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Algorithmic methods for Fuchsian systems of linear partial differential equations

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Introduction.

A generalization of the notion of regular singularity for linear ordinary differential equations to (single) partial differential equations was introduced by Baouendi and Goulaouic [**1**]. They called such equations Fuchsian partial differential equations with respect to a hypersurface. In [**12**], [**21**], Kashiwara and Oshima called the same equations ones with regular singularities in a weak sense along a hypersurface and studied the boundary value problem for such equations.

Recently, the notion of Fuchsian partial differential equation of [**1**] has been generalized to that of Fuchsian system of linear partial differential equations along a submanifold Y of arbitrary codimension by Laurent and Monteiro Fernandes [**13**]. Especially, it has been proved in [**13**] for Fuchsian systems that any power series solution which converges with respect to the variables tangent to Y and formal with respect to the variable(s) normal to Y converges with respect to all the variables. It is also known that the holonomic system with regular singularities in the sense of Kashiwara and Kawai is Fuchsian along any submanifold (cf. [**11**], [**13**]). Thus Fuchsian systems constitute a nice and substantially wide class of systems containing many interesting examples.

Suppose that a system of linear partial differential equations

$$\mathcal{M} : P_1 u = \cdots = P_s u = 0$$

for an unknown function u in an open subset of $\boldsymbol{C}^{n+1}$ and a non-singular complex analytic hypersurface Y are given. (For example, if $\mathcal{M}$ is holonomic, then we take as Y an irreducible component of the "singular locus" of $\mathcal{M}$.) Then, from the computational point of view, we have the following basic problems about $\mathcal{M}$:

A. Is $\mathcal{M}$ Fuchsian along Y?

B. If so, find the structure of the space of multi-valued analytic (or hyperfunction, etc.) solutions of $\mathcal{M}$ around Y.

If the system $\mathcal{M}$ is Fuchsian, we can define its characteristic exponents as in the case of ordinary differential equations, and the "boundary values" of (multi-valued) analytic solutions of $\mathcal{M}$, which are analytic functions on Y. (Boundary values can be also defined for hyperfunction solutions (cf. [**12**], [**21**], [**18**], [**15**]). However, in the present paper, we restrict ourselves to analytic solutions for the sake of simplicity.) Then a somewhat vague problem B is reduced substantially to the more concrete one:

C. If $\mathcal{M}$ is Fuchsian along Y, compute its characteristic exponents and the system of equations which their boundary values satisfy (i.e., the induced, or the tangential system of $\mathcal{M}$ along Y).

The purpose of this paper is to present effective algorithmic methods which solve the problem C completely and partially solve the problem A. For this purpose, we introduce a new notion of Gröbner basis for the ring of differential operators with respect to a filtration of Kashiwara [**10**] attached to the hypersurface Y.

The method of Gröbner basis was first introduced by Buchberger [**4**] (cf. [**7**]) for the polynomial ring, and has been extended to various rings of differential operators by several authors (e.g., [**8**], [**6**], [**17**], [**23**]). In particular, the singular locus and the rank (i.e., the dimension of the solution space) of a holonomic system are effectively computed by using the Gröbner basis algorithm for the ring of differential operators of polynomial or rational function coefficients (cf. [**23**], [**25**], [**19**]). The Gröbner basis for the ring of differential operators with analytic coefficients, which is more directly related to the analytic theory of systems of differential equations, was studied in [**6**], [**20**].

In this paper, we introduce variants of Gröbner bases for rings of differential operators with analytic or rational function coefficients. The analytic version, which we call the FD-Gröbner basis (F for filtration, and D for the ring of differential operators with analytic coefficients), solves the problem C completely and the problem A partially, but it would be difficult to carry out actual computation in case of more than two variables. On the other hand, the rational version, which we call the FR-Gröbner basis (R for the ring of differential operators with rational coefficients), has an algebraic and global nature and is more suited to actual computation by computers. Furthermore, it is shown that an FR-Gröbner basis is also an FD-Gröbner basis at a generic point of Y. These Gröbner bases are defined by a new total order among (exponents of) monomials of differential operators, and the fact that this order is not a well-order makes the situation slightly more complicated than in the usual theory of Gröbner basis.

In Section 1, we recall the definition of the Fuchsian system with one unknown function along a hypersurface and define their characteristic exponents.

In Section 2, we give the notion and fundamental properties of the FD-Gröbner basis for left ideals of the ring of partial differential operators with power series coefficients. This constitutes the theoretical foundation for the rest of the paper. In Section 3, we introduce a more practical notion of FR-Gröbner basis for the ring of differential operators with rational function coefficients and give an algorithm of finding FR-Gröbner bases. It is proved that an FR-Gröbner basis is also an FD-Gröbner basis at a generic point of the hypersurface Y. In Sections 4 and 5, we describe algorithmic methods for computing the characteristic exponents and the induced systems by using FD- or FR-Gröbner bases. Finally in Section 6, we give some examples of actual computation.

After the main part of the present work had been completed the author was informed that Takayama [**26**] proposed a different method (a kind of Hensel construction) for solving the problem C with the purpose of finding connection formulas of special functions of several variables.

1. Fuchsian system of partial differential equations along a hypersurface.

Let $(t, x)=(t, x_1, \cdots, x_n)$ be a coordinate system of the $(n+1)$-dimensional complex Euclidean space $X=\boldsymbol{C}^{n+1}$ (with $n\geqq 1$) and we use the notation $\partial_t=\partial/\partial t$ and $\partial_x=(\partial_1, \cdots, \partial_n)$ with $\partial_i=\partial/\partial x_i$. We write $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\partial_x^\beta=\partial_1^{\beta_1}\cdots\partial_n^{\beta_n}$, $|\alpha|=\alpha_1+\cdots+\alpha_n$ for multi-indices $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\beta=(\beta_1, \cdots, \beta_n)\in\boldsymbol{N}^n$ with $\boldsymbol{N}=\{0, 1, 2, \cdots\}$.

1.1. A filtration of $\mathcal{D}_0$ and Fuchsian operators. Let $\mathcal{D}$ be the sheaf on X of rings of linear partial differential operators with holomorphic coefficients. We denote by $\mathcal{D}_0$ the stalk of the sheaf $\mathcal{D}$ at the origin $0\in X$. Put $Y=\{(t, x)\in X\,|\,t=0\}$. The following arguments apply likewise to the stalk $\mathcal{D}_p$ of $\mathcal{D}$ at $p\in Y$. An element P of $\mathcal{D}_0$ is a linear partial differential operator whose coefficients are holomorphic at 0; i.e., convergent power series of (t, x). Thus P is written in the form

$$P=\sum_{\nu\geqq 0,\,\beta\in\boldsymbol{N}^n} a_{\nu,\beta}(t, x)\partial_t{}^\nu\partial_x{}^\beta=\sum_{\nu,\,\mu\geqq 0,\,\beta,\,\alpha\in\boldsymbol{N}^n} a_{\mu,\nu,\alpha,\beta}t^\mu x^\alpha\partial_t{}^\nu\partial_x{}^\beta, \tag{1.1}$$

where the sum is finite with respect to ν and β. The order $\mathrm{ord}(P)$ of P is defined as the maximum of $\nu+|\beta|$ such that $a_{\nu,\beta}(t, x)$ is non-zero as a power series.

We introduce a filtration $\{\mathcal{F}_m\}_{m\in\boldsymbol{Z}}$ of $\mathcal{D}_0$ as follows: For each integer m, put

$$\mathcal{F}_m=\left\{P=\sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta}t^\mu x^\alpha\partial_t{}^\nu\partial_x{}^\beta\in\mathcal{D}_0\,\middle|\,a_{\mu,\nu,\alpha,\beta}=0 \text{ if } \nu-\mu>m\right\}.$$

Then $\mathcal{F}_m$ is a $\boldsymbol{C}$-subspace of $\mathcal{D}_0$ and satisfies

$$\cdots \mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \cdots, \quad \bigcup_{m \in \boldsymbol{Z}} \mathcal{F}_m = \mathcal{D}_0 .$$

For a nonzero $P \in \mathcal{D}_0$, its *F-order* $\mathrm{ord}_F(P)$ is defined as the minimum integer m satisfying $P \in \mathcal{F}_m$. If the F-order of the operator P written as (1.1) is m, then we put

$$\hat{\sigma}(P) = \hat{\sigma}_m(P) = \sum_{\nu-\mu=m} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t{}^\nu \partial_x{}^\beta \in \mathcal{D}_0$$

and call it the *formal symbol* of P after [**14**]. (We put $\hat{\sigma}(0)=0$.) It is easy to see that $\hat{\sigma}(PQ) = \hat{\sigma}(P)\hat{\sigma}(Q)$ holds for $P, Q \in \mathcal{D}_0$ although $\hat{\sigma}(Q)\hat{\sigma}(P) \neq \hat{\sigma}(P)\hat{\sigma}(Q)$ in general.

The filtration defined above was introduced by Kashiwara [**10**] and was used systematically with the formal symbol for the study of induced systems by Laurent and Schapira [**14**].

The following definition is a slight generalization of that of [**1**].

Definition 1.1 ([**1**]). $P \in \mathcal{D}_0$ is called a *Fuchsian operator* along $Y = \{t=0\}$ at 0 if and only if P satisfies the following two conditions (FC1) and (FC2):

(FC1) There exist non-negative integers k, m and holomorphic functions $a_j(x)$ with $a_0(0) \neq 0$ such that

$$\hat{\sigma}(P) = \sum_{j=0}^{\min\{k,m\}} a_j(x) t^{k-j} \partial_t{}^{m-j} .$$

(FC2) The order of $\hat{\sigma}(P)$ is equal to the order of P.

Definition 1.2. $P \in \mathcal{D}_0$ is said to be *formally Fuchsian* along Y at 0 if it satisfies the condition (FC1).

We remark that the notion of formally Fuchsian operator (or system) was introduced in [**14**] under the name of ellipticity along Y.

Let Y be a non-singular complex hypersurface of X. Then Definitions 1.1 and 1.2 also apply to such Y with a local coordinate (t, x) such that $Y = \{t=0\}$. Then these definitions are independent of the choice of such a local coordinate system.

1.2. Fuchsian systems of partial differential equations. We consider the system of linear partial differential equations

$$\mathcal{M} : P_1 u = \cdots = P_s u = 0$$

for an unknown function u, where $P_1, \cdots, P_s$ are linear partial differential operators whose coefficients are holomorphic functions on an open subset Ω of $X = \boldsymbol{C}^{n+1}$. (In the sequel, we assume $0 \in \Omega$.)

In order to study the system $\mathcal{M}$, it is natural to consider the sheaf of ideals $\mathcal{J}$ of $\mathcal{D}$ generated by $P_1, \cdots, P_s$; i.e., $\mathcal{J}=\mathcal{D}P_1+\cdots+\mathcal{D}P_s$ and regard the system $\mathcal{M}$ as a coherent sheaf of $\mathcal{D}$-modules $\mathcal{D}/\mathcal{J}$ (cf. [**9**]). Let Y be a non-singular complex analytic hypersurface in Ω and p be a point of Y.

DEFINITION 1.3. In the notation above, the system $\mathcal{M}$ is called a *Fuchsian system* along Y at p after [**13**] if there exists an element (section) P of $\mathcal{J}$ which is a Fuchsian operator along Y at p. Moreover, $\mathcal{M}$ is said to be *formally Fuchsian* along Y at p if there exists $P\in\mathcal{J}$ which is formally Fuchsian along Y at p.

1.3. Characteristic exponents of a Fuchsian system. Let $\bar{\mathcal{D}}_0$ be the graded ring associated with the filtration $\{\mathcal{F}_m\}$; i.e., $\bar{\mathcal{D}}_0=\bigoplus_{m\in \boldsymbol{Z}}\mathcal{F}_m/\mathcal{F}_{m-1}$. Note that $\bar{\mathcal{D}}_0$ is a non-commutative ring. For each $m\in\boldsymbol{Z}$, the formal symbol induces a homomorphism

$$\hat{\sigma}=\hat{\sigma}_m:\mathcal{F}_m\longrightarrow \mathcal{F}_m/\mathcal{F}_{m-1}\subset\bar{\mathcal{D}}_0\,.$$

We shall define an injective ring homomorphism ψ of $\bar{\mathcal{D}}_0$ into the ring

$$\mathcal{D}_0'[\theta,\tau,\tau^{-1}]:=\bigoplus_{m\in\boldsymbol{Z}}\mathcal{D}_0'[\theta]\tau^m\,,$$

where τ and θ are indeterminates and $\mathcal{D}_0'[\theta]$ denotes the polynomial ring in θ with coefficients in the ring $\mathcal{D}_0'=\boldsymbol{C}\{x\}\langle\partial_x\rangle$ of differential operators in x with convergent power series coefficients. We give a ring structure to $\mathcal{D}_0'[\theta,\tau,\tau^{-1}]$ by

$$(P(\theta,x,\partial_x)\tau^j)\cdot(Q(\theta,x,\partial_x)\tau^k):=P(\theta-k,x,\partial_x)Q(\theta,x,\partial_x)\tau^{j+k}\,.$$

For any element P of $\mathcal{F}_m\setminus\mathcal{F}_{m-1}$, $\hat{\sigma}(P)$ can be written uniquely in the form

$$\hat{\sigma}(P)=t^{-m}\hat{P}(t\partial_t,x,\partial_x)\,.$$

Then we put $\psi(P)=\hat{P}(\theta,x,\partial_x)\tau^m$. This defines a map $\psi:\mathcal{F}_m/\mathcal{F}_{m-1}\to\mathcal{D}_0'[\theta]\tau^m$ for each m. Moreover, it is easy to see that ψ is injective for all m and bijective for $m\leqq 0$. Thus we easily get

LEMMA 1.4. *$\psi:\bar{\mathcal{D}}_0\to\mathcal{D}_0'[\theta,\tau,\tau^{-1}]$ is an injective ring homomorphism.*

Now assume the system $\mathcal{M}$ above is Fuchsian along $Y=\{t=0\}$ at 0. (In fact, it suffices to assume $\mathcal{M}$ is formally Fuchsian along Y for the following definitions.) Let $\mathcal{J}_0$ be the stalk at 0 of the sheaf of left ideals $\mathcal{J}=\mathcal{D}P_1+\cdots+\mathcal{D}P_s$. Let us define a left ideal $\bar{\mathcal{J}}_0$ of $\mathcal{D}_0'[\theta,\tau,\tau^{-1}]$ by $\bar{\mathcal{J}}_0:=\bigoplus_{m\in\boldsymbol{Z}}\hat{\sigma}_m(\mathcal{J}_0\cap\mathcal{F}_m)$. Put

$$\mathcal{O}_0'[\theta,\tau,\tau^{-1}]=\bigoplus_{m\in\boldsymbol{Z}}\mathcal{O}_0'[\theta]\tau^m\subset\mathcal{D}_0'[\theta,\tau,\tau^{-1}]$$

with $\mathcal{O}_0'=\boldsymbol{C}\{x\}$ (the ring of convergent power series in x). Let $\mathcal{G}$ be the smallest left ideal of $\mathcal{O}'_0[\theta, \tau, \tau^{-1}]$ that contains $\psi(\mathcal{J}_0)\cap\mathcal{O}'_0[\theta, \tau, \tau^{-1}]$ and put $\mathcal{J}_Y(\mathcal{M}, 0)=\mathcal{J}\cap\mathcal{O}'_0[\theta]$, which is an ideal of the commutative ring $\mathcal{O}'_0[\theta]$. Then it is easy to see that $\mathcal{J}$ is generated by $\mathcal{J}_Y(\mathcal{M}, 0)$ over $\mathcal{O}'_0[\theta, \tau, \tau^{-1}]$. Moreover we can easily verify

LEMMA 1.5.

$$\mathcal{J}_Y(\mathcal{M}, 0) = \{f(\theta, x)\in\mathcal{O}'_0[\theta] \mid f(\theta, x)\tau^{-m}\in\psi(\mathcal{J}_0)\cap\mathcal{O}'_0[\theta]\tau^{-m} \text{ for some } m\geqq 0\}.$$

The ideal $\mathcal{J}_Y(\mathcal{M}, p)$ of $\mathcal{O}'_p[\theta]$ is defined likewise with 0 replaced by a point p of Y, where $\mathcal{O}'_p$ denotes the ring of germs of holomorphic functions in x at p.

DEFINITION 1.6. For a point p of Y we call the set

$$e_Y(\mathcal{M}, p) := \{\theta\in\boldsymbol{C} \mid f(\theta, p)=0 \text{ for any } f\in\mathcal{J}_Y(\mathcal{M}, p)\}$$

the *set of the characteristic exponents* of $\mathcal{M}$ along Y at p.

DEFINITION 1.7. We define another ideal $\tilde{\mathcal{J}}_Y(\mathcal{M}, p)$ of $\mathcal{O}'_p[\theta]$ by

$$\tilde{\mathcal{J}}_Y(\mathcal{M}, p) = \{f\in\mathcal{O}'_p[\theta] \mid af\in\mathcal{J}_Y(\mathcal{M}, p) \text{ for some } a\in\mathcal{O}'_p\},$$

and the *set of the strong characteristic exponents* of $\mathcal{M}$ along Y at $p\in Y$ by

$$\tilde{e}_Y(\mathcal{M}, p) = \{\theta\in\boldsymbol{C} \mid f(\theta, p)=0 \text{ for any } f\in\tilde{\mathcal{J}}_Y(\mathcal{M}, p)\}.$$

LEMMA 1.8. *Suppose that the system $\mathcal{M}$ is formally Fuchsian along Y at 0. Then the ideal $\tilde{\mathcal{J}}_Y(\mathcal{M}, 0)$ is generated by a polynomial $f\in\tilde{\mathcal{J}}_Y(\mathcal{M}, 0)$ monic in θ.*

PROOF. We denote by $\mathcal{K}'_0$ the quotient field of the ring $\mathcal{O}'_0$. Note that $\mathcal{O}'_0$ is a unique factorization domain. Let $\mathcal{L}$ be the ideal of $\mathcal{K}'_0[\theta]$ generated by $\tilde{\mathcal{J}}_Y(\mathcal{M}, 0)$. Then we have

$$\mathcal{L} = \{cf \mid f\in\tilde{\mathcal{J}}_Y(\mathcal{M}, 0),\ c\in\mathcal{K}'_0\}.$$

Let f be an element of $\tilde{\mathcal{J}}_Y(\mathcal{M}, 0)$ with least degree with respect to θ. (We may assume that f is primitive.) Then $\mathcal{L}$ is generated by f. Since $\mathcal{M}$ is formally Fuchsian along Y, there exists a monic polynomial $g\in\mathcal{J}_Y(\mathcal{M}, 0)$. Then f divides g in $\mathcal{K}'_0[\theta]$. The Gauss lemma implies that f divides g in $\mathcal{O}'_0[\theta]$ and f is a monic polynomial in θ. Now let h be an element of $\tilde{\mathcal{J}}_Y(\mathcal{M}, 0)$. Then f divides h in $\mathcal{K}'_0[\theta]$, hence in $\mathcal{O}'_0[\theta]$. This completes the proof.

EXAMPLE 1.9. Put $n=1$, $x=x_1$ and let us consider the system

$$\mathcal{M} : (t\partial_t-a)(t\partial_t-b)u = x(t\partial_t-a)u = 0$$

with distinct constants $a, b\in\boldsymbol{C}$. Then we have

$$\mathcal{I}_Y(\mathcal{M}, 0) = \mathcal{O}_0'[\theta](\theta-a)(\theta-b)+\mathcal{O}_0'[\theta]x(\theta-a),$$

$$\tilde{\mathcal{I}}_Y(\mathcal{M}, 0) = \mathcal{O}_0'[\theta](\theta-a)$$

and hence

$$e_Y(\mathcal{M}, 0) = \{a, b\}, \qquad \tilde{e}_Y(\mathcal{M}, 0) = \{a\}.$$

Note that any multi-valued analytic solution of $\mathcal{M}$ is in the form $u=v(x)t^a$ with v holomorphic, whereas, in the real domain $\mathcal{M}$ has a distribution solution $t_+^a+\delta(x)t_+^b$.

1.4. Boundary value problem for Fuchsian systems. Here we recall some facts on the structure of analytic solutions of a Fuchsian system. First, let us recall the notion of induced (or tangential) system. Let $\mathcal{M}$ and $\mathcal{I}$ be as in Section 1.2. Then the induced system $\mathcal{M}_Y$ of $\mathcal{M}$ along $Y=\{t=0\}$ is the sheaf of $\mathcal{D}'$-modules

$$\mathcal{M}_Y := \mathcal{M}/t\mathcal{M} = \mathcal{D}/(t\mathcal{D}+\mathcal{I}),$$

where $\mathcal{D}'$ denotes the sheaf on Y of the ring of linear differential operators with holomorphic functions in x as coefficients. It is shown in [**14**] that $\mathcal{M}_Y$ is a coherent $\mathcal{D}'$-module if $\mathcal{M}$ is formally Fuchsian along Y.

PROPOSITION 1.10 ([**13**, Théorème 3.2.2]). *Assume that the system $\mathcal{M}$ is Fuchsian along Y. Then there exists a canonical sheaf isomorphism*

$$\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O})|_Y \cong \mathcal{H}om_{\mathcal{D}'}(\mathcal{M}_Y, \mathcal{O}'),$$

where $\mathcal{O}$ and $\mathcal{O}'$ denote the sheaves of holomorphic functions in (t, x) and in x respectively, and $\mathcal{H}om$ the sheaf of homomorphisms.

The following proposition follows from [**22**, Theorem 1.3.9] (see also [**21**]):

PROPOSITION 1.11. *Assume that the system $\mathcal{M}$ is Fuchsian along Y at 0 and there exists a Fuchsian operator $P\in\mathcal{I}_0$ whose characteristic exponents $\theta_1, \cdots, \theta_m$ are all constant with multiplicity one. Assume also that $\theta_i-\theta_j$ is not an integer for any $i\neq j$. Put $S=\{i\in\{1, \cdots, m\} \mid \theta_i\in\tilde{e}_Y(\mathcal{M}, 0)\}$. Then any (multi-valued) analytic solution u of $\mathcal{M}$ on $U\setminus Y$ with U being a neighborhood of $0\in X$ can be written in the form*

$$u = \sum_{i\in S} v_i(t, x)t^{\theta_i}$$

with holomorphic functions v_i on a neighborhood of $U\cap Y$.

2. FD-Gröbner basis—analytic and local algorithmic method.

In this section we develop the theory of FD-Gröbner bases for left ideals of the ring $\mathcal{D}_0$ of differential operators with analytic coefficients. Instead of

$\mathscr{D}_0$, the following arguments apply also to the stalk $\mathscr{D}_p$ of the sheaf $\mathscr{D}$ at $p\in Y=\{(t, x)\mid t=0\}$.

Let $\prec$ be a lexicographic (or an inverse lexicographic) order of $\boldsymbol{N}^n$ with $\boldsymbol{N}:=\{0, 1, 2, \cdots\}$. We define a total order $\prec_{FD}$ of the set $\boldsymbol{N}^{2n+2}$, which we call the *FD-order*, as follows: For two indices $(\mu, \nu, \alpha, \beta)$ and $(\mu', \nu', \alpha', \beta')\in \boldsymbol{N}\times\boldsymbol{N}\times\boldsymbol{N}^n\times\boldsymbol{N}^n$,

$$\begin{aligned}(\mu, \nu, \alpha, \beta)\prec_{FD}(\mu', \nu', \alpha', \beta') \quad &\text{if and only if } (\nu-\mu<\nu'-\mu')\\ &\text{or } (\nu-\mu=\nu'-\mu',\ |\beta|<|\beta'|)\\ &\text{or } (\nu-\mu=\nu'-\mu',\ |\beta|=|\beta'|,\ \nu<\nu')\\ &\text{or } (\nu=\nu',\ \mu=\mu',\ |\beta|=|\beta'|,\ \beta\prec\beta')\\ &\text{or } (\nu=\nu',\ \mu=\mu',\ \beta=\beta',\ |\alpha|>|\alpha'|)\\ &\text{or } (\nu=\nu',\ \mu=\mu',\ \beta=\beta',\ |\alpha|=|\alpha'|,\ \alpha\prec\alpha').\end{aligned}$$

Let the *FR-order* $\prec_{FR}$ be the order of $\boldsymbol{N}^{n+2}$ induced by $\prec_{FD}$; i.e., we define

$$(\mu, \nu, \beta)\prec_{FR}(\mu', \nu', \beta') \quad \text{if and only if } (\mu, \nu, 0, \beta)\prec_{FD}(\mu', \nu', 0, \beta').$$

It is easy to see that any subset of $\{(\mu, \nu, \alpha, \beta)\in\boldsymbol{N}^{2n+2}\mid\nu+|\beta|\leqq m\}$ has a maximum element with respect to the FD-order, and any subset of $\{(\mu, \nu, \beta)\in \boldsymbol{N}^{n+2}\mid\nu-\mu\geqq m\}$ has a minimum element with respect to the FR-order for any m. (This definition of the FD-order can be generalized to some extent, but we do not discuss this problem here.) For an element $P\in\mathscr{D}_0$ of the form

$$P=\sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta}t^{\mu}x^{\alpha}\partial_t{}^{\nu}\partial_x{}^{\beta},$$

we define the set of exponents, leading exponent, leading coefficient, leading term of P with respect to the FD-order by

$$\begin{aligned}&\mathrm{exps}_{FD}(P)=\{(\mu, \nu, \alpha, \beta)\mid a_{\mu,\nu,\alpha,\beta}\neq 0\},\\ &\mathrm{lexp}_{FD}(P)=\max{}_{FD}(\mathrm{exps}_{FD}(P)),\\ &\mathrm{lcoef}_{FD}(P)=a_{\mu,\nu,\alpha,\beta} \quad \text{with } (\mu, \nu, \alpha, \beta):=\mathrm{lexp}_{FD}(P),\\ &\mathrm{lterm}_{FD}(P)=a_{\mu,\nu,\alpha,\beta}t^{\mu}x^{\alpha}\partial_t{}^{\nu}\partial_x{}^{\beta} \quad \text{with } (\mu, \nu, \alpha, \beta):=\mathrm{lexp}_{FD}(P),\end{aligned}$$

where $\max_{FD}$ denotes the maximum with respect to the FD-order. (If $P=0$, then we put $\mathrm{lexp}_{FD}(P)=(\infty, 0, 0, 0)$, and suppose $(\infty, 0, 0, 0)\prec_{FD}(\mu, \nu, \alpha, \beta)$ for any $(\mu, \nu, \alpha, \beta)\in\boldsymbol{N}^{2n+2}$.) Let $\varpi:\boldsymbol{N}^{2n+2}\to\boldsymbol{N}^{n+2}$ be the projection defined by $\varpi(\mu, \nu, \alpha, \beta)=(\mu, \nu, \beta)$. Then through this projection, we also define the leading exponent, leading coefficient and leading term of P with respect to the FR-order by

$$\mathrm{lexp}_{FR}(P) = \varpi(\mathrm{lexp}_{FD}(P)),$$

$$\mathrm{lcoef}_{FR}(P) = \sum_{\alpha\in N^n} a_{\mu_0,\nu_0,\alpha,\beta_0} x^\alpha \quad \text{with } (\mu_0, \nu_0, \beta_0) := \mathrm{lexp}_{FR}(P),$$

$$\mathrm{lterm}_{FR}(P) = \mathrm{lcoef}_{FR}(P) t^{\mu_0}\partial_t^{\nu_0}\partial_x^{\beta_0} \quad \text{with } (\mu_0, \nu_0, \beta_0) := \mathrm{lexp}_{FR}(P).$$

Moreover, for an exponent $(\mu, \nu, \beta)\in \boldsymbol{N}^{n+2}$, we set

$$\mathrm{coef}_{FR}(P, (\mu, \nu, \beta)) = \sum_{\alpha} a_{\mu,\nu,\alpha,\beta} x^\alpha .$$

Recall that the principal symbol of P (of order m) is defined by

$$\sigma_m(P) = \sum_{\mu\in N,\, \alpha\in N^n,\, \nu+|\beta|=m} a_{\mu,\nu,\alpha,\beta} t^\mu \tau^\nu x^\alpha \xi^\beta$$

regarded as an element of the ring of the convergent power series $\boldsymbol{C}\{t, \tau, x, \xi\}$ with $\xi=(\xi_1, \cdots, \xi_n)$ and $\xi^\beta=\xi_1^{\beta_1}\cdots\xi_n^{\beta_n}$ if P is of order $\leqq m$. We also write $\sigma(P)=\sigma_m(P)$ if P is precisely of order m.

The following two lemmas are easily proved.

LEMMA 2.1. *For $P, Q\in\mathcal{D}_0$ we have*

$$\mathrm{lexp}_{FD}(PQ) = \mathrm{lexp}_{FD}(P)+\mathrm{lexp}_{FD}(Q),$$

$$\mathrm{lcoef}_{FD}(PQ) = \mathrm{lcoef}_{FD}(P)\mathrm{lcoef}_{FD}(Q),$$

$$\mathrm{lexp}_{FR}(PQ) = \mathrm{lexp}_{FR}(P)+\mathrm{lexp}_{FR}(Q),$$

$$\mathrm{lcoef}_{FR}(PQ) = \mathrm{lcoef}_{FR}(P)\mathrm{lcoef}_{FR}(Q).$$

LEMMA 2.2. *$P\in\mathcal{D}_0$ is formally Fuchsian along Y at* 0 *if and only if* $\mathrm{lexp}_{FD}(P)=(\mu, \nu, 0, 0)\in \boldsymbol{N}\times\boldsymbol{N}\times\boldsymbol{N}^n\times\boldsymbol{N}^n$ *with some $\mu, \nu\in\boldsymbol{N}$.*

LEMMA 2.3 (A division theorem). *Let P and $P_1, \cdots, P_s$ be elements of $\mathcal{D}_0$. Then for any integer m, there exist elements $Q_1, \cdots, Q_s$ and R of $\mathcal{D}_0$ such that*

$$P = \sum_{i=1}^{s} Q_i P_i + R,$$

$$\mathrm{exps}_{FD}(R)\cap \bigcup_{i=1}^{s} (\mathrm{lexp}_{FD}(P_i)+\boldsymbol{N}^{2n+2}) \subset \mathrm{lexp}_{FD}(\mathcal{F}_m),$$

$$\mathrm{lexp}_{FD}(Q_iP_i) \leqq_{FD} \mathrm{lexp}_{FD}(P), \qquad \mathrm{lexp}_{FD}(R) \leqq_{FD} \mathrm{lexp}_{FD}(P).$$

We denote such R, which is not necessarily unique, by $\mathrm{red}_{FD}(P, \{P_1, \cdots, P_s\}, m)$.

PROOF. Given $P, P_1, \cdots, P_s$, put $E_i=\mathrm{lexp}_{FD}(P_i)+\boldsymbol{N}^{2n+2}$ and $E=\bigcup_{i=1}^{s} E_i$. We consider all the possible expressions of the form

$$P = \sum_{i=1}^{s} Q_i P_i + R \tag{2.1}$$

with Q_i, $R\in\mathscr{D}_0$ satisfying

$$\mathrm{lexp}_{FD}(Q_iP_i) \leqq_{FD} \mathrm{lexp}_{FD}(P) \qquad \text{for any } i. \tag{2.2}$$

Let $\mathrm{redlexp}_{FR}(R)$ be the maximum element of the set

$$\{(\mu,\nu,\beta)\in \boldsymbol{N}\times\boldsymbol{N}\times\boldsymbol{N}^n \mid \nu-\mu\geqq m+1,\ (\mu,\nu,\beta)\in\varpi(\mathrm{exps}_{FD}(R)\cap E)\}$$

with respect to the FR-order. (Put $\mathrm{redlexp}_{FR}(R)=(\infty, 0, 0)$ if the above set is empty.) Now take an expression that minimizes $\mathrm{redlexp}_{FR}(R)$ (in the FR-order) among all the expressions of the form (2.1) with the condition (2.2). Suppose $\mathrm{redlexp}_{FR}(R)\neq(\infty, 0, 0)$ and put $(\mu_0, \nu_0, \beta_0)=\mathrm{redlexp}_{FR}(R)$, $(\mu_i, \nu_i, \beta_i)=\mathrm{lexp}_{FR}(P_i)$ $(i=1, \cdots, s)$. Write

$$R=\sum_{\mu,\nu,\beta} a_{\mu,\nu,\beta}(x)t^{\mu}\partial_t^{\nu}\partial_x^{\beta}, \qquad P_i=\sum_{\mu,\nu,\beta} a_{i,\mu,\nu,\beta}(x)t^{\mu}\partial_t^{\nu}\partial_x^{\beta}.$$

Put $S=\{i\in\{1,\cdots,s\}\mid(\mu_0,\nu_0,\beta_0)\in\varpi(E_i)\}$ and $a(x)=a_{\mu_0,\nu_0,\beta_0}(x)$, $a_i(x)=a_{i,\mu_i,\nu_i,\beta_i}(x)$ for $i\in S$. By the Weierstrass-Hironaka division theorem for convergent power series (cf. [**3**]), there exist convergent power series $q_i(x)$, $r(x)$ such that

$$a(x)=\sum_{i\in S} q_i(x)a_i(x)+r(x), \qquad r(x)=\sum_{\alpha} r_\alpha x^\alpha,$$

$$\mathrm{lexp}_{FD}(q_i(x)a_i(x)) \leqq_{FD} \mathrm{lexp}(a(x)) \qquad \text{for } i\in S,$$

$$r_\alpha=0 \qquad \text{if } \alpha\in\bigcup_{i\in S}(\alpha_i+\boldsymbol{N}^n).$$

Put

$$Q_i'=q_i(x)t^{\mu_0-\mu_i}\partial_t^{\nu_0-\nu_i}\partial_x^{\beta_0-\beta_i}, \qquad R'=R-\sum_{i\in S}Q_i'P_i.$$

Then it follows $\mathrm{redlexp}_{FR}(R') \prec_{FR} \mathrm{redlexp}_{FR}(R)$ since we have $\mathrm{redlexp}_{FR}(R')\leqq_{FR}(\mu_0, \nu_0, \beta_0)$ and

$$\mathrm{coef}_{FR}(R', (\mu_0,\nu_0,\beta_0)) = a_{\mu_0,\nu_0,\beta_0}(x)-\sum_{i\in S}q_i(x)a_i(x)=r(x)$$

with $\mathrm{exps}_{FD}(r(x)t^{\mu_0}\partial_t^{\nu_0}\partial_x^{\beta_0})\cap E=\emptyset$. Moreover we have

$$P=\sum_{i=1}^{s}Q_iP_i+\sum_{i\in S}Q_i'P_i+R'.$$

This contradicts the minimum property assumed above. This completes the proof.

DEFINITION 2.4. Let $\mathscr{I}_0$ be a left ideal of $\mathscr{D}_0$. Then a finite subset $\boldsymbol{G}=\{P_1, \cdots, P_s\}$ of $\mathscr{I}_0$ is called an *FD-Gröbner basis* of $\mathscr{I}_0$ (along Y) if it satisfies the following two conditions:

(1) $\boldsymbol{G}$ generates $\mathscr{I}_0$, i.e., $\mathscr{I}_0=\mathscr{D}_0P_1+\cdots+\mathscr{D}_0P_s$.

(2) Put $E_{FD}(\mathscr{I}_0)=\{\mathrm{lexp}_{FD}(P)\mid P\in\mathscr{I}_0\}$. Then we have

$$E_{FD}(\mathcal{I}_0) = \bigcup_{P \in G} (\mathrm{lexp}_{FD}(P) + \boldsymbol{N}^{2n+2}) .$$

DEFINITION 2.5. For $P, Q \in \mathcal{D}_0$ with

$$\mathrm{lexp}_{FD}(P) = (\mu, \nu, \alpha, \beta), \qquad \mathrm{lexp}_{FD}(Q) = (\mu', \nu', \alpha', \beta'),$$

the S-polynomial (or S-operator) of P and Q is defined by

$$\begin{aligned}\mathrm{sp}_{FD}(P, Q) = {}& \mathrm{lcoef}_{FD}(Q) t^{\mu \vee \mu' - \mu} \partial_t^{\nu \vee \nu' - \nu} x^{\alpha \vee \alpha' - \alpha} \partial_x^{\beta \vee \beta' - \beta} P \\ & - \mathrm{lcoef}_{FD}(P) t^{\mu \vee \mu' - \mu'} \partial_t^{\nu \vee \nu' - \nu'} x^{\alpha \vee \alpha' - \alpha'} \partial_x^{\beta \vee \beta' - \beta'} Q ,\end{aligned}$$

where we use the notation

$$\nu \vee \nu' := \max\{\nu, \nu'\}, \qquad \alpha \vee \alpha' := (\max\{\alpha_1, \alpha_1'\}, \cdots, \max\{\alpha_n, \alpha_n'\})$$

for $\nu, \nu' \in \boldsymbol{N}$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha' = (\alpha_1', \cdots, \alpha_n') \in \boldsymbol{N}^n$.

THEOREM 2.6. *Let $\mathcal{I}_0$ be a left ideal of $\mathcal{D}_0$ and $\boldsymbol{G} = \{P_1, \cdots, P_s\}$ be a set of generators of $\mathcal{I}_0$. Then the following two conditions for $\boldsymbol{G}$ are equivalent:*

(1) *$\boldsymbol{G}$ is an FD-Gröbner basis of $\mathcal{I}_0$.*

(2) *For any i, j with $1 \leqq i < j \leqq s$ and for any $m \in \boldsymbol{Z}$, there exist $Q_{ij1}, \cdots, Q_{ijs} \in \mathcal{D}_0$ and $R_{ij} \in \mathcal{F}_m$ such that*

$$\mathrm{sp}_{FD}(P_i, P_j) = \sum_{k=1}^{s} Q_{ijk} P_k + R_{ij}$$

with $\mathrm{lexp}_{FD}(Q_{ijk} P_k) \prec_{FD} \mathrm{lexp}_{FD}(P_i) \vee \mathrm{lexp}_{FD}(P_j)$ *for any k.*

PROOF. Without loss of generality we may assume $\mathrm{lcoef}_{FD}(P_k) = 1$ for $k = 1, \cdots, s$. Assume (1) and let m be an arbitrary integer. Then in view of Lemma 2.3, there exist $Q_1, \cdots, Q_s, R \in \mathcal{D}_0$ such that $\mathrm{sp}_{FD}(P_i, P_j) = \sum_{k=1}^{s} Q_k P_k + R$ with $\mathrm{exps}_{FD}(R) \cap E \subset E_{FD}(\mathcal{F}_m)$ and

$$\mathrm{lexp}_{FD}(Q_k P_k) \leqq_{FD} \mathrm{lexp}_{FD}(\mathrm{sp}_{FD}(P_i, P_j)) \prec_{FD} \mathrm{lexp}_{FD}(P_i) \vee \mathrm{lexp}_{FD}(P_j) ,$$

where $E := \bigcup_{i=1}^{s} (\mathrm{lexp}_{FD}(P_i) + \boldsymbol{N}^{2n+2})$. Since $R \in \mathcal{I}_0$ and $\boldsymbol{G}$ is an FD-Gröbner basis, we have $\mathrm{lexp}_{FD}(R) \in E \cap \mathrm{exps}_{FD}(R) \subset E_{FD}(\mathcal{F}_m)$. This implies (2).

Next, assume (2). In order to prove (1), let P be an arbitrary element of $\mathcal{I}_0$. In the course of the proof, we use the ring $\hat{\mathcal{D}}_0 := \boldsymbol{C}[[t, x]]\langle \partial_t, \partial_x \rangle$ of differential operators with formal power series coefficients. Note that lexp_{FD}, etc., are also defined for the elements of $\hat{\mathcal{D}}_0$. We shall prove $\mathrm{lexp}_{FD}(P) \in E$ in two steps.

(1st step) Fix an integer $m < 0$ such that $\mathrm{ord}_F(P) > m$. We consider all the possible expressions for P of the form

$$P = \sum_{k=1}^{s} Q_k P_k + R \tag{2.3}$$

with $Q_k \in \hat{\mathcal{D}}_0$ and $R \in \hat{\mathfrak{F}}_m$, where

$$\hat{\mathfrak{F}}_m := \left\{ P = \sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t^\nu \partial_x^\beta \in \hat{\mathcal{D}}_0 \,\middle|\, a_{\mu,\nu,\alpha,\beta} = 0 \text{ if } \nu - \mu > m \right\}.$$

There is at least one such expression since $P \in \mathcal{J}_0$. Among the expressions of the form (2.3) let us choose one that minimizes $\max_{FR}\{\mathrm{lexp}_{FR}(Q_k P_k) \mid k=1, \cdots, s\}$ with respect to the FR-order. In the sequel, we suppose that the expression (2.3) has this minimum property. Put $\mathrm{lexp}_{FD}(P_k) = (\mu_k, \nu_k, \beta_k)$. Our aim to show

$$(2.4) \qquad (\mu, \nu, \beta) := \max_{FR}\{\mathrm{lexp}_{FR}(Q_k P_k) \mid k=1, \cdots, s\} = \mathrm{lexp}_{FR}(P).$$

Suppose $(\mu, \nu, \beta) \succ_{FR} \mathrm{lexp}_{FR}(P)$. Write the S-polynomial explicitly as

$$\mathrm{sp}_{FD}(P_i, P_j) = S_{ij} P_i - S_{ji} P_j$$

for $i<j$, where we put $S_{ij} = t^{\mu_{ij}} \partial_t^{\nu_{ij}} x^{\alpha_{ij}} \partial_x^{\beta_{ij}}$ with $\mu_{ij} := \mu_i \vee \mu_j - \mu_i$, etc. for $i \neq j$. Then using the assumption (2) with m replaced by $m' := m - \nu - 1 - \max\{\mu_i \vee \mu_j \mid 1 \leqq i < j \leqq s\}$, we have, for $1 \leqq i < j \leqq s$,

$$S_{ij} P_i - S_{ji} P_j = \sum_{k=1}^{s} Q_{ijk} P_k + R_{ij}$$

with the same conditions as in (2) with m replaced by m'. Put $p_i = \sigma(\mathrm{lterm}_{FR}(P_i))$, $s_{ij} = \sigma(S_{ij})$, and

$$q_{ijk} = \begin{cases} \sigma(\mathrm{lterm}_{FR}(Q_{ijk})) & \text{if } \mathrm{lexp}_{FR}(Q_{ijk} P_k) = (\mu_i \vee \mu_j,\ \nu_i \vee \nu_j,\ \beta_i \vee \beta_j) \\ 0 & \text{otherwise,} \end{cases}$$

which are all monomials in t, τ, ξ with power series in x as coefficients. Then we have the relations

$$(2.5) \qquad s_{ij} p_i - s_{ji} p_j = \sum_{k=1}^{s} q_{ijk} p_k \qquad (1 \leqq i < j \leqq s)$$

in the ring of formal power series $\boldsymbol{C}[[t, \tau, x, \xi]]$. Hence $\{p_1, \cdots, p_s\}$ constitutes a Gröbner (or standard) basis of the ideal which they generate in $\boldsymbol{C}[[t, \tau, x, \xi]]$ with respect to the order $\prec_O$ in $\boldsymbol{N}^{2n+2}$ defined as follows: For two indices $(\mu, \nu, \alpha, \beta)$ and $(\mu', \nu', \alpha', \beta') \in \boldsymbol{N}^{2n+2}$,

$(\mu, \nu, \alpha, \beta) \prec_O (\mu', \nu', \alpha', \beta')$ if and only if

$\mu + \nu + |\alpha| + |\beta| > \mu' + \nu' + |\alpha'| + |\beta'|$

or $(\mu + \nu + |\alpha| + |\beta| = \mu' + \nu' + |\alpha'| + |\beta'|$

and $(\nu, \mu, \beta, \alpha) \prec (\nu', \mu', \beta', \alpha')$ in the lexicographic order $\prec$ of $\boldsymbol{N}^{2n+2}$).

The theory of Gröbner basis for (formal) power series (cf. [6]) shows that the submodule

$$\left\{(q_1, \cdots, q_s) \in (\boldsymbol{C}[[t, \tau, x, \xi]])^s \,\middle|\, \sum_{k=1}^{s} q_k p_k = 0\right\}$$

of $\boldsymbol{C}[[t, \tau, x, \xi]]^s$ is generated by the relations (2.5), i.e., generated by vectors of power series

$$\vec{v}_{ij} := (0, \cdots, \overset{(i)}{s_{ij}}, \cdots, -\overset{(j)}{s_{ji}}, \cdots, 0) - (q_{ij1}, \cdots, q_{ijs})$$

for $1 \leqq i < j \leqq s$. Here note that the k-th component of $\vec{v}_{ij}$ is written in the form $v_{ijk}(x) t^{\mu_i \vee \mu_j - \mu_k} \tau^{\nu_i \vee \nu_j - \nu_k} \xi^{\beta_i \vee \beta_j - \beta_k}$ with some $v_{ijk}(x) \in \boldsymbol{C}\{x\}$.

Now returning to the expression (2.3) with the minimum property assumed above, put

$$q_k = \begin{cases} \sigma(\mathrm{lterm}_{FR}(Q_k)) & \text{if } \mathrm{lexp}_{FR}(Q_k P_k) = (\mu, \nu, \beta) \\ 0 & \text{otherwise.} \end{cases}$$

Then expression (2.3) and the assumption $(\mu, \nu, \beta) \succ_{FR} \mathrm{lexp}_{FR}(P)$ imply $\sum_{k=1}^{s} q_k p_k = 0$. Hence there exist $u_{ij} \in \boldsymbol{C}[[t, \tau, x, \xi]]$ such that

$$(q_1, \cdots, q_s) = \sum_{1 \leq i < j \leq s} u_{ij} \vec{v}_{ij}. \tag{2.6}$$

Moreover, considering the monomials in t, τ, ξ of exponent $(\mu - \mu_k, \nu - \nu_k, \beta - \beta_k)$ with coefficients in $\boldsymbol{C}[[x]]$ in the k-th component of the both sides of (2.6), we may assume that u_{ij} is of the form

$$u_{ij} = c_{ij}(x) t^{\mu - \mu_i \vee \mu_j} \tau^{\nu - \nu_i \vee \nu_j} \xi^{\beta - \beta_i \vee \beta_j}$$

with some $c_{ij}(x) \in \boldsymbol{C}[[x]]$. (It follows $c_{ij}(x) = 0$ if $(\mu, \nu, \beta) \neq (\mu_i \vee \mu_j, \nu_i \vee \nu_j, \beta_i \vee \beta_j)$.) Put

$$U_{ij} = c_{ij}(x) t^{\mu - \mu_i \vee \mu_j} \partial_t^{\nu - \nu_i \vee \nu_j} \partial_x^{\beta - \beta_i \vee \beta_j},$$

$$\vec{V}_{ij} = (0, \cdots, \overset{(i)}{S_{ij}}, \cdots, -\overset{(j)}{S_{ji}}, \cdots, 0) - (Q_{ij1}, \cdots, Q_{ijs}),$$

$$(Q_1', \cdots, Q_s') = (Q_1, \cdots, Q_s) - \sum_{i<j} U_{ij} \vec{V}_{ij}.$$

Then from (2.3) we get

$$P = \sum_{k=1}^{s} Q_k' P_k + \sum_{i<j} U_{ij} \vec{V}_{ij} \cdot (P_1, \cdots, P_s) + R$$

$$= \sum_{k=1}^{s} Q_k' P_k + \sum_{i<j} U_{ij} R_{ij} + R.$$

Here it follows $\sum_{i<j} U_{ij} R_{ij} + R \in \mathfrak{F}_m$ from the assumptions above. Moreover, (2.6) implies $\mathrm{lexp}_{FR}(Q_k' P_k) \prec_{FR} (\mu, \nu, \beta)$. This contradicts the minimum property of the expression (2.3) assumed above. Hence we may assume (2.4) for the expression (2.3).

(2nd step) Put $(\mu_0, \nu_0, \alpha_0, \beta_0) = \mathrm{lexp}_{FD}(P)$ and $m = \nu_0 - \mu_0 - 1$. We consider all the possible expressions of the form (2.3) with $Q_k \in \hat{\mathcal{D}}_0$ and $R \in \hat{\mathcal{I}}_m$ satisfying the condition (2.4). By virtue of (1st step), there exists at least one such expression. Moreover, for such expression (2.3) that satisfies (2.4), we have $|\alpha| \leqq |\alpha_0|$ if we put

$$(\mu_0, \nu_0, \alpha, \beta_0) = \max_{FD}\{\mathrm{lexp}_{FD}(Q_k P_k) \mid k=1, \cdots, s\} \tag{2.7}$$

since $(\mu_0, \nu_0, \alpha, \beta_0) \geqq_{FD} (\mu_0, \nu_0, \alpha_0, \beta_0)$. Hence there exists an expression (2.3) that minimizes the exponent (2.7) with respect to the FD-order. In the sequel, let us assume that (2.3) has this mimimum property. Then our aim is to show $\alpha = \alpha_0$.

Assume $\alpha \neq \alpha_0$. Then we have $(\mu_0, \nu_0, \alpha, \beta_0) \succ_{FD} (\mu_0, \nu_0, \alpha_0, \beta_0)$. By a permutation of the indices we may assume

$$\mathrm{lexp}_{FD}(Q_k P_k) = (\mu_0, \nu_0, \alpha, \beta_0) \qquad \text{for } 1 \leqq k \leqq \sigma\,,$$

$$\mathrm{lexp}_{FD}(Q_k P_k) \prec_{FD} (\mu_0, \nu_0, \alpha, \beta_0) \qquad \text{for } \sigma+1 \leqq k \leqq s$$

for some $\sigma \geqq 2$. Put $c_k = \mathrm{lcoef}_{FD}(Q_k)$ and $Q'_k = \mathrm{lterm}_{FD}(Q_k)$ for $k=1, \cdots, \sigma$. Then we have $Q'_k = c_k t^{\mu'_k} \partial_t^{\nu'_k} x^{\alpha'_k} \partial_x^{\beta'_k}$ with

$$\mu'_k := \mu_0 - \mu_k, \quad \nu'_k := \nu_0 - \nu_k, \quad \alpha'_k := \alpha - \alpha_k, \quad \beta'_k := \beta_0 - \beta_k.$$

Put also $Q''_k = Q_k - Q'_k$ for $k=1, \cdots, \sigma$. Then we get

$$P = \sum_{k=1}^{\sigma} Q'_k P_k + \sum_{k=1}^{\sigma} Q''_k P_k + \sum_{k=\sigma+1}^{s} Q_k P_k\,. \tag{2.8}$$

Here note that $\mathrm{lexp}_{FD}(Q''_k P_k) \prec_{FD} (\mu_0, \nu_0, \alpha, \beta_0)$ for $k=1, \cdots, \sigma$. The first term of the right hand side of (2.8) can be rewritten as

$$\begin{aligned} P' := \sum_{k=1}^{\sigma} Q'_k P_k &= \sum_{k=1}^{\sigma} c_k t^{\mu'_k} \partial_t^{\nu'_k} x^{\alpha'_k} \partial_x^{\beta'_k} P_k \\ &= \sum_{k=1}^{\sigma-1} (c_1 + \cdots + c_k)(t^{\mu'_k} \partial_t^{\nu'_k} x^{\alpha'_k} \partial_x^{\beta'_k} P_k - t^{\mu'_{k+1}} \partial_t^{\nu'_{k+1}} x^{\alpha'_{k+1}} \partial_x^{\beta'_{k+1}} P_{k+1}) \\ &\quad + (c_1 + \cdots + c_\sigma) t^{\mu'_\sigma} \partial_t^{\nu'_\sigma} x^{\alpha'_\sigma} \partial_x^{\beta'_\sigma} P_\sigma\,. \end{aligned} \tag{2.9}$$

Since $\mathrm{lcoef}_{FD}(P_k) = 1$ we have

$$\mathrm{lterm}_{FD}(P) = \mathrm{lterm}_{FD}(P') = (c_1 + \cdots + c_\sigma)\mathrm{lterm}_{FD}(t^{\mu'_\sigma} \partial_t^{\nu'_\sigma} x^{\alpha'_\sigma} \partial_x^{\beta'_\sigma} P_\sigma)$$

with $\mathrm{lexp}_{FD}(P) = (\mu_0, \nu_0, \alpha, \beta_0)$ if $c_1 + \cdots + c_\sigma \neq 0$. Hence it follows $c_1 + \cdots + c_\sigma = 0$ from the assumption.

On the other hand, using the Leibniz formula, we have

(2.10)
$$t^{\mu'_k}\partial_t^{\nu'_k}x^{\alpha'_k}\partial_x^{\beta'_k}P_k-t^{\mu'_{k+1}}\partial_t^{\nu'_{k+1}}x^{\alpha'_{k+1}}\partial_x^{\beta'_{k+1}}P_{k+1}$$
$$=t^{\mu_0-\mu_k\vee\mu_{k+1}}\partial_t^{\nu_0-\nu_k\vee\nu_{k+1}}x^{\alpha-\alpha_k\vee\alpha_{k+1}}\partial_x^{\beta_0-\beta_k\vee\beta_{k+1}}\mathrm{sp}_{FD}(P_k,\ P_{k+1})$$
$$+S_kP_k+T_{k+1}P_{k+1}$$

with some S_k, $T_{k+1}\in\hat{\mathcal{D}}_0$ such that

$$\mathrm{lexp}_{FD}(S_kP_k),\ \mathrm{lexp}_{FD}(T_{k+1}P_{k+1})\prec_{FD}(\mu_0,\ \nu_0,\ \alpha,\ \beta_0).$$

Put

$$m'=m-1-\max\{\nu_0-\mu_0-\nu_k\vee\nu_{k+1}+\mu_k\vee\mu_{k+1}\,|\,k=1,\ \cdots,\ \sigma-1\}.$$

Combining (2.8), (2.9), (2.10) and the assumption (2) with m replaced by m', we get

$$P=\sum_{k=1}^{\sigma-1}(c_1+\cdots+c_k)t^{\mu_0-\mu_k\vee\mu_{k+1}}\partial_t^{\nu_0-\nu_k\vee\nu_{k+1}}x^{\alpha-\alpha_k\vee\alpha_{k+1}}\partial_x^{\beta_0-\beta_k\vee\beta_{k+1}}$$
$$\cdot\Big(\sum_{k=1}^{s}Q_{ijk}P_k+R_{ij}\Big)+\sum_{k=1}^{\sigma-1}S_kP_k+\sum_{k=2}^{\sigma}T_kP_k+\sum_{k=1}^{\sigma}Q''_kP_k+\sum_{k=\sigma+1}^{s}Q_kP_k\,.$$

This contradicts the minimum property of the expression (2.3) with respect to the FD-order.

Now we have proved that there exists an expression (2.3) with some $Q_k\in\hat{\mathcal{D}}_0$ and $R\in\hat{\mathcal{F}}_m$ such that $\mathrm{lexp}_{FD}(Q_kP_k)\leqq_{FD}\mathrm{lexp}_{FD}(P)$ for any k. Thus $\mathrm{lexp}_{FD}(P)=\mathrm{lexp}_{FD}(Q_k)+\mathrm{lexp}_{FD}(P_k)\in E$ holds for some k. This completes the proof of Theorem 2.6.

Theorem 2.6 together with Lemma 2.3 enables us to give an algorithm to compute, at least theoretically, an FD-Gröbner basis of a given left ideal of $\mathcal{D}_0$.

ALGORITHM 2.7 (FD-Gröbner basis). Given a finite set $\boldsymbol{G}$ of generators of a left ideal $\mathcal{J}_0$ of $\mathcal{D}_0$, finds an FD-Gröbner basis of $\mathcal{J}_0$.

$m:=\min\{\mathrm{ord}_F(P)\,|\,P\in\boldsymbol{G}\}$;
$\boldsymbol{G}_m:=\boldsymbol{G}$;
REPEAT
  $\boldsymbol{G}_{m-1}:=\boldsymbol{G}_m$;
  $m:=m-1$;
  REPEAT
    FOR each pair $(P,\ Q)$ of elements of $\boldsymbol{G}_m$ DO {
      $R:=\mathrm{red}_{FD}(\mathrm{sp}_{FD}(P,\ Q),\ \boldsymbol{G}_m,\ m)$;
      IF $R\notin\mathcal{F}_m$ THEN $\boldsymbol{G}_m:=\boldsymbol{G}_m\cup\{R\}$;
    }
  UNTIL $\mathrm{red}_{FD}(\mathrm{sp}_{FD}(P,\ Q),\ \boldsymbol{G}_m,\ m)\in\mathcal{F}_m$ for any $P,\ Q\in\boldsymbol{G}_m$;
UNTIL $\boldsymbol{G}_m$ becomes stationary, i.e., $\boldsymbol{G}_m=\boldsymbol{G}_\mu$ for any $\mu<m$;
RETURN $\boldsymbol{G}_m$;

The output of this algorithm is indeed an FD-Gröbner basis by virtue of Theorem 2.6. The termination condition of this algorithm is fulfilled a priori in finitely many steps because of the Noetherian property of monoideals (or monomial ideals) generated by the leading exponents of elements of $\boldsymbol{G}_m$ (cf. [**7**, pp. 68-72]).

The FD-Gröbner basis solves the problem C completely (see Sections 4, 5) and the problem A partially as follows. (It is known that a holonomic system is always formally Fuchsian (cf. [**11**]).)

THEOREM 2.8. *Let $\mathcal{M}$ and $\mathcal{J}$ be as in Section* 1.2 *and let $\mathcal{J}_0$ be the stalk of the sheaf $\mathcal{J}$ at* 0. *Assume that $\boldsymbol{G}$ is an FD-Gröbner basis of the left ideal $\mathcal{J}_0$ of $\mathcal{D}_0$. Then $\mathcal{M}$ is formally Fuchsian along $Y=\{(t, x) \mid t=0\}$ at* 0 *if and only if there exists $P \in \boldsymbol{G}$ such that* $\operatorname{lexp}_{FD}(P)=(\mu, \nu, 0, 0)$ *with some $\mu, \nu \in \boldsymbol{N}$.*

PROOF. If there exists $P \in \boldsymbol{G}$ such that $\operatorname{lexp}_{FD}(P)=(\mu, \nu, 0, 0)$, then P, and consequently $\mathcal{M}$ is formally Fuchsian along Y. Now assume that $\mathcal{M}$ is formally Fuchsian along Y. Then there exists $A \in \mathcal{J}_0$ which is formally Fuchsian along Y. Hence we have $(\mu', \nu', 0, 0) \in E_{FD}(\mathcal{J}_0)$ for some $\mu', \nu' \in \boldsymbol{N}$. Since $\boldsymbol{G}$ is an FD-Gröbner basis of $\mathcal{J}_0$, we have by definition

$$E_{FD}(\mathcal{J}_0) = \bigcup_{P \in \boldsymbol{G}} (\operatorname{lexp}_{FD}(P) + \boldsymbol{N}^{2n+2}) \ni (\mu', \nu', 0, 0) .$$

Hence there exists $P \in \boldsymbol{G}$ such that $\operatorname{lexp}_{FD}(P)=(\mu, \nu, 0, 0)$ with some $\mu, \nu \in \boldsymbol{N}$. This completes the proof.

3. FR-Gröbner basis—algebraic and global algorithmic method.

3.1. FR-Gröbner basis. In order to carry out actual computation, we introduce the ring $\mathcal{D}_R$ of differential operators whose coefficients are polynomials of t with rational functions of x as coefficients:

$$\begin{aligned} \mathcal{D}_R &:= \boldsymbol{C}(x)[t]\langle \partial_t, \partial_x \rangle \\ &= \Big\{ P = \sum_{\mu, \nu, \beta} a_{\mu,\nu,\beta}(x) t^\mu \partial_t^\nu \partial_x^\beta \,\Big|\, a_{\mu,\nu,\beta}(x) \text{ is a rational function of } x \Big\}, \end{aligned}$$

where the sum is finite with respect to μ, ν, β.

For an operator $P \in \mathcal{D}_R$ of the form

$$P = \sum_{\mu, \nu, \beta} a_{\mu,\nu,\beta}(x) t^\mu \partial_t^\nu \partial_x^\beta ,$$

we define its leading exponent, leading coefficient, leading term (in the FR-order) by

$$\operatorname{lexp}_{FR}(P) = \max_{FR} \{(\mu, \nu, \beta) \mid a_{\mu,\nu,\beta}(x) \neq 0\},$$

$$\mathrm{lcoef}_{FR}(P) = a_{\mu,\nu,\beta}(x) \quad \text{with } (\mu, \nu, \beta) := \mathrm{lexp}_{FR}(P),$$

$$\mathrm{lterm}_{FR}(P) = a_{\mu,\nu,\beta}(x) t^{\mu}\partial_t^{\nu}\partial_x^{\beta} \quad \text{with } (\mu, \nu, \beta) := \mathrm{lexp}_{FR}(P).$$

In the same way as Lemma 2.1 we get

LEMMA 3.1. *For $P, Q \in \mathcal{D}_R$ we have*

$$\mathrm{lexp}_{FR}(PQ) = \mathrm{lexp}_{FR}(P) + \mathrm{lexp}_{FR}(Q),$$

$$\mathrm{lcoef}_{FR}(PQ) = \mathrm{lcoef}_{FR}(P)\mathrm{lcoef}_{FR}(Q).$$

DEFINITION 3.2. Let I be a left ideal of $\mathcal{D}_R$. Then a finite subset $\boldsymbol{G} = \{P_1, \cdots, P_s\}$ of $\mathcal{D}_R$ is said to be an *FR-Gröbner basis* of I (along $Y = \{t=0\}$) if it satisfies the following two conditions:

(1) $\boldsymbol{G}$ generates I, i.e., $I = \mathcal{D}_R P_1 + \cdots + \mathcal{D}_R P_s$.

(2) Put $E_{FR}(I) := \{\mathrm{lexp}_{FR}(P) \mid P \in I\}$. Then we have

$$E_{FR}(I) = E_{FR}(\boldsymbol{G}) := \bigcup_{P \in \boldsymbol{G}} (\mathrm{lexp}_{FR}(P) + \boldsymbol{N}^{n+2}).$$

DEFINITION 3.3. For $P, Q \in \mathcal{D}_R$ with

$$\mathrm{lexp}_{FR}(P) = (\mu, \nu, \beta), \qquad \mathrm{lexp}_{FR}(Q) = (\mu', \nu', \beta'),$$

the S-polynomial (or the S-operator) of P and Q is defined by

$$\begin{aligned} \mathrm{sp}_{FR}(P, Q) := \ & \mathrm{lcoef}_{FR}(Q) t^{\mu\vee\mu'-\mu}\partial_t^{\nu\vee\nu'-\nu}\partial_x^{\beta\vee\beta'-\beta} P \\ & - \mathrm{lcoef}_{FR}(P) t^{\mu\vee\mu'-\mu'}\partial_t^{\nu\vee\nu'-\nu'}\partial_x^{\beta\vee\beta'-\beta'} Q. \end{aligned}$$

As in the previous section, we define a filtration of $\mathcal{D}_R$ by

$$\mathcal{F}_m = \left\{ P = \sum_{\mu,\nu,\beta} a_{\mu,\nu,\beta}(x) t^{\mu}\partial_t^{\nu}\partial_x^{\beta} \in \mathcal{D}_R \,\middle|\, a_{\mu,\nu,\beta}(x) = 0 \text{ if } \nu - \mu > m \right\}$$

for any integer m (we use the same notation as for the filtration of $\mathcal{D}_0$).

DEFINITION 3.4. Let $\boldsymbol{G} = \{P_1, \cdots, P_s\}$ be a finite subset of $\mathcal{D}_R$ and m be an arbitrary integer. For an element P of $\mathcal{D}_R$,

(1) P is said to be *$\mathcal{F}_m$-reducible* with respect to $\boldsymbol{G}$ if and only if

$$\mathrm{lexp}_{FR}(P) \in \left(\bigcup_{i=1}^{s} (\mathrm{lexp}_{FR}(P_i) + \boldsymbol{N}^{n+2}) \right) \setminus E_{FR}(\mathcal{F}_m).$$

P is said to be *$\mathcal{F}_m$-irreducible* with respect to $\boldsymbol{G}$ if it is not $\mathcal{F}_m$-reducible.

(2) Let P be $\mathcal{F}_m$-reducible. Then an *$\mathcal{F}_m$-reduction step* for P by $\boldsymbol{G}$ is a procedure to replace P by

$$P - \frac{\mathrm{lcoef}_{FR}(P)}{\mathrm{lcoef}_{FR}(P_i)} t^{\mu-\mu_i}\partial_t^{\nu-\nu_i}\partial_x^{\beta-\beta_i} P_i$$

with an arbitrary $i \in \{1, \cdots, s\}$ such that $\mathrm{lexp}_{FR}(P) \in \mathrm{lexp}_{FR}(P_i) + \boldsymbol{N}^{n+2}$, where $(\mu, \nu, \beta) = \mathrm{lexp}_{FR}(P)$ and $(\mu_i, \nu_i, \beta_i) = \mathrm{lexp}_{FR}(P_i)$.

(3) An *$\mathcal{F}_m$-reduction procedure* for P by $\boldsymbol{G}$ is a sequence of $\mathcal{F}_m$-reduction steps so that its final output becomes $\mathcal{F}_m$-irreducible. We denote the output by $\mathrm{red}_{FR}(P, \boldsymbol{G}, m)$ although it is not uniquely determined by P, $\boldsymbol{G}$, m.

Note that a sequence of $\mathcal{F}_m$-reduction steps always terminates in finitely many steps because the FR-order defines a well-order on $\{(\mu, \nu, \beta) \in \boldsymbol{N}^{n+2} \mid \nu - \mu > m\}$.

DEFINITION 3.5. Let I be a left ideal of $\mathcal{D}_R$ and m be an integer. Then a finite subset $\boldsymbol{G} = \{P_1, \cdots, P_s\}$ of $\mathcal{D}_R$ is said to be a *set of $\mathcal{F}_m$-generators* of I if it satisfies the following two conditions:

(1) $\boldsymbol{G}$ generates I, i.e., $I = \mathcal{D}_R P_1 + \cdots + \mathcal{D}_R P_s$,

(2) For any distinct $i, j \in \{1, \cdots, s\}$, the output of *some* $\mathcal{F}_m$-reduction procedure for $\mathrm{sp}(P_i, P_j)$ by $\boldsymbol{G}$ belongs to $\mathcal{F}_m$.

THEOREM 3.6. *Let I be a left ideal of $\mathcal{D}_R$ and $\boldsymbol{G}$ be a finite set of generators of I. Then the following three conditions are equivalent:*

(1) *$\boldsymbol{G}$ is an FR-Gröbner basis of I.*

(2) *$\boldsymbol{G}$ is a set of $\mathcal{F}_m$-generators of I for any integer m.*

(3) *For any $P \in I$ and any integer m, the output of an arbitrary $\mathcal{F}_m$-reduction procedure for P by $\boldsymbol{G}$ belongs to $\mathcal{F}_m$.*

PROOF. Condition (2) implies that for any $m \in \boldsymbol{Z}$ and distinct $i, j \in \{1, \cdots, s\}$, there exist $Q_1, \cdots, Q_s \in \mathcal{D}_R$ and $R \in \mathcal{F}_m$ such that

$$\mathrm{sp}_{FR}(P_i, P_j) = Q_1 P_1 + \cdots + Q_s P_s + R$$

with $\mathrm{lexp}_{FR}(Q_k P_k) \prec_{FR} \mathrm{lexp}_{FR}(P_i) \vee \mathrm{lexp}_{FR}(P_j)$ for any $k = 1, \cdots, s$. Hence it is proved that (2) implies (1) in the same (and easier) way as the proof of Theorem 2.6.

Now assume (1) and choose arbitrary $P \in I$ and $m \in \boldsymbol{Z}$. Let R be the output of an arbitrary $\mathcal{F}_m$-reduction procedure for P by $\boldsymbol{G}$. Then $R \in I$ and hence $\mathrm{lexp}_{FR}(R) \in E_{FR}(I) = E_{FR}(\boldsymbol{G})$. It follows $R \in \mathcal{F}_m$ since R is $\mathcal{F}_m$-irreducible. This proves (3).

Finally (3) implies (2) since $\mathrm{sp}_{FR}(P_i, P_j) \in I$. This completes the proof.

ALGORITHM 3.7 (FR-Gröbner basis). Given a finite set $\boldsymbol{G}$ of generators of a left ideal I of $\mathcal{D}_R$, finds an FR-Gröbner basis of I.

$m := \min\{\mathrm{ord}_F(P) \mid P \in \boldsymbol{G}\}$;

$\boldsymbol{G}_m := \boldsymbol{G}$;

REPEAT

$G_{m-1} := G_m$;
$m := m-1$;
REPEAT
FOR each pair (P, Q) of elements of G_m DO {
$R := \mathrm{red}_{FR}(\mathrm{sp}_{FR}(P, Q), G_m, m)$;
IF $R \notin \mathcal{F}_m$ THEN $G_m := G_m \cup \{R\}$;
}
UNTIL $\mathrm{red}_{FR}(\mathrm{sp}_{FR}(P, Q), G_m, m) \in \mathcal{F}_m$ for any $P, Q \in G_m$;
UNTIL G_m becomes stationary, i. e., $G_m = G_\mu$ for any $\mu < m$;
RETURN G_m;

The output of Algorithm 3.7 is indeed an FR-Gröbner basis in view of Theorem 3.6. The termination condition of Algorithm 3.7 is satisfied for some m although we do not know when in general. In order to overcome this difficulty, we introduce the method of homogenization in the next section.

Let us denote by $A_{n+1} = C[t, x]\langle \partial_t, \partial_x \rangle$ the Weyl algebra, or the ring of differential operators with polynomial coefficients (cf. [**2**]).

For an operator $P \in \mathcal{D}_R$, there exists a polynomial $b(x)$ of least total degree such that $b(x)P \in A_{n+1}$ and we denote such $b(x)$ by den(P) (the denominator of P).

An FR-Gröbner basis provides an FD-Gröbner basis at a generic point of Y as follows:

THEOREM 3.8. *Assume that a subset $G = \{P_1, \cdots, P_s\}$ of A_{n+1} is an FR-Gröbner basis of the left ideal $I := \mathcal{D}_R P_1 + \cdots + \mathcal{D}_R P_s$ of $\mathcal{D}_R$. Put*

$$a(x) = \mathrm{lcoef}_{FR}(P_1)(x) \cdots \mathrm{lcoef}_{FR}(P_s)(x)$$

and assume $a(x_0) \neq 0$. Put $p = (0, x_0)$. Then G is also an FD-Gröbner basis of the left ideal $\mathcal{J}_p := \mathcal{D}_p P_1 + \cdots + \mathcal{D}_p P_s$ of $\mathcal{D}_p$.

PROOF. We may assume $x_0 = 0$. Put

$$(\mu_i, \nu_i, \beta_i) = \mathrm{lexp}_{FR}(P_i), \qquad a_i(x) = \mathrm{lcoef}_{FR}(P_i) \in C[x]$$

for $i = 1, \cdots, s$. Take arbitrary $i, j \in \{1, \cdots, s\}$ with $i \neq j$. Then we have

$$\mathrm{sp}_{FR}(P_i, P_j) = a_j(x) S_{ij} P_i - a_i(x) S_{ji} P_j \tag{3.1}$$

with $S_{ij} = t^{\mu_{ij}} \partial_t^{\nu_{ij}} \partial_x^{\beta_{ij}}$, where $\mu_{ij} = \mu_i \vee \mu_j - \mu_i$, etc.. On the other hand, since $a_i(0) a_j(0) \neq 0$, we have

$$\mathrm{sp}_{FD}(P_i, P_j) = a_j(0) S_{ij} P_i - a_i(0) S_{ji} P_j \,. \tag{3.2}$$

Let R_{ij} be the output of an $\mathcal{F}_m$-reduction procedure for $\mathrm{sp}_{FR}(P_i, P_j)$ by G. Then there exist $Q_1, \cdots, Q_s \in \mathcal{D}_R$ and $R \in \mathcal{F}_m$ such that

$$\mathrm{sp}_{FR}(P_i, P_j) = Q_1 P_1 + \cdots + Q_s P_s + R \tag{3.3}$$

with $\mathrm{lexp}_{FR}(P_iQ_i) \prec_{FR} \mathrm{lexp}_{FR}(P_i) \vee \mathrm{lexp}_{FR}(P_j)$. Moreover, by the definition of $\mathcal{F}_m$-reduction procedure there exists a positive integer k such that $a(x)^k Q_i$ ($i=1, \cdots, s$) and $a(x)^k R$ belong to the Weyl algebra A_{n+1}. Hence Q_i and R can be regarded as elements of $\mathcal{D}_0$ and as such

$$\mathrm{lexp}_{FD}(P_kQ_k) \prec_{FD} \mathrm{lexp}_{FD}(P_i) \vee \mathrm{lexp}_{FD}(P_j) \qquad (1 \leqq k \leqq s)$$

holds. From (3.1), (3.2), (3.3) it follows

$$\mathrm{sp}_{FD}(P_i, P_j) = Q_1P_1 + \cdots + Q_sP_s + R$$
$$-(a_j(x) - a_j(0))S_{ij}P_i + (a_i(x) - a_i(0))S_{ji}P_j .$$

Hence Theorem 2.6 assures that $\boldsymbol{G}$ is an FD-Gröbner basis since

$$\mathrm{lexp}_{FD}((a_j(x) - a_j(0))S_{ij}P_i) \prec_{FD} \mathrm{lexp}_{FD}(S_{ij}P_i) = \mathrm{lexp}_{FD}(P_i) \vee \mathrm{lexp}_{FD}(P_j) .$$

This completes the proof.

COROLLARY 3.9. *Let $\boldsymbol{G} = \{P_1, \cdots, P_s\}$ be a subset of A_{n+1} and let*

$$\boldsymbol{G}_m = \{P_1, \cdots, P_s, P_{s+1}, \cdots, P_\sigma\}$$

be the output of Algorithm 3.7 with the input $\boldsymbol{G}$. Put $\mathrm{lcoef}_{FR}(P_j) = a_j(x)/b_j(x)$ with polynomials $a_j(x)$, $b_j(x)$ relatively prime to each other. If a point $p=(0, x_0)$ of Y satisfies $a_1(x_0) \cdots a_\sigma(x_0) \neq 0$, then $\boldsymbol{G}$ constitutes an FD-Gröbner basis of the left ideal $\mathcal{J}_p = \mathcal{D}_pP_1 + \cdots + \mathcal{D}_pP_s$ of $\mathcal{D}_p$.

PROOF. First note that, for any $j=s+1, \cdots, \sigma$, P_{j+1} is the output of some $\mathcal{F}_m$-reduction procedure for the S-polynomial of some pair of elements of $\boldsymbol{G}_j = \{P_1, \cdots, P_j\}$ by $\boldsymbol{G}_j$. Hence as polynomial $\mathrm{den}(P_{s+1})$ divides $a_1(x) \cdots a_s(x)$ and $\mathrm{den}(P_{j+1})$ divides $a_1(x) \cdots a_j(x)$ by induction. This implies that all P_j's are regarded as elements of $\mathcal{D}_p$ and contained in the ideal $\mathcal{J}_p$ generated by $\boldsymbol{G}$. Combined with the preceeding theorem, this completes the proof.

3.2. FR-Gröbner basis through homogenization.

In this section, we present a modification of Algorithm 3.7 by using a kind of homogenization. This compensates the lack of the termination condition in Algorithm 3.7. Let us denote by $\mathcal{D}_R[z]$ the (non-commutative) ring of the polynomials of z with coefficients in $\mathcal{D}_R$. Hence z can be regarded as a parameter for an element P of $\mathcal{D}_R[z]$. We define a filtration $\{\mathcal{F}_m^h\}$ of $\mathcal{D}_R[z]$ by

$$\mathcal{F}_m^h = \left\{ \sum_{\mu,\nu,\beta,\zeta} a_{\mu,\nu,\beta,\zeta}(x) t^\mu \partial_t^\nu \partial_x^\beta z^\zeta \in \mathcal{D}_R[z] \,\middle|\, a_{\mu,\nu,\beta,\zeta}(x) = 0 \text{ if } \nu - \mu - \zeta > m \right\}$$

for each integer m.

DEFINITION 3.10. An element P of $\mathcal{D}_R[z]$ is called *F-homogeneous* (of order

m) if there exists an integer m so that P is written in the form

$$P=\sum_{\nu-\mu-\zeta=m} a_{\mu,\nu,\beta,\zeta}(x)t^{\mu}\partial_t{}^{\nu}\partial_x{}^{\beta}z^{\zeta}\,.$$

DEFINITION 3.11. For an operator $P\in\mathcal{D}_R$ of the form

$$P=\sum_{\mu,\nu,\beta} a_{\mu,\nu,\beta}(x)t^{\mu}\partial_t{}^{\nu}\partial_x{}^{\beta}\,,$$

we define its *F-homogenization* P^h by

$$P^h=\sum_{\mu,\nu,\beta} a_{\mu,\nu,\beta}(x)t^{\mu}\partial_t{}^{\nu}\partial_x{}^{\beta}z^{\nu-\mu-m}\in\mathcal{D}_R[z]$$

with $m:=\min\{\nu-\mu\,|\,a_{\mu,\nu,\beta}(x)\neq 0\}$.

It is easy to see that P^h is F-homogeneous.

Now we introduce a total order $\prec_{FRH}$ in $\boldsymbol{N}^{n+3}$ by

$$(\mu,\nu,\beta,\zeta)\prec_{FRH}(\mu',\nu',\beta',\zeta') \text{ if and only if } (\nu-\mu-\zeta<\nu'-\mu'-\zeta')$$
$$\text{or } (\nu-\mu-\zeta=\nu'-\mu'-\zeta' \text{ and } (\mu,\nu,\beta)\prec_{FR}(\mu',\nu',\beta'))$$

for $(\mu,\nu,\beta),(\mu',\nu',\beta')\in\boldsymbol{N}^{n+2}$ and $\zeta,\zeta'\in\boldsymbol{N}$. Then with respect to this order, leading exponent and leading coefficient are defined by

$$\mathrm{lexp}_{FRH}(P)=\max_{FRH}\{(\mu,\nu,\beta,\zeta)\,|\,a_{\mu,\nu,\beta,\zeta}(x)\neq 0\}\,,$$
$$\mathrm{lcoef}_{FRH}(P)=a_{\mu,\nu,\beta,\zeta}(x) \quad \text{with } (\mu,\nu,\beta,\zeta):=\mathrm{lexp}_{FRH}(P)$$

for an operator $P\in\mathcal{D}_R[z]$ of the form

$$P=\sum_{\mu,\nu,\beta,\zeta} a_{\mu,\nu,\beta,\zeta}(x)t^{\mu}\partial_t{}^{\nu}\partial_x{}^{\beta}z^{\zeta}\,.$$

DEFINITION 3.12. Let I be a left ideal of $\mathcal{D}_R[z]$ and $\boldsymbol{G}=\{P_1,\cdots,P_s\}$ be a finite subset of I. Then $\boldsymbol{G}$ is said to be an *FRH-Gröbner basis* of I (along $Y=\{t=0\}$) if it satisfies the following two conditions:

(1) $\boldsymbol{G}$ generates I, i.e., $I=\mathcal{D}_R[z]P_1+\cdots+\mathcal{D}_R[z]P_s$.

(2) Put $E_{FRH}(I):=\{\mathrm{lexp}_{FRH}(P)\,|\,P\in I\}$. Then we have

$$E_{FRH}(I)=E_{FRH}(\boldsymbol{G}):=\bigcup_{P\in\boldsymbol{G}}(\mathrm{lexp}_{FRH}(P)+\boldsymbol{N}^{n+3})\,.$$

DEFINITION 3.13. For $P,Q\in\mathcal{D}_R[z]$ with

$$\mathrm{lexp}_{FRH}(P)=(\mu,\nu,\beta,\zeta)\,,\qquad \mathrm{lexp}_{FRH}(Q)=(\mu',\nu',\beta',\zeta')\,,$$

the S-polynomial (or the S-operator) of P and Q is defined by

$$\mathrm{sp}_{FRH}(P, Q) := \mathrm{lcoef}_{FRH}(Q) t^{\mu\vee\mu'-\mu}\partial_t^{\nu\vee\nu'-\nu}\partial_x^{\beta\vee\beta'-\beta} z^{\zeta\vee\zeta'-\zeta} P$$
$$-\mathrm{lcoef}_{FRH}(P) t^{\mu\vee\mu'-\mu'}\partial_t^{\nu\vee\nu'-\nu'}\partial_x^{\beta\vee\beta'-\beta'} z^{\zeta\vee\zeta'-\zeta'} Q .$$

DEFINITION 3.14. Let $G=\{P_1, \cdots, P_s\}$ be a finite subset of $\mathcal{D}_R[z]$ consisting of F-homogeneous operators. For an F-homogeneous operator $P\in\mathcal{D}_R[z]$,

(1) P is said to be *reducible* with respect to G if and only if

$$\mathrm{lexp}_{FRH}(P) \in \bigcup_{i=1}^{s} (\mathrm{lexp}_{FRH}(P_i)+\boldsymbol{N}^{n+3}) .$$

P is said to be *irreducible* with respect to G if it is not reducible.

(2) Let P be reducible with respect to G. Then a *reduction step* for P by G is a procedure to replace P by

$$P-\frac{\mathrm{lcoef}_{FRH}(P)}{\mathrm{lcoef}_{FRH}(P_i)} t^{\mu-\mu_i}\partial_t^{\nu-\nu_i}\partial_x^{\beta-\beta_i} z^{\zeta-\zeta_i} P_i$$

with an arbitrary $i\in\{1, \cdots, s\}$ such that $\mathrm{lexp}_{FRH}(P)\in\mathrm{lexp}_{FRH}(P_i)+\boldsymbol{N}^{n+3}$, where $(\mu, \nu, \beta, \zeta)=\mathrm{lexp}_{FRH}(P)$ and $(\mu_i, \nu_i, \beta_i, \zeta_i)=\mathrm{lexp}_{FRH}(P_i)$.

(3) A *reduction procedure* for P by G is a sequence of reduction steps so that its final output becomes irreducible or zero. We denote the output by $\mathrm{red}_{FRH}(P, G)$ although it is not uniquely determined by P and G.

LEMMA 3.15. *In the same notation as in Definition* 3.14, *every reduction procedure for P by G terminates in finitely many steps.*

PROOF. Suppose that P is F-homogeneous of order m and reducible with respect to G. Let P' be the output of a reduction step for P by G. Then we have by definition $\mathrm{lexp}_{FRH}(P')\prec_{FRH}\mathrm{lexp}_{FRH}(P)$ and P' is F-homogeneous of order m if $P'\neq 0$. Hence every reduction step does not change the F-homogeneous order as long as the output is not zero. This implies that for every sequence of reduction steps, the output becomes zero or irreducible in finitely many steps since the order $\prec_{FRH}$ is a well-order restricted to the set $\{(\mu, \nu, \beta, \zeta)\in\boldsymbol{N}^{n+3}\mid \nu-\mu-\zeta=m\}$. This completes the proof.

THEOREM 3.16. *Let G be a finite set of $\mathcal{D}_R[z]$ consisting of F-homogeneous operators. Let I be the left ideal of $\mathcal{D}_R[z]$ generated by G. Then the following two conditions are equivalent:*

(1) *G is an FRH-Gröbner basis of I.*

(2) *For any pair (P, Q) of distinct elements of G, the output of some reduction procedure for $\mathrm{sp}_{FRH}(P, Q)$ by G is zero.*

PROOF. Since $\mathrm{sp}_{FRH}(P, Q)$ is also F-homogeneous, this theorem can be proved in the same way as Theorem 3.6 by replacing the $\mathcal{F}_m$-reduction procedure by the reduction procedure.

THEOREM 3.17. *Let S be a finite set of generators of a left ideal I of $\mathcal{D}_R$. Put $S^h=\{P^h \mid P\in S\}$ and I^h be the left ideal of $\mathcal{D}_R[z]$ generated by S^h. Let $\boldsymbol{G}^h$ be an FRH-Gröbner basis of I^h consisting of F-homogeneous operators. Then $\boldsymbol{G}:=\{\mathrm{subst}(P, z, 1) \mid P\in \boldsymbol{G}^h\}$ is an FR-Gröbner basis of I, where $\mathrm{subst}(P, z, 1)$ denotes the result of substitution $z=1$ in P.*

PROOF. First let us show that $\boldsymbol{G}$ is a set of generators of I. Put $\boldsymbol{G}^h=\{P_1, \cdots, P_s\}$ and let P be an element of I. Then there exist $Q_1, \cdots, Q_s\in\mathcal{D}_R[z]$ such that

$$z^{\zeta}P^h = Q_1P_1+ \cdots +Q_sP_s$$

for some $\zeta\in\boldsymbol{N}$. Then it follows

$$\begin{aligned} P &= \mathrm{subst}(z^{\zeta}P^h, z, 1) \\ &= \mathrm{subst}(Q_1, z, 1)\mathrm{subst}(P_1, z, 1)+ \cdots +\mathrm{subst}(Q_s, z, 1)\mathrm{subst}(P_s, z, 1). \end{aligned}$$

Hence I is generated by $\boldsymbol{G}$.

Next, let us show

$$\{\mathrm{lexp}_{FR}(P) \mid P\in I\} = \bigcup_{P\in \boldsymbol{G}} \mathrm{lexp}_{FR}(P)+\boldsymbol{N}^{n+2}. \tag{3.4}$$

Suppose $P\in I$. Then it is easy to see that $\mathrm{lexp}_{FRH}(P^h)=(\mathrm{lexp}_{FR}(P), \zeta)$ with some $\zeta\in\boldsymbol{N}$. On the other hand, since $\boldsymbol{G}^h$ is an FRH-Gröbner basis, we have

$$\mathrm{lexp}_{FRH}(z^{\zeta'}P^h) \in \bigcup_{i=1}^{s} \mathrm{lexp}_{FRH}(P_i)+\boldsymbol{N}^{n+3}$$

with some $\zeta'\in\boldsymbol{N}$. This implies

$$\mathrm{lexp}_{FR}(P) \in \bigcup_{i=1}^{s} \mathrm{lexp}_{FR}(\mathrm{subst}(P_i, z, 1))+\boldsymbol{N}^{n+2}.$$

This proves (3.4), and at the same time, completes the proof of Theorem 3.17.

This theorem yields the following algorithm of computing FR-Gröbner basis.

ALGORITHM 3.18 (FR-Gröbner basis). Given a finite set $\boldsymbol{G}$ of generators of a left ideal I of $\mathcal{D}_R$, finds an FR-Gröbner basis of I.

$\boldsymbol{G}:=\{P^h \mid P\in\boldsymbol{G}\}$;
REPEAT
    FOR each pair (P, Q) of elements of $\boldsymbol{G}$ DO {
        $R:=\mathrm{red}_{FRH}(\mathrm{sp}_{FRH}(P, Q), \boldsymbol{G})$;
        IF $R\neq 0$ THEN $\boldsymbol{G}:=\boldsymbol{G}\cup\{R\}$;
    }
UNTIL $\mathrm{red}_{FRH}(\mathrm{sp}_{FRH}(P, Q), \boldsymbol{G})=0$ for any $P, Q\in\boldsymbol{G}$;
$\boldsymbol{G}:=\{\mathrm{subst}(P, z, 1) \mid P\in\boldsymbol{G}\}$;
RETURN $\boldsymbol{G}$;

The output of Algorithm 3.18 is indeed an FR-Gröbner basis in view of Theorem 3.17.

In the same way as was pointed out by Buchberger [5] for the polynomial ring, we can often save computation in Algorithm 3.18 by the following criterion (the proof is similar and omitted):

PROPOSITION 3.19. *Let $\boldsymbol{G}$ be a finite subset of $\mathcal{D}_R[z]$ consisting of F-homogeneous operators and P, Q be two distinct elements of $\boldsymbol{G}$. Assume that there exists a sequence $\{P_1, \cdots, P_k\}$ of elements of $\boldsymbol{G}$ such that*

(1) $P_1 = P,\ P_k = Q,$

(2) $\mathrm{lexp}_{FRH}(P_1) \vee \cdots \vee \mathrm{lexp}_{FRH}(P_k) = \mathrm{lexp}_{FRH}(P) \vee \mathrm{lexp}_{FRH}(Q),$

(3) $\mathrm{red}_{FRH}(\mathrm{sp}_{FRH}(P_j, P_{j+1}), \boldsymbol{G})=0$ *by some reduction procedure for any* $j=1, \cdots, k-1$.

Then the output of some reduction procedure for $\mathrm{sp}_{FRH}(P, Q)$ *becomes zero.*

4. Computation of the characteristic exponents.

We use the same notation as in Section 1. In particular, let $\mathcal{J}$ be a left ideal of $\mathcal{D}_0$ associated with a Fuchsian system $\mathcal{M}$ as in Section 1.2. We assume $Y=\{(t, x) \mid t=0\}$. In fact, we can treat any non-singular complex analytic hypersurface Y for the (theoretical) computation of Algorithm 2.7. For the (practical) computation of Algorithm 3.6, we can treat any hypersurface Y that can be brought into the hyperplane $t=0$ by a birational transformation of $\boldsymbol{C}^{n+1}$.

THEOREM 4.1. *Assume that the system $\mathcal{M}$ is Fuchsian along Y at 0 with $P_1, \cdots, P_s \in \mathcal{D}_0$. Let $\boldsymbol{G}$ be an FD-Gröbner basis of $\mathcal{J}_0 := \mathcal{D}_0 P_1 + \cdots + \mathcal{D}_0 P_s$. Put*

$$\boldsymbol{G}' = \{P \in \boldsymbol{G} \mid \mathrm{lexp}_{FD}(P)=(\mu, \nu, \alpha, 0) \text{ for some } \mu, \nu \in \boldsymbol{N} \text{ and some } \alpha \in \boldsymbol{N}^n\}.$$

Then the set of the characteristic exponents of $\mathcal{M}$ at 0 is given by

$$e_Y(\mathcal{M}, 0) = \{\theta \in \boldsymbol{C} \mid \psi(\hat{\sigma}(P))(\theta, 0)=0 \text{ for any } P \in \boldsymbol{G}'\}. \tag{4.1}$$

Moreover, let P be an element of $\boldsymbol{G}'$ with minimum order with respect to ∂_t. Then there exist a monic polynomial $f(\theta, x) \in \mathcal{O}_0'[\theta]$ and $a(x) \in \mathcal{O}_0'$ such that $\psi(\hat{\sigma}(P))=a(x)f(\theta, x)\tau^k$ with some $k \in \boldsymbol{Z}$, and the ideal $\tilde{\mathcal{J}}_Y(\mathcal{M}, 0)$ is generated by f. In particular, we have

$$\tilde{e}_Y(\mathcal{M}, 0) = \{\theta \in \boldsymbol{C} \mid f(\theta, 0)=0\}.$$

PROOF. To prove (4.1) it suffices to show that the ideal $\mathcal{J}_Y(\mathcal{M}, 0)$ is generated by

$$\{\tau^{-\mathrm{ord}_F(P)} \cdot \psi(\hat{\sigma}(P)) \mid P \in \boldsymbol{G}'\}.$$

It is easy to see by definition that this set is contained in $\mathcal{J}_Y(\mathcal{M}, 0)$. Suppose

$g \in \mathcal{J}_Y(\mathcal{M}, 0)$. Then in view of Lemma 1.5, there exist $P \in \mathcal{J}_0$ and $k \in \boldsymbol{N}$ such that $\psi(\hat{\sigma}(P)) = g(\theta, x)\tau^{-k}$. This implies $\mathrm{lexp}_{FD}(P) = (\nu+k, \nu, \alpha, 0)$ with some $\nu \in \boldsymbol{N}$ and $\alpha \in \boldsymbol{N}^n$. We may assume $\boldsymbol{G} = \{P_1, \cdots, P_s\}$ with $\mathrm{ord}_F(P_i) = k_i$ for $i=1, \cdots, s$. Set $\psi(\hat{\sigma}(P_i)) = f_i(\theta, x)\tau^{k_i}$ for $P_i \in \boldsymbol{G}'$.

Since $P \in \mathcal{J}_0$ and $\boldsymbol{G}$ is an FD-Gröbner basis, there exist $Q_1, \cdots, Q_s \in \mathcal{D}_0$ and $R \in \mathcal{F}_{-k-1}$ so that

$$P = Q_1P_1 + \cdots + Q_sP_s + R, \qquad \mathrm{lexp}_{FD}(Q_iP_i) \leqq_{FD} (\nu+k, \nu, \alpha, 0)$$

in view of Theorem 2.6. This implies $\mathrm{ord}_F(Q_i) \leqq -k-k_i$, and if $\mathrm{ord}_F(Q_i) = -k-k_i$, we have $P_i \in \boldsymbol{G}'$ and $q_i(\theta, x) := \tau^{k+k_i} \cdot \psi(\hat{\sigma}(Q_i)) \in \mathcal{O}_0'[\theta]$. Put $S = \{i \in \{1, \cdots, s\} \mid \mathrm{ord}_F(Q_i) = -k-k_i\}$. Then we have

$$\hat{\sigma}(P) = \sum_{i \in S} \hat{\sigma}(Q_i)\hat{\sigma}(P_i)$$

and hence

$$g(\theta, x) = \sum_{i \in S} q_i(\theta - k_i, x) f_i(\theta, x).$$

This implies that $\mathcal{J}_Y(\mathcal{M}, 0)$ is generated by $\{f_i(\theta, x) \mid P_i \in \boldsymbol{G}'\}$. This proves (4.1).

Let $f(\theta, x) \in \mathcal{O}_0'[\theta]$ be the monic polynomial which generates $\tilde{\mathcal{J}}_Y(\mathcal{M}, 0)$ as in Lemma 1.8. Put $S' = \{i \in \{1, \cdots, s\} \mid P_i \in \boldsymbol{G}'\}$. Then since $\boldsymbol{G}$ is an FD-Gröbner basis, there exist $a(x) \in \mathcal{O}_0'$ and $r_i(\theta, x) \in \mathcal{O}_0'[\theta]$ such that

$$a(x) f(\theta, x) = \sum_{i \in S'} r_i(\theta - k_i) f_i(\theta, x)$$

and that the degree of $r_i(\theta, x) f_i(\theta, x)$ in θ is less than or equal to that of $f(\theta, x)$, which we denote by m. Hence, if $r_i \neq 0$, the degree of f_i in θ must be m, which implies $f_i(\theta, x) = a_i(x) f(\theta, x)$ with some $a_i(x) \in \mathcal{O}_0'$ since f divides f_i in $\mathcal{O}_0'[\theta]$. This completes the proof.

On generic points, we can compute the characteristic exponents from an FR-Gröbner basis. In fact, the following is an immediate consequence of Theorems 3.8 and 4.1.

COROLLARY 4.2. *Under the same assumptions as in Theorem* 3.8, *put*

$$S = \{i \in \{1, \cdots, s\} \mid \mathrm{lexp}_{FR}(P_i) = (\mu_i, \nu_i, 0) \text{ with some } \mu_i, \nu_i \in \boldsymbol{N}\}.$$

Among the set $\{P_i \mid i \in S\}$, *let* P_{i_0} *have minimum degree with respect to* ∂_t *and set* $\psi(\hat{\sigma}(P_{i_0})) = f_{i_0}(\theta, x)\tau^k$. *Then we have*

$$\mathcal{J}_Y(\mathcal{M}, p) = \tilde{\mathcal{J}}_Y(\mathcal{M}, p) = \mathcal{O}_p'[\theta] f_{i_0}(\theta, x).$$

5. Computation of the induced system.

Here we use the same notation as above and assume the system $\mathcal{M}$ (as in Section 1.2) is (formally) Fuchsian along $Y=\{(t, x) \mid t=0\}$ at 0. We study the structure of the induced system $\mathcal{M}_Y=\mathcal{D}/(\mathcal{J}+t\mathcal{D})$ of $\mathcal{M}$ along Y. The induced system is a system which the restriction to Y of the holomorphic solutions of $\mathcal{M}$ satisfy. Our purpose is to determine the structure of the stalk $\mathcal{M}_{Y,0}$ of $\mathcal{M}_Y$ at $0 \in Y$ as a module over $\mathcal{D}_0'=\boldsymbol{C}\{x\}\langle\partial_x\rangle$. We denote by u the modulo class of $1 \in \mathcal{D}$ in $\mathcal{M}=\mathcal{D}/\mathcal{J}$, and for $P \in \mathcal{D}$, we denote by $[Pu]$ the modulo class of $P \in \mathcal{D}$ in $\mathcal{M}_Y$.

Let us begin with the following general result:

THEOREM 5.1. *Assume $\mathcal{M}$ is formally Fuchsian along Y at 0 and*

$$\{k \in \boldsymbol{N} \mid k \geqq k_0\} \cap e_Y(\mathcal{M}, 0)=\emptyset$$

for some $k_0 \in \boldsymbol{N}$. Then $\mathcal{M}_{Y,0}$ is generated by $[\partial_t{}^j u]$ with $0 \leqq j \leqq k_0-1$ as a $\mathcal{D}_0'$-module. In particular, we have $\mathcal{M}_{Y,0}=0$ if $k_0=0$.

PROOF. By definition, $\mathcal{M}_{Y,0}$ is generated by $[\partial_t{}^j u]$ with $j \geqq 0$ over $\mathcal{D}_0'$. Now assume $k \geqq k_0$. Then there exists $P \in \mathcal{J}_0$ such that $\psi(\hat{\sigma}(P))=f(\theta, x)\tau^{-j}$ with $j \geqq 0$ and $f \in \mathcal{O}_0'[\theta]$ satisfying $f(k, 0) \neq 0$. Hence P can be written in the form

$$P=t^j f(t\partial_t, x)+t^{j+1}P'(t, \partial_t, x, \partial_x)$$

with $P' \in \mathcal{F}_0$. From this we get

$$\begin{aligned}\partial_t{}^{j+k}P &= \partial_t{}^{j+k}(t^j f(t\partial_t, x)+t^{j+1}P') \\ &= (t\partial_t+k+1)(t\partial_t+k+2)\cdots(t\partial_t+k+j)f(t\partial_t+k, x)\partial_t{}^k+\partial_t{}^{j+k}t^{j+1}P'.\end{aligned}$$

This implies in $\mathcal{M}_{Y,0}$

$$0=[\partial_t{}^{j+k}Pu]=(k+1)(k+2)\cdots(k+j)f(k, x)[\partial_t{}^k u]+[\partial_t{}^{j+k}t^{j+1}P'u].$$

Since $\partial_t{}^{j+k}t^{j+1}P' \in \mathcal{F}_{k-1}$, there exists $Q(\partial_t, x, \partial_x) \in \mathcal{D}_0$ with order less than k with respect to ∂_t so that

$$\partial_t{}^{j+k}t^{j+1}P'-Q(\partial_t, x, \partial_x) \in t\mathcal{D}_0 .$$

Thus we get

$$[\partial_t{}^k u] \in \mathcal{D}_0'[u]+\mathcal{D}_0'[\partial_t u]+\cdots+\mathcal{D}_0'[\partial_t{}^{k-1}u] .$$

This proves the statement of Theorem 5.1 by induction on k.

In view of this theorem, $\mathcal{M}_Y$ represents the relations among the restrictions

$$u(0, x), \partial_t u(0, x), \cdots, \partial_t{}^{k_0-1}u(0, x)$$

of a holomorphic solution $u(t, x)$ of $\mathcal{M}$ on a neighborhood of Y.

Now let us describe an effective method to compute the induced system $\mathcal{M}_{Y,0}$ under some moderate condition, which is always satisfied at a generic point of Y. (See [**24**] for a different general method not based on Theorem 5.1.)

Assume that the system $\mathcal{M}$ satisfies the same assumptions as in Theorem 5.1. Let $\boldsymbol{G}$ be a finite set of generators of the left ideal $\mathcal{I}_0$ of $\mathcal{D}_0$. We assume that there exists an element P_0 of $\boldsymbol{G}$ such that $\psi(\hat{\sigma}(P_0))=f(\theta, x)\tau^{-j_0}$ and that $f(k, 0)\neq 0$ for any integer $k\geqq k_0$. (We may assume $j_0\geqq 0$.) In view of Corollary 4.2, this assumption is fulfilled if $\boldsymbol{G}$ satisfies the conditions of Theorem 3.8 at 0; i.e., if $\boldsymbol{G}$ consists of elements of A_{n+1} with $\mathrm{lcoef}_{FR}(P)(0)\neq 0$ for any $P\in\boldsymbol{G}$, and if $\boldsymbol{G}$ is an FR-Gröbner basis of the ideal which it generates over $\mathcal{D}_R$.

We define a $\mathcal{D}_0'$-homomorphism $\rho:\mathcal{D}_0\to\mathcal{D}_0'[\partial_t]$ as follows: Write $P\in\mathcal{D}_0$ explicitly as (1.1). Then we put

$$\rho(P)=\sum_{\nu,\alpha,\beta} a_{0,\nu,\alpha,\beta}x^\alpha\partial_x^\beta\partial_t^\nu \in \mathcal{D}_0'[\partial_t].$$

For an element P of $\mathcal{D}_0'[\partial_t]$, its F-order $\nu=\mathrm{ord}_F(P)$ is the order of P with respect to ∂_t and its formal symbol is of the form $\hat{\sigma}(P)=A(x, \partial_x)\partial_t^\nu$ with some $A\in\mathcal{D}_0'$. Let us denote this A by $\mathrm{coef}(P, \partial_t, \nu)$.

By the proof of Theorem 5.1, we have, for any $k\geqq k_0$,

$$\hat{\sigma}(\rho(\partial_t^{j_0+k}P_0)) = p_k(x)\partial_t^k$$

with some $p_k(x)\in\boldsymbol{C}\{x\}$ such that $p_k(0)\neq 0$. By using this P_0 we define $\mathrm{ind}(P, P_0)$ $\in\mathcal{D}_0'[\partial_t]$ for each $P\in\mathcal{D}_0'[\partial_t]$ by the following algorithm:

ALGORITHM 5.2 (Definition of $\mathrm{ind}(P, P_0)$). Given $P\in\mathcal{D}_0'[\partial_t]$ returns $\mathrm{ind}(P, P_0)$ $\in\mathcal{D}_0'[\partial_t]$.
INPUT $P\in\mathcal{D}_0'[\partial_t]$;
WHILE $\nu:=\mathrm{ord}_F(P)\geqq k_0$ DO
  $P:=P-(\mathrm{coef}(P, \partial_t, \nu)/p_\nu)\rho(\partial_t^{j_0+\nu}P_0)$;
RETURN P;

Put $\mathcal{D}_0'^{(k_0)}=\bigoplus_{k=0}^{k_0-1}\mathcal{D}_0'\partial_t^k\subset\mathcal{D}_0'[\partial_t]$. Then $\mathrm{ind}(\cdot, P_0)$ defines a $\mathcal{D}_0'$-homomorphism of $\mathcal{D}_0'[\partial_t]$ to $\mathcal{D}_0'^{(k_0)}$. For an element $Q=\sum_{k=0}^{k_0-1}Q_k(x, \partial_x)\partial_t^k$ of $\mathcal{D}_0'^{(k_0)}$, we write $[Qu]=\sum_{k=0}^{k_0-1}Q_k(x, \partial_x)[\partial_t^k u]\in\mathcal{M}_{Y,0}$.

THEOREM 5.3. *Under the assumptions above, there exists an integer* $j_0\geqq 0$ *such that* $\mathcal{M}_{Y,0}$ *is explicitly given by the system of equations*

$$[\mathrm{ind}(\rho(\partial_t^j P), P_0)u]=0 \qquad \textit{for any } P\in\boldsymbol{G} \textit{ and any } j=0, 1, \cdots, j_0$$

for unknowns $[u], \cdots, [\partial_t^{k_0-1}u]$.

PROOF. Suppose $P\in\boldsymbol{G}$ and $j\geqq 0$ and put $\nu=\mathrm{ord}_F(P)$. Then by Algorithm 5.2 we have

$$\rho(\partial_t{}^j P) = \sum_{k=k_0}^{\nu+j} Q_k(x, \partial_x)\rho(\partial_t{}^{j_0+k}P_0) + \mathrm{ind}(\partial_t{}^j P, P_0)$$

with some $Q_k(x, \partial_x) \in \mathcal{D}'_0$. Thus $\mathrm{ind}(\rho(\partial_t{}^j P), P_0)$ belongs to $\mathcal{L} := \rho(\mathcal{I}_0)$. This implies $[\mathrm{ind}(\rho(\partial_t{}^j P), P_0)u] = 0$.

Let us denote the $\mathcal{D}'_0$-homomorphism $\mathrm{ind}(\cdot, P_0)$ simply by ind. Then ind: $\mathcal{D}'_0[\partial_t] \to \mathcal{D}'^{(k_0)}_0$ is surjective since $\mathrm{ind}(\partial_t{}^j) = \partial_t{}^j$ if $j < k_0$. Moreover the inverse image $\mathrm{ind}^{-1}(\mathrm{ind}(\mathcal{L}))$ is again contained in $\mathcal{L}$. In fact, for $P \in \mathcal{D}'_0[\partial_t]$ with $\nu := \mathrm{ord}_F(P)$ we have

$$P = \sum_{k=k_0}^{\nu} Q_k(x, \partial_x)\rho(\partial_t{}^{j_0+k}P_0) + \mathrm{ind}(P)$$

with some $Q_k \in \mathcal{D}'_0$, and hence $\mathrm{ind}(P) \in \mathcal{L}$ if and only if $P \in \mathcal{L}$. This implies

$$\mathcal{L} \subset \mathrm{ind}^{-1}(\mathrm{ind}(\mathcal{L})) \subset \mathrm{ind}^{-1}(\mathcal{L} \cap \mathcal{D}'^{(k_0)}_0) \subset \mathcal{L}\,.$$

Hence we have $\mathcal{D}'_0$-module isomorphisms induced by ρ and ind,

$$\mathcal{M}_{Y,0} \cong \mathcal{D}'_0[\partial_t]/\mathcal{L} \cong \mathcal{D}'^{(k_0)}_0/\mathrm{ind}(\mathcal{L})\,.$$

Since $\mathcal{L}$ is a $\mathcal{D}'_0$-submodule of $\mathcal{D}'_0[\partial_t]$ generated by the set $\{\rho(\partial_t{}^j P) \mid P \in \boldsymbol{G}, j \geqq 0\}$, $\mathrm{ind}(\mathcal{L})$ is a $\mathcal{D}'_0$-submodule of $\mathcal{D}'^{(k_0)}_0$ generated by the set $\{\mathrm{ind}(\rho(\partial_t{}^j P)) \mid P \in \boldsymbol{G}, j \geqq 0\}$. Since $\mathcal{D}'_0$ is a Noetherian ring, there exists j_0 so that $\mathrm{ind}(\mathcal{L})$ is generated by $\{\mathrm{ind}(\rho(\partial_t{}^j P)) \mid P \in \boldsymbol{G}, 0 \leqq j \leqq j_0\}$. This completes the proof.

Finally, let us assume that the Fuchsian system $\mathcal{M}$ has a constant characteristic exponent $\lambda \in \boldsymbol{C}$; i.e., $\lambda \in e_Y(\mathcal{M}, p)$ for any $p \in Y$. Then we can define the system $\mathcal{M}^{(\lambda)}$ for the unknown $t^{-\lambda}u$ as follows: Let $P_1, \cdots, P_s$ be as in Section 1.2 and put $k_i = \mathrm{ord}_F(P_i)$ and $k_i^+ = \max\{k_i, 0\}$. Put $Q_i = t^{-\lambda + k_i^+} P_i t^{\lambda} \in \mathcal{D}_0$ and define the system $\mathcal{M}^{(\lambda)}$ by

$$\mathcal{M}^{(\lambda)} : Q_1 u = \cdots = Q_s u = 0\,.$$

Then its induced system $\mathcal{M}_Y^{(\lambda)}$ represents the relations among $v(0, x)$, $\partial_t v(0, x)$, $\cdots$ for analytic solutions u of $\mathcal{M}$ of the form $u = v(t, x)t^{\lambda}$ with $v(t, x)$ holomorphic on a neighborhood of Y.

6. Examples of actual computation.

We have implemented the algorithms presented so far on a computer algebra system Risa/asir (cf. [**16**]). For example, by using this implementation, we can compute the characteristic exponents and the induced systems along each irreducible component of the singular loci of the systems for Appell's hypergeometric functions of two variables. Since these systems are holonomic, they are formally Fuchsian by a theorem of Kashiwara, but it does not seem obvious that they are Fuchsian along their singular loci.

In the sequel we put $n=1$ and use the notation $\partial_x=\partial/\partial x$, $\partial_y=\partial/\partial y$ with $(x, y)\in \boldsymbol{C}^2$ as well as $(t, x)\in \boldsymbol{C}^2$ as in the preceding sections. Let us describe briefly the computation for the systems for Appell's F_3 and F_4 (see e.g., [**25**] for computation of F_1-F_3 based on the concrete representation of solutions). Our computation is purely algorithmic.

EXAMPLE 6.1 (System for Appell's F_3). The system $\mathcal{M}_3$ for Appell's hypergeometric function F_3 is defined by

$$\mathcal{M}_3: P_{31}u = P_{32}u = 0\,,$$

where

$$P_{31} := x(1-x)\partial_x^2+y\partial_x\partial_y+\{\gamma-(\alpha+\beta+1)x\}\partial_x-\alpha\beta\,,$$

$$P_{32} := y(1-y)\partial_y^2+x\partial_x\partial_y+\{\gamma-(\alpha'+\beta'+1)y\}\partial_y-\alpha'\beta'$$

with parameters $\alpha, \alpha', \beta, \beta', \gamma\in \boldsymbol{C}$. (We assume these parameters take generic values.) It is well-known that $\mathcal{M}_3$ is a holonomic system of rank 4 and its singular loci are defined by $xy(x-1)(y-1)(xy-x-y-1)=0$. (See [**19**] for the algorithmic verification.)

Put $Y=\{(x, y)\,|\,x=0\}$ and $I=\mathcal{D}_R P_{31}+\mathcal{D}_R P_{32}$. Then Algorithm 3.18 returns $\boldsymbol{G} := \{P_{31}, P_{32}, P_{33}\}$ as a FR-Gröbner basis for I_3 along Y; here P_{33} is a Fuchsian operator given by

$$\begin{aligned} P_{33} = {} & (1-x)yx^2\partial_x{}^3+(y-1)yx^2\partial_y\partial_x{}^2 \\ & +\{(-\alpha+\alpha'-\beta+\beta'-\gamma-3)x+(-\alpha'-\beta'+2\gamma+1)\}yx\partial_x{}^2 \\ & +(\alpha+\beta+1)(y-1)yx\partial_y\partial_x \\ & +[\{(\alpha'-\beta+\beta'-\gamma-1)\alpha+(\beta+1)\alpha'+(\beta'-\gamma-1)\beta+\beta'-\gamma-1\}x \\ & +(\beta'-\gamma)\alpha'-\gamma\beta'+\gamma^2]y\partial_x+\alpha\beta(y-1)y\partial_y+\alpha\beta(\alpha'+\beta'-\gamma)y\,. \end{aligned}$$

This implies that $\mathcal{M}_3$ is Fuchsian along Y on $\{(0, y)\in Y\,|\,y\neq 0, 1\}$. (We can also verify that $\mathcal{M}_3$ is also Fuchsian along Y at $(0, 0)$ and $(0, 1)$ by Algorithm 2.7.) We get

$$e_Y(\mathcal{M}_3, p) = \tilde{e}_Y(\mathcal{M}_3, p) = \{0, \alpha'-\gamma+1, \beta'-\gamma+1\}$$

for any $p\in Y\setminus\{(0, 0), (0, 1)\}$. Hence any multi-valued analytic solution u of $\mathcal{M}_3$ around Y is expressed in the form

$$u = v_1(x, y)+v_2(x, y)x^{\alpha'-\gamma+1}+v_3(x, y)x^{\beta'-\gamma+1}$$

with v_1, v_2, v_3 holomorphic on a neighborhood of $Y\setminus\{(0, 0), (0, 1)\}$. Moreover, by Algorithm 5.2 $v_1(0, y)$, $v_2(0, y)$, $v_3(0, y)$ satisfy the equations

$$\{y(1-y)\partial_y{}^2+(\gamma-(\alpha'+\beta'+1)y)\partial_y-\alpha'\beta'\}v_1(0, y) = 0\,,$$

$$(y\partial_y+\alpha')v_2(0,\ y)=0\,,\qquad (y\partial_y+\beta')v_3(0,\ y)=0\,.$$

We know that these systems give precisely the induced systems because the sum of the rank of these systems equals the rank of the system $\mathcal{M}_3$.

EXAMPLE 6.2 (System for Appell's F_4). The system $\mathcal{M}_4$ for Appell's F_4 is defined by

$$P_{41}u=P_{42}u=0\,,$$

where

$$P_{41}:=x(1-x)\partial_x^2-2xy\partial_x\partial_y-y^2\partial_y^2+\{\gamma-(\alpha+\beta+1)x\}\partial_x-(\alpha+\beta+1)y\partial_y-\alpha\beta,$$

$$P_{42}:=y(1-y)\partial_y^2-2xy\partial_x\partial_y-x^2\partial_x^2+\{\gamma'-(\alpha+\beta+1)y\}\partial_y-(\alpha+\beta+1)x\partial_x-\alpha\beta$$

with parameters $\alpha,\ \beta,\ \gamma,\ \gamma'\in\boldsymbol{C}$. This is a holonomic system of rank 4 with singular locus $xy(x^2+y^2-2xy-2x-2y+1)=0$. Put $I=\mathcal{D}_R P_{41}+\mathcal{D}_R P_{42}$ and

$$Y=\{(x,\ y)\,|\,x^2+y^2-2xy-2x-2y+1=0\}.$$

We make a birational coordinate transformation

$$t=x^2+y^2-2xy-2x-2y+1\,,\qquad x=x-y$$

and rewrite P_{41}, P_{42} in the new coordinate system $(t,\ x)$.

Inputting $\{P_{41},\ P_{42}\}$ to Algorithm 3.7, we get, as the output of the algorithm stopped when $m=-1$, $\boldsymbol{G}=\{P_{41},\ P_{42},\ P_{43},\ P_{44}\}$ with

$$\operatorname{lterm}_{FR}(P_{41})=(x+1)(x-1)^2\partial_t\partial_x,\qquad \operatorname{lterm}_{FR}(P_{42})=(x+1)^2(x-1)\partial_t\partial_x,$$

$$\operatorname{lterm}_{FR}(P_{43})=2(x+1)(x-1)t\partial_t^2,\qquad \operatorname{lterm}_{FR}(P_{44})=\frac{1}{2}(x+1)^3(x-1)^2\partial_x^3.$$

Moreover, P_{43} is Fuchsian along Y on $Y\setminus\{(0,\ 1),\ (0,\ -1)\}$ (hence so is $\mathcal{M}_4$). (By using Algorithm 2.7 we can verify that $\mathcal{M}_4$ is also Fuchsian along Y at $(0,\ \pm1)$). We do not know if $\boldsymbol{G}$ is indeed an FR-Gröbner basis of I along Y. In any case, we know by the algorithms that any multi-valued analytic solution u of $\mathcal{M}_4$ around Y is written in the form

$$u=v_1(t,\ x)+v_2(t,\ x)t^{\gamma+\gamma'-\alpha-\beta-1/2}$$

with v_1, v_2 holomorphic on a neighborhood of $Y\setminus\{(0,\ 1),\ (0,\ -1)\}$ satisfying $R_1v_1(0,\ x)=R_2v_2(0,\ x)=0$, where

$$\begin{aligned}R_1=&\,(x-1)^2(x+1)^2\partial_x{}^3\\ &+(x-1)(x+1)\{(2\alpha+2\beta+\gamma+\gamma'+2)x-3\gamma+3\gamma'\}\partial_x{}^2\\ &+[\{(4\beta+2\gamma+2\gamma')\alpha+(2\gamma+2\gamma')\beta+\gamma+\gamma'\}x^2-2(\gamma-\gamma')(2\alpha+2\beta+1)x\\ &+(-4\beta+2\gamma+2\gamma'-4)\alpha+(2\gamma+2\gamma'-4)\beta+(-8\gamma'+5)\gamma+5\gamma'-4]\partial_x\\ &+4\alpha\beta\{(\gamma+\gamma'-1)x-\gamma+\gamma'\},\end{aligned}$$

$$R_2=(x-1)(x+1)\partial_x+\{(3\gamma+3\gamma'-2\alpha-2\beta-2)x-\gamma+\gamma'\}.$$

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