J. Math. Soc. Japan
Vol. 47, No. 2, 1995

Standard generalized vectors for algebras of unbounded operators

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(Received Apr. 22, 1993)
(Revised Aug. 11, 1993)

§ 1. Introduction.

In [7, 8] we defined the notion of generalized vectors for O^*-algebras which is a generalization of cyclic vectors to study the structure of O^*-algebras. In particular, we defined and studied the notion of (full) standard generalized vectors which makes it possible to develop the Tomita-Takesaki theory in O^*-algebras. In this paper we shall continue such a study for O^*-algebras (called generalized von Neumann algebra) which are an unbounded generalization of von Neumann algebras. Let λ be a standard generalized vector for a generalized von Neumann algebra $\mathcal{M}$ on $\mathcal{D}$. Then it has been shown in [7] that a one-parameter group $\{\sigma_t^\lambda\}_{t\in\boldsymbol{R}}$ of $*$-automorphisms of $\mathcal{M}$ is defined and λ satisfies the KMS-condition with respect to $\{\sigma_t^\lambda\}$. We shall show a Radon-Nikodym type property which establishes a link between the modular automorphism groups $\{\sigma_t^\lambda\}$ and $\{\sigma_t^\mu\}$ of $\mathcal{M}$ for two full standard generalized vectors λ and μ: There uniquely exists a strongly continuous map $t\in\boldsymbol{R}\rightarrow U_t\in\mathcal{M}_u\equiv\{U\in\mathcal{M}\,;\,\bar{U}$ is unitary$\}$ such that $U_{t+s}=U_t\sigma_s^\lambda(U_s)$ and $\sigma_t^\mu(X)=U_t\sigma_t^\lambda(X)U_t^*$ for all $s, t\in\boldsymbol{R}$ and $X\in\mathcal{M}$. The map $t\in\boldsymbol{R}\rightarrow U_t\in\mathcal{M}$ is called the cocycle associated with μ with respect to λ and denoted by $[D\mu : D\lambda]$. Further, we shall show that $\{[D\mu : D\lambda]_t\}_{t\in\boldsymbol{R}}$ is a one-parameter group if and only if the domain $D(\mu)$ of μ is $\{\sigma_t^\lambda\}$-invariant and $\|\mu(\sigma_t^\lambda(X))\|=\|\mu(X)\|$ for all $X\in D(\mu)$ and $t\in\boldsymbol{R}$ if and only if $\{[D\mu : D\lambda]_t\}\subset\mathcal{M}_b^{\sigma^\lambda}$ $\equiv\{A\in\mathcal{M}\,;\,\bar{A}$ is bounded and $\sigma_t^\lambda(A)=A$ for all $t\in\boldsymbol{R}\}$. Then, we say that μ commutes with λ. These results are generalization of the Connes cocycle theorem [1, 17] for von Neumann algebras. We shall extend the Pedersen-Takesaki Radon-Nikodym theorem [12, 17] for von Neumann algebras to generalized von Neumann algebras. Let λ be a full standard generalized vector for $\mathcal{M}$ and $\mathcal{M}_\eta^{\sigma^\lambda}$ the set of all non-singular positive self-adjoint operators A in $\mathcal{H}$ satisfying $\{E_A(t)\,;\,-\infty<t<\infty\}''\lceil\mathcal{D}\subset\mathcal{M}_b^{\sigma^\lambda}$, where $\{E_A(t)\}$ is the spectral resolutions of A. For every $A\in\mathcal{M}_\eta^{\sigma^\lambda}$ we can define a standard generalized vector λ_A for $\mathcal{M}$ satisfying $\sigma_t^{\lambda_A}(X)=A^{2it}\sigma_t^\lambda(X)A^{-2it}$ and $[D(\lambda_A)_\sigma : D\lambda]_t=A^{2it}\lceil\mathcal{D}$ for all $X\in\mathcal{M}$ and $t\in\boldsymbol{R}$, and so $(\lambda_A)_\sigma$ commutes with λ, where $(\lambda_A)_\sigma$ is the full extension of λ_A. Conversely suppose a full standard generalized vector μ commutes with λ. By the Stone theorem there exists a non-singular positive self-adjoint operator $A_{\lambda,\mu}$ in $\mathcal{H}$ such that $A_{\lambda,\mu}^{it} \lceil \mathcal{D}=[D\mu : D\lambda]_t$ for all $t\in \boldsymbol{R}$. We remark that $A_{\lambda,\mu}^{it} \lceil \mathcal{D}\in \mathcal{M}_b^{\sigma^\lambda}$ for all $t\in \boldsymbol{R}$, but $A_{\lambda,\mu}$ does not necessarily belong to $\mathcal{M}_\eta^{\sigma^\lambda}$. We shall show that if $A_{\lambda,\mu}\in \mathcal{M}_\eta^{\sigma^\lambda}$ then μ is identical with the full extension $(\lambda_{A_{\lambda,\mu}})_\sigma$ of $\lambda_{A_{\lambda,\mu}}$. In particular, if $\mathcal{M}$ is an EW*-algebra then μ commutes with λ if and only if $\mu=(\lambda_A)_\sigma$ for some non-singular positive self-adjoint operator A affiliated with the von Neumann algebra $\overline{\mathcal{M}_b^{\sigma^\lambda}}$. Furthermore, we shall show that μ satisfies the KMS-condition with respect to $\{\sigma_t^\lambda\}$ if and only if $\sigma_t^\mu=\sigma_t^\lambda$ for all $t\in \boldsymbol{R}$ if and only if $\mu=(\lambda_A)_\sigma$ for some non-singular positive self-adjoint operator A affiliated with the center of the von Neumann algebra $(\mathcal{M}_w')'$ where $\mathcal{M}_w'$ is the weak commutant of $\mathcal{M}$.

§ 2. Standard generalized vectors.

In this section we state some of definitions and the basic properties of generalized von Neumann algebras and the standard generalized vectors.

Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$ with inner product $(\ |\ )$ and $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all linear operators X from $\mathcal{D}$ to $\mathcal{H}$ satisfying $\mathcal{D}(X^*) \supset \mathcal{D}$. $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a *-invariant vector space with the usual operations and the involution $X^\dagger \equiv X^* \lceil \mathcal{D}$. We introduce the locally convex topology t_s^* on $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ defined by the systems $\{p_\xi^*(\cdot)\,;\,\xi\in\mathcal{D}\}$ of seminorms: $p_\xi^*(X)=\|X\xi\|+\|X^\dagger\xi\|$, $X\in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, and it is said to be the *strong* topology*. $(\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}), t_s^*)$ is a complete locally convex space. We put

$$\mathcal{L}^\dagger(\mathcal{D}) = \{X\in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})\,;\, X\mathcal{D}\subset\mathcal{D} \text{ and } X^*\mathcal{D}\subset\mathcal{D}\}.$$

Then $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra with the usual operations and the involution $X\to X^\dagger \equiv X^* \lceil \mathcal{D}$. A *-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$ is called an *O^*-algebra* on $\mathcal{D}$. Throughout this paper we assume that an O^*-algebra has always an identity operator. Let $\mathcal{M}$ be an O^*-algebra on $\mathcal{D}$. A locally convex topology on $\mathcal{D}$ defined by a family $\{\|\ \|_X\,;\,X\in\mathcal{M}\}$ of the seminorms: $\|\xi\|_X=\|X\xi\|$ $(\xi\in\mathcal{D})$ is called the *induced topology* on $\mathcal{D}$, and denoted by $t_\mathcal{M}$. If the locally convex space $(\mathcal{D}, t_\mathcal{M})$ is complete, then $\mathcal{M}$ is said to be *closed*. We put

$$\tilde{\mathcal{D}}(\mathcal{M}) = \bigcap_{X\in\mathcal{M}} \mathcal{D}(\bar{X}) \quad\text{and}\quad \tilde{X} = \bar{X} \lceil \tilde{\mathcal{D}}(\mathcal{M}) \quad (X\in\mathcal{M}).$$

Then $\tilde{\mathcal{D}}(\mathcal{M})$ is identical with the completion of $(\mathcal{D}, t_\mathcal{M})$ and $\tilde{\mathcal{M}}\equiv\{\tilde{X}\,;\,X\in\mathcal{M}\}$ is a closed O^*-algebra on $\tilde{\mathcal{D}}(\mathcal{M})$ which is the smallest closed extension of $\mathcal{M}$ and it is called the *closure* of $\mathcal{M}$. Hense $\mathcal{M}$ is closed if and only if $\mathcal{D}=\tilde{\mathcal{D}}(\mathcal{M})$. If $\mathcal{D}^*(\mathcal{M})\equiv\bigcap_{X\in\mathcal{M}}\mathcal{D}(X^*)=\tilde{\mathcal{D}}(\mathcal{M})$, then $\mathcal{M}$ is said to be *essentially self-adjoint*, and if $\mathcal{D}^*(\mathcal{M})=\mathcal{D}$, then $\mathcal{M}$ is said to be *self-adjoint*. We define the *weak commutant*

$\mathcal{M}'_w$ of †-invariant subset $\mathcal{M}$ of $\mathcal{L}^\dagger(\mathcal{D})$ as follows:

$$\mathcal{M}'_w = \{C \in \mathcal{B}(\mathcal{H})\,;\, (CX\xi|\eta)=(C\xi|X^\dagger\eta) \text{ for all } \xi,\ \eta \in \mathcal{D} \text{ and } X \in \mathcal{M}\},$$

where $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. Then $\mathcal{M}'_w$ is a $*$-invariant weakly closed subspace of $\mathcal{B}(\mathcal{H})$, but it is not necessarily an algebra. If $\mathcal{M}$ is self-adjoint, then $\mathcal{M}'_w\mathcal{D} \subset \bar{\mathcal{D}}$, and further $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$ if and only if $\mathcal{M}'_w$ is a von Neumann algebra and $\bar{X}$ is affiliated with $(\mathcal{M}'_w)'$ for all $X \in \mathcal{M}$. For the general theory of O^*-algebra we refer to [**2**, **11**, **13**, **15**].

We introduce two notions which are unbounded generalizations of von Neumann algebras. A closed O^*-algebra $\mathcal{M}$ on $\mathcal{D}$ is said to be a *generalized von Neumann algebra* if $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$ and $\mathcal{M} = \mathcal{M}''_{wc} \equiv \{X \in \mathcal{L}^\dagger(\mathcal{D})\,;\, XC\xi = CX\xi,\ \forall C \in \mathcal{M}'_w,\ \forall \xi \in \mathcal{D}\}$. Suppose $\mathcal{M}$ is a closed O^*-algebra on $\mathcal{D}$ such that $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{M}$ is a generalized von Neumann algebra if and only if $\mathcal{M}$ is t_s^*-closed if and only if $\mathcal{M} = \{X \in \mathcal{L}^\dagger(\mathcal{D})\,;\, \bar{X} \text{ is affiliated with } (\mathcal{M}'_w)'\}$ [**5**]. A generalized von Neumann algebra $\mathcal{M}$ on $\mathcal{D}$ is said to be an *EW*-algebra* if $(\mathcal{M}'_w)'\mathcal{D} \subset \mathcal{D}$ [**4**].

We next introduce the notion of generalized vectors which is a generalization of cyclic vectors for O^*-algebras [**7**]. Let $\mathcal{M}$ be an O^*-algebra on $\mathcal{D}$ such that $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$. A map λ of $\mathcal{M}$ into $\mathcal{D}$ is said to be a *generalized vector* for $\mathcal{M}$ if the domain $D(\lambda)$ of λ is a left ideal of $\mathcal{M}$, λ is a linear map of $D(\lambda)$ into $\mathcal{D}$ and $\lambda(XA) = X\lambda(A)$ for all $X \in \mathcal{M}$ and $A \in D(\lambda)$. Suppose that a generalized vector λ for $\mathcal{M}$ satisfies the condition:

(i) $\lambda((D(\lambda) \cap D(\lambda)^\dagger)^2)$ is total in $\mathcal{H}$.

Then we define the commutant λ^c of λ which is a generalized vector for the von Neumann algebra $\mathcal{M}'_w$ as follows:

$$D(\lambda^c) = \{K \in \mathcal{M}'_w\,;\, \exists \xi_K \in \mathcal{D} \text{ s.t. } K\lambda(X) = X\xi_K \text{ for all } X \in D(\lambda)\},$$

$$\lambda^c(K) = \xi_K, \qquad K \in D(\lambda^c).$$

Definition 2.1. A generalized vector λ for $\mathcal{M}$ is said to be cyclic and separating if the above condition (i) and the following condition (ii) hold:

(ii) $\lambda^c((D(\lambda^c) \cap D(\lambda^c)^*)^2)$ is total in $\mathcal{H}$.

Proposition 2.2. *Suppose λ is a cyclic and separating generalized vector for $\mathcal{M}$. Put*

$$D(\lambda_\sigma) = \{X \in \mathcal{M}\,;\, \exists \xi_X \in \mathcal{D} \text{ s.t. } X\lambda^c(K) = K\xi_X \text{ for all } K \in D(\lambda^c)\},$$

$$\lambda_\sigma(X) = \xi_X, \qquad X \in D(\lambda_\sigma).$$

Then λ_σ is a cyclic and separating vector for $\mathcal{M}$ satisfying

(1) *$\lambda \subset \lambda_\sigma$, that is $D(\lambda) \subset D(\lambda_\sigma)$ and $\lambda(X) = \lambda_\sigma(X)$ for all $X \in D(\lambda)$,*

(2) *λ is equivalent to λ_σ, that is, $\lambda^c = \lambda^c_\sigma$.*

PROOF. This is easily shown and so we omit the proof.

DEFINITION 2.3. A cyclic and separating generalized vector λ for $\mathcal{M}$ is said to be full if $\lambda=\lambda_{\sigma}$.

Suppose λ is a cyclic and separating generalized vector for $\mathcal{M}$ and put

$$D(\lambda^{cc})=\{A\in(\mathcal{M}_w')'\,;\,\exists\xi_A\in\mathcal{H}\text{ s.t. }A\lambda^c(K)=K\xi_A\text{ for all }K\in D(\lambda^c)\},$$

$$\lambda^{cc}(A)=\xi_A,\qquad A\in D(\lambda^{cc}).$$

Then λ^{cc} is a cyclic and separating generalized vector for the von Neumann algebra $(\mathcal{M}_w')'$. So, the maps $\lambda(X)\to\lambda(X^{\dagger})$, $X\in D(\lambda)$ and $\lambda^{cc}(A)\to\lambda^{cc}(A^*)$, $A\in D(\lambda^{cc})$ are closable in $\mathcal{H}$ and their closures are denoted by S_{λ} and $S_{\lambda^{cc}}$, respectively. Let $S_{\lambda}=J_{\lambda}\Delta_{\lambda}^{1/2}$ and $S_{\lambda^{cc}}=J_{\lambda^{cc}}\Delta_{\lambda^{cc}}^{1/2}$ be the polar decompositions of S_{λ} and $S_{\lambda^{cc}}$, respectively. Then we see that $S_{\lambda}\subset S_{\lambda^{cc}}$, and $J_{\lambda^{cc}}(\mathcal{M}_w')'J_{\lambda^{cc}}=\mathcal{M}_w'$ and $\Delta_{\lambda^{cc}}^{it}(\mathcal{M}_w')'\cdot\Delta_{\lambda^{cc}}^{-it}=(\mathcal{M}_w')'$ for all $t\in\boldsymbol{R}$ by the Tomita fundamental theorem. But, we do not know how the unitary group $\{\Delta_{\lambda^{cc}}^{it}\}_{t\in\boldsymbol{R}}$ acts on the O^*-algebra $\mathcal{M}$ and so we define a system which has the best condition:

DEFINITION 2.4. A generalized vector λ for $\mathcal{M}$ is said to be standard if the following conditions hold:

(i) λ is cyclic and separating.

(ii) $\Delta_{\lambda^{cc}}^{it}\mathcal{D}\subset\mathcal{D}$ and $\Delta_{\lambda^{cc}}^{it}\mathcal{M}\Delta_{\lambda^{cc}}^{-it}=\mathcal{M}$, $t\in\boldsymbol{R}$.

(iii) $\Delta_{\lambda^{cc}}^{it}(D(\lambda)\cap D(\lambda)^{\dagger})\Delta_{\lambda^{cc}}^{-it}=D(\lambda)\cap D(\lambda)^{\dagger}$, $t\in\boldsymbol{R}$.

A standard generalized vector λ is said to be full if $\lambda=\lambda_{\sigma}$.

It follows from Proposition 2.2 that if λ is a standard generalized vector for $\mathcal{M}$, then λ_{σ} is a full standard generalized vector for $\mathcal{M}$ such that $J_{\lambda}=J_{\lambda_{\sigma}}=J_{\lambda^{cc}}$ and $\Delta_{\lambda}=\Delta_{\lambda_{\sigma}}=\Delta_{\lambda^{cc}}$.

THEOREM 2.5. *Suppose λ is a standard generalized vector for $\mathcal{M}$. Then the following statements hold:*

(1) *$S_{\lambda}=S_{\lambda^{cc}}$, and so $J_{\lambda}=J_{\lambda^{cc}}$ and $\Delta_{\lambda}=\Delta_{\lambda^{cc}}$.*

(2) *$\{\sigma_t^{\lambda}\}_{t\in\boldsymbol{R}}$ is a one-parameter group of $*$-automorphisms of $\mathcal{M}$, where $\sigma_t^{\lambda}(X)=\Delta_{\lambda}^{it}X\Delta_{\lambda}^{-it}$, $X\in\mathcal{M}$, $t\in\boldsymbol{R}$.*

(3) *λ satisfies the KMS-condition with respect to $\{\sigma_t^{\lambda}\}$, that is, for each $X, Y\in D(\lambda)\cap D(\lambda)^{\dagger}$ there exists an element $f_{X,Y}$ of $A(0, 1)$ such that*

$$f_{X,Y}(t)=(\lambda(\sigma_t^{\lambda}(X))\,|\,\lambda(Y))\quad\text{and}\quad f_{X,Y}(t+i)=(\lambda(Y^{\dagger})\,|\,\lambda(\sigma_t^{\lambda}(X^{\dagger}))$$

for all $t\in\boldsymbol{R}$, where $A(0, 1)$ is the set of all complex-valued functions, bounded and continuous on $0\leqq\operatorname{Im}z\leqq1$ and analytic in the interior.

(4) *Suppose λ is full. Then $\sigma_t^{\lambda}(D(\lambda))\subset D(\lambda)$ and $\lambda(\sigma_t^{\lambda}(X))=\Delta_{\lambda}^{it}\lambda(X)$ for all $X\in D(\lambda)$ and $t\in\boldsymbol{R}$.*

PROOF. The statements (1), (2), and (3) follow from ([7] Theorem 5.5, 5.6). We show the statement (4). We put

$$D(\lambda')=\{K\in\mathcal{M}'_w;\ \exists\xi_K\in\bigcap_{X\in D(\lambda)}\mathcal{D}(\bar{X})\ \text{s.t.}\ K\lambda(X)=\bar{X}\xi_K,\ \forall X\in D(\lambda)\},$$

$$\lambda'(K)=\xi_K,\qquad K\in D(\lambda').$$

By ([7] Proposition 4.1) λ' is a generalized vector for $\mathcal{M}'_w$ satisfying $\lambda^c\subset\lambda'$ and we have by ([7] Lemma 5.1, Theorem 5.6)

$$\begin{aligned}\sigma_t^\lambda(X)\lambda^c(K)&=\Delta_\lambda^{it}X\Delta_\lambda^{-it}\lambda'(K)=\Delta_\lambda^{it}X\lambda'(\sigma_{-t}^\lambda(K))\\&=\Delta_\lambda^{it}\sigma_{-t}^\lambda(K)\lambda(X)\\&=K\Delta_\lambda^{it}\lambda(X)\end{aligned}$$

for all $X\in D(\lambda)$ and $K\in D(\lambda^c)$. Since λ is full, it follows that $\sigma_t^\lambda(X)\in D(\lambda)$ and $\lambda(\sigma_t^\lambda(X))=\Delta_\lambda^{it}\lambda(X)$ for all $X\in D(\lambda)$ and $t\in\boldsymbol{R}$. This completes the proof.

§ 3. Generalized Connes cocycle theorem.

In this section we generalize the Connes cocycle theorem for von Neumann algebras to generalized von Neumann algebras. Let $\mathcal{M}$ be a generalized von Neumann algebra on $\mathcal{D}$ in $\mathcal{H}$. Let $\mathcal{K}_4$ be a four-dimensional Hilbert space with an orthonormal basis $\{\eta_{ij}\}_{i,j=1,2}$ and $\mathcal{F}_2$ a 2×2-matrix algebra generated by the matrices E_{ij} which are defined by $E_{ij}\eta_{kl}=\delta_{jk}\eta_{kl}$. Identifying $\zeta=\zeta_1\otimes\eta_{11}+\zeta_2\otimes\eta_{21}+\zeta_3\otimes\eta_{12}+\zeta_4\otimes\eta_{22}\in\mathcal{H}\otimes\mathcal{K}_4$ with $\zeta=(\zeta_1,\zeta_2,\zeta_3,\zeta_4)\in\mathcal{H}^4\equiv\mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H}$, $\mathcal{M}\otimes\mathcal{F}_2$ is regarded as the matrix algebra on $\mathcal{D}^4\equiv\mathcal{D}\oplus\mathcal{D}\oplus\mathcal{D}\oplus\mathcal{D}$:

$$\left\{\begin{pmatrix}X_{11}&X_{12}&0&0\\X_{21}&X_{22}&0&0\\0&0&X_{11}&X_{12}\\0&0&X_{21}&X_{22}\end{pmatrix};\ X_{ij}\in\mathcal{M}\right\}.$$

Suppose λ and μ are cyclic and separating generalized vectors for $\mathcal{M}$. We put

$$D(\theta_{\lambda,\mu})=\left\{X=\begin{pmatrix}X_{11}&X_{12}&0&0\\X_{21}&X_{22}&0&0\\0&0&X_{11}&X_{12}\\0&0&X_{21}&X_{22}\end{pmatrix};\ \begin{matrix}X_{11},X_{21}\in D(\lambda)\\X_{12},X_{22}\in D(\mu)\end{matrix}\right\},$$

$$\theta_{\lambda,\mu}(X)=\begin{pmatrix}\lambda(X_{11})\\\lambda(X_{21})\\\mu(X_{12})\\\mu(X_{22})\end{pmatrix},\qquad X\in D(\theta_{\lambda,\mu}).$$

Then it is easily shown that $\theta_{\lambda,\mu}$ is a cyclic and separating generalized vector for $\mathcal{M}\otimes\mathcal{F}_2$ satisfying

$$D(\theta^c_{\lambda,\mu})=\left\{K=\begin{pmatrix}K_{11} & 0 & K_{12} & 0\\ 0 & K_{11} & 0 & K_{12}\\ K_{21} & 0 & K_{22} & 0\\ 0 & K_{21} & 0 & K_{22}\end{pmatrix};\ \begin{matrix}K_{11},K_{21}\in D(\lambda^c)\\ K_{12},K_{22}\in D(\mu^c)\end{matrix}\right\},$$

$$\theta^c_{\lambda,\mu}(K)=\begin{pmatrix}\lambda^c(K_{11})\\ \mu^c(K_{21})\\ \lambda^c(K_{12})\\ \mu^c(K_{22})\end{pmatrix},\qquad K\in D(\theta^c_{\lambda,\mu});\tag{3.1}$$

$$D(\theta^{cc}_{\lambda,\mu})=\left\{A=\begin{pmatrix}A_{11} & A_{12} & 0 & 0\\ A_{21} & A_{22} & 0 & 0\\ 0 & 0 & A_{11} & A_{12}\\ 0 & 0 & A_{21} & A_{22}\end{pmatrix};\ \begin{matrix}A_{11},A_{21}\in D(\lambda^{cc})\\ A_{12},A_{22}\in D(\mu^{cc})\end{matrix}\right\},$$

$$\theta^{cc}_{\lambda,\mu}(A)=\begin{pmatrix}\lambda^{cc}(A_{11})\\ \lambda^{cc}(A_{21})\\ \mu^{cc}(A_{12})\\ \mu^{cc}(A_{22})\end{pmatrix},\qquad A\in D(\theta^{cc}_{\lambda,\mu}).\tag{3.2}$$

PROPOSITION 3.1. *Let λ and μ be cyclic and separating generalized vectors for $\mathcal{M}$. The following statements are equivalent.*

(1) *λ and μ are (full) standard generalized vectors for $\mathcal{M}$.*

(2) *$\theta_{\lambda,\mu}$ is a (full) standard generalized vector for $\mathcal{M}\otimes\mathcal{F}_2$.*

PROOF. (1)$\Rightarrow$(2) By (3.1) and (3.2) we have

$$S_{\theta^{cc}_{\lambda,\mu}}=\begin{pmatrix}S_{\lambda cc} & 0 & 0 & 0\\ 0 & 0 & S_{\lambda cc\mu cc} & 0\\ 0 & S_{\mu cc\lambda cc} & 0 & 0\\ 0 & 0 & 0 & S_{\mu cc}\end{pmatrix},$$

and so

$$\Delta_{\theta^{cc}_{\lambda,\mu}}=\begin{pmatrix}\Delta_{\lambda cc} & 0 & 0 & 0\\ 0 & \Delta_{\mu cc\lambda cc} & 0 & 0\\ 0 & 0 & \Delta_{\lambda cc\mu cc} & 0\\ 0 & 0 & 0 & \Delta_{\mu cc}\end{pmatrix},\tag{3.3}$$

where $S_{\lambda cc\mu cc}$ is the closure of the conjugate linear operator $\mu^{cc}(A)\to\lambda^{cc}(A^*)$, $A\in D(\mu^{cc})\cap D(\lambda^{cc})^*$ and $S_{\lambda cc\mu cc}=J_{\lambda cc\mu cc}\Delta^{1/2}_{\lambda cc\mu cc}$ is the polar decomposition of $S_{\lambda cc\mu cc}$, and $S_{\mu cc\lambda cc}$, $J_{\mu cc\lambda cc}$ and $\Delta_{\mu cc\lambda cc}$ are operators defined similarly. Since

$$\Delta^{it}_{\theta^{cc}_{\lambda,\mu}}(\mathcal{M}'_w)'\otimes\mathcal{F}_2\Delta^{-it}_{\theta^{cc}_{\lambda,\mu}}=(\mathcal{M}'_w)'\otimes\mathcal{F}_2,\qquad t\in\boldsymbol{R},$$

it follows from (3.3) that

$$\sigma_t^{\lambda^{cc}}(A_{11}) = \Delta_{\lambda^{cc}\mu^{cc}}^{it} A_{11} \Delta_{\lambda^{cc}\mu^{cc}}^{-it}, \tag{3.4}$$

$$\Delta_{\lambda^{cc}}^{it} A_{12} \Delta_{\mu^{cc}\lambda^{cc}}^{-it} = \Delta_{\lambda^{cc}\mu^{cc}}^{it} A_{12} \Delta_{\mu^{cc}}^{-it}, \tag{3.5}$$

$$\Delta_{\mu^{cc}\lambda^{cc}}^{it} A_{21} \Delta_{\lambda^{cc}}^{-it} = \Delta_{\mu^{cc}}^{it} A_{21} \Delta_{\lambda^{cc}\mu^{cc}}^{-it}, \tag{3.6}$$

$$\sigma_t^{\mu^{cc}}(A_{22}) = \Delta_{\mu^{cc}\lambda^{cc}}^{it} A_{22} \Delta_{\mu^{cc}\lambda^{cc}}^{-it} \tag{3.7}$$

for all A_{11}, A_{12}, A_{21}, $A_{22} \in (\mathcal{M}_w')'$ and $t \in \boldsymbol{R}$. We now denote by $[D\mu^{cc} : D\lambda^{cc}]_t$ the Connes cocycle associated with the weight $\varphi_{\mu^{cc}}$ with respect to the weight $\varphi_{\lambda^{cc}}$, that is,

$$[D\mu^{cc} : D\lambda^{cc}]_t = \Delta_{\mu^{cc}}^{it} \Delta_{\lambda^{cc}\mu^{cc}}^{-it}, \qquad t \in \boldsymbol{R}. \tag{3.8}$$

By (3.6) we have

$$[D\mu^{cc} : D\lambda^{cc}]_t = \Delta_{\mu^{cc}\lambda^{cc}}^{it} \Delta_{\lambda^{cc}}^{-it} \in (\mathcal{M}_w')', \qquad t \in \boldsymbol{R} \tag{3.9}$$

and by (3.4) and (3.7)

$$\Delta_{\lambda^{cc}}^{it} \Delta_{\lambda^{cc}\mu^{cc}}^{-it}, \qquad \Delta_{\mu^{cc}}^{it} \Delta_{\mu^{cc}\lambda^{cc}}^{-it} \in \mathcal{M}_w', \qquad t \in \boldsymbol{R}. \tag{3.10}$$

Since $\Delta_{\lambda^{cc}}^{it}\mathcal{D} \subset \mathcal{D}$, $\Delta_{\mu^{cc}}^{it}\mathcal{D} \subset \mathcal{D}$ ($\forall t \in \boldsymbol{R}$) and $\mathcal{M}_w'\mathcal{D} \subset \mathcal{D}$, it follows from (3.10) that

$$\Delta_{\mu^{cc}\lambda^{cc}}^{it}\mathcal{D} = \Delta_{\mu^{cc}}^{it} \Delta_{\mu^{cc}}^{-it} \Delta_{\mu^{cc}\lambda^{cc}}^{it}\mathcal{D} \subset \Delta_{\mu^{cc}}^{it}\mathcal{M}_w'\mathcal{D} \subset \mathcal{D}, \tag{3.11}$$

and similarly

$$\Delta_{\lambda^{cc}\mu^{cc}}^{it}\mathcal{D} \subset \mathcal{D}, \tag{3.12}$$

which implies by (3.8), (3.9) and (3.3) that

$$[D\mu^{cc} : D\lambda^{cc}]_t \lceil \mathcal{D} \in \mathcal{M} \quad \text{and} \quad \Delta_{\theta_{\lambda\mu}^{it}}^{it}\mathcal{D}^4 \subset \mathcal{D}^4, \qquad t \in \boldsymbol{R}. \tag{3.13}$$

Furthermore, it follows from (3.4)$\sim$(3.7), (3.11) and (3.12) that

$$\sigma_t^{\lambda^{cc}}(X_{11}) = \Delta_{\lambda^{cc}\mu^{cc}}^{it} X_{11} \Delta_{\lambda^{cc}\mu^{cc}}^{-it},$$

$$\Delta_{\lambda^{cc}}^{it} X_{12} \Delta_{\mu^{cc}\lambda^{cc}}^{-it} = \Delta_{\lambda^{cc}\mu^{cc}}^{it} X_{12} \Delta_{\mu^{cc}}^{-it},$$

$$\Delta_{\mu^{cc}\lambda^{cc}}^{it} X_{21} \Delta_{\lambda^{cc}}^{-it} = \Delta_{\mu^{cc}}^{it} X_{21} \Delta_{\lambda^{cc}\mu^{cc}}^{-it},$$

$$\sigma_t^{\mu^{cc}}(X_{22}) = \Delta_{\mu^{cc}\lambda^{cc}}^{it} X_{22} \Delta_{\mu^{cc}\lambda^{cc}}^{-it}$$

for each X_{11}, X_{12}, X_{21}, $X_{22} \in \mathcal{M}$ and $t \in \boldsymbol{R}$, and they belong to $\mathcal{M}$ since $\mathcal{M}$ is a generalized von Neumann algebra. Hence we have

$$\sigma_t^{\mu^{cc}}(X_{11}) = [D\mu^{cc} : D\lambda^{cc}]_t \sigma_t^{\lambda^{cc}}(X_{11}) [D\mu^{cc} : D\lambda^{cc}]_t^*$$
$$X_{11} \in \mathcal{M}, \qquad t \in \boldsymbol{R}, \tag{3.14}$$

and so

$$\sigma_t^{\theta^{cc}_{\lambda,\mu}}(X)$$

$$= \begin{pmatrix} \sigma_t^{\lambda^{cc}}(X_{11}) & [D\mu^{cc} : D\lambda^{cc}]_t^* \sigma_t^{\mu^{cc}}(X_{12}) & 0 & 0 \\ [D\mu^{cc} : D\lambda^{cc}]_t \sigma_t^{\lambda^{cc}}(X_{21}) & \sigma_t^{\mu^{cc}}(X_{22}) & 0 & 0 \\ 0 & 0 & \sigma_t^{\lambda^{cc}}(X_{11}) & \sigma_t^{\lambda^{cc}}(X_{12})[D\mu^{cc} : D\lambda^{cc}]_t^* \\ 0 & 0 & \sigma_t^{\mu^{cc}}(X_{21})[D\mu^{cc} : D\lambda^{cc}]_t & \sigma_t^{\mu^{cc}}(X_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_t^{\lambda^{cc}}(X_{11}) & [D\mu^{cc} : D\lambda^{cc}]_t^* \sigma_t^{\mu^{cc}}(X_{12}) & 0 & 0 \\ [D\mu^{cc} : D\lambda^{cc}]_t \sigma_t^{\lambda^{cc}}(X_{21}) & \sigma_t^{\mu^{cc}}(X_{22}) & 0 & 0 \\ 0 & 0 & \sigma_t^{\lambda^{cc}}(X_{11}) & [D\mu^{cc} : D\lambda^{cc}]_t^* \sigma_t^{\mu^{cc}}(X_{12}) \\ 0 & 0 & [D\mu^{cc} : D\lambda^{cc}]_t \sigma_t^{\lambda^{cc}}(X_{21}) & \sigma_t^{\mu^{cc}}(X_{22}) \end{pmatrix}$$

$$\in \mathcal{M} \otimes \mathcal{F}_2 \tag{3.15}$$

for each $X \in \mathcal{M} \otimes \mathcal{F}_2$ and $t \in \boldsymbol{R}$. Therefore $\theta_{\lambda,\mu}$ is a standard generalized vector for $\mathcal{M} \otimes \mathcal{F}_2$.

(2) $\Rightarrow$ (1) This is trivial.

It is easily shown by (3.1) that $\theta_{\lambda,\mu}$ is full if and only if λ and μ are full.

REMARK 3.2. Suppose λ and μ are standard generalized vectors for $\mathcal{M}$. We put

$$S_{\mu\lambda}\lambda(X) = \mu(X^\dagger), \qquad X \in D(\lambda) \cap D(\mu)^\dagger,$$

$$S_{\lambda\mu}\mu(X) = \lambda(X^\dagger), \qquad X \in D(\lambda) \cap D(\mu)^\dagger.$$

Then $S_{\lambda\mu}$ and $S_{\lambda\mu}$ are closable operators in $\mathcal{H}$ whose closures denoted by the same $S_{\mu\lambda}$ and $S_{\lambda\mu}$, respectively. Let $S_{\mu\lambda} = J_{\mu\lambda}\Delta_{\mu\lambda}^{1/2}$ and $S_{\lambda\mu} = J_{\lambda\mu}\Delta_{\lambda\mu}^{1/2}$ be polar decompositions of $S_{\mu\lambda}$ and $S_{\lambda\mu}$, respectively. By Proposition 3.1 $\theta_{\lambda,\mu}$ is a standard generalized vector for $\mathcal{M} \otimes \mathcal{F}_2$, and so by ([7] Theorem 5.5) $S_{\theta^{cc}_{\lambda,\mu}} = S_{\theta_{\lambda,\mu}}$. Therefore, we have

$$S_{\lambda^{cc}} = S_\lambda, \quad S_{\mu^{cc}} = S_\mu, \quad S_{\mu^{cc}\lambda^{cc}} = S_{\mu\lambda}, \quad S_{\lambda^{cc}\mu^{cc}} = S_{\lambda\mu}$$

and so

$$\Delta_{\lambda^{cc}} = \Delta_\lambda, \quad \Delta_{\mu^{cc}} = \Delta_\mu, \quad \Delta_{\mu^{cc}\lambda^{cc}} = \Delta_{\mu\lambda}, \quad \Delta_{\lambda^{cc}\mu^{cc}} = \Delta_{\lambda\mu}.$$

Hence we have

$$[D\mu^{cc} : D\lambda^{cc}]_t = \Delta_\mu^{it}\Delta_{\lambda\mu}^{-it} = \Delta_{\mu\lambda}^{it}\Delta_\lambda^{-it}, \qquad t \in \boldsymbol{R}. \tag{3.16}$$

THEOREM 3.3. *Suppose λ and μ are full standard generalized vectors for $\mathcal{M}$. Then there uniquely exists a strongly continuous map $t \in \boldsymbol{R} \to U_t \in \mathcal{M}$ such that*

(i) *$\bar{U}_t$ is unitary, $t \in \boldsymbol{R}$;*

(ii) *$U_{t+s} = U_t \sigma_t^\lambda(U_s)$, $s, t \in \boldsymbol{R}$;*

(iii) *$\sigma_t^\mu(X) = U_t \sigma_t^\lambda(X) U_t^\dagger$, $X \in \mathcal{M}$, $t \in \boldsymbol{R}$;*

(iv) *for each $X \in D(\mu) \cap D(\lambda)^\dagger$ and $Y \in D(\lambda) \cap D(\mu)^\dagger$ there exists an element $F_{X,Y} \in A(0, 1)$ such that*

$$F_{X,Y}(t)=(\lambda(U_t\sigma_t^{\lambda}(Y))\,|\,\lambda(X^{\dagger})),\ F_{X,Y}(t+i)=(\mu(X)\,|\,\mu(U_t^{\dagger}\sigma_t^{\mu}(Y^{\dagger})))$$
$$\text{for all } t\in\boldsymbol{R}.$$

PROOF. We put

$$U_t=[D\mu^{cc}:D\lambda^{cc}]_t\lceil\mathcal{D},\qquad t\in\boldsymbol{R}.$$

Then it follows from Proposition 3.1, (3.14), (3.15) and ([**17**] Theorem 3.1) that $t\in\boldsymbol{R}\to U_t\in\mathcal{M}$ is a strongly continuous map satisfying (i)$\sim$(iv). We show the uniqueness of $\{U_t\}_{t\in\boldsymbol{R}}$. Let $t\in\boldsymbol{R}\to V_t\in\mathcal{M}$ be a strongly continuous map satisfying (i)$\sim$(iv). We put

$$\delta_t\left(\begin{pmatrix}X_{11}&X_{12}&0&0\\X_{21}&X_{22}&0&0\\0&0&X_{11}&X_{12}\\0&0&X_{21}&X_{22}\end{pmatrix}\right)$$
$$=\begin{pmatrix}\sigma_t^{\lambda}(X_{11})&V_t^*\sigma_t^{\mu}(X_{12})&0&0\\V_t\sigma_t^{\lambda}(X_{21})&\sigma_t^{\mu}(X_{22})&0&0\\0&0&\sigma_t^{\lambda}(X_{11})&V_t^*\sigma_t^{\mu}(X_{12})\\0&0&V_t\sigma_t^{\lambda}(X_{21})&\sigma_t^{\mu}(X_{22})\end{pmatrix},$$
$$X\in\mathcal{M}\otimes\mathcal{F}_2,\qquad t\in\boldsymbol{R}.$$

Then $\{\delta_t\}$ is a strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{M}\otimes\mathcal{F}_2$ such that $\delta_t(D(\theta)\cap D(\theta)^{\dagger})\subset D(\theta)\cap D(\theta)^{\dagger}$ for each $t\in\boldsymbol{R}$, where $\theta\equiv\theta_{\lambda,\mu}$, and θ satisfies the KMS-condition with respect to $\{\delta_t\}$. By ([**7**] Theorem 5.5) we have $\delta_t=\sigma_t^{\theta}$ for all $t\in\boldsymbol{R}$, which implies by (3.15) that $V_t=[D\mu^{cc}:D\lambda^{cc}]_t\lceil\mathcal{D}$ $=U_t$ for all $t\in\boldsymbol{R}$. This completes the proof.

Let λ and μ be *full* standard generalized vectors for $\mathcal{M}$. The map $t\in\boldsymbol{R}\to U_t\in\mathcal{M}$, uniquely determined by the above theorem, is called *the cocycle associated with μ with respect to λ*, and is denoted by $[D\mu:D\lambda]$. Suppose standard generalized vectors λ and μ are not necessarily full, then we put $[D\mu:D\lambda]_t=[D\mu_\sigma:D\lambda_\sigma]_t$, $t\in\boldsymbol{R}$. Then $t\to[D\mu:D\lambda]_t$ is a strongly continuous map satisfying the conditions (i)$\sim$(iv) in Theorem 3.3 and it is called the cocycle associated with μ with respect to λ. This equals the Connes cocycle $[D\mu^{cc}:D\lambda^{cc}]$ associated with μ^{cc} with respect to λ^{cc}.

§ 4. Generalized Pedersen and Takesaki Radon-Nikodym theorem.

In this section we construct the standard generalized vector λ_A associated with a given full standard generalized vector λ and a given positive self-adjoint operator A affiliated with the centralizer of λ, and consider when a full standard generalized vector μ is represented as the full extension of such a λ_A.

Let $\mathcal{M}$ be a generalized von Neumann algebra on $\mathcal{D}$ in a Hilbert space $\mathcal{H}$ and λ a standard generalized vector for $\mathcal{M}$. We put

$$\mathcal{M}^{\sigma^\lambda} = \{A \in \mathcal{M} ; A\Delta_\lambda^{it} \supset \Delta_\lambda^{it} A, \forall t \in \boldsymbol{R}\}, \qquad \mathcal{M}_b^{\sigma^\lambda} = \mathcal{M}_b \cap \mathcal{M}^{\sigma^\lambda}.$$

Then $\mathcal{M}^{\sigma^\lambda}$ and $\mathcal{M}_b^{\sigma^\lambda}$ are O^*-subalgebras of $\mathcal{M}$.

LEMMA 4.1. *Let λ be a full standard generalized vector for $\mathcal{M}$. Then the following statements hold.*

(1) *Suppose $A \in \mathcal{M}_b$ such that $\Delta_\lambda^{1/2} A^\dagger \Delta_\lambda^{-1/2}$ is bounded. Then $XA \in D(\lambda) \cap D(\lambda)^\dagger$ and $\lambda(XA) = J_\lambda \Delta_\lambda^{1/2} A^\dagger \Delta_\lambda^{-1/2} J_\lambda \lambda(X)$ for each $X \in D(\lambda) \cap D(\lambda)^\dagger$.*

(2) *Suppose $X \in D(\lambda) \cap D(\lambda)^\dagger$ and $A \in \mathcal{M}$ such that $XA \in D(\lambda) \cap D(\lambda)^\dagger$. Then $\lambda(XA) = J_\lambda \Delta_\lambda^{1/2} A^\dagger \Delta_\lambda^{-1/2} J_\lambda \lambda(X)$.*

(3) *Suppose $A \in \mathcal{M}_b^{\sigma^\lambda}$. Then $XA \in D(\lambda)$ and $\lambda(XA) = J_\lambda A^* J_\lambda \lambda(X)$ for each $X \in D(\lambda)$.*

PROOF. (1) Since $\Delta_\lambda^{1/2} A^\dagger \Delta_\lambda^{-1/2}$ is bounded, it follows that

$$A^\dagger \lambda(X^\dagger) \in \mathcal{D}(S_\lambda) \quad \text{and} \quad S_\lambda A^\dagger \lambda(X^\dagger) = J_\lambda \Delta_\lambda^{1/2} A^\dagger \Delta_\lambda^{-1/2} J_\lambda \lambda(X),$$

which implies

$$\begin{aligned}
(XA\lambda^c(K) \mid \lambda^c(K_1)) &= (\lambda^c(K) \mid A^\dagger X^\dagger \lambda^c(K_1)) \\
&= (\lambda^c(K) \mid K_1 \lambda(A^\dagger X^\dagger)) \\
&= (\lambda^c(K_1^* K) \mid \lambda(A^\dagger X^\dagger)) \\
&= (S_\lambda^* \lambda^c(K^* K_1) \mid \lambda(A^\dagger X^\dagger)) \\
&= (S_\lambda \lambda(A^\dagger X^\dagger) \mid \lambda^c(K^* K_1)) \\
&= (K J_\lambda \Delta_\lambda^{1/2} A^\dagger \Delta_\lambda^{-1/2} J_\lambda \lambda(X) \mid \lambda^c(K_1))
\end{aligned}$$

for each $K, K_1 \in D(\lambda^c) \cap D(\lambda^c)^*$. Hence we have

$$XA\lambda^c(K) = K J_\lambda \Delta_\lambda^{1/2} A^\dagger \Delta_\lambda^{-1/2} J_\lambda \lambda(X)$$

for each $K \in D(\lambda^c) \cap D(\lambda^c)^*$. Since λ is full, it follows that $XA \in D(\lambda)$ and $\lambda(XA) = J_\lambda \Delta_\lambda^{1/2} A^\dagger \Delta_\lambda^{-1/2} J_\lambda \lambda(X)$.

(2) This follows from

$$\begin{aligned}
(\lambda(XA) \mid \lambda^c(K^* K_1)) &= (XA\lambda^c(K) \mid \lambda^c(K_1)) \\
&= (\lambda^c(K_1^* K) \mid A^\dagger \lambda(X^\dagger)) \\
&= (S_\lambda^* \lambda^c(K^* K_1) \mid A^\dagger \lambda(X^\dagger)) \\
&= (S_\lambda A^\dagger \lambda(X^\dagger) \mid \lambda^c(K^* K_1)) \\
&= (J_\lambda \Delta_\lambda^{1/2} A^\dagger \Delta_\lambda^{-1/2} J_\lambda \lambda(X) \mid \lambda^c(K^* K_1))
\end{aligned}$$

for each $X\in D(\lambda)\cap D(\lambda)^{\dagger}$ and $K, K_1\in D(\lambda^c)\cap D(\lambda^c)^*$.

(3) We first show

$$A\lambda^c(K)\in \mathcal{D}(S_\lambda^*) \quad\text{and}\quad S_\lambda^* A\lambda^c(K)=J_\lambda \bar{A}\lambda_\lambda\lambda^c(K^*) \tag{4.1}$$

for each $K\in D(\lambda^c)\cap D(\lambda^c)^*$. This follows from

$$\begin{aligned}(S_\lambda\lambda(Y)\,|\,A\lambda^c(K)) &= (A^*\Delta_\lambda^{-1/2}J_\lambda\lambda(Y)\,|\,\lambda^c(K))\\ &= (\Delta_\lambda^{-1/2}A^*J_\lambda\lambda(Y)\,|\,\lambda^c(K))\\ &= (\lambda^c(K^*)\,|\,J_\lambda A^*J_\lambda\lambda(Y))\\ &= (J_\lambda\bar{A}J_\lambda\lambda^c(K^*)\,|\,\lambda(Y))\end{aligned}$$

for each $Y\in D(\lambda)\cap D(\lambda)^{\dagger}$. By (4.1) we have

$$\begin{aligned}(XA\lambda^c(K)\,|\,\lambda(Y)) &= (A\lambda^c(K)\,|\,\lambda(X^{\dagger}Y))\\ &= (\lambda(Y^{\dagger}X)\,|\,S_\lambda^*A\lambda^c(K))\\ &= (\lambda(Y^{\dagger}X)\,|\,J_\lambda\bar{A}J_\lambda\lambda^c(K^*))\\ &= (\lambda(X)\,|\,J_\lambda\bar{A}J_\lambda K^*\lambda(Y))\\ &= (KJ_\lambda A^*J_\lambda\lambda(X)\,|\,\lambda(Y))\end{aligned}$$

for each $K\in D(\lambda^c)\cap D(\lambda^c)^*$ and $Y\in D(\lambda)\cap D(\lambda)^{\dagger}$, which implies by the fullness of λ that $XA\in D(\lambda)$ and $\lambda(XA)=J_\lambda A^*J_\lambda\lambda(X)$.

THEOREM 4.2. *Let $\mathcal{M}$ be a generalized von Neumann algebra on $\mathcal{D}$ in $\mathcal{H}$ and λ and μ full standard generalized vectors for $\mathcal{M}$. Then the following statements are equivalent.*

(1) *$D(\mu)$ is $\{\sigma_t^\lambda\}$-invariant and $\|\mu(\sigma_t^\lambda(X))\|=\|\mu(X)\|$ for all $X\in D(\mu)$.*

(1)′ *$D(\lambda)$ is $\{\sigma_t^\mu\}$-invariant and $\|\lambda(\sigma_t^\mu(X))\|=\|\lambda(X)\|$ for all $X\in D(\lambda)$.*

(2) $[D\mu: D\lambda]_t\in\mathcal{M}^{\sigma^\mu}$, $\forall t\in\boldsymbol{R}$.

(2)′ $[D\mu: D\lambda]_t\in\mathcal{M}^{\sigma^\lambda}$, $\forall t\in\boldsymbol{R}$.

(3) *$\{[D\mu: D\lambda]_t\}_{t\in\boldsymbol{R}}$ is a strongly continuous one-parameter group of unitary elements of $\mathcal{M}$.*

PROOF. The equivalence of (2), (2)′ and (3) follows from Theorem 3.3.

(1) $\Rightarrow$ (2) We now put $U_t=[D\mu: D\lambda]_t$, $t\in\boldsymbol{R}$. Take an arbitrary $t\in\boldsymbol{R}$ and put $A=\sigma^\mu_{-t}(U_t)$. Then it follows from the assumption (1) that

$$XA\in D(\mu)\cap D(\mu)^{\dagger} \quad\text{and}\quad (\mu(X)\,|\,\mu(Y))=(\mu(XA)\,|\,\mu(YA)) \tag{4.2}$$

for all $X, Y\in D(\mu)\cap D(\mu)^{\dagger}$, and further by Lemma 4.1, (2)

$$\begin{aligned}\|\mu(X)\| = \|\mu(\sigma_t^\lambda(X))\| &= \|\mu(U_t^*\sigma_t^\mu(X)U_t)\| \\ &= \|\mu(XA)\| \\ &= \|J_\mu\Delta_\mu^{1/2}A^\dagger\Delta_\mu^{-1/2}J_\mu\mu(X)\|\end{aligned}$$

for all $X\in D(\mu)\cap D(\mu)^\dagger$. Hence, $J_\mu\Delta_\mu^{1/2}A^\dagger\Delta_\mu^{-1/2}J_\mu$ is bounded. Furthermore, since $D(\mu)\cap D(\mu)^\dagger$ is $\{\sigma_t^\mu\}$-invariant and $\{\sigma_t^\lambda\}$-invariant, it follows from Theorem 3.3 that

$$XU_s^* = U_s^*\sigma_s^\mu(\sigma_{-s}^\lambda(X))\in D(\mu)\cap D(\mu)^\dagger$$

for all $X\in D(\mu)\cap D(\mu)^\dagger$ and $s\in\boldsymbol{R}$, which implies

$$X\sigma_s^\mu(U_s^*) \in D(\mu)\cap D(\mu)^\dagger$$

for all $X\in D(\mu)\cap D(\mu)^\dagger$ and $s\in\boldsymbol{R}$. Hence, by (4.2) we have

$$XA^\dagger \in D(\mu)\cap D(\mu)^\dagger \quad \text{and} \quad (\mu(X)\,|\,\mu(YA)) = (\mu(XA^\dagger)\,|\,\mu(Y))$$

for all $X, Y\in D(\mu)\cap D(\mu)^\dagger$, which implies by Lemma 4.1, (2) that

$$\begin{aligned}(\mu(X)\,|\,J_\mu\Delta_\mu^{1/2}A^\dagger\Delta_\mu^{-1/2}J_\mu\mu(Y)) &= (\mu(X)\,|\,\mu(YA)) \\ &= (\mu(XA^\dagger)\,|\,\mu(Y)) \\ &= (J_\mu\Delta_\mu^{1/2}A\Delta_\mu^{-1/2}J_\mu\mu(X)\,|\,\mu(Y)) \\ &= (\mu(X)\,|\,(J_\mu\Delta_\mu^{1/2}A\Delta_\mu^{-1/2}J_\mu)^*\mu(Y))\end{aligned}$$

for each $X, Y\in D(\mu)\cap D(\mu)^\dagger$. Hence we have

$$\overline{J_\mu\Delta_\mu^{1/2}A^\dagger\Delta_\mu^{-1/2}J_\mu} = (J_\mu\Delta_\mu^{1/2}A\Delta_\mu^{-1/2}J_\mu)^*,$$

which implies $\bar{A}\Delta_\mu\subset\Delta_\mu\bar{A}$. Therefore it follows that $U_t\in\mathcal{M}^{\sigma^\mu}$ for all $t\in\boldsymbol{R}$.

(2) $\Rightarrow$ (1) It follows from Theorem 3.3 and Lemma 4.1, (3) that

$$\sigma_t^\lambda(X) = U_t^*\sigma_t^\mu(X)U_t \in D(\mu)$$

and

$$\begin{aligned}\|\mu(\sigma_t^\lambda(X))\| &= \|\mu(U_t^*\sigma_t^\mu(X)U_t)\| \\ &= \|J_\mu U_t^*J_\mu\mu(\sigma_t^\mu(X))\| \\ &= \|\mu(X)\|\end{aligned}$$

for each $X\in D(\mu)$ and $t\in\boldsymbol{R}$.

(1)′ $\Leftrightarrow$ (2) This is proved in similar to the equivalence of (1) and (2).

This completes the proof.

If the equivalent conditions in Theorem 4.2 are satisfied, we say that μ *commutes with* λ. If μ commutes with λ, then

$$\sigma_t^\lambda \circ \sigma_t^\mu = \sigma_t^\mu \circ \sigma_t^\lambda, \qquad t \in \boldsymbol{R}.$$

But, the converse is not necessarily true even in the bounded case ([**17**] 4.15). We next present the canonical construction and the properties of the generalized vector λ_A associated with a given full standard generalized vector λ and a given positive self-adjoint operator A affiliated with the centralizer of λ. We investigate when a full standard generalized vector μ for $\mathcal{M}$ which commutes with λ is represented as $(\lambda_A)_\sigma$.

Let λ be a full standard generalized vector for $\mathcal{M}$ and $\mathcal{M}_\eta^{\sigma^\lambda}$ the set of all non-singular positive self-adjoint operators A in $\mathcal{H}$ satisfying $\{E_A(t); -\infty<t<\infty\}'' \lceil \mathcal{D} \subset \mathcal{M}_b^{\sigma^\lambda}$, *where* $\{E_A(t)\}$ *is the spectral resolutions of* A. Let $A \in \mathcal{M}_\eta^{\sigma^\lambda}$ and put

$$D(\lambda_A) = \{X \in D(\lambda); \lambda(YX) \in \mathcal{D}(J_\lambda A J_\lambda) \text{ for all } Y \in \mathcal{M}\},$$

$$\lambda_A(X) = J_\lambda A J_\lambda \lambda(X), \qquad X \in D(\lambda_A).$$

Then we have the following

LEMMA 4.3. *λ_A is a standard generalized vector for $\mathcal{M}$ satisfying*

$$\sigma_t^{\lambda_A}(X) = A^{2it} \sigma_t^\lambda(X) A^{-2it},$$

$$[D\lambda_A : D\lambda]_t \equiv [D(\lambda_A)_\sigma : D\lambda]_t = A^{2it} \lceil \mathcal{D}, \quad X \in \mathcal{M}, \quad t \in \boldsymbol{R}.$$

PROOF. It is clear that λ_A is generalized vector for $\mathcal{M}$. Since

$$\{E_A(n')YE_A(m')E_A(n)XA^{-1}E_A(m); X, Y \in D(\lambda) \cap D(\lambda)^\dagger, m, n, m', n' \in N\}$$
$$\subset (D(\lambda_A) \cap D(\lambda_A)^\dagger)^2$$

and

$$\lambda_A(E_A(n')YE_A(m')E_A(n)XA^{-1}E_A(m)) = E_A(n')YE_A(m')E_A(n)J_\lambda E_A(m)J_\lambda \lambda(X)$$
$$\longrightarrow Y\lambda(X), \qquad (m, n, m', n' \to \infty)$$

it follows that

$$\lambda_A((D(\lambda_A) \cap D(\lambda_A)^\dagger)^2) \text{ is dense in } \mathcal{H}. \tag{4.3}$$

We put

$$\mathcal{K} = \{K \in D(\lambda^c); \lambda^c(K) \in \mathcal{D}(A) \cap \mathcal{D}(J_\lambda A^{-1} J_\lambda) \text{ and } A\lambda^c(K) \in \mathcal{D}\}.$$

Then we have

$$\mathcal{K} \subset D(\lambda_A^c) \quad \text{and} \quad \lambda_A^c(K) = A\lambda^c(K), \qquad \forall K \in \mathcal{K}. \tag{4.4}$$

In fact, this follows from

$$\begin{aligned} K\lambda_A(X) = KJ_\lambda AJ_\lambda\lambda(X) &= \lim_{n\to\infty} KJ_\lambda AE_A(n)J_\lambda\lambda(X) \\ &= \lim_{n\to\infty} K\lambda(XAE_A(n)) \\ &= \lim_{n\to\infty} XAE_A(n)\lambda^c(K) \\ &= XA\lambda^c(K) \end{aligned}$$

for each $X\in D(\lambda_A)$ and $K\in\mathcal{K}$. We put

$$K_{mn} = J_\lambda E_A(m)J_\lambda KJ_\lambda E_A(n)J_\lambda$$

for $K\in D(\lambda^c)\cap D(\lambda^c)^*$ and $m, n\in N$. Then we have

$$\begin{aligned} K_{mn}\lambda(X) &= (J_\lambda E_A(m)J_\lambda)K(J_\lambda E_A(n)J_\lambda)\lambda(X) \\ &= (J_\lambda E_A(m)J_\lambda)K\lambda(XE_A(n)) \\ &= (J_\lambda E_A(m)J_\lambda)XE_A(n)\lambda^c(K) \\ &= X(J_\lambda E_A(m)J_\lambda)E_A(n)\lambda^c(K) \end{aligned}$$

and

$$K^*_{mn}\lambda(X) = X(J_\lambda E_A(n)J_\lambda)E_A(m)\lambda^c(K^*)$$

for each $X\in D(\lambda)$, and so

$$\begin{aligned} &K_{mn}\in \mathcal{K}\cap\mathcal{K}^*, \\ &\lambda^c(K_{mn}) = (J_\lambda E_A(m)J_\lambda)E_A(n)\lambda^c(K), \\ &\lambda^c(K^*_{mn}) = (J_\lambda E_A(n)J_\lambda)E_A(m)\lambda^c(K^*). \end{aligned} \tag{4.5}$$

Hence we have

$$\begin{aligned} &C_{mn}K_{mn}\in \mathcal{K}\cap\mathcal{K}^*, \\ &\lim_{m,n\to\infty}\lambda^c(C_{mn}K_{mn}) = \lim_{m,n\to\infty} C_{mn}\lambda^c(K_{mn}) \\ &\qquad = \lim_{m,n\to\infty} C_{mn}(J_\lambda E_A(m)J_\lambda)E_A(n)\lambda^c(K) \\ &\qquad = C\lambda^c(K) = \lambda^c(CK), \\ &\lim_{m,n\to\infty}\lambda^c((C_{mn}K_{mn})^*) = \lambda^c((CK)^*) \end{aligned}$$

for each $C, K\in D(\lambda^c)\cap D(\lambda^c)^*$, which implies that

$$\lambda^c((\mathcal{K}\cap\mathcal{K}^*)^2) \text{ is dense in the Hilbert space } \mathcal{D}(S^*_\lambda). \tag{4.6}$$

For each $K\in\mathcal{K}\cap\mathcal{K}^*$ and $n\in N$ we have

$$K_n \equiv KJ_\lambda A^{-1}E_A(n)J_\lambda \in \mathcal{K}\cap\mathcal{K}^*,$$

$$\lambda^c(K_n) = A^{-1}E_A(n)\lambda^c(K),$$
$$\lambda^c(K_n^*) = J_\lambda A^{-1}E_A(n)J_\lambda\lambda^c(K^*)$$

and so by (4.4)

$$\lim_{n\to\infty}\lambda_A^c(CK_n) = \lim_{n\to\infty}C\lambda_A^c(K_n) = \lim_{n\to\infty}CE_A(n)\lambda^c(K)$$
$$= C\lambda^c(K)$$

for each $C\in\mathcal{K}\cap\mathcal{K}^*$. Hence it follows from (4.6) that $\lambda_A^c((\mathcal{K}\cap\mathcal{K}^*)^2)$ is dense in $\mathcal{H}$, which implies by (4.4) that

$$\lambda_A^c((D(\lambda_A^c)\cap D(\lambda_A^c)^*)^2) \text{ is dense in } \mathcal{H}. \tag{4.7}$$

By (4.3) and (4.7) λ_A is a cyclic and separating vector for $\mathcal{M}$. For each $K\in\mathcal{K}\cap\mathcal{K}^*$ we have by (4.4) and (4.5)

$$\lim_{m,n\to\infty}\lambda_A^c(K_{mn}) = \lim_{m,n\to\infty}AE_A(n)J_\lambda E_A(m)J_\lambda\lambda^c(K)$$
$$= A\lambda^c(K) = \lambda_A^c(K),$$
$$\lim_{m,n\to\infty}\lambda_A^c(K_{mn}^*) = \lim_{m,n\to\infty}AE_A(m)J_\lambda E_A(n)J_\lambda\lambda^c(K^*)$$
$$= \lambda_A^c(K^*).$$

Furthermore, for each $C\in D(\lambda_A^c)\cap D(\lambda_A^c)^*$ and $m, n\in N$ we put

$$C_{mn} = (J_\lambda E_A(m)J_\lambda)C(J_\lambda E_A(n)J_\lambda).$$

Then we have

$$C_{mn}\lambda(X) = (J_\lambda E_A(m)J_\lambda)C\lambda_A(XA^{-1}E_A(n))$$
$$= (J_\lambda E_A(m)J_\lambda)XA^{-1}E_A(n)\lambda_A^c(C)$$
$$= X(J_\lambda E_A(m)J_\lambda)A^{-1}E_A(n)\lambda_A^c(C),$$
$$C_{mn}^*\lambda(X) = X(J_\lambda E_A(n)J_\lambda)A^{-1}E_A(m)\lambda_A^c(C^*),$$

and so by (4.4) and (4.5)

$$C_{mn}\in\mathcal{K}\cap\mathcal{K}^*,$$
$$\lim_{m,n\to\infty}\lambda_A^c(C_{mn}) = \lim_{m,n\to\infty}(J_\lambda E_A(m)J_\lambda)E_A(n)\lambda_A^c(C)$$
$$= \lambda_A^c(C)$$
$$\lim_{m,n\to\infty}\lambda_A^c(C_{mn}^*) = \lambda_A^c(C^*).$$

Therefore it follows that

$$\{\lambda_A^c(K_{mn}) : K \in \mathcal{K} \cap \mathcal{K}^*,\ m,\ n \in N\}$$

is dense in the Hilbert space $\mathcal{D}(S^*_{\lambda_A^{cc}})$. (4.8)

For each $K \in \mathcal{K} \cap \mathcal{K}^*$ and $m, n \in N$ we have by (4.4) and (4.5)

$$\begin{aligned} S^*_{\lambda_A^{cc}} \lambda_A^c(K_{mn}) &= AE_A(n) J_\lambda E_A(m) J_\lambda \lambda^c(K^*) \\ &= AE_A(n) J_\lambda E_A(m) J_\lambda S_\lambda^* \lambda^c(K) \\ &= S_\lambda^* J_\lambda A E_A(n) J_\lambda E_A(m) \lambda^c(K) \\ &= S_\lambda^* J_\lambda A E_A(n) J_\lambda A^{-1} E_A(m) \lambda_A^c(K)\,, \end{aligned}$$

and so

$$\lambda_A^c(K) \in \mathcal{D}(\overline{S_\lambda^* J_\lambda A J_\lambda A^{-1}}) \quad \text{and} \quad \overline{S_\lambda^* J_\lambda A J_\lambda A^{-1}} \lambda_A^c(K) = S^*_{\lambda_A^{cc}} \lambda_A^c(K)$$

for each $K \in \mathcal{K} \cap \mathcal{K}^*$. By (4.8) we have

$$S^*_{\lambda_A^{cc}} \subset \overline{S_\lambda^* J_\lambda A J_\lambda A^{-1}}\,. \tag{4.9}$$

Similarly we have

$$S_\lambda^* \subset \overline{S^*_{\lambda_A^{cc}} J_\lambda A^{-1} J_\lambda A}\,. \tag{4.10}$$

By (4.9) and (4.10) we have

$$S^*_{\lambda_A^{cc}} = \overline{S_\lambda^* J_\lambda A J_\lambda A^{-1}} = \overline{J_\lambda \Delta_\lambda^{-1/2} J_\lambda A J_\lambda A^{-1}}\,. \tag{4.11}$$

Since A is affiliated with $(\mathcal{M}_w'')^{\sigma^\lambda}$, it follows that the two self-adjoint operators $\Delta_\lambda^{-1/2}$ and $\overline{J_\lambda A J_\lambda A^{-1}}$ are strongly commuting, that is, the spectral projections of the two self-adjoint operators are mutually commuting, and so $\overline{\Delta_\lambda^{-1/2} J_\lambda A J_\lambda A^{-1}}$ is self-adjoint and it equals $\overline{J_\lambda A J_\lambda A^{-1} \Delta_\lambda^{-1/2}}$. Hence, it follows from (4.11) and the uniqueness of the polar decomposition of $S^*_{\lambda_A^{cc}}$, it follows that

$$J_{\lambda_A^{cc}} = J_\lambda \quad \text{and} \quad \Delta_{\lambda_A^{cc}}^{-1/2} = \overline{\Delta_\lambda^{-1/2} J_\lambda A J_\lambda A^{-1}} = \overline{J_\lambda A J_\lambda A^{-1} \Delta_\lambda^{-1/2}}\,,$$

which implies

$$\Delta_{\lambda_A^{cc}}^{it} = J_\lambda A^{-2it} J_\lambda A^{2it} \Delta_\lambda^{it} \quad \text{and} \quad \sigma_t^{\lambda_A^{cc}}(X) = A^{2it} \sigma_t^\lambda(X) A^{-2it}$$

for $X \in \mathcal{M}$ and $t \in \boldsymbol{R}$. Hence it follows from Lemma 4.1, (3) that

$$\sigma_t^{\lambda_A^{cc}}(D(\lambda_A) \cap D(\lambda_A)^\dagger) \subset D(\lambda_A) \cap D(\lambda_A)^\dagger\,, \qquad t \in \boldsymbol{R}\,.$$

Therefore λ_A is a standard generalized vector for $\mathcal{M}$. Further, it follows from Theorem 3.3 that $[D\lambda_A : D\lambda]_t \equiv [D(\lambda_A)_\sigma : D\lambda]_t = A^{2it} \lceil \mathcal{D}$ for $t \in \boldsymbol{R}$. This completes the proof.

LEMMA 4.4. *Let λ, μ_1 and μ_2 be full standard generalized vectors for $\mathcal{M}$. Suppose $[D\mu_1 : D\lambda]_t = [D\mu_2 : D\lambda]_t$ for all $t \in \boldsymbol{R}$. Then $\mu_1 = \mu_2$.*

PROOF. By ([**17**] Corollary 3.6) we have $\mu_1^{cc}=\mu_2^{cc}$, and so $\mu_1^c=\mu_2^c$. Take an arbitrary $X\in D(\mu_1)$. By ([**7**] Proposition 4.3) there exists a sequence $\{X_n\}$ in $D(\mu_1^{cc})=D(\mu_2^{cc})$ such that $\lim_{n\to\infty}X_n\xi=X\xi$ for each $\xi\in D$ and $\lim_{n\to\infty}\mu_2^{cc}(X_n)=\lim_{n\to\infty}\mu_1^{cc}(X_n)=\mu_1(X)$. Hence we have

$$\begin{aligned} K\mu_1(X) &= \lim_{n\to\infty} K\mu_2^{cc}(X_n) = \lim_{n\to\infty} X_n\mu_2^c(K) \\ &= X\mu_2^c(K) \end{aligned}$$

for all $K\in D(\mu_2^c)\cap D(\mu_2^c)^*$, which implies by the fullness of μ_2 that $\mu_1\subset\mu_2$. Similarly we can show $\mu_2\subset\mu_1$.

Let λ and μ be full standard generalized vectors for $\mathcal{M}$. Suppose μ commutes with λ. Then it follows from Theorem 4.2 that $\{[D\mu : D\lambda]_t\}_{t\in\boldsymbol{R}}$ is a strongly continuous one-parameter group of unitary operators in $(\mathcal{M}_w'')^{\sigma^\lambda}$, and so by the Stone theorem there exists a unique non-singular positive self-adjoint operator $A_{\lambda,\mu}$ affiliated with $(\mathcal{M}_w'')^{\sigma^\lambda}$ such that $[D\mu ; D\lambda]_t=A_{\lambda,\mu}^{2it}\lceil\mathcal{D}$ for all $t\in\boldsymbol{R}$. By Lemma 4.3, 4.4 we have the following

THEOREM 4.5. *Let $\mathcal{M}$ be a generalized von Neumann algebra on $\mathcal{D}$ in $\mathcal{H}$ and λ and μ full standard generalized vectors for $\mathcal{M}$. Suppose $A_{\lambda,\mu}\in\mathcal{M}_\eta^{\sigma^\lambda}$. Then $\mu=(\lambda_{A_{\lambda,\mu}})_\sigma$.*

COROLLARY 4.6. *Let $\mathcal{M}$ be an EW^*-algebra on $\mathcal{D}$ in $\mathcal{H}$ and λ and μ full standard generalized vectors for $\mathcal{M}$. Then μ commutes with λ if and only if $\mu=(\lambda_A)_\sigma$ for some non-singular positive self-adjoint operator A affiliated with $(\mathcal{M}_w'')^{\sigma^\lambda}$.*

PROOF. Suppose μ commutes with λ. Since $\mathcal{M}$ is an EW^*-algebra on $\mathcal{D}$ in $\mathcal{H}$, we have $A_{\lambda,\mu}\in\mathcal{M}_\eta^{\sigma^\lambda}$, and so $\mu=(\lambda_{A_{\lambda,\mu}})_\sigma$ by Theorem 4.5. The converse follows from Lemma 4.3.

THEOREM 4.7. *Let $\mathcal{M}$ be a generalized von Neumann algebra on $\mathcal{D}$ in $\mathcal{H}$ and λ and μ full standard generalized vectors for $\mathcal{M}$. Then the following statements are equivalent.*

(1) *μ satisfies the KMS-condition with respect to $\{\sigma_t^\lambda\}$.*

(2) *$\sigma_t^\mu=\sigma_t^\lambda$ for each $t\in\boldsymbol{R}$.*

(3) *There exists a non-singular positive self-adjoint operator A affiliated with the center of $(\mathcal{M}_w')'$ such that $\mu=(\lambda_A)_\sigma$.*

PROOF. (1) $\Rightarrow$ (2) This follows from ([**17**] Corollary 4.11).

(2) $\Rightarrow$ (1) This is trivial.

(2) $\Rightarrow$ (3) This follows from

$$\sigma_t^\lambda(X) = \sigma_t^\mu(X) = A_{\lambda,\mu}^{2it}\sigma_t^\lambda(X)A_{\lambda,\mu}^{-2it}, \qquad X\in\mathcal{M},\ t\in\boldsymbol{R}.$$

(3) $\Rightarrow$ (2) Since A is affiliated with the center of $(\mathcal{M}'_w)'$, it follows that $A \in \mathcal{M}_\eta^{\sigma^\lambda}$, which implies by Lemma 4.3 that λ_A is well-defined and

$$\sigma_t^\mu(X) = \sigma_t^{(\lambda_A)\sigma}(X) = A^{2it}\sigma_t^\lambda(X)A^{-2it} = \sigma_t^\lambda(X)$$

for all $X \in \mathcal{M}$ and $t \in \boldsymbol{R}$. This completes the proof.

Let $\mathcal{M}$ be a generalized von Neumann algebra on $\mathcal{D}$ in $\mathcal{H}$. A generalized vector λ for $\mathcal{M}$ is said to be *tracial* if $\|\lambda(X)\| = \|\lambda(X^\dagger)\|$ for all $X \in D(\lambda) \cap D(\lambda)^\dagger$. It is clear that a cyclic and separating tracial generalized vector λ is standard and $\Delta_\lambda = 1$. If there exists a cyclic and separating tracial generalized vector λ for $\mathcal{M}$, then $\mathcal{M}$ is said to be *spatially semifinite*.

PROPOSITION 4.8. *Let $\mathcal{M}$ be a generalized von Neumann algebra on $\mathcal{D}$ in $\mathcal{H}$. The following statements hold.*

(1) *Suppose $\mathcal{M}$ is spatially semifinite. Then, for each full standard generalized vector λ for $\mathcal{M}$ there exists a non-singular positive self-adjoint operator A affiliated with $(\mathcal{M}''_w)^{\sigma^\lambda}$ such that $\sigma_t^\lambda(X) = A^{2it}XA^{-2it}$ for all $X \in \mathcal{M}$ and $t \in \boldsymbol{R}$.*

(2) *Conversely suppose there exist a full standard generalized vector λ for $\mathcal{M}$ and a non-singular positive self-adjoint operator $A \in \mathcal{M}_\eta^{\sigma^\lambda}$ such that $\sigma_t^\lambda(X) = A^{2it}XA^{-2it}$ for all $X \in \mathcal{M}$ and $t \in \boldsymbol{R}$. Then $\mathcal{M}$ is spatially semifinite.*

PROOF. (1) Since $\mathcal{M}$ is spatially semifinite, there exists a full standard generalized vector μ for $\mathcal{M}$ such that $\Delta_\mu = 1$. Hence it follows from Theorem 4.2 that $\sigma_t^\lambda(X) = A_{\mu,\lambda}^{2it}\sigma_t^\mu(X)A_{\mu,\lambda}^{-2it} = A_{\mu,\lambda}^{2it}XA_{\mu,\lambda}^{-2it}$ for all $X \in \mathcal{M}$ and $t \in \boldsymbol{R}$.

(2) Since $A^{-1} \in \mathcal{M}_\eta^{\sigma^\lambda}$, it follows from Lemma 4.3 that $\mu \equiv \lambda_{A^{-1}}$ is well-defined and $\sigma_t^\mu(X) = A^{-2it}\sigma_t^\lambda(X)A^{2it} = X$ for all $X \in \mathcal{M}$ and $t \in \boldsymbol{R}$. Therefore, μ is tracial, and so $\mathcal{M}$ is spatially semifinite.

COROLLARY 4.9. *An EW^*-algebra $\mathcal{M}$ is spatially semifinite if and only if there exist a standard generalized vector λ for $\mathcal{M}$ and a non-singular positive self-adjoint operator A affiliated with $(\mathcal{M}'_w)'$ such that $\sigma_t^\lambda(X) = A^{2it}XA^{-2it}$ for all $X \in \mathcal{M}$ and $t \in \boldsymbol{R}$.*

PROOF. This follows from $(\mathcal{M}'_w)'\mathcal{D} \subset \mathcal{D}$ and Proposition 4.8.

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