J. Math. Soc. Japan
Vol. 47, No. 2, 1995

Asymptotic expansions of the solutions to a class of quasilinear hyperbolic initial value problems

By Atsushi YOSHIKAWA

(Received May 27, 1992)
(Revised July 12, 1993)

0. Introduction.

Let us consider the initial value problem related to the following quasilinear positive symmetric strictly hyperbolic system:

$$A_0(u)\frac{\partial}{\partial t}u+\sum_{\nu=1}^{n}A_\nu(u)\frac{\partial}{\partial x_\nu}u+B(u)u=0. \tag{0.1}$$

Thus, $A_0(u), \cdots, A_n(u)$ are $m\times m$ symmetric matrices depending smoothly on $u\in\boldsymbol{R}^m$, and $A_0(u)$ is positive definite while $B(u)$ may be any $m\times m$ smooth matrix. Strict hyperbolicity means that, for any $\xi=(\xi_1, \cdots, \xi_n)\neq 0$, the matrix

$$M(u, \xi)=\sum_{\nu=1}^{n}\xi_\nu A_0(u)^{-1}A_\nu(u) \tag{0.2}$$

has m distinct real eigenvalues $p_1(u, \xi), \cdots, p_m(u, \xi)$. We assume some of these eigenvalues actually depend on u because of quasi-linearity of the system (0.1).

We are interested in how hyperbolicity and non-linearity interact. To begin with, we seek an analogy of the oscillatory initial value problem which is basic in linear hyperbolic equations.

We choose as the initial data an m-vector of the form

$$u=\lambda^{-1}g(\lambda x\cdot\eta, x) \quad \text{at } t=0, \tag{0.3}$$

where $\lambda>0$ is a large parameter, $x\cdot\eta$ the scalar product of x and $\eta\in\boldsymbol{R}^n$, η being a fixed n-vector $\neq 0$, and $g(s, x)$ is a given m-vector valued smooth function with compact support in s, x, i.e., $g\in C_0^\infty(\boldsymbol{R}^{n+1})^m$.

The following is a convenient assumption on the initial data:

$$\int_{\boldsymbol{R}} g(s, x)ds=0. \tag{0.4}$$

We may rewrite (0.3) as

This research was partly supported by Grant-in-Aid for Scientific Research, Ministry of Education, Science and Culture, Japanese Government, No. 03640169.

$$(0.5)\qquad u=\lambda^{-1}g(\lambda x\cdot\eta,\ x)=\lambda^{-1}\sum_{j=1}^{m}g_j(\lambda x\cdot\eta,\ x)r_j(0,\ \eta)\qquad\text{at}\quad t=0$$

with appropriate scalar functions $g_j(s, x)$. Here $r_j(u, \eta)$ are eigenvectors of the matrix $M(u, \eta)$ corresponding to the eigenvalues $p_j(u, \eta)$, $j=1, \cdots, m$. (0.4) thus means the vanishing of mean with respect to s of each $g_j(s, x)$. In passing, we recall that the initial data we have previously considered are in one, e.g., the first, characteristic direction so that

$$(0.6)\qquad g_j(s,\ x)=0,\qquad j\geqq 2$$

([**7**][**8**]. See [**9**] for a summary. See § 5 for a discussion).

The initial data (0.3) (0.4) can be interpreted as a certain slightly oscillatory infinitesimal state which is represented by $\lambda\rightarrow+\infty$. The factor λ^{-1} is just good for the balance of non-linearity and hyperbolicity. Note that such factors are of no significance in linear problems. The solution of the initial value problem should then represent a certain infinitesimal state, which presumably reflects essential characters of the original system provided solutions exist at least in a time interval independent of $\lambda\rightarrow+\infty$. Though our choice of the linear initial phase $x\cdot\eta$ and the requirements on $g(s, x)$ make our discussions considerably simpler, we still see how quasi-linearity dictates the solution in its first order terms in λ^{-1}.

Let

$$(0.7)\qquad X_j={}^t r_j(u,\ \eta)\cdot\nabla_u,\qquad j=1,\ \cdots,\ m,$$

be characteristic vector fields. Here ^{t} denotes the transpose and $\nabla_u=(\partial/\partial u_1, \cdots, \partial/\partial u_m)$ is the gradient operation. We say that the system (0.1) satisfies Hypothesis (H) if, for each pair X_j, X_k, $j\neq k$, of characteristic vector fields, the commutator $[X_j, X_k]=X_jX_k-X_kX_j$ is a linear combination of X_j and X_k:

$$(0.8)\qquad [X_j,\ X_k]=a_{jk}(u,\ \eta)X_j-a_{kj}(u,\ \eta)X_k,\quad j,\ k=1,\ \cdots,\ m,\quad j\neq k,$$

a_{jk} and a_{kj} being smooth scalar functions.

REMARK. When Hypothesis (H) is satisfied, we can choose $r_j(u, \eta)$, $j=1, \cdots, m$, so that $a_{jk}=a_{kj}=0$, or

$$(0.9)\quad {}^t r_j(u,\ \eta)\cdot\nabla_u r_k(u,\ \eta)={}^t r_k(u,\ \eta)\cdot\nabla_u r_j(u,\ \eta),\quad j,\ k=1,\ \cdots,\ m,\quad j\neq k$$

(see § 2). We will assume (0.9) whenever we discuss systems satisfying Hypothesis (H) below.

The system of equations of the 2-D isentropic fluid flow is a standard example of systems satisfying Hypothesis (H) (see § 1).

Now one of the results in the present paper is the following

THEOREM 1. *Suppose the system* (0.1) *satisfies Hypothesis* (*H*). *Let* $n=2$ *or* 3 *and* $g\in C_0^\infty(\boldsymbol{R}^{n+1})^m$ *with* (0.4). *Then for some* $\lambda_1>0$ *and* $T_1>0$, *independent of* $\lambda\geqq\lambda_1$, *there is a uniquely determined solution* $u(x, t, \lambda)$, $x\in\boldsymbol{R}^n$, $0\leqq t\leqq T_1$, $\lambda\geqq\lambda_1$, *of the problem* (0.1) (0.3) *such that*

$$u(\cdot, t, \lambda)\in L^\infty([0, T_1]; H^3(\boldsymbol{R}^n))\cap C([0, T_1]; H^\sigma(\boldsymbol{R}^n)) \tag{0.10}$$

and

$$\frac{\partial}{\partial t}u(\cdot, t, \lambda)\in L^\infty([0, T_1]; H^2(\boldsymbol{R}^n))\cap C([0, T_1]; H^{\sigma-1}(\boldsymbol{R}^n)) \tag{0.11}$$

for any $\sigma<3$. *Here* $H^\rho(\boldsymbol{R}^n)$ *is the Sobolev space over* $\boldsymbol{R}^n$ *of exponent* ρ.

REMARK. The Sobolev ρ-norm of the initial data (0.3) is of order $\lambda^{\rho-1}$ as $\lambda\to\infty$. General discussions imply that the local solutions are expected to exist in a time interval of length $\lambda^{1-\rho}$ as $\lambda\to+\infty$.

Actually, when restricted to linear phases an analogue of Theorem 1 holds without the restriction on n or without Hypothesis (H) (see Joly-Metivier-Rauch [**2**], Schochet [**6**]). As will be seen below (Theorem 2), our discussions imply at the same time how the solutions behave as $\lambda\to+\infty$ in a fixed interval valid for all λ. Our method of proof is to use an approximate solution with so nice an error that its Sobolev 3-norm can be evaluated (see [**7**] in particular). This method yields to the restriction on n, but is basically valid for non-linear phases once approximate solutions are worked out.

To the problem (0.1) (0.3) we have a formal asymptotic solution of the form

$$\begin{aligned} U(x, t, \lambda) &= \lambda^{-1}\sum_{j=1}^{m} a_j(\lambda S_j(x, t), x, t)r_j(0, \eta) \\ &\quad +\lambda^{-2}\sum_{i,j,k=1}^{m} b_{ijk}(\lambda S_i(x, t), \lambda S_j(x, t), x, t)r_k(0, \eta). \end{aligned} \tag{0.12}$$

Here

$$S_j(x, t) = -p_j(0, \eta)t+x\cdot\eta, \qquad j = 1, \cdots, m, \tag{0.13}$$

are the planar phase functions, and $a_j(s_j, x, t)$ are determined from the following partial differential equations of first order (essentially of Burgers' type):

$$\frac{\partial}{\partial t}a_j+\sum_{\nu=1}^{n} p_j^{(\nu)}(0, \eta)\frac{\partial}{\partial x_\nu}a_j+(X_jp_j)(0, \eta)\frac{\partial}{\partial s_j}\left(\frac{1}{2}a_j^2\right)+\beta_{jj}a_j = 0 \tag{0.14}$$

with

$$a_j(s_j, x, 0) = g_j(s_j, x), \tag{0.15}$$

$j=1, \cdots, m$. Here $p_j^{(\nu)}(0, \eta)=\partial p_j(0, \eta)/\partial\eta_\nu$ and β_{jj} are certain constants. We see here how non-linearity and hyperbolicity are coupled even though compactness of the support of $g(s, x)$ with respect to s comes in to throw away non-local terms (see §5).

$b_{ijk}(s_i, s_j, x, t)$ are also determined from simple partial differential equations which will be given shortly.

Then, in fact, $U(x, t, \lambda)$ is asymptotic to $u(x, t, \lambda)$ in the following sense.

THEOREM 2. *Suppose the system* (0.1) *satisfies Hypothesis* (*H*). *Let* $u(x, t, \lambda)$ *be the solution to the problem* (0.1) (0.3) (0.4) *given in Theorem* 1. *Then, for* $U(x, t, \lambda)$ *of* (0.12),

$$\|u(\cdot, t, \lambda)-U(\cdot, t, \lambda)\|_s \leqq K\lambda^{s-3}, \qquad 0 \leqq s \leqq 3, \tag{0.16}$$

and

$$\left\|\frac{\partial}{\partial t}u(\cdot, t, \lambda)-\frac{\partial}{\partial t}U(\cdot, t, \lambda)\right\|_s \leqq K_1\lambda^{s-2}, \qquad 0 \leqq s \leqq 2, \tag{0.17}$$

for $\lambda \geqq \lambda_1$, $0 \leqq t \leqq T_1$. *Here* K *and* K_1 *are constants independent of* λ *and* t, *and* $\|\cdot\|_s$ *are the Sobolev s-norms.*

REMARK. Let $r_j^*(u, \eta)$, $j=1, \cdots, m$, be left (or row) eigenvectors of $M(u, \eta)$ corresponding to the eigenvalues $p_j(u, \eta)$ and normalized so that the relations

$$r_j^*(u, \eta)\cdot r_k(u, \eta) = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{0.18}$$

hold. Then $\beta_{jj}=r_j^*(0, \eta)\cdot A_0(0)^{-1}B(0)r_j(0, \eta)$ in (0.14). If the system (0.1) satisfies Hypothesis (H) and $r_j(u, \eta)$ are chosen to fulfill (0.9), then, in (0.13),

$$b_{ijk}(s_i, s_j, x, t) = \frac{1}{2}\gamma_{ijk}a_i(s_i, x, t)a_j(s_j, x, t), \qquad i \neq j, \tag{0.19}$$

and, for $j \neq k$,

$$\begin{aligned}(p_j(0, \eta)-p_k(0, \eta))\frac{\partial}{\partial s_j}(b_{jjk}(s_j, s_j, x, t))\\ = \gamma_{jjk}\frac{\partial}{\partial s_j}\frac{1}{2}a_j(s_j, x, t)^2+\beta_{jk}a_j(s_j, x, t)+\sum_{\nu=1}^{n}\alpha_{jk}^{\nu}\frac{\partial}{\partial x_\nu}a_j(s_j, x, t),\end{aligned} \tag{0.20}$$

where $\gamma_{ijk} = r_k^*(0, \eta)\cdot(X_ir_j)(0, \eta)$, $\beta_{jk} = r_k^*(0, \eta)\cdot A_0(0)^{-1}B(0)r_j(0, \eta)$, and $\alpha_{jk}^{\nu} = r_k^*(0, \eta)\cdot A_0(0)^{-1}A_\nu(0)r_j(0, \eta)$, $i, j, k=1, \cdots, m$, $\nu=1, \cdots, n$. The requirement (0.4) is called upon here to solve b_{jjk}.

Finally, $b_{kkk}(s_k, s_k, x, t)$ are determined from

$$\frac{\partial}{\partial t}b_{kkk}+\sum_{\nu=1}^{n}p_k^{(\nu)}(0, \eta)\frac{\partial}{\partial x_\nu}b_{kkk}+(X_kp_k)(0, \eta)\frac{\partial}{\partial s_k}(a_kb_{kkk})+\beta_{kk}b_{kkk} = h_{kk}, \tag{0.21}$$

with

$$b_{kkk}(s_k, s_k, x, 0) = -\sum_{(i,j)\neq(k,k)} b_{ijk}(s_k, s_k, x, 0). \tag{0.22}$$

Here $h_{kk}=h_{kk}(s_k, x, t)$ are known terms computed from a_k, b_{kkj}, $k\neq j$. If, on the other hand, the initial data are in one characteristic direction, then only one a_j, say a_1, survives, and b_{ijk}, $i\neq j$, all disappear (and thus required computations are considerably simpler).

The equation (0.14) shows that $a_j(s_j, x, t)$ develops shocks and loses smoothness in a finite time provided $X_j p_j(0, \eta)\neq 0$. The interval $[0, T_1]$ in Theorems 1 and 2 above, though uniform with respect to λ, lies within the interval of t where all the a_j's and hence $U(x, t, \lambda)$ remain smooth. Although a_j's and $U(x, t, \lambda)$ make sense up to $t=+\infty$ at the expense of their regularity, say, in the class of BV functions and their derivatives, we are yet unable to exploit this fact.

Theorems 1 and 2 will be proved in the sequel. We have previously discussed the case of initial data in one characteristic direction, compactly supported in s, and satisfying (0.4), though without Hypothesis (H) ([**7**]). As for the system satisfying Hypothesis (H), we have discussed the case of the initial data in one characteristic direction, periodic in the phase variable ([**8**]). The proofs of Theorems 1 and 2 are in spirit quite close to those in the above cases. To extend our results to more general situations, it would be necessary to analyze formal solutions proposed by Hunter, Majda and Rosales in full detail. ([**1**], [**3**], [**4**]. See also §5 below).

1. Supplementary observations on the system.

We begin by supplementing technical assumptions on the system:

$$A_0(u)\frac{\partial}{\partial t}u+\sum_{\nu=1}^{n}A_\nu(u)\frac{\partial}{\partial x_\nu}u+B(u)u=0. \tag{0.1}$$

Basic assumptions are stated in §0. The coefficient matrices $A_0(v), \cdots, B(v)$ depend C^∞ smoothly on $v\in \boldsymbol{R}^m$, and for a technical reason we suppose each of them is a sum of an $m\times m$ matrix of rapidly decreasing ($\mathcal{S}(\boldsymbol{R}^m)$) entries and one with constant entries (i.e., a constant matrix). Thus, for instance, $A_0(v)=A_0^c+A_0^d(v)$ with constant A_0^c and rapidly decreasing $A_0^d(v)$. Since our solutions will be shown to be bounded, this assumption is not quite restrictive. (For more details, see [**7**]).

Now $A_0(v), \cdots, A_n(v)$ are symmetric, and $A_0(v)$ is positive definite. More precisely, we have positive constants γ and Γ such that

$$\gamma y\cdot y\leqq y\cdot A_0(v)y\leqq \Gamma y\cdot y\,. \tag{1.1}$$

for all $y\in\boldsymbol{R}^m$, $v\in\boldsymbol{R}^m$.

As for the matrix $M(v, \xi)$, $v\in\boldsymbol{R}^m$, $\xi\in\boldsymbol{R}^n$, $\xi\neq 0$ (see (0.2)), we suppose its eigenvalues $p_1(v, \xi), \cdots, p_m(v, \xi)$ and right eigenvectors $r_1(v, \xi), \cdots, r_m(v, \xi)$ are

C^∞ smooth in v and $\xi\neq 0$. Similarly we suppose its left eigenvectors $r_1^*(v,\xi)$, $\cdots$, $r_m^*(v,\xi)$ are C^∞ smooth in v and $\xi\neq 0$. (See also (0.18)).

We have stated Hypothesis (H) in § 0. If $m\geqq 3$, this hypothesis is nontrivial. Here is a standard example.

EXAMPLE 1.1 (Equations of the isentropic fluid flow). Consider the equations for $u=(u_0, u_1, \cdots, u_n)$, $u_0>0$:

$$(1.2)\qquad \begin{cases} \dfrac{\partial}{\partial t}u_0+\dfrac{\partial}{\partial x_1}u_1+\cdots+\dfrac{\partial}{\partial x_n}u_n=0, \\ \dfrac{\partial}{\partial t}u_k+\sum_{i=1}^{n}\dfrac{\partial}{\partial x_i}\Big(\dfrac{u_iu_k}{u_0}\Big)+\dfrac{\partial}{\partial x_k}P(u_0)=0, \quad k=1,\cdots,n, \end{cases}$$

where $P(u_0)$ is a smooth scalar function satisfying

$$(1.3)\qquad P'(u_0)>0 \qquad (\text{and } P''(u_0)>0).$$

Physically, u_0 represents the density of the fluid, $(u_1, \cdots, u_n)$ the velocity vector, and $P(u_0)$ the pressure. Thus, $m=n+1$,

$$(1.4)\qquad A_0(u)=\begin{pmatrix} \dfrac{u_1^2+\cdots+u_n^2}{u_0^2}+P'(u_0) & -\dfrac{u_1}{u_0} & \cdots & -\dfrac{u_n}{u_0} \\ -\dfrac{u_1}{u_0} & 1 & 0 & \\ \vdots & & \ddots & \\ & 0 & & \\ -\dfrac{u_n}{u_0} & \cdots & & 1 \end{pmatrix},$$

$$(1.5)\qquad A_k(u)=\frac{u_k}{u_0}A_0(u)+P'(u_0)\begin{pmatrix} -\dfrac{2u_k}{u_0} & 0 & \cdots\cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & & & & & & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \\ 1 & & & & \mathbf{0} & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \end{pmatrix},$$

$k=1, \cdots, n$, and (1.2) takes the form (0.1). (In the second term of $A_k(u)$, 1 appears only in the $(k+1)$st place of the first column and of the first row). Note

$$(1.6)\qquad M(u,\xi)=\begin{pmatrix} 0 & \xi_1 & \cdots & \xi_n \\ -\dfrac{u_1}{u_0}\dfrac{u\cdot\xi}{u_0}+\xi_1P'(u_0) & \dfrac{u\cdot\xi}{u_0}+\xi_1\dfrac{u_1}{u_0} & \cdots & \xi_n\dfrac{u_1}{u_0} \\ \vdots & \vdots & & \vdots \\ -\dfrac{u_n}{u_0}\dfrac{u\cdot\xi}{u_0}+\xi_nP'(u_0) & \xi_1\dfrac{u_n}{u_0} & \cdots & \dfrac{u\cdot\xi}{u_0}+\xi_n\dfrac{u_n}{u_0} \end{pmatrix},$$

where $u\cdot\xi=u_1\xi_1+\cdots+u_n\xi_n$. Then

$$p_\pm(u,\xi)=\frac{u\cdot\xi}{u_0}\pm|\xi|\sqrt{P'(u_0)}, \tag{1.7}$$

and

$$p(u,\xi)=\frac{u\cdot\xi}{u_0} \tag{1.8}$$

are its eigenvalues ($p(u,\xi)$ being $(n-1)$-ple if $n\geqq 3$).

$$r_\pm(u,\xi)=\begin{pmatrix} u_0/P'(u_0) \\ u_1/P'(u_0)\pm\xi_1/\sqrt{P'(u_0)}|\xi| \\ \vdots \\ u_n/P'(u_0)\pm\xi_n/\sqrt{P'(u_0)}|\xi| \end{pmatrix} \tag{1.9}$$

and

$$r_i(u,\xi)=u_0\begin{pmatrix} 0 \\ c_1^i \\ \vdots \\ c_n^i \end{pmatrix}, \qquad i=1,\cdots,n-1, \tag{1.10}$$

are corresponding eigenvectors. Here c_j^i are constants satisfying

$$c_1^i\xi_1+\cdots+c_n^i\xi_n=0, \qquad (c_1^i,\cdots,c_n^i)\neq 0, \tag{1.11}$$

and

$$\begin{vmatrix} c_1^1 & \cdots & c_1^{n-1} & \xi_1 \\ \vdots & & \vdots & \vdots \\ c_n^1 & \cdots & c_n^{n-1} & \xi_n \end{vmatrix}\neq 0. \tag{1.12}$$

The characteristic vector fields corresponding to our choice of eigenvectors are

$$X_\pm=\frac{u_0}{P'(u_0)}\frac{\partial}{\partial u_0}+\sum_{k=1}^n\Big(\frac{u_k}{P'(u_0)}\pm\frac{\xi_k}{\sqrt{P'(u_0)}|\xi|}\Big)\frac{\partial}{\partial u_k} \tag{1.13}$$

and

$$X_i=u_0\sum_{k=1}^n c_k^i\frac{\partial}{\partial u_k}, \qquad i=1,\cdots,n-1. \tag{1.14}$$

It is immediately seen that each pair of the vector fields $X_1,\cdots,X_{n-1},X_+,X_-$ commute.

We also have

$$X_ip=X_ip_\pm=c^i\cdot\xi u_0=0, \qquad i=1,\cdots,n-1, \tag{1.15}$$

$$X_\pm p=\pm\frac{|\xi|}{u_0\sqrt{P'(u_0)}}, \tag{1.16}$$

$$X_\pm p_\pm=\pm\frac{|\xi|}{u_0\sqrt{P'(u_0)}}\Big(1+\frac{u_0^2P''(u_0)}{2P'(u_0)}\Big), \tag{1.17}$$

(1.18) $$X_{\pm}p_{\mp} = \pm\frac{|\xi|}{u_0\sqrt{P'(u_0)}}\left(1-\frac{u_0^2P''(u_0)}{2P'(u_0)}\right).$$

Thus, if

(1.19) $$P(u_0) = c_0\int_1^{u_0} e^{-2/r}dr+c_1\,,$$

$$c_1 > c_0\int_0^1 e^{-2/r}dr > 0\,,$$

then $X_{\pm}p_{\mp}=0$.

REMARK. To meet our technical requirement on the system, we have to modify (1.4) and (1.5) for large $|u|$ and u_0 near $u_0=0$ or $u_0\leqq 0$. But for our present purpose, this is not serious. On the other hand, the system (1.2) is strictly hyperbolic only when $n=2$.

2. Discussions on Hypothesis (H).

Suppose the system (0.1) satisfies Hypothesis (H). We show that we can then choose eigenvectors $r_k(v, \xi)$, $k=1, \cdots, m$, of $M(v, \xi)$ so that characteristic fields $X_1, \cdots, X_m$ commute each other (see Example 1.1). In Appendix A, we will indicate certain peculiarities of such systems in case they are of conservation laws as in Example 1.1.

LEMMA 2.1. *Suppose the vector fields $X_1, \cdots, X_m$ defined by* (0.7) *satisfy the commutator relation* (0.8) (*with u replaced by v*). *Then there are non-vanishing smooth functions $b_1(v, \xi), \cdots, b_m(v, \xi)$ such that*

(2.1) $$X_kb_j = a_{jk}b_j, \qquad j,\ k = 1, \cdots . m, \qquad j \neq k\,,$$

hold (*at least locally*).

COROLLARY 2.2. *Let*

(2.2) $$Y_j = b_j(v, \xi)X_j, \qquad j = 1, \cdots, m\,.$$

Then, for any i, $j=1, \cdots, m$,

(2.3) $$[Y_i, Y_j] = Y_iY_j-Y_jY_i = 0, \qquad i,\ j = 1, \cdots, m\,,$$

REMARK. Y_j corresponds to the eigenvector $b_j(v, \xi)r_j(v, \xi)$.

PROOF OF LEMMA 2.1. For each j, (2.1) is an overdetermined system of $m-1$ equations for a single unknown b_j. First, note the following relations among a_{jk}'s:

$$\left\{\begin{array}{l} X_i a_{jk} + a_{ki} a_{jk} = X_k a_{ji} + a_{ik} a_{ji}, \\ X_j a_{ki} + a_{ij} a_{ki} = X_i a_{kj} + a_{ji} a_{kj}, \\ X_k a_{ij} + a_{jk} a_{ij} = X_j a_{ik} + a_{kj} a_{ik}, \end{array}\right. \tag{2.4}$$

$i, j, k=1, \cdots, m, i \neq j \neq k \neq i$. In fact, (2.4) follows from Jacobi's identity:

$$[[X_i, X_j], X_k] + [[X_j, X_k], X_i] + [[X_k, X_i], X_j] = 0$$

and (0.8) (u replaced by v). Now, for j fixed, let

$$w_i = X_i b_j - a_{ji} b_j, \qquad i = 1, \cdots, m, \qquad i \neq j.$$

(2.4) yields to

$$X_i w_k + (a_{ki} - a_{ji}) w_k = X_k w_i + (a_{ik} - a_{jk}) w_i, \tag{2.5}$$

$i, k=1, \cdots, m, i \neq j \neq k \neq i$. Therefore, if $w_i=0$ holds and if $w_k=0$ on a hypersurface transversal to X_i, then w_k also vanishes everywhere. To fix the idea, let $j=1$, and suppose $w_m=0$. We have to show $w_k=0$, $k=2, \cdots, m-1$, on a hypersurface S_m transversal to X_m. Since the vector fields $X_1, \cdots, X_{m-1}$ are in involution, we can choose the surface S_m to be their integral manifold. Now if $w_k=0$, $k=2, \cdots, m-1$, hold on S_m, then since this means the values of b_1 are known on S_m, b_1 is determined by $w_m=0$ in a neighborhood of S_m. (2.5) then automatically yields to $w_k=0$ outside S_m, $k=2, \cdots, m-1$. Similar discussions are valid on S_m for the fields $X_1, \cdots, X_{m-1}$ and functions $w_2, \cdots, w_{m-1}$ restricted there. So we only need to verify the bottom case, that is, $m=3$. Let S_3 be an integral surface of X_1 and X_2, to which X_3 is transversal. Let C_1 be an integral curve of X_1 lying on the surface S_3, to which X_2 is transversal. We can then determine b_1 on S_3 through the equation $w_2=X_2 b_1 - a_{12} b_1 =0$ restricted to S_3 by specifying the values of b_1 arbitrarily, thus non-vanishing, on C_1. Using thus determined values of b_1 on S_3, solve b_1 outside S_3 by $X_3 b_1 - a_{13} b_1 = 0$, or $w_3=0$. By (2.5), we see $w_2=0$ outside S_3 too.

In the following discussions, we choose eigenvectors $r_1(v, \xi), \cdots, r_m(v, \xi)$ of the matrix $M(v, \xi)$ so that their corresponding characteristic vector fields $X_1, \cdots, X_m$ all commute each other, or $X_j X_k = X_k X_j$ hold for $j, k=1, \cdots, m$. In other words,

$$dr_j(v, \xi)[r_k(v, \xi)] = dr_k(v, \xi)[r_j(v, \xi)] \tag{0.9}$$

for $j, k=1, \cdots, m$. Here

$$dr_j(v, \xi)[w] = \frac{d}{d\varepsilon} r_j(v+\varepsilon w, \xi)|_{\varepsilon=0} = {}^t w \cdot \nabla r_j(v, \xi)$$

is the Fréchet-Gâteaux derivative of $r_j(v, \xi)$.

3. Formal solutions.

Let us return to the initial value problem (0.1) (0.3). The corresponding system of partial differential operators depending on $v\in \boldsymbol{R}^m$ is given by

$$\mathcal{L}(v)=\frac{\partial}{\partial t}+\sum_{\nu=1}^{n}\tilde{A}_\nu(v)\frac{\partial}{\partial x_\nu}+\tilde{B}(v), \tag{3.1}$$

where $\tilde{A}_\nu(v)=A_0(v)^{-1}A_\nu(v)$, $\tilde{B}(v)=A_0(v)^{-1}B(v)$.

Let us choose an m-vector valued function $V=V(x, t, \lambda)$, roughly of order λ^{-1} as $\lambda\to+\infty$, such that

$$V(x, 0, \lambda)=\lambda^{-1}g(\lambda x\cdot\eta, x) \tag{3.2}$$

and

$$\mathcal{L}(V)V=F, \qquad F=F(x, t, \lambda), \tag{3.3}$$

is to be interpreted as of order λ^{-3}. More precisely, we require that $V(x, t, \lambda)$ be smooth in

$$D=D_{T_0,\lambda_0}=\{(x, t, \lambda);\ x\in \boldsymbol{R}^n,\ 0\leqq t\leqq T_0,\ \lambda\geqq\lambda_0\}$$

for some $T_0>0$ and $\lambda_0>0$, be compactly supported with respect to x, and satisfy the estimates:

$$\sup_D \lambda^{-k-|\alpha|+1}|\partial_t^k\partial_x^\alpha V(x, t, \lambda)|\leqq C_{k,\alpha}<+\infty \tag{3.4}$$

for non-negative integers k and multi-indices $\alpha=(\alpha_1, \cdots, \alpha_n)$, together with

$$\sup_\Lambda \lambda^{-k-s+1}\|\partial_t^k V(\cdot, t, \lambda)\|_s\leqq C_s<+\infty \tag{3.5}$$

for $s\geqq 0$. Here $\|\cdot\|_s$ is the Sobolev norm of exponent s and

$$\Lambda=\Lambda_{T_0,\lambda_0}=\{(t, \lambda);\ 0\leqq t\leqq T_0,\ \lambda\geqq\lambda_0\}.$$

$F(x, t, \lambda)$, on the other hand, is required to be smooth in D, compactly supported with respect to x, and to satisfy slightly different estimates:

$$\sup_D \lambda^{-k-|\alpha|+3}|\partial_t^k\partial_x^\alpha F(x, t, \lambda)|\leqq C_{k,\alpha}<+\infty \tag{3.6}$$

and

$$\sup_\Lambda \lambda^{-k-s+3}\|\partial_t^k F(\cdot, t, \lambda)\|_s\leqq C_s<+\infty. \tag{3.7}$$

Now let

$$\tilde{A}_\nu(v)=\tilde{A}_\nu(0)+d\tilde{A}_\nu(0)[v]+\frac{1}{2}d^2\tilde{A}_\nu(0)[v, v]+R_3(\tilde{A}_\nu)(0;v), \quad \nu=1, \cdots, n,$$

and

$$\tilde{B}(v)=\tilde{B}(0)+d\tilde{B}(0)[v]+R_2(\tilde{B})(0;v)$$

be the Taylor expansions of $\tilde{A}_\nu(v)$ and $\tilde{B}(v)$ around $v=0$. Then

$$\mathcal{L}(V)V=\mathcal{L}^0(V)V+F_1(x,t,\lambda),$$

where

$$\mathcal{L}^0(V)V=\mathcal{L}(0)V+\sum_{\nu=1}^{n}\left\{d\tilde{A}_\nu(0)[V]+\frac{1}{2}d^2\tilde{A}_\nu(0)[V,V]\right\}\frac{\partial}{\partial x_\nu}V+d\tilde{B}(0)[V]V \tag{3.8}$$

and

$$F_1(x,t,\lambda)=\sum_{\nu=1}^{n}R_3(\tilde{A}_\nu)(0;V)\frac{\partial}{\partial x_\nu}V+R_2(\tilde{B})(0;V)V. \tag{3.9}$$

$F_1(x,t,\lambda)$ satisfies the same estimates as (3.6)(3.7) provided $V(x,t,\lambda)$ does (3.4)(3.5) (see discussions in [**5**]). We put

$$V(x,t,\lambda)=\lambda^{-1}u_1(x,t,\lambda)+\lambda^{-2}u_2(x,t,\lambda)+\lambda^{-3}u_3(x,t,\lambda) \tag{3.10}$$

with appropriate u_1, u_2, u_3 in order that (3.4) and (3.5) be fulfilled (see Appendix B). Substitute (3.10) into (3.8). We get

$$\begin{aligned}\mathcal{L}^0(V)V=&\ \lambda^{-1}\mathcal{L}(0)u_1+\lambda^{-2}\left\{\mathcal{L}(0)u_2+\sum_{\nu=1}^{n}d\tilde{A}_\nu(0)[u_1]\frac{\partial}{\partial x_\nu}u_1+d\tilde{B}(0)[u_1]u_1\right\}\\&+\lambda^{-3}\left\{\mathcal{L}(0)u_3+\sum_{\nu=1}^{n}\left(d\tilde{A}_\nu(0)[u_1]\frac{\partial}{\partial x_\nu}u_2+d\tilde{A}_\nu(0)[u_2]\frac{\partial}{\partial x_\nu}u_1\right)\right.\\&\left.\qquad+\frac{1}{2}\sum_{\nu=1}^{n}d\tilde{A}_\nu(0)[u_1,u_1]\frac{\partial}{\partial x_\nu}u_1+d\tilde{B}(0)[u_1]u_2+d\tilde{B}(0)[u_2]u_1\right\}\\&+\sum_{k=4}^{9}\lambda^{-k}G_k(x,t,\lambda),\end{aligned} \tag{3.11}$$

where $G_k(x,t,\lambda)$, $k\geq 4$, are basically known as computed from u_1, u_2, u_3 and their first derivatives (see Appendix B for explicit computations). Therefore, our task is to choose u_1, u_2, u_3 so that we may reduce $\mathcal{L}^0(V)V-\sum_{k=4}^{9}\lambda^{-k}G(x,t,\lambda)$ as far as possible.

Fix $\eta\in\boldsymbol{R}^n$, $\eta\neq 0$, and let $S_j(x,t)=-p_j(0,\eta)t+x\cdot\eta$, $j=1,\cdots,m$ (recall (0.15)). Note

$$S_{j,t}+p_j(0,S_{j,x})=0,\qquad S_j(x,0)=x\cdot\eta \tag{3.12}$$

since $S_{j,x}=\eta$ for all j. We now set

$$u_1(x,t,\lambda)=\sum_{i=1}^{m}a_i(\lambda S_i(x,t),x,t)r_i(0,\eta), \tag{3.13}$$

$$u_2(x,t,\lambda)=\sum_{i,j,k=1}^{m}b_{ijk}(\lambda S_i(x,t),\lambda S_j(x,t),x,t)r_k(x,t), \tag{3.14}$$

$$u_3(x,t,\lambda)=\sum_{i,j,k,l=1}^{m}c_{ijkl}(\lambda S_i(x,t),\lambda S_j(x,t),\lambda S_k(x,t),x,t)r_l(0,\eta), \tag{3.15}$$

by choosing suitable functions $a_i(s_i, x, t)$, $b_{ijk}(s_i, s_j, x, t)$, $c_{ijkl}(s_i, s_j, s_k, x, t)$, i, j, k, $l=1, \cdots, m$, to be determined. Here we assume

$$b_{ijk}=b_{jik}, \qquad c_{ijkl}=c_{jkil}=c_{kijl}=c_{jikl}=c_{kjil}=c_{ikjl}$$

for $i, j, k, l=1, \cdots, m$. Now we can rewrite $\mathcal{L}^0(V)V-\sum_{k=4}^9\lambda^{-k}G_k(x, t, \lambda)$ rather formally as

$$\mathcal{L}^0(V)V-\sum_{k=4}^9\lambda^{-k}G_k=\lambda^{-1}G_1^0+\lambda^{-2}G_2^0+\lambda^{-3}G_3$$

where, at $s_j=\lambda S_j(x, t)$, $j=1, \cdots, m$,

$$\begin{aligned}G_1^0= & \sum_{i=1}^m\{a_{i,t}+M(0, a_{i,x})+a_i\tilde{B}(0)\}r_i(0, \eta)\\ & +\sum_{i,j=1}^m a_i\frac{\partial}{\partial s_j}a_j dM(0, \eta)[r_i(0, \eta)]r_j(0, \eta)\\ & +\sum_{i,j,k=1}^m\Big\{(-p_i(0, \eta)+p_k(0, \eta))\frac{\partial}{\partial s_i}b_{ijk}\\ & \qquad +(-p_j(0, \eta)+p_k(0, \eta))\frac{\partial}{\partial s_j}b_{ijk}\Big\}r_k(0, \eta),\end{aligned}\tag{3.16}$$

$$\begin{aligned}G_2^0= & \sum_{i,j=1}^m a_i dM(0, a_{j,x})[r_i(0, \eta)]r_j(0, \eta)+\sum_{i,j=1}^m a_ia_j d\tilde{B}(0)[r_i(0, \eta)]r_j(0, \eta)\\ & +\frac{1}{2}\sum_{i,j,k=1}^m a_ia_j\frac{\partial}{\partial s_k}a_k d^2M(0, \eta)[r_i(0, \eta), r_j(0, \eta)]r_k(0, \eta)\\ & +\sum_{i,j,k,l=1}^m\Big\{a_i\Big(\frac{\partial}{\partial s_j}b_{jkl}+\frac{\partial}{\partial s_k}b_{jkl}\Big)dM(0, \eta)[r_i(0, \eta)]r_l(0, \eta)\\ & \qquad +\frac{\partial a_i}{\partial s_i}b_{jkl}dM(0, \eta)[r_l(0, \eta)]r_i(0, \eta)\Big\}\\ & +\sum_{i,j,k=1}^m\{b_{ijk,t}+M(0, b_{ijk,x})+b_{ijk}\tilde{B}(0)\}r_k(0, \eta)\\ & +\sum_{i,j,k,l=1}^m\Big\{(-p_i(0, \eta)+p_l(0, \eta))\frac{\partial}{\partial s_i}c_{ijkl}+(-p_j(0, \eta)\\ & \qquad +p_l(0, \eta))\frac{\partial}{\partial s_j}c_{ijkl}+(-p_k(0, \eta)+p_l(0, \eta))\frac{\partial}{\partial s_k}c_{ijkl}\Big\}r_l(0, \eta),\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}G_3= & \frac{1}{2}\sum_{i,j,k=1}^m a_ia_jd^2M(0, a_{k,x})[r_i(0, \eta), r_j(0, \eta)]r_k(0, \eta)\\ & +\sum_{i,j,k,l=1}^m\{a_idM(0, b_{jkl,x})[r_i(0, \eta)]r_l(0, \eta)+b_{jkl}dM(0, a_{i,x})[r_l(0, \eta)]r_i(0, \eta)\\ & \qquad +a_ib_{jkl}(d\tilde{B}(0)[r_i(0, \eta)]r_l(0, \eta)+d\tilde{B}(0)[r_l(0, \eta)]r_i(0, \eta))\}\end{aligned}\tag{3.18}$$

$$+\sum_{i,j,k,l=1}^{m}\{c_{ijkl,t}+M(0,\ c_{ijkl,x})+c_{ijkl}\tilde{B}(0)\}r_l(0,\ \eta).$$

Now we have to choose nicely behaving a_i, b_{ijk}, c_{ijkl} which furthermore make $G_1^0=0$ and $G_2^0=0$. Note that (3.2) requires

$$(3.19)\qquad u_1(x,\ 0,\ \lambda)=g(\lambda x\cdot\eta,\ x),\qquad u_2(x,\ 0,\ \lambda)=0,\qquad u_3(x,\ 0,\ \lambda)=0.$$

Let us compute $r_i^*(0,\ \eta)\cdot G_1^0=0$, $i=1,\ \cdots,\ m$. Then

$$(3.20)\qquad a_{i,t}+\sum_{\nu=1}^{n}p_i^{(\nu)}(0,\ \eta)a_{i,x_\nu}+\sum_{j=1}^{m}\beta_{ji}a_j+(X_ip_i)(0,\ \eta)\frac{\partial}{\partial s_i}\Big(\frac{1}{2}a_i^2\Big)$$
$$+\sum_{j,k=1}^{m}(p_j(0,\ \eta)-p_i(0,\ \eta))\gamma_{kji}\frac{\partial}{\partial s_j}a_ja_k+\sum_{\substack{j\neq i\\ j=1}}^{m}\sum_{\nu=1}^{n}\alpha_{ji}^{\nu}a_{j,x_\nu}$$
$$+\sum_{j,k=1}^{m}\Big\{(-p_j(0,\ \eta)+p_i(0,\ \eta))\frac{\partial}{\partial s_j}b_{jki}+(-p_k(0,\ \eta)+p_i(0,\ \eta))\frac{\partial}{\partial s_k}b_{jki}\Big\}=0.$$

Here $\alpha_{ji}^{\nu}=r_i^*(0,\ \eta)\cdot\tilde{A}_\nu(0)r_j(0,\ \eta)$, $\beta_{ji}=r_i^*(0,\ \eta)\cdot\tilde{B}(0)r_j(0,\ \eta)$, and $\gamma_{kji}=r_i^*(0,\ \eta)\cdot dr_j(0,\ \eta)[r_k(0,\ \eta)]=r_i^*(0,\ \eta)\cdot(X_kr_j)(0,\ \eta)$. Separating functions a_i, b_{ijk}, etc. according to the phases, we stipulate

$$(3.21)\qquad a_{i,t}+\sum_{\nu=1}^{n}p_i^{(\nu)}(0,\ \eta)a_{i,x_\nu}+\beta_{ii}a_i+(X_ip_i)(0,\ \eta)\frac{\partial}{\partial s_i}\Big(\frac{1}{2}a_i^2\Big)=0$$

with

$$(3.22)\qquad a_i(s_i,\ x,\ 0)=g_i(s_i,\ x)$$

$i=1,\ \cdots,\ m$ (see (0.14) (0.15)),

$$(3.23)\qquad (-p_j(0,\ \eta)+p_i(0,\ \eta))\frac{\partial}{\partial s_j}(b_{jji})+\sum_{\nu=1}^{n}\alpha_{ji}^{\nu}a_{i,x_\nu}+\beta_{ji}a_j$$
$$+(p_j(0,\ \eta)-p_i(0,\ \eta))\gamma_{jji}\frac{\partial}{\partial s_j}\Big(\frac{1}{2}a_j^2\Big)=0,$$

$j\neq i$, $i,j=1,\ \cdots,\ m$ (see (0.20)), and

$$(3.24)\qquad (p_j(0,\ \eta)-p_i(0,\ \eta))\gamma_{kji}\frac{\partial}{\partial s_j}a_ja_k+(p_k(0,\ \eta)-p_i(0,\ \eta))\gamma_{jki}\frac{\partial}{\partial s_k}a_ja_k$$
$$+2\Big\{(-p_j(0,\ \eta)+p_i(0,\ \eta))\frac{\partial}{\partial s_j}+(-p_k(0,\ \eta)+p_j(0,\ \eta))\frac{\partial}{\partial s_k}\Big\}b_{kji}=0,$$

$j\neq k$, $i,\ j,\ k=1,\ \cdots,\ m$.

REMARK. The Ansatz by Hunter-Majda-Rosales [**1**] does not appeal to separation of phases. We will discuss on the matter in §5.

The results concerning the equations (3.21)-(3.24) are summarized in the following

PROPOSITION 3.1. (i) *Let* $g_i(s_i,\ x)\in C_0^\infty(\boldsymbol{R}^{n+1})$, $i=1,\ \cdots,\ m$. *For each* i, *there is a unique solution* $a_i(s_i, x, t)$ *to the equations* (3.21) (3.22), *which is smooth with respect to* $(s_i, x, t)\in \boldsymbol{R}^{n+1}\times[0, T]$ *for some* $T>0$. $a_i(s_i, x, t)$ *is compactly supported with respect to* s_i, x, *and furthermore*

$$\int_{\boldsymbol{R}} a_i(s_i, x, t)ds_i = 0 \tag{3.25}$$

provided $\int_{\boldsymbol{R}} g_i(s_i, x)ds_i=0$ *holds* (*see* (0.4)).

(ii) *For each pair* i, j, $i\neq j$, *there is a solution* $b_{jji}(s_j, s_j, x, t)$ *to the equation* (3.23), *smooth in* $(s_j, x, t)\in \boldsymbol{R}^{n+1}\times[0, T]$, *compactly supported with respect to* x, *while bounded in* s_j. *Furthermore, if* (3.25) *holds* (*with* i *replaced by* j), *there is a unique* $b_{jji}(s_j, s_j, x, t)$ *which is compactly supported with respect to* s_j.

(iii) *For any pair* j, k, $j\neq k$, *there is a solution* $b_{kji}(s_k, s_j, x, t)$ *to the equation* (3.24), *smooth in* (s_k, s_j, x, t), *compactly supported with respect to* x, *and bounded with respect to* s_k, s_j. *Furthermore, if Hypothesis* (*H*) *holds, then we can take* $b_{kji}=(1/2)\gamma_{kji}a_j a_k$ (*see* (0.19)).

PROOF. Quite obvious. As for (iii), Hypothesis (H) and our choice of eigenvectors satisfying (0.9) imply $\gamma_{kji}=\gamma_{jki}$. Then (3.24) reduces to

$$\Big\{(p_j(0, \eta)-p_i(0, \eta))\frac{\partial}{\partial s_j}+(p_k(0, \eta)-p_i(0, \eta))\frac{\partial}{\partial s_k}\Big\}\Big(b_{kji}-\frac{1}{2}\gamma_{kji}a_k a_j\Big)=0\,.$$

For the remaining part of (iii), we appeal to the following

LEMMA 3.2. *Let* $\alpha_1, \cdots, \alpha_N$ *be real and* $\alpha_1\neq 0$. *Suppose* $G(t_1, \cdots, t_N)$ *is smooth and uniformly bounded together with all its derivatives. If the* t_1-*projection of the support of* G *is compact, then*

$$\Big(\alpha_1\frac{\partial}{\partial t_1}+\cdots+\alpha_N\frac{\partial}{\partial t_N}\Big)F=G, \qquad (t_1, \cdots, t_N)\in \boldsymbol{R}^N\,, \tag{3.26}$$

has a solution, which is bounded together with all its derivatives.

PROOF. We may suppose $\alpha_1=1$. Then

$$F(t_1, \cdots, t_N)=\int_{-\infty}^{t_1} G(s, t_2+\alpha_2(s-t_1), \cdots, t_N+\alpha_N(s-t_1))ds$$

is a solution. Let $[a, b]$ be a bounded interval which contains the t_1-projection of supp G. Then $F=0$ for $t_1\leqq a$, and if $t_1>a$, then

$$|F(t_1, \cdots, t_N)|\leqq M(b-a), \qquad M=\sup|G(t_1, \cdots, t_N)|\,.$$

Similar arguments are valid for the derivatives of F.

REMARKS. 1. If $(X_i p_i)(0, \eta)\neq 0$, then (3.21) is essentially of Burgers' type. Its solution $a_i(s_i, x, t)$ develops shocks however smooth its initial data $g_i(s_i, x)$ may be. A non-regular solution, the entropy solution, then makes sense beyond shocks up to $t=+\infty$.

2. If the initial data are in one characteristic direction, and $g_i=0$, $i\geqq 2$, then $a_i\equiv 0$ for $i\geqq 2$, $b_{iik}\equiv 0$, $i\geqq 2$, $i\neq k$, $b_{ijk}\equiv 0$, $i\neq j$.

So far we have determined a_i, $i=1, \cdots, m$, b_{jji}, $i\neq j$, $i, j=1, \cdots, m$, and b_{jki}, $j\neq k$, $i, j, k=1, \cdots, m$. To get a complete description of $u_2(x, t, \lambda)$, we still have to determine b_{iii}, $i=1, \cdots, m$. We will also need to know c_{ijkl} for controlling $u_3(x, t, \lambda)$. Thus, suppose $G_2^0=0$.

Separating phases as before, we then stipulate

$$\begin{aligned}(3.27)\quad &\{b_{iii,t}+M(0, b_{iii,x})+b_{iii}\tilde{B}(0)\}r_i(0, \eta)+\frac{\partial}{\partial s_i}(a_i b_{iii})dM(0, \eta)[r_i(0, \eta)]r_i(0, \eta)\\ &+\sum_{\substack{k=1\\k\neq i}}^{m}(-p_i(0, \eta)+p_k(0, \eta))\frac{\partial}{\partial s_i}(c_{iiik})r_k(0, \eta)+f_i=0,\end{aligned}$$

$$\begin{aligned}(3.28)\quad &3\sum_{k=1}^{m}\Big\{(-p_i(0, \eta)+p_k(0, \eta))\frac{\partial}{\partial s_i}(c_{iijk})\\ &+(-p_j(0, \eta)+p_k(0, \eta))\frac{\partial}{\partial s_j}c_{iijk}\Big\}r_k(0, \eta)+f_{ij}=0, \qquad i\neq j,\end{aligned}$$

$$\begin{aligned}(3.29)\quad &6\sum_{l=1}^{m}\Big\{(-p_i(0, \eta)+p_l(0, \eta))\frac{\partial}{\partial s_i}c_{ijkl}+(-p_j(0, \eta)+p_l(0, \eta))\frac{\partial}{\partial s_j}c_{ijkl}\\ &+(-p_k(0, \eta)+p_l(0, \eta))\frac{\partial}{\partial s_k}c_{ijkl}\Big\}r_l(0, \eta)+f_{ijk}=0, \quad i\neq j\neq k\neq i.\end{aligned}$$

Here $f_i=f_i(s_i, x, t)$, $f_{ij}=f_{ij}(s_i, s_j, x, t)$, $f_{ijk}=f_{ijk}(s_i, s_j, s_k, x, t)$ are computed from so far determined functions a_i, b_{ijk}, $i\neq j$ or $i=j\neq k$. (See Appendix C for explicit computations.)

As the initial data for $b_{iii}(s_i, s_i, x, t)$, we take

$$(3.30)\qquad b_{iii}(s_i, s_i, x, 0)=-\sum_{\substack{j,k=1\\j\neq k}}^{m}b_{jki}(s_i, s_i, x, 0)-\sum_{\substack{j=1\\j\neq i}}^{m}b_{jji}(s_i, s_i, x, 0)$$

because of (3.19). Then we solve b_{iii} from

$$(3.31)\qquad b_{iii,t}+\sum_{\nu=1}^{m}p_i^{(\nu)}(0, \eta)b_{iii,x_\nu}+(X_i p_i)(0, \eta)\frac{\partial}{\partial s_i}(a_i b_{iii})+\beta_{ii}b_{iii}=h_{ii},$$

where $h_{ii}=-r_i^*(0, \eta)\cdot f_i(s_i, x, t)$ (see (0.21)). We also handle

$$\begin{aligned}(3.32)\quad &(-p_i(0, \eta)+p_k(0, \eta))\frac{\partial}{\partial s_i}(c_{iiik})+\sum_{\nu=1}^{m}\alpha_{ik}^{\nu}b_{iii,x_\nu}+\beta_{ik}b_{iii}\\ &+(p_i(0, \eta)-p_k(0, \eta))\gamma_{iik}\frac{\partial}{\partial s_i}(a_i b_{iii})=h_{ik}\end{aligned}$$

for $i\neq k$, where $h_{ik}=-r_k^*(0, \eta)\cdot f_i(s_i, x, t)$,

$$(3.33)\quad (-p_i(0, \eta)+p_k(0, \eta))\frac{\partial}{\partial s_i}(c_{iijk})+(-p_j(0, \eta)+p_k(0, \eta))\frac{\partial}{\partial s_j}(c_{iijk})=h_{ijk},$$

for $i \neq j$, where $h_{ijk}=-(1/3)r_k^*(0, \eta)\cdot f_{ij}(s_i, s_j, x, t)$, and

$$(3.34) \quad (-p_i(0, \eta)+p_l(0, \eta))\frac{\partial}{\partial s_i}c_{ijkl}+(-p_j(0, \eta)+p_l(0, \eta))\frac{\partial}{\partial s_j}c_{ijkl} \\ +(-p_k(0, \eta)+p_l(0, \eta))\frac{\partial}{\partial s_k}c_{ijkl} = h_{ijkl},$$

$i \neq j \neq k \neq i$, where $h_{ijkl}=-(1/6)r_l^*(0, \eta)\cdot f_{ijk}(s_i, s_j, x, t)$.

As for the solutions to these equations, we have the following

PROPOSITION 3.3. (i) *For each i, the solution $b_{iii}(s_i, s_i, x, t)$ to the equation* (3.30) (3.31) *is smooth in $(s_i, x, t)\in \boldsymbol{R}^{n+1}\times[0, T]$, compactly supported with respect to x. b_{iii} is bounded with respect to s_i, and if* (0.4) *is fulfilled, b_{iii} is compactly supported with respect to s_i.*

(ii) *Suppose* (0.4) *holds. Then for each i, k, $i \neq k$, there is a solution c_{iiik} to the equation* (3.32), *smooth in $(s_i, x, t)\in \boldsymbol{R}^{n+1}\times[0, T]$, compactly supported with respect to x, and bounded in s_i. If* (0.4) *fails to hold, then c_{iiik} grows like $|s_i|$ as $|s_i|\to\infty$.*

(iii) *Suppose the system* (0.1) *satisfies Hypothesis* (H). *Then for each triplet i, j, k, $i \neq j \neq k \neq i$, there is a solution c_{iijk} to the equation* (3.33), *smooth in $(s_i, s_j, x, t) \in \boldsymbol{R}^{n+2}\times[0, T]$, compactly supported with respect to x, and bounded with respect to s_i, s_j. If* (0.4) *holds, this is also the case for c_{iijj}, c_{iiji}.*

(iv) *For each quadruplet i, j, k, l, $i \neq j \neq k \neq i$, there is a solution c_{ijkl} to the equation* (3.34), *smooth in $(s_i, s_j, s_k, x, t)\in \boldsymbol{R}^{n+3}\times[0, T]$, compactly supported with respect to x, and bounded in s_i, s_j, s_k.*

(v) *Suppose* (0.6) *holds. Then $c_{iijk}=0$, $i \neq 1$, $j \neq 1$, and $c_{ijkl}=0$, $i \neq j \neq k \neq i$. On the other hand, c_{iijk}, $i \neq j$, i or $j=1$, $k \neq 1$, can be chosen bounded with respect to s_i and s_j, and compactly supported in x. c_{iiik}, $k \neq i$, are compactly supported in x, but grow like $|s_i|$ as $|s_i|\to\infty$ (unless* (0.4) *holds).*

PROOF. Obvious from Proposition 3.1 and Lemma 3.2 (see also Appendix C).

REMARKS. 1. In (v), c_{iiik}, $i \geqq 2$, $k \neq i$, can be written as

$$c_{iiik} = s_i\int_{-\infty}^{s_i}\tilde{b}_{ik}(s, x, t)ds-\int_{-\infty}^{s_i}s\tilde{b}_{ik}(s, x, t)ds,$$

where $\tilde{b}_{ik}(s, x, t)$ are compactly supported with respect to s, x. On the other hand, c_{iiji}, $j \neq 1$, contains a term of the form

$$-\frac{1}{3}\frac{\partial}{\partial s_1}a_1\frac{dp_1(0, \eta)[r_j(0, \eta)]}{p_1(0, \eta)-p_j(0, \eta)}\int_{-\infty}^{s_j}b_{jjj}(s, s, x, t)ds.$$

Note

$$\int_{-\infty}^{s_j}b_{jjj}(s, s, x, t)ds = s_j\int_{-\infty}^{s_j}\tilde{b}_j(s, x, t)ds-\int_{-\infty}^{s_j}\tilde{b}_j(s, x, t)ds, \qquad j \neq 1,$$

with $\tilde{b}_j(s, x, t)$ compactly supported in s, x.

2. In (iii), if (0.4) fails to hold, $c_{iijj}(=c_{jjij})$ contains terms of the form

$$-\frac{1}{3}\sum_{k=1}^{m}\frac{dp_j(0, \eta)[r_k(0, \eta)]}{p_j(0, \eta)-p_i(0, \eta)}\frac{\partial}{\partial s_j}a_j\int_{-\infty}^{s_i}b_{iik}ds,$$

which are linear in s_i.

Finally, we are content with realizing $u_3(x, 0, \lambda)=0$ and just solve c_{iiii} from

$$c_{iiii,t}+\sum_{\nu=1}^{n}p_i^{(\nu)}(0, \eta)c_{iiii,x_\nu}+\beta_{ii}c_{iiii}=0 \tag{3.35}$$

with

$$c_{iiii}(s_i, s_i, s_i, x, 0)=-\sum_{(j,k,l)\neq(i,i,i)}c_{jkli}(s_i, s_i, s_i, x, 0). \tag{3.36}$$

Then c_{iiii} are smooth in $(s_i, x, t)\in\boldsymbol{R}^{n+1}\times[0, T]$, compactly supported with respect to x. The behaviors of c_{iiii} with respect to s_i inherit those of initial data, and thus when Hypothesis (H) is satisfied and (0.4) holds, s_{iiii} are bounded in s_i. Summarizing, we have shown

PROPOSITION 3.4. *Suppose Hypothesis* (*H*) *holds. If* (0.4) *is satisfied, then we have a formal solution* $V(x, t, \lambda)$ *of the problem* (3.1) (3.2) *given by* (3.10) (3.13) (3.14) (3.15). $F(x, t, \lambda)$ *in* (3.3) *is given by*

$$F(x, t, \lambda)=F_1(x, t, \lambda)+\sum_{k=3}^{9}\lambda^{-k}G_k(x, t, \lambda)$$

(*Recall* (3.9) (3.10) (3.18)).

4. Proofs of Theorems 1 and 2.

Theorem 1 follows from Theorem 2, and Theorem 2 is proved as in the previous case (see [7], §3). Namely, for some $T_1>0$ and $\lambda_1>0$, we can find $v=v(x, t, \lambda)$ valid for $x\in\boldsymbol{R}^n$, $n=2$ or 3, $0\leqq t<T_1$, $\lambda\geqq\lambda_1$, such that

$$\mathcal{L}(V+v)(V+v)=0, \tag{4.1}$$

$$v(x, 0, \lambda)=0, \tag{4.2}$$

with

$$\|v(\cdot, x, \lambda)\|_s\leqq L\lambda^{s-3}, \qquad 0\leqq s\leqq 3, \tag{4.3}$$

$$\left\|\frac{\partial}{\partial t}v(\cdot, x, \lambda)\right\|_s\leqq L_1\lambda^{s-2}, \qquad 0\leqq s\leqq 2. \tag{4.4}$$

for $0\leqq t\leqq T_1$, $\lambda\geqq\lambda_1$. This is a consequence of estimates (3.4) (3.5) (3.6) (3.7) and a series of energy estimates combined with the iteration procedure:

$$\mathcal{L}(V+v^{k-1})(V+v^k)=0, \qquad v^k(x, 0, \lambda)=0, \qquad k\geqq 1$$

starting from $v^0(x, t, \lambda)=0$. T_1 and λ_1 are chosen so that v^k converge to v in the metric space corresponding to (4.3) (4.4). Then for $U=U(x, t, \lambda)$ of (0.12)

$$u-U=v+\lambda^{-3}u_3$$

and the estimates (0.16) (0.17) hold. We refer further details to [7] (cf. [4]).

5. Discussions.

In § 3, we have derived (3.21) (3.23) (3.24) from (3.20) by regrouping terms in (3.20) according to phases involved. This procedure leads to Hypothesis (H) to integrate b_{jki}, $j\neq k$. We have applied similar separation procedures to handle $G_2^0=0$. On the other hand, Hunter-Majda-Rosales [1], in order to ensure boundedness of b_{jki}, proposed an equation for a_j more complicated than (3.21) as involving non-local interaction terms. However, since boundedness in the s_i's of a_i, b_{jki} and c_{ijkl} is what we need in our discussions, an eventual removal of Hypothesis (H) or relaxation of (0.4) would be certainly welcoming.

Nevertheless, requirement that a_i and b_{jki} be compactly supported in the s-variables is almost identical to Hypothesis (H) together with (0.4).

PROPOSITION 5.1. *Suppose* (3.20) *holds for a_i and b_{jki} which are compactly supported with respect to s_i and to s_j, s_k, respectively* ($i, j, k=1, \cdots, m$) *Then* (3.21), (3.23) *and* (3.24) *hold.*

In fact, let $J\neq i$, and integrate (3.20) in s_J from $s_J=-L$ to $s_J=L$, $L>0$ large enough. Then terms not involving s_J are multiplied by $2L$ while terms containing s_J are integrated to become bounded or vanishing terms. Thus, (3.20) reduces to the case without $j=J$. In this way, we obtain (3.21), and returning to a prior step

$$\beta_{ji}a_j+(p_j(0, \eta)-p_i(0, \eta))\left\{\gamma_{jji}\frac{\partial}{\partial s_j}\left(\frac{a_j^2}{2}\right)+\gamma_{iji}\frac{\partial}{\partial s_j}a_ja_i+\gamma_{jii}\frac{\partial}{\partial s_j}a_ia_j\right\}$$
$$+\sum_{\nu=1}^{n}\alpha_{ji}^{\nu}a_{j,x_\nu}+(-p_j(0, \eta)+p_i(0, \eta))\left\{\frac{\partial}{\partial s_j}(b_{jji})+\frac{\partial}{\partial s_j}b_{jii}\right\}=0,$$

for $j\neq i$. Integrate this equality now in s_i from $s_i=-L$ to $s_i=L$, and we obtain (3.23). Handling a further prior step in the same way, we get (3.24).

In order to ensure compactness in s_j of the support of b_{jji}, we have

$$\sum_{\nu=1}^{n}\left(\alpha_{ji}^{\nu}\frac{\partial}{\partial x_\nu}+\beta_{ji}\right)\int_R g_j(s_j, x)ds_j=0, \qquad i\neq j, \tag{5.1}$$

for $i=1, \cdots, m$, in view of (3.21). (5.1) is somewhat weaker than (0.4) but much more complicated.

In order to ensure compactness in s_j, s_k of supp b_{jki}, $j\neq k$, we have

$$\int_R \Big\{\gamma_{kji}\frac{d}{d\theta}a_j(s_j(\theta),\ x,\ t)\cdot a_k(s_k(\theta),\ x,\ t)$$

$$+\gamma_{jki}a_j(s_j(\theta),\ x,\ t)\frac{d}{d\theta}a_k(s_k(\theta),\ x,\ k)\Big\}d\theta=0,$$

or

$$(5.2)\qquad (\gamma_{kji}-\gamma_{jki})\int_R a_j(s_j(\theta),\ x,\ t)\frac{d}{d\theta}a_k(s_k(\theta),\ x,\ t)d\theta=0.$$

Here $s_j(\theta)=(p_j(0,\ \eta)-p_i(0,\ \eta))\theta+s_j$, etc. (for each fixed i). So instead of Hypothesis (H) we might try to assure

$$\int_R a_j(s_j(\theta),\ x,\ t)\frac{d}{d\theta}a_k(s_k(\theta),\ x,\ t)d\theta=0.$$

But this is hard to realize unless all but one a_j's vanish.

Now we roughly recall what seems the spirit of reasonings of Hunter-Majda-Rosales [1]. Let $s_j(\theta)=s_j+(p_j(0,\ \eta)-p_i(0,\ \eta))\theta$, $j\neq i$, for each fixed i, and replace s_j's by $s_j(\theta)$'s in (3.20). Then the last sum in the left hand side of (3.20) turns out to be

$$-\frac{d}{d\theta}\sum_{j,k=1}^{m}b_{jki}(s_j(\theta),\ s_k(\theta),\ x,\ t).$$

So if $\sum_{j,k=1}^{m}b_{jki}(s_j,\ s_k,\ x,\ t)$ are bounded (or sublinear) with respect to s_j and s_k, we have

$$\lim_{L,L'\to+\infty}\frac{1}{L+L'}\sum_{j,k=1}^{m}\{b_{jki}(s_j(L),\ s_k(L),\ x,\ t)-b_{jki}(s_j(-L'),\ s_k(-L'),\ x,\ t)\}=0.$$

Thus, from (3.20), we obtain

$$(5.3)\quad a_{i,t}+\sum_{\nu=1}^{n}p_i^{(\nu)}(0,\ \eta)a_{i,x_\nu}+\beta_{ii}a_i+(X_ip_i)(0,\ \eta)\frac{\partial}{\partial s_i}\Big(\frac{1}{2}a_i^2\Big)$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{m}\Big\{\beta_{ji}\lim_{L,L'\to\infty}\frac{1}{L+L'}\int_{-L'}^{L}a_j(s_j(\theta),\ x,\ t)d\theta$$

$$+\sum_{\nu=1}^{n}\alpha_{ji}^{\nu}\lim_{L,L'\to\infty}\frac{1}{L+L'}\int_{-L'}^{L}a_{j,x_\nu}(s_j(\theta),\ x,\ t)d\theta\Big\}$$

$$+\sum_{\substack{j,k=1\\ j\neq i\neq k\neq j}}^{m}\gamma_{kji}\lim_{L,L'\to\infty}\frac{1}{L+L'}\int_{-L'}^{L}\frac{d}{d\theta}a_j(s_j(\theta),\ x,\ t)\cdot a_k(s_k(\theta),\ x,\ t)d\theta$$

$$=0.$$

If a_j are known to be compactly supported in s_j, then the non-local terms vanish, and (5.3) reduces to (3.21). Non-local terms also vanish when the initial data are reduced in a single characteristic direction, that is, $g_j(s,\ x)=0$ except for $j=1$, say.

REMARK. Note that if $\bar{a}_j(x,\ t)=\lim_{M,M'\to+\infty}\frac{1}{M+M'}\int_{-M'}^{M}a_j(s,\ x,\ t)ds$ exists, then

$$\lim_{L,L'\to+\infty}\frac{1}{L+L'}\int_{-L'}^{L}a_j(s_j(\theta),\,x,\,t)d\theta=\bar{a}_j(x,\,t)$$

is independent of s_j. Similarly, if $j\neq k\neq i\neq j$ and

$$\overline{W}(s_j,\,s_k)=\lim_{L,L'\to+\infty}\frac{1}{L+L'}\int_{-L'}^{L}W(s_j(\theta),\,s_k(\theta))d\theta$$

for a bounded function $W(s_j,\,s_k)$, then

$$\overline{W}(s_j,\,s_k)=\overline{W}(s_j(\rho),\,s_k(\rho))$$

for any real ρ. Observe that if $S_j(x,\,t)$ are given by (0.15), then

$$\begin{aligned}&(p_k(0,\,\eta)-p_i(0,\,\eta))S_j(x,\,t)-(p_j(0,\,\eta)-p_i(0,\,\eta))S_k(x,\,t)\\&=(p_k(0,\,\eta)-p_j(0,\,\eta))S_i(x,\,t).\end{aligned}$$

Thus, when we admit non-local terms, we have to supplement the relations in the s-space:

(5.4) $\quad (p_i(0,\,\eta)-p_j(0,\,\eta))s_k+(p_j(0,\,\eta)-p_k(0,\,\eta))s_i+(p_k(0,\,\eta)-p_i(0,\,\eta))s_j=0\,,$

$i,\,j,\,k=1,\,\cdots,\,m$. It follows $s_j(\rho)=s_k(\rho)$ if $\rho=-(s_j-s_k)/(p_j(0,\,\eta)-p_k(0,\,\eta))$.

Therefore, if

(5.5) $$\tilde{a}_{jk}((p_k-p_j)s_i,\,x,\,t)=\lim_{L,L'\to\infty}\frac{1}{L+L'}\int_{-L'}^{L}a_j(s_j(\theta),\,x,\,t)a_k(s_k(\theta),\,x,\,t)d\theta$$

makes sense, then

$$\tilde{a}_{jk}((p_k-p_j)s_i,\,x,\,t)=\tilde{a}_{kj}((p_k-p_j)s_i,\,x,\,t)$$

and since

$$s_j(\theta)=\frac{(p_j(0,\,\eta)-p_k(0,\,\eta))s_i}{p_i(0,\,\eta)-p_k(0,\,\eta)}+\frac{(p_j(0,\,\eta)-p_i(0,\,\eta))s_k(\theta)}{p_k(0,\,\eta)-p_i(0,\,\eta)},$$

(5.6) $$\begin{aligned}&\lim_{L,L'\to\infty}\frac{1}{L+L'}\int_{-L'}^{L}\frac{d}{d\theta}(a_j(s_j(\theta),\,x,\,t))\cdot a_k(s_k(\theta),\,x,\,t)d\theta\\&=\frac{(p_j-p_i)(p_k-p_i)}{p_k-p_j}\frac{\partial}{\partial s_i}\tilde{a}_{jk}((p_k-p_j)s_i,\,x,\,t),\end{aligned}$$

$i\neq j\neq k\neq i$ (at least weakly). That is, (5.3) is a system of conservation laws involving non-local terms. As a consequence, we see that $\bar{a}(x,\,t)=\sum_{j=1}^{m}\bar{a}_j(x,\,t)r_j(0,\,\eta)$ satisfies the equation

(5.7) $$\frac{\partial}{\partial t}\bar{a}+\sum_{\nu=1}^{n}\tilde{A}_\nu(0)\frac{\partial}{\partial x_\nu}\bar{a}+\tilde{B}(0)\bar{a}=0$$

with

(5.8) $$\bar{a}(x,\,0)=\bar{g}(x),$$

provided

$$\bar{g}(x) = \lim_{L, L' \to \infty} \frac{1}{L+L'} \int_{-L'}^{L} g(s, x) ds$$

exists.

Since, under (0.4), we only need (3.21), what is more important is to realize $c_{ijkl}(s_i, s_j, s_k, x, t)$ bounded or at least sublinear in s_i, s_j, s_k. In a similar manner to the above, we obtain from $G_2^0 = 0$ (see (3.17)) equations for b_{iii}, which are linear. However, since we still need precise growth estimates of a_i, b_{jki} and c_{ijkl}, we suspend our discussions for the time being.

Appendix.

A. Hypothesis (H) and systems of conservation laws.

Suppose, in particular, the system (0.1) is of conservation laws:

$$A_0(v)^{-1}A_\nu(v)w = dQ_\nu(v)[w], \qquad w \in \boldsymbol{R}^m, \tag{A.1}$$

hold for some smooth m-vector valued functions $Q_\nu(v)$, $v \in \boldsymbol{R}^m$, $\nu=1, \cdots, n$. Here d stands for the Fréchet-Gâteaux differentiation:

$$dQ_\nu(v)[w] = \frac{\partial}{\partial \varepsilon} Q_\nu(v+\varepsilon w)\Big|_{\varepsilon=0}.$$

Let

$$Q(v, \xi) = \sum_{\nu=1}^{n} \xi_\nu Q_\nu(v), \qquad \xi = (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n. \tag{A.2}$$

Then

$$dQ(v, \xi)[w] = M(v, \xi)w \tag{A.3}$$

$$d^2Q(v, \xi)[w^1, w^2] = dM(v, \xi)[w^1]w^2 = dM(v, \xi)[w^2]w^1 \tag{A.4}$$

for w, w^1, $w^2 \in \boldsymbol{R}^m$. Note the symmetry in (A.4).

PROPOSITION A.1. *Suppose* (A.1) *holds. Let eigenvectors $r_j(v, \xi)$ and $r_k(v, \xi)$, $j \neq k$, satisfy* (0.9) *(u, η replaced by v, ξ). If the corresponding eigenvalues $p_j(v, \xi)$ and $p_k(v, \xi)$ are distinct, then*

$$d^2Q[r_j, r_k] = dp_j(v, \xi)[r_k]r_j + dp_k(v, \xi)[r_j]r_k \tag{A.5}$$

and

$$dr_j(v, \xi)[r_k] = dr_k(v, \xi)[r_j] = c_{jk}r_j + c_{kj}r_k, \tag{A.6}$$

where

$$c_{jk} = c_{jk}(v, \xi) = r_j^* \cdot dr_j(v, \xi)[r_k] = \frac{dp_j(v, \xi)[r_k]}{p_k(v, \xi) - p_j(v, \xi)}. \tag{A.7}$$

PROOF. Since

$$M(v, \xi)r_j(v, \xi) = p_j(v, \xi)r_j(v, \xi), \tag{A.8}$$

we have, for $w \in \boldsymbol{R}^m$,

$$\begin{aligned} & dM(v, \xi)[w]r_j(v, \xi)+M(v, \xi)dr_j(v, \xi)[w] \\ & = dp_j(v, \xi)[w]r_j(v, \xi)+p_j(v, \xi)dr_j(v, \xi)[w]. \end{aligned} \tag{A.9}$$

Let $w=r_k$, $k \neq j$, and apply (A.4) and (0.9). Then

$$dp_j(v, \xi)[r_k]r_j+p_j(v, \xi)dr_j(v, \xi)[r_k] = dp_k(v, \xi)[r_j]r_k+p_k(v, \xi)dr_k(v, \xi)[r_j],$$

from which (A.6) (A.7) follow provided $p_j(v, \xi) \neq p_k(v, \xi)$. Combining (A.8) (A.9) and (A.3) (A.6) (A.7), we get (A.5).

REMARK. If $r_j(v, \xi)$ and $r_k(v, \xi)$, $j \neq k$, correspond to the same eigenvalue $p(v, \xi)=p_j(v, \xi)=p_k(v, \xi)$ (at $(v, \xi)=(0, \eta)$, say), then (0.9) implies $dp_j(0, \eta)[r_k(0, \eta)]=0$, $dp_k(0, \eta)[r_j(0, \eta)]=0$.

We can also compute higher order differentials such as $d^3Q[r_i, r_j, r_k]$, $d^2p_i[r_j, r_k]$ and $d^2r_i[r_j, r_k]$, $i \neq j \neq k \neq i$, $i, j, k=1, \cdots, m$.

PROPOSITION A.2. *Suppose all the eigenvalues are distinct, and* (0.9) *holds for corresponding eigenvectors. Then we have*

$$\begin{aligned} & d^3Q(v, \xi)[r_i, r_j, r_k] \\ & = d^2p_i(v, \xi)[r_j, r_k]r_i+d^2p_j(v, \xi)[r_k, r_i]r_j+d^2p_k(v, \xi)[r_i, r_j]r_k, \end{aligned} \tag{A.10}$$

$i \neq j \neq k \neq i$. Furthermore,

$$\begin{aligned} d^2p_i(v, \xi)[r_j, r_k] &= \frac{1}{p_j(v, \xi)-p_i(v, \xi)} dp_j(v, \xi)[r_k]dp_i(v, \xi)[r_j] \\ & + \frac{1}{p_k(v, \xi)-p_i(v, \xi)} dp_k(v, \xi)[r_j]dp_i(v, \xi)[r_k] \\ & + \left(\frac{1}{p_i(v, \xi)-p_j(v, \xi)} + \frac{1}{p_i(v, \xi)-p_k(v, \xi)}\right) dp_i(v, \xi)[r_j]dp_i(v, \xi)[r_k] \\ & = (p_j-p_k)c_{ik}c_{kj}+(p_k-p_j)c_{ij}c_{jk}+(2p_i-p_j-p_k)c_{ij}c_{ik}. \end{aligned} \tag{A.11}$$

PROOF. Differentiating (A.5), we obtain

$$\begin{aligned} & d^3Q(v, \xi)[w, r_j, r_k]+d^2Q(v, \xi)[dr_j[w], r_k]+d^2Q(v, \xi)[r_j, dr_k[w]] \\ & = d^2p_j(v \ \xi)[w, r_k]r_j+dp_j(v, \xi)[dr_k[w]]r_j+dp_j(v, \xi)[r_k]dr_j[w] \\ & \quad +d^2p_k(v, \xi)[w, r_j]r_k+dp_k(v, \xi)[dr_j[w]]r_k+dp_k(v, \xi)[r_j]dr_k[w], \end{aligned} \tag{A.12}$$

$w \in \boldsymbol{R}^m$. Taking $w=r_i$, $i \neq j \neq k \neq i$, and applying (A.5) (A.6), together with symmetry in i, j, k, we have (A.10) and (A.11).

REMARK. In a similar manner, we have

(A.13) $$d^2r_i(v,\xi)[r_j(v,\xi),\ r_k(v,\xi)] = (c_{ji}c_{ik}+c_{jk}c_{ki}-c_{ji}c_{jk})r_j(v,\xi)+(c_{ki}c_{ij}+c_{kj}c_{ji}-c_{ki}c_{kj})r_k(v,\xi),$$

$i\neq j\neq k\neq i$.

On the other hand, $d^3Q(v,\xi)[r_i, r_i, r_k]$, $i\neq k$, or $d^3Q(v,\xi)[r_i, r_i, r_i]$ do not admit particularly handy representations, for $dr_i(v,\xi)[r_i]$ does generally not reduce to a simple form such as (A.6).

B. Explicit forms of $G_k(x, t, \lambda)$.

In the construction of the formal solution to the problem (0.1) (0.3), we have grouped rather harmless terms in the form of $\sum_{k=4}^{9}\lambda^{-k}G_k(x, t, \lambda)$ (see (3.11)). G_k are explicitly computed as follows.

(B.1) $$G_4(x,t,\lambda)=\sum_{\nu=1}^{n}\left\{d\tilde{A}_\nu(0)[u_1]\frac{\partial}{\partial x_\nu}u_3+d\tilde{A}_\nu(0)[u_2]\frac{\partial}{\partial x_\nu}u_2+d\tilde{A}_\nu(0)[u_3]\frac{\partial}{\partial x_\nu}u_1\right\}$$
$$+\frac{1}{2}\sum_{\nu=1}^{n}\left\{d^2\tilde{A}_\nu(0)[u_1, u_1]\frac{\partial}{\partial x_\nu}u_2+2d^2\tilde{A}_\nu(0)[u_1, u_2]\frac{\partial}{\partial x_\nu}u_1\right\}$$
$$+d\tilde{B}(0)[u_1]u_3+d\tilde{B}(0)[u_2]u_2+d\tilde{B}(0)[u_3]u_1,$$

(B.2) $$G_5(x,t,\lambda)=\sum_{\nu=1}^{n}\left\{d\tilde{A}_\nu(0)[u_2]\frac{\partial}{\partial x_\nu}u_3+d\tilde{A}_\nu(0)[u_3]\frac{\partial}{\partial x_\nu}u_2\right\}$$
$$+\frac{1}{2}\sum_{\nu=1}^{n}\left\{d^2\tilde{A}_\nu(0)[u_1, u_1]\frac{\partial}{\partial x_\nu}u_3+d^2\tilde{A}_\nu(0)[u_2, u_2]\frac{\partial}{\partial x_\nu}u_1\right.$$
$$\left.+2d^2\tilde{A}_\nu(0)[u_1, u_2]\frac{\partial}{\partial x_\nu}u_2+2d^2\tilde{A}_\nu(0)[u_1, u_3]\frac{\partial}{\partial x_\nu}u_1\right\}$$
$$+d\tilde{B}(0)[u_2]u_3+d\tilde{B}(0)[u_3]u_2,$$

(B.3) $$G_6(x,t,\lambda)=\sum_{\nu=1}^{n}d\tilde{A}_\nu(0)[u_3]\frac{\partial}{\partial x_\nu}u_3+\frac{1}{2}\sum_{\nu=1}^{n}\left\{d^2\tilde{A}_\nu(0)[u_2, u_2]\frac{\partial}{\partial x_\nu}u_2\right.$$
$$\left.+2d^2\tilde{A}_\nu(0)[u_1, u_2]\frac{\partial}{\partial x_\nu}u_3+2d^2\tilde{A}_\nu(0)[u_2, u_3]\frac{\partial}{\partial x_\nu}u_1\right\}+d\tilde{B}(0)[u_3]u_3,$$

(B.4) $$G_7(x,t,\lambda)=\frac{1}{2}\sum_{\nu=1}^{n}\left\{d^2\tilde{A}_\nu(0)[u_2, u_2]\frac{\partial}{\partial x_\nu}u_3+d^2\tilde{A}_\nu(0)[u_3, u_3]\frac{\partial}{\partial x_\nu}u_1\right.$$
$$\left.+2d^2\tilde{A}_\nu(0)[u_1, u_3]\frac{\partial}{\partial x_\nu}u_3+2d^2\tilde{A}_\nu(0)[u_2, u_3]\frac{\partial}{\partial x_\nu}u_2\right\},$$

(B.5) $$G_8(x,t,\lambda)=\frac{1}{2}\sum_{\nu=1}^{n}\left\{d^2\tilde{A}_\nu(0)[u_3, u_3]\frac{\partial}{\partial x_\nu}u_2+2d^2\tilde{A}_\nu(0)[u_2, u_3]\frac{\partial}{\partial x_\nu}u_3\right\}$$

and

(B.6) $$G_9(x,t,\lambda)=\frac{1}{2}\sum_{\nu=1}^{n}d^2\tilde{A}_\nu(0)[u_3, u_3]\frac{\partial}{\partial x_\nu}u_3.$$

Therefore, if $u_1(x, t, \lambda)$, $u_2(x, t, \lambda)$, $u_3(x, t, \lambda)$ are smooth in $x \in \boldsymbol{R}^n$, $0 \leqq t \leqq T_0$, $\lambda \geqq \lambda_0$, compactly supported with respect to x, and furthermore

$$\text{(B.7)} \qquad \sup_D \lambda^{-k-|\alpha|} |\partial_t^k \partial_x^\alpha u_j(x, t, \lambda)| \leqq C_{k,\alpha} < \infty, \qquad j = 1, 2, 3,$$

and

$$\text{(B.8)} \qquad \sup_\Lambda \lambda^{-k-s} \|\partial_t^k u_j(\cdot, t, \lambda)\|_s \leqq C_s < \infty,$$

then $G(x, t, \lambda) = \sum_{i=4}^{9} \lambda^{-i} G_i(x, t, \lambda)$ satisfies the estimates

$$\text{(B.9)} \qquad \sup_D \lambda^{3-k-|\alpha|} |\partial_t^k \partial_x^\alpha G(x, t, \lambda)| \leqq C_{k,\alpha} < \infty,$$

and

$$\text{(B.10)} \qquad \sup_\Lambda \lambda^{3-k-s} \|\partial_t^k G(\cdot, t, \lambda)\|_s \leqq C_s < \infty.$$

Here $k = 0, 1, 2, \cdots$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\alpha_j = 0, 1, 2, \cdots$, and $s \geqq 0$.

C. Explicit forms of $f_i(s_i, x, t)$, $f_{ij}(s_i, s_j, x, t)$ and $f_{ijk}(s_i, s_j, s_k, x, t)$.

In computing $G_2^0 = 0$ in § 3, we have remainder terms f_i, f_{ij}, f_{ijk} (see (3.27) (3.28) (3.29)). Here are their details:

$$\text{(C.1)} \qquad f_i(s_i, x, t) = \sum_{\substack{k=1\\k \neq i}}^{m} \{b_{iik,t} + M(0, b_{iik,x}) + b_{iik}\tilde{B}(0)\} r_k(0, \eta)$$
$$+ \sum_{\substack{k=1\\k \neq i}}^{m} \Big\{ a_i \frac{\partial}{\partial s_i}(b_{iik}) dM(0, \eta)[r_i(0, \eta)] r_k(0, \eta)$$
$$+ b_{iik} \frac{\partial}{\partial s_i} a_i dM(0, \eta)[r_k(0, \eta)] r_i(0, \eta) \Big\}$$
$$+ a_i dM(0, a_{i,x})[r_i(0, \eta)] r_i(0, \eta) + a_i^2 d\tilde{B}(0)[r_i(0, \eta)] r_i(0, \eta)$$
$$+ \frac{1}{6} \frac{\partial}{\partial s_i}(a_i^3) d^2 M(0, \eta)[r_i(0, \eta), r_i(0, \eta)] r_i(0, \eta), \qquad i = 1, \cdots, m,$$

$$\text{(C.2)} \qquad f_{ij}(s_i, s_j, x, t) = 2 \sum_{k=1}^{m} \{b_{ijk,t} + M(0, b_{ijk,x}) + b_{ijk}\tilde{B}(0)\} r_k(0, \eta)$$
$$+ \sum_{k=1}^{m} \Big\{ a_i \frac{\partial}{\partial s_j}(b_{jjk}) dM(0, \eta)[r_i(0, \eta)] r_k(0, \eta)$$
$$+ a_j \frac{\partial}{\partial s_i}(b_{iik}) dM(0, \eta)[r_j(0, \eta)] r_k(0, \eta)$$
$$+ b_{jjk} \frac{\partial}{\partial s_i} a_i dM(0, \eta)[r_k(0, \eta)] r_i(0, \eta) + b_{iik} \frac{\partial}{\partial s_j} a_j dM(0, \eta)[r_k(0, \eta)] r_j(0, \eta) \Big\}$$
$$+ 2 \sum_{k=1}^{m} \Big\{ a_i \Big(\frac{\partial}{\partial s_i} b_{ijk} + \frac{\partial}{\partial s_j} b_{ijk} \Big) dM(0, \eta)[r_i(0, \eta)] r_k(0, \eta)$$

$$+a_j\Big(\frac{\partial}{\partial s_j}b_{ijk}+\frac{\partial}{\partial s_i}b_{ijk}\Big)dM(0,\ \eta)[r_j(0,\ \eta)]r_k(0,\ \eta)$$

$$+\frac{\partial a_i}{\partial s_i}b_{ijk}dM(0,\ \eta)[r_k(0,\ \eta)]r_i(0,\ \eta)+\frac{\partial a_j}{\partial s_j}b_{ijk}dM(0,\ \eta)[r_k(0,\ \eta)]r_j(0,\ \eta)\Big\}$$

$$+a_i dM(0,\ a_{j,x})[r_i(0,\ \eta)]r_j(0,\ \eta)+a_j dM(0,\ a_{i,x})[r_j(0,\ \eta)]r_i(0,\ \eta)$$

$$+a_i a_j\{d\tilde{B}(0)[r_i(0,\ \eta)]r_j(0,\ \eta)+d\tilde{B}(0)[r_j(0,\ \eta)]r_i(0,\ \eta)\}$$

$$+\frac{1}{2}\frac{\partial}{\partial s_j}(a_i^2 a_j)d^2M(0,\ \eta)[r_i(0,\ \eta),\ r_i(0,\ \eta)]r_j(0,\ \eta)$$

$$+\frac{1}{2}\frac{\partial}{\partial s_i}(a_i a_j^2)d^2M(0,\ \eta)[r_j(0,\ \eta),\ r_j(0,\ \eta)]r_i(0,\ \eta)$$

$$+\frac{1}{2}\frac{\partial}{\partial s_j}(a_i a_j^2)d^2M(0,\ \eta)[r_i(0,\ \eta),\ r_j(0,\ \eta)]r_j(0,\ \eta)$$

$$+\frac{1}{2}\frac{\partial}{\partial s_i}(a_i^2 a_j)d^2M(0,\ \eta)[r_i(0,\ \eta),\ r_j(0,\ \eta)]r_i(0,\ \eta),\quad i\neq j,\ i,j=1,\ \cdots,\ m,$$

$$\text{(C.3)}\qquad f_{ijk}(s_i,\ s_j,\ s_k,\ x,\ t)=a_i a_j\frac{\partial}{\partial s_k}a_k d^2M(0,\ \eta)[r_i(0,\ \eta),\ r_j(0,\ \eta)]r_k(0,\ \eta)$$

$$+a_j a_k\frac{\partial}{\partial s_i}a_i d^2M(0,\ \eta)[r_j(0,\ \eta),\ r_k(0,\ \eta)]r_i(0,\ \eta)$$

$$+a_k a_i\frac{\partial}{\partial s_j}a_j d^2M(0,\ \eta)[r_k(0,\ \eta),\ r_i(0,\ \eta)]r_j(0,\ \eta)$$

$$+2\sum_{l=1}^{m}\Big\{a_i\Big(\frac{\partial}{\partial s_j}b_{jkl}+\frac{\partial}{\partial s_k}b_{jkl}\Big)dM(0,\ \eta)[r_i(0,\ \eta)]r_l(0,\ \eta)$$

$$+a_k\Big(\frac{\partial}{\partial s_i}b_{ijl}+\frac{\partial}{\partial s_j}b_{ijl}\Big)dM(0,\ \eta)[r_k(0,\ \eta)]r_l(0,\ \eta)$$

$$+a_j\Big(\frac{\partial}{\partial s_k}b_{kil}+\frac{\partial}{\partial s_i}b_{kil}\Big)dM(0,\ \eta)[r_j(0,\ \eta)]r_l(0,\ \eta)\Big\}$$

$$+2\sum_{l=1}^{m}\Big\{\frac{\partial a_i}{\partial s_i}b_{jkl}dM(0,\eta)[r_l(0,\eta)]r_i(0,\eta)+\frac{\partial a_k}{\partial s_k}b_{ijl}dM(0,\eta)[r_l(0,\eta)]r_k(0,\eta)$$

$$+\frac{\partial a_j}{\partial s_j}b_{kil}dM(0,\ \eta)[r_l(0,\ \eta)]r_j(0,\ \eta)\Big\},\quad i\neq j\neq k\neq i,\quad i,\ j,\ k,\ l=1,\ \cdots,\ m.$$

Thus, $f_{ij}=f_{ji}$, $i\neq j$, and $f_{ijk}=f_{jki}=f_{kij}=f_{kji}=f_{ikj}=f_{jik}$ (because of the choice of b_{ijk}).

If (0.6) and (0.10) hold, then

$$\text{(C.4)}\qquad f_i=0,\qquad i\geqq 2,$$

$$\text{(C.5)}\qquad f_{ij}=0,\qquad i\neq j\neq 1\neq i.$$

(C.6) $f_{i1}(s_i, s_1, x, t) = a_1(s_1, x, t)\frac{\partial}{\partial s_i}(b_{iii}(s_i, s_i, x, t))dM(0, \eta)[r_1(0, \eta)]r_i(0, \eta)$

$$+b_{iii}(s_i, s_i, x, t)\frac{\partial}{\partial s_1}a_1(s_1, x, t)dM(0, \eta)[r_i(0, \eta)]r_1(0, \eta),$$

$i \geqq 2$. Furthermore,

$$f_{ijk}(s_i, s_j, s_k, x, t) = 0, \qquad i \neq j \neq k \neq i. \tag{C.7}$$

On the other hand, if the system (0.1) is of conservation laws and thus (A.1) is fulfilled, then (C.1) (C.2) (C.3) are somewhat simplified. Such simplifications are not quite useful for our purpose in the present paper.

References

[1] J. Hunter, A. Majda and R. Rosales, Resonantly interacting weakly nonlinear hyperbolic waves, II: Several space variables, Stud. Appl. Math., **75** (1986), 187-226.

[2] J.-L. Joly, J. Metivier and J. Rauch, Remarques sur l'optique géometrique non linéaire multidimensionnelle, Sém. E.D.P., 1990-1991, Ecole Polytechnique.

[3] A. Majda, Nonlinear geometric optics for hyperbolic systems of conservation laws, In Oscillation theory, computation, and methods of compactness, IMA, vol. **2**, Springer, New York, 1986, pp. 115-165.

[4] A. Majda, Compressible fluid flow and systems of conservation laws in several space variables, Appl. Math. Ser., **53**, Springer, Berlin-Heidelberg-New York, 1984.

[5] A. Majda, The interaction of nonlinear analysis and modern applied mathematics, Proc. ICM 1990, vol. **1**, The Mathematical Society of Japan, Springer, Tokyo-Berlin-Heidelberg-New York-London-Hong Kong-Barcelona-Budapest, 1991, pp. 175-191.

[6] S. Schochet, Fast singular limits of hyperbolic PDEs, (preprint 1992).

[7] A. Yoshikawa, Solutions containing a large parameter of a quasi-linear hyperbolic system of equations and their non-linear geometric optics approximation, Trans. Amer. Math. Soc., **340** (1993), 103-126.

[8] A. Yoshikawa, Quasilinear oscillations and geometric optics, Adv. Math. Sci. Appl., **4** (1994), 381-398.

[9] A. Yoshikawa, Quasilinear geometric optics approximation, Lecture Notes in Math., **1540**, Springer, 1993, pp. 403-413.

Atsushi YOSHIKAWA
Graduate School of Mathematics
Kyushu University 36
Hakozaki, Fukuoka 812-81
Japan
(e-mail: yskw-gsm@mbox.nc.Kyushu-u.ac.jp)