B_3/2)$ is one of the following.

No.	$\frac{1}{2}B_3$	Y
1	*	a smooth divisor on $\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$ of tridegree (1, 1, 1, 1)
2	1	the blowup of the cone over a smooth quadric surface S in $\boldsymbol{P}^3$ with center a disjoint union of the vertex and an elliptic curve on S

No. 1.

There is an example. We define an involution θ as type $(-1)\times(-1)\times(-1)\times(-1)$ and set $Y=V(\sum_{\substack{i+j+k+l=\\ 0 \text{ or } 2 \text{ or } 4}} a_I X_i Y_j Z_k W_l)$, $a_I\neq 0$. Then θ fixes just 8 points.

No. 2.

Let D_1 be a smooth quadric in $\boldsymbol{P}^3$ and $Y_2 \subset \boldsymbol{P}^4$ the cone over D_1. Let Y_1 be the blowup of Y_2 with center the vertex and D_2 the exceptional divisor. Let D_3 be the strict transform of the cone over an elliptic curve on D_1, Y the blowup of Y_1 with center C and D_4 the exceptional divisor. We denote R_i ($i=1, 2, 3, 4$) the extremal ray associated with D_i. The $h^{p,q}$ data are as follows.

0	0	0	1
0	1	4	0
0	4	1	0
1	0	0	0

Case trace $\theta^*|_{H^{1,2}}=1$.

In this case θ is the lift of an involution θ_1 of Y_1 since θ fixes $\operatorname{Pic} Y$. The $h^{p,q}$ data of Y_1 are as follows.

0	0	0	1
0	0	3	0
0	3	0	0
1	0	0	0

Hence θ_1 fixes $\operatorname{Pic} Y_1$ and θ_1 is the lift of an involution θ_2 of Y_2. The dimension of the fixed locus of θ_2 is not less than 1, so this case never occur.

Case trace $\theta^*|_{H^{1,2}}=-1$.

In this case trace $\theta^*|_{\operatorname{Pic} Y \otimes C}=2$. θ desides a permutation of the extremal rays, but it fixes the each type. The type of R_1 and R_2, R_3 and R_4 are the same. It is easy to show that the case trace $\theta^*|_{\operatorname{Pic} Y \otimes C}=2$ cannot occur by considering the configulation of D_i's.

5. Case $\rho(Y) \geqq 5$.

The smooth Fano 3-fold with $\rho(Y) \geqq 5$, $4|(-K_Y)^3$ is one of the following.

No.	$\rho(Y)$	Y	one of the formers
1	5	*	$\boldsymbol{Q}$
2	5	*	$\boldsymbol{P}^3$
3	5	$\boldsymbol{P}^1 \times S_6$	*
4	7	$\boldsymbol{P}^1 \times S_4$	*
5	9	$\boldsymbol{P}^1 \times S_2$	*

No. 1, 2.

This case cannot occur.

No. 3, 4, 5.

Recall that S_d ($d=2, 4, 6$) can be obtained by blowing up of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. It is easy to check that Y has the involution fixing just 8 points as lift of the involution of $\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$ of type $(-1)\times(-1)\times(-1)$. ■

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References

[CM] A. Conte and J.P. Murre, Algebraic varieties of dimension three whose hyperplane sections are Enriques surfaces, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 12 (1985), 43-80.

[GH] P. Griffiths and J. Harris, Principle of Algebraic Geometry, Wiley-interscience publication.

[Is] V.A. Iskovskih, Fano 3-folds I, II, Math. USSR-Izv., 11 (1977), 485-527, ibid., 12 (1978), 469-506.

[Ka] Y. Kawamata, On Plurigenera of minimal algebraic 3-folds with $K\equiv 0$, Math. Ann., 275 (1986), 539-546.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem, Adv. Stud. Pure Math., 10 (1987), 283-360.

[MM] S. Mori and S. Mukai, Classification of Fano 3-fold with $B_2 \geqq 2$, Manuscripta Math., 36 (1981), 147-162.

[Re] M. Reid, Young person's guide to canonical singularities, In Algebraic geometry Bowdoin 1985, part I, Proc. Sympos. Pure Math., 46 (1987), 345-414.

[Sa] T. Sano, Classification of non-Gorenstein $\boldsymbol{Q}$-Fano d-folds of Fano index greater than $d-2$, preprint, (1991).

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