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# Generalized standard Auslander-Reiten components

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Let $A$ be an artin algebra, $\operatorname{mod} A$ the category of finitely generated right $A$-modules, $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ the infinite radical of $\operatorname{mod} A$, and $\Gamma_A$ the Auslander-Reiten quiver of $A$. It is well-known (see [**6**]) that $\Gamma_A$ describes the quotient category $\operatorname{mod} A/\operatorname{rad}^{\infty}(\operatorname{mod} A)$. We are interested in the behaviour of the connected components of $\Gamma_A$ in the category $\operatorname{mod} A$.

In the representation theory of finite dimensional algebras over an algebraically closed field $k$, an important role is played by the standard Auslander-Reiten components. Recall that following [**12**], [**36**], a connected component $\mathcal{C}$ of the Auslander-Reiten quiver $\Gamma_\Lambda$ of a finite dimensional $k$-algebra $\Lambda$ is called standard if the full subcategory of $\operatorname{mod} \Lambda$ formed by all modules from $\mathcal{C}$ is equivalent to the mesh-category $k(\mathcal{C})$ of $\mathcal{C}$. If $\Lambda$ is representation-finite (basic, connected), then $\Gamma_\Lambda$ is *standard* if and only if $\Lambda$ admits a simply connected Galois covering [**12**], [**13**]. Moreover, if $k$ is of characteristic 2, then there are representation-finite, basic, connected $k$-algebras $\Lambda$ with $\Gamma_\Lambda$ nonstandard [**32**]. Examples of infinite standard components are the preprojective components, preinjective components and connecting components over representation-infinite tilted algebras as well as all tubes over tame tilted and tubular algebras [**36**], [**24**]. In the study of simply connected $k$-algebras of polynomial growth (in the sense of [**39**]) appeared a natural generalization of the notion of tube, called a coil, and then a more general concept of a multicoil, being a glueing of a finite number of coils by directed parts (see [**3**], [**4**], [**41**]). It is shown in [**41**] that a strongly simply connected $k$-algebra $\Lambda$ is of polynomial growth if and only if every connected component of $\Gamma_\Lambda$ containing an oriented cycle is a standard multicoil.

The aim of this paper is to introduce a natural generalization of the notion of standard component, called generalized standard component, which is simpler and makes sense for any artin algebra. We shall prove some basic facts on generalized standard Auslander-Reiten components and on artin algebras whose Auslander-Reiten quiver has such components. In particular, we solve (Corollary 2.5) a Ringel's problem [**37**, Problem 3] on the shape of regular standard

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components.

DEFINITION. Let $A$ be an artin algebra and $\mathcal{C}$ be a connected component of $\Gamma_A$. We say that $\mathcal{C}$ is *generalized standard* if $\mathrm{rad}^{\infty}(X, Y)=0$ for all modules $X$ and $Y$ from $\mathcal{C}$.

The above notion is motivated by some recent investigations of modules over arbitrary artin algebras [**29**], [**30**], [**40**], [**42**], [**43**], [**44**], [**45**], and by the fact that all preprojective components, all preinjective components, and the connecting components of all tilted artin algebras are generalized standard (see (1.2) and (1.3)).

The paper is organized as follows. In Section 1 we recall those facts about artin algebras and their module categories that will be needed in the paper. In Section 2 we show that a generalized standard component admits at most finitely many nonperiodic $DTr$-orbits, and then at most finitely many modules of any given length, which gives a partial solution of the Ringel's Problem 1 in [**37**]. We also describe the shapes of regular generalized standard components. Section 3 is devoted to a complete description of semi-regular generalized standard components without oriented cycles. In particular, we show that almost all generalized standard components of the Auslander-Reiten quiver of an artin algebra are stable tubes. We show also that the preprojective components, preinjective components and tubes are unique generalized standard semi-regular components of the Auslander-Reiten quivers of tame algebras. Section 4 contains an example showing that there are algebras whose Auslander-Reiten quiver admits more than one sincere regular generalized standard component without oriented cycles. In Section 5 we investigate generalized standard stable tubes. We obtain some characterizations of such components and some bounds for their ranks.

## 1. Preliminaries.

**1.1. Notation.** Throughout this paper, $A$ will denote a fixed artin algebra over a commutative artin ring $R$ and $n$ be the rank of the Grothendieck group $K_0(A)$ of $A$. By a module is usually meant a finitely generated right module. We shall denote by $\mathrm{mod}\,A$ the category of all (finitely generated right) $A$-modules, by $\mathrm{rad}(\mathrm{mod}\,A)$ the radical of the category $\mathrm{mod}\,A$, and by $\mathrm{rad}^{\infty}(\mathrm{mod}\,A)$ the intersection of all powers $\mathrm{rad}^{i}(\mathrm{mod}\,A)$, $i\geqq 0$, of $\mathrm{rad}(\mathrm{mod}\,A)$. From the existence of the Auslander-Reiten sequences in $\mathrm{mod}\,A$ we know that $\mathrm{rad}(\mathrm{mod}\,A)$ is generated by the irreducible maps as a left and as a right ideal [**7**], [**8**]. It is known that $A$ is representation-finite if and only if $\mathrm{rad}^{\infty}(\mathrm{mod}\,A)=0$ (see [**5**], [**26**, p. 332]). We denote be $D$ the standard duality $\mathrm{Hom}_R(-, I)$, where $I$ is the injective envelope of $R/\mathrm{rad}\,R$ in $\mathrm{mod}\,R$. A *path* in $\mathrm{mod}\,A$ is a sequence of non-zero non-isomorphisms

$$(*) \qquad M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \longrightarrow M_{t-1} \xrightarrow{f_t} M_t$$

where the $M_i$ are indecomposable. Such a path (*) is called *infinite* if $f_i$ belongs to $\mathrm{rad}^\infty(\mathrm{mod}\,A)$ for some $1 \leqq i \leqq t$. Further, it is a *cycle* if $M_0 \cong M_t$. Moreover, a path (resp. cycle) (*) with $t \leqq 2$ is said to be *short*. An indecomposable $A$-module $X$ is called *directing* (see [**36**]) if it lies on no cycle in $\mathrm{mod}\,A$. Finally, an $A$-module $X$ with $\mathrm{End}_A(X)$ a division ring is said to be a *brick*.

**1.2. Auslander-Reiten components.** We denote by $\Gamma_A$ the *Auslander-Reiten quiver* of $A$ and by $\tau_A$ and $\tau_A^-$ the Auslander-Reiten operators $DTr$ and $TrD$, respectively. We will not distinguish between an indecomposable $A$-module, its isomorphism class and the vertex of $\Gamma_A$ corresponding to it. By a *component* of $\Gamma_A$ we mean a connected component of $\Gamma_A$. For a component $\mathcal{C}$ of $\Gamma_A$ we denote by $\mathrm{ann}\,\mathcal{C}$ the annihilator of $\mathcal{C}$ in $A$, that is, the intersection of the annihilators $\mathrm{ann}\,X$ of all modules $X$ in $\mathcal{C}$. If $\mathrm{ann}\,\mathcal{C}=0$, then $\mathcal{C}$ is called *faithful*. Moreover, $\mathcal{C}$ is called *sincere* if any simple $A$-module occurs as a simple composition factor of some module in $\mathcal{C}$. We say that two components $\mathcal{C}$ and $\mathcal{D}$ of $\Gamma_A$ are *orthogonal* if $\mathrm{Hom}_A(X, Y)=0=\mathrm{Hom}_A(Y, X)$ for all modules $X$ from $\mathcal{C}$ and $Y$ from $\mathcal{D}$.

A component $\mathcal{C}$ of $\Gamma_A$ is said to be *regular* if it contains neither a projective nor an injective module. Moreover, a component $\mathcal{C}$ of $\Gamma_A$ is said to be *semi-regular* if it does not contain both a projective and an injective module. A regular component of the form $\boldsymbol{Z}A_\infty/(\tau^r)$, where $\tau$ is the translation of $\boldsymbol{Z}A_\infty$ and $r$ some positive integer, is called a *stable tube* of rank $r$. The $\tau_A$-orbit of a stable tube $\mathcal{T}$ of $\Gamma_A$ formed by the modules with only one direct predecessor is called the *mouth* of $\mathcal{T}$. A component $\mathcal{C}$ of $\Gamma_A$ is said to be *preprojective* if $\mathcal{C}$ contains no oriented cycle and each module in $\mathcal{C}$ belongs to the $\tau_A$-orbit of a projective module, *preinjective* if $\mathcal{C}$ contains no oriented cycle and each module in $\mathcal{C}$ belongs to the $\tau_A$-orbit of an injective module. It follows from [**36**, (2.4)] that if $\mathcal{P}$ is a preprojective component of $\Gamma_A$ then each module in $\mathcal{P}$ has only finitely many predecessors and any path in $\mathrm{mod}\,A$ with the target module from $\mathcal{P}$ consists entirely of modules from $\mathcal{P}$, and consequently $\mathcal{P}$ is a generalized standard component of $\Gamma_A$. Dually, any preinjective component of $\Gamma_A$ is generalized standard.

Let $X$ be an indecomposable $A$-module. Then $X$ is said to be *left stable* if $\tau_A^m X \neq 0$ for all positive integers $m$, *right stable* if $\tau_A^m X \neq 0$ for all negative integers $m$, and *stable* if $\tau_A^m X \neq 0$ for all integers $m$. Moreover, $X$ is called *periodic* if $\tau_A^m X \cong X$ for some $m \geqq 1$. A $\tau_A$-orbit of $\Gamma_A$ consisting of periodic modules is called *periodic*. A path $X_0 \to X_1 \to \cdots \to X_s$ in $\Gamma_A$ is called *sectional* if $X_i \not\cong \tau_A X_{i+2}$ for all $0 \leqq i \leqq s-2$. Following [**9**] and [**29**], an $A$-module $M$ is said to be the *middle of a short chain* if there exists an indecomposable $A$-module $X$ with $\mathrm{Hom}_A(X, M) \neq 0$ and $\mathrm{Hom}_A(M, \tau_A X) \neq 0$.

For basic results of Auslander-Reiten theory we refer to [7], [8] and [**36**].

**1.3. Tilting theory.** Let $H$ be a hereditary artin algebra of type $\Delta$, $T$ a tilting $H$-module and $B = \mathrm{End}_H(T)$ the associated tilted algebra. Then $T$ determines a torsion theory $(\mathcal{F}(T), \mathcal{G}(T))$ in mod $H$, where $\mathcal{F}(T) = \{X_H \mid \mathrm{Hom}_H(T, X) = 0\}$ and $\mathcal{G}(T) = \{Y_H \mid \mathrm{Ext}_H^1(T, Y) = 0\}$, and a splitting torsion theory $(\mathcal{Y}(T), \mathcal{X}(T))$ in mod $B$, where $\mathcal{Y}(T) = \{N_B \mid \mathrm{Tor}_1^B(N, T) = 0\}$ and $\mathcal{X}(T) = \{M_B \mid M \otimes_B T = 0\}$. Then by the theorem of Brenner and Butler the functor $F = \mathrm{Hom}_H(T, -)$ induces an equivalence between $\mathcal{G}(T)$ and $\mathcal{Y}(T)$, and $F' = \mathrm{Ext}_H^1(T, -)$ an equivalence between $\mathcal{F}(T)$ and $\mathcal{X}(T)$. Then $DH$ belongs to $\mathcal{G}(T)$, and all indecomposable direct summands of $F(DH)$ belong to one connected component of $\Gamma_B$, called the *connecting component* of $\Gamma_B$ corresponding to $T$. This connecting component consists of two parts: the torsion-free part formed by all its modules belonging to $\mathcal{Y}(T)$, which is closed under predecessors, and the torsion part formed by all its modules belonging to $\mathcal{X}(T)$, which is closed under successors (see [**22**], [**24**], [**36**]). Since $\mathrm{Hom}_B(X, Y) = 0$ for all modules $X \in \mathcal{X}(T)$ and $Y \in \mathcal{Y}(T)$, this connecting component is a generalized standard component of $\Gamma_B$. Moreover, the connecting component $\mathcal{C}$ of $\Gamma_B$ is regular (resp. does not contain projective modules) if and only if $T$ is regular (resp. $T$ has no preinjective direct summands). In case $T$ is regular, $\mathcal{C}$ is of the form $Z\Delta^{op}$, where $\Delta^{op}$ is the opposite quiver of $\Delta$, and consists entirely of directing modules (see [**36**] and [**37**]). It was proved in [**38**] that $H$ admits a regular tilting module if and only if $\Delta$ has at least three vertices and is neither of Dynkin nor of Euclidean type.

We shall use in the paper the following lemma proved in [**30**, (1.6)].

LEMMA. *Let $M$ be a faithful $A$-module with $pd_A M \leqq 1$, $\mathrm{id}_A M \leqq 1$, $\mathrm{Ext}_A^1(M, M) = 0$ and with the property that if $\mathrm{Hom}_A(M, X) \neq 0$ for some indecomposable $A$-module $X$ which is not direct summand of $M$ then $\mathrm{Hom}_A(\tau_A^- M, X) \neq 0$. Then $M$ is a tilting and cotilting $A$-module.*

For a tilting-theoretical background we refer to [1], [**11**], [**22**] and [**36**].

**1.4. Tame and wild algebras.** Following [**16**] $A$ is said to be *strictly wild* if there are $A$-modules $X$ and $Y$ whose endomorphism rings $\mathrm{End}_A(X)$ and $\mathrm{End}_A(Y)$ are division rings, with $\mathrm{Hom}_A(X, Y) = 0 = \mathrm{Hom}_A(Y, X)$, and the product

$$\dim_{\mathrm{End}_A(Y)} \mathrm{Ext}_A^1(X, Y) \cdot \dim \mathrm{Ext}_A^1(X, Y)_{\mathrm{End}_A(X)}$$

is at least 5. If $R$ is a field $k$, then by [**33**], $A$ is strictly wild if and only if there exists a finite field extension $K$ of $k$ and a $K\langle x, y\rangle$-$A$-bimodule $M$ which is finitely generated projective over $K\langle x, y\rangle$ and such that the tensor product functor $-\otimes_{K\langle x,y\rangle}M$: Mod $K\langle x, y\rangle\to$Mod $A$ is fully faithful. Here, $K\langle x, y\rangle$ denotes the free associative $K$-algebra in two generators, and Mod $K\langle x, y\rangle$ and Mod $A$ the categories of all right $K\langle x, y\rangle$-modules and all right $A$-modules, respectively. Moreover, following [**15**], we say that $A$ is *wild* if there exists a $k\langle x, y\rangle$-$A$-bimodule $M$, finitely generated and projective as a left $k\langle x, y\rangle$-module, such that the functor $-\otimes_{k\langle x,y\rangle}M$: Mod $k\langle x, y\rangle\to$Mod $A$ preserves indecomposability and isomorphism classes of modules. It is known that a wild hereditary $k$-algebra is strictly wild (see [**33**], [**16**]) but in general the converse is not true. Assume now that $R$ is an algebraically closed field $k$. Then following [**18**], $A$ is said to be *tame* if for all $d\in N$ there are a finite number of $k[x]$-$A$-bimodules $M_1, \cdots, M_r$ which are free of rank $d$ as left $k[x]$-modules, and such that every indecomposable $A$-module of dimension $d$ is isomorphic to $k[x]/(x-\lambda)\otimes_{k[x]}M_i$ for some $1\leqq i\leqq r$ and $\lambda\in k$. By the well-known Drozd theorem [**18**] (see also [**14**]) a finite dimensional $k$-algebra $A$ is either tame or wild, and not both. Moreover, $A$ is tame if and only if $A$ is generically tame, that is, for each $d\in N$, there are only finitely many isomorphism classes of generic right $A$-modules of endolength $d$ (see [**15**]). Recall that a right $A$-module $Z$ is called *generic* if $Z$ is indecomposable, of infinite length over $A$, but finite endolength (which is the length of $Z$ as an $\mathrm{End}_A(Z)$-module).

## 2. Properties of generalized standard components.

**2.1.** We shall use the following lemma proved in [**40**]. For a convenience of the reader we present its short proof also here.

LEMMA. *Let $M_1, \cdots, M_r$ be pairwise nonisomorphic indecomposable $A$-modules such that* $\mathrm{Hom}_A(M_i, \tau_A M_j)=0$ *for all $1\leqq i, j\leqq r$. Let $M=M_1\oplus\cdots\oplus M_r$, $I$ be the annihilator of $M$ in $A$, and $B=A/I$. Then $M$ is a partial tilting $B$-module. In particular, we have $r\leqq n$.*

PROOF. From our assumption we have $\mathrm{Hom}_A(M, \tau_A M)=0$, and then $\mathrm{Hom}_B(M, \tau_B M)=0$ because $\tau_B M$ is a submodule of $\tau_A M$. Moreover, $M$ is a faithful $B$-module and $\mathrm{Ext}_B^1(M, M)\cong D\,\overline{\mathrm{Hom}}_B(M, \tau_B M)=0$. This implies that $pd_B M\leqq 1$ (see [**30**, (1.5)]). Indeed, since $M$ is faithful in mod $B$, there is an epimorphism $M^s\to DB$ for some $s\geqq 1$. Then $\mathrm{Hom}_B(DB, \tau_B M)=0$, because $\mathrm{Hom}_B(M, \tau_B M)=0$, and hence $pd_B M\leqq 1$ (see [**36**, (2.4)]). Therefore, $M$ is a partial tilting $B$-module and, according to a result of Bongartz [**11**], $M$ may be extended to a tilting $B$-module. Hence $r$ is less than or equal to the rank of

$K_0(B)$ which is again less than or equal to $n$.

**2.2.** Recall that a brick $X$ in $\mathrm{mod}\,A$ with $\mathrm{Ext}_A^1(X, X)=0$ is called a *Schur module* (see [**47**]). A Schur module $X$ with $pd_A X \leqq 1$ is called a *stone* (see [**25**]). We propose the following notion. An indecomposable $A$-module $X$ with $\mathrm{Hom}_A(X, \tau_A X)=0$ is said to be a *rock*. Clearly, each stone is a rock. Moreover, if $X$ is both a rock and brick, then it is a Schur module, and it is a stone in $\mathrm{mod}\,B$, where $B=A/I$ and $I=\mathrm{ann}\,X$. One can prove (see [**29**], [**42**]) that, if $X$ and $Y$ are two nonisomorphic rocks in $\mathrm{mod}\,A$ having the same simple composition factors, then there is a short cycle $X \to Y \to X$ in $\mathrm{mod}\,A$. We have also the following consequence of the above lemma.

COROLLARY. *The maximal number of pairwise orthogonal components of $\Gamma_A$ containing rocks is less than or equal to $n$.*

The above remarks suggest that it might be interesting to study properties of rocks (see also (5.15)).

**2.3.** The following theorem gives some informations on the $\tau_A$-orbits of generalized standard components of $\Gamma_A$.

THEOREM. *Let $\mathcal{C}$ be a generalized standard component of $\Gamma_A$. Then $\mathcal{C}$ admits at most finitely many nonperiodic $\tau_A$-orbits.*

PROOF. Suppose that the number of nonperiodic $\tau_A$-orbits in $\mathcal{C}$ is infinite. Consider the right stable part $\mathcal{C}_r$ of $\mathcal{C}$, obtained from $\mathcal{C}$ by removing the $\tau_A$-orbits of injective modules and the arrows attached to them. Since $\mathcal{C}$ is locally finite, there exists a connected component $\mathcal{D}$ of $\mathcal{C}_r$ which admits infinitely many nonperiodic $\tau_A$-orbits. Then clearly $\mathcal{D}$ does not contain periodic modules. Moreover, since the number of $\tau_A$-orbits in $\mathcal{D}$ is infinite, we infer, by the dual of [**28**, (2.3)], that $\mathcal{D}$ has no oriented cycle. Then, by [**28**, (3.7)], there exists an infinite, locally finite valued quiver $\Delta$ containing no oriented cycle such that $\mathcal{D}$ is isomorphic to a full translation subquiver of $\boldsymbol{Z}\Delta$ which is closed under successors. Fix a copy of $\Delta$ in $\mathcal{D}$ such that there is no path in $\mathcal{C}$ with the source in $\Delta$ and the target in an injective module. Then, since $\mathcal{C}$ is generalized standard, we get that $\mathrm{Hom}_A(X, \tau_A Y)=0$ for all $X$ and $Y$ from $\Delta$. Hence, by Lemma 2.1, $\Delta$ is finite, a contradiction. This finishes the proof.

A general theorem about the shapes of regular components of $\Gamma_A$ is given by the Happel-Preiser-Ringel theorem [**21**] and Zhang's theorem [**48**]. The Happel-Preiser-Ringel theorem says that all regular components of $\Gamma_A$ containing periodic modules are stable tubes. The Zhang's theorem says that if $\mathcal{C}$ is a regular component of $\Gamma_A$ without periodic modules then $\mathcal{C}$ is isomorphic to

$\boldsymbol{Z}\Delta$ for some valued quiver $\Delta$ without oriented cycles. Combining this with the above theorem we get the following description of the shapes of regular generalized standard components.

COROLLARY 2.4. *Let $C$ be a regular generalized standard component of $\Gamma_A$. Then $C$ is either a stable tube or of the form $\boldsymbol{Z}\Delta$ for some finite valued quiver $\Delta$ without oriented cycles.*

Moreover, we get the following corollary which solves Problem 3 in [37].

COROLLARY 2.5. *Let $R$ be an algebraically closed field $k$ and $C$ be a standard regular component of $\Gamma_A$. Then $C$ is either a stable tube or of the form $\boldsymbol{Z}\Delta$ for some finite quiver $\Delta$ without oriented cycles.*

PROOF. Assume that $C$ is not a stable tube. Then, by [48], $C$ is isomorphic to $\boldsymbol{Z}\Delta$ for some quiver $\Delta$ without oriented cycles. Since $C$ is standard, we get $\mathrm{Hom}_A(X, \tau_A Y)=0$ for all $X$ and $Y$ from $\Delta$. Hence $\Delta$ is finite, by Lemma 2.1.

PROPOSITION 2.6. *Let $C$ be a component of $\Gamma_A$ which admits at most finitely many nonperiodic $\tau_A$-orbits. Then for each given positive integer $d$, there are at most finitely many modules of length $d$ in $C$.*

PROOF. Suppose that for some positive integer $d$ there are infinitely many modules of length $d$ in $C$. We claim that then there is a $\tau_A$-orbit $\mathcal{O}$ in $C$ containing infinitely many modules of length $d$. Indeed, if this is not the case, there is a connected component $\mathcal{D}$ of the stable part $C_s$ of $C$ having infinitely many $\tau_A$-orbits containing modules of length $d$. From our assumption, $\mathcal{D}$ contains periodic modules and hence is a stable tube. Then there is a sectional path in $\mathcal{D}$

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \cdots \to X_i \xrightarrow{f_i} X_{i+1} - \cdots$$

containing infinitely many modules of length $d$. Since all compositions $f_i \cdots f_1$, $i \geqq 1$, are nonzero, there is a path in mod $A$

$$X_{i_1} \xrightarrow{g_1} X_{i_2} \xrightarrow{g_2} X_{i_3} \to \cdots \to X_{i_r} \xrightarrow{g_r} X_{i_{r+1}} - \cdots$$

such that, for each $r \geqq 1$, the composition $g_r \cdots g_1$ is nonzero, and all modules $X_{i_r}$ are indecomposable of length $d$. This contradicts to the lemma of Harada and Sai [23] (see also [35]). Therefore, there is a nonperiodic $\tau_A$-orbit $\mathcal{O}$ in $C$ containing infinitely many modules of length $d$. Without loss of generality, we may assume that $\mathcal{O}$ consists of left stable modules. Then, by [27, (3.9)], there is an infinite sectional path $\cdots \to Z_{j+1} \to Z_j \to \cdots \to Z_2 \to Z_1$ in $C$ formed by nonperiodic modules and which meets each $\tau_A$-orbit in $C$ at most once. But this is impossible because $\mathcal{C}$ admits at most finitely many nonperiodic $\tau_A$-orbits. The proof is completed.

We obtain the following consequence of Theorem 2.3 and Proposition 2.6.

COROLLARY 2.7. *Let $\mathcal{C}$ be a generalized standard component of $\Gamma_A$. Then for each given positive integer $d$, there are at most finitely many modules of length $d$ in $\mathcal{C}$.*

## 3. Generalized standard components without oriented cycles.

**3.1.** Recently, Liu described in [28] the shapes of semi-regular components of $\Gamma_A$. He proved that, if $\mathcal{C}$ is a semi-regular component of $\Gamma_A$ containing an oriented cycle then $\mathcal{C}$ is a tube in the sense of [36]. Moreover, if $\mathcal{C}$ is a semi-regular component of $\Gamma_A$ without oriented cycles, then there is a valued quiver $\Delta$ without oriented cycles such that $\mathcal{C}$ is isomorphic to a full translation subquiver of $\boldsymbol{Z}\Delta$ which is closed under predecessors or closed under successors. We shall now describe the structure of generalized standard semi-regular components without oriented cycles.

THEOREM. *Let $\mathcal{C}$ be a generalized standard component of $\Gamma_A$ without projective modules and oriented cycles. Let $I=\operatorname{ann}\mathcal{C}$ and $B=A/I$. Then*

(i) *$B$ is a tilted algebra of the form $B=\operatorname{End}_H(T)$ for some hereditary artin algebra $H$ and a tilting $H$-module $T$ without preinjective direct summands.*

(ii) *$\mathcal{C}$ is the connecting component of $\Gamma_B$.*

PROOF. From [28] and Theorem 2.3, there exists a finite valued quiver $\Delta$ without oriented cycles such that $\mathcal{C}$ is isomorphic to a full translation subquiver of $\boldsymbol{Z}\Delta$ which is closed under predecessors. Since $A$ is an artin algebra there is some $A$-module $U$ which is a direct sum of modules in $\mathcal{C}$ such that $I=\operatorname{ann}U$. Let $M$ be the direct sum of all modules corresponding to the vertices of some fixed $\Delta$ in $\mathcal{C}$ such that every indecomposable direct summand of $U$ is a successor in $\mathcal{C}$ of some indecomposable direct summand of $M$. Then $I=\operatorname{ann}M$, $M$ is a faithful $B$-module, $\mathcal{C}$ consists entirely of $B$-modules and is a generalized standard component of $\Gamma_B$. This implies that $\operatorname{Hom}_B(M, \tau_B M)=0$ and $\operatorname{Hom}_B(\tau_B^- M, M)=0$. In particular, $\operatorname{Ext}_B^1(M, M)\cong D\,\overline{\operatorname{Hom}}_B(M, \tau_B M)=0$. Since $M$ is a faithful $B$-module, there are an epimorphism $M^r\to DB$ and a monomorphism $B\to M^s$ for some $r$ and $s$. Then $\operatorname{Hom}_B(DB, \tau_B M)=0$, $\operatorname{Hom}_B(\tau_B^- M, B)=0$, and therefore $pd_B M\leqq 1$ and $\operatorname{id}_B M\leqq 1$ (see [**36**, (2.4)]). Clearly, if $\operatorname{Hom}_B(M, X)\neq 0$ for some indecomposable $B$-module $X$ which is not direct summand of $M$, then $\operatorname{Hom}_B(\tau_B^- M, X)\neq 0$. Then, by Lemma 1.3, $M$ is a tilting $B$-module. Moreover, since $\mathcal{C}$ is generalized standard, $H=\operatorname{End}_B(M)$ is a hereditary artin algebra of type $\Delta^{op}$. Hence $B=\mathrm{End}_H(T)$ for some tilting $H$-module $T$, and $\mathcal{C}$ is the connecting component of $\Gamma_B$ corresponding to $T$. Moreover, since $\mathcal{C}$ has no projective modules, $T$ has no preinjective direct summands.

Dually we obtain the following

THEOREM 3.2. *Let $\mathcal{C}$ be a generalized standard component of $\Gamma_A$ without injective modules and oriented cycles. Let $I=\mathrm{ann}\,\mathcal{C}$ and $B=A/I$. Then*

(i) *$B$ is a tilted algebra of the form $B=\mathrm{End}_H(T)$ for some hereditary artin algebra $H$ and a tilting $H$-module $T$ without preprojective direct summands.*

(ii) *$\mathcal{C}$ is the connecting component of $\Gamma_B$.*

COROLLARY 3.3. *Let $\mathcal{C}$ be a regular generalized standard component of $\Gamma_A$ without periodic modules, $I=\mathrm{ann}\,\mathcal{C}$ and $B=A/I$. Then $B$ is a tilted algebra of the form $\mathrm{End}_H(T)$, for some (wild) hereditary artin algebra $H$ and a regular tilting $H$-module $T$, and $\mathcal{C}$ is the connecting component of $\Gamma_B$.*

**3.4.** It is known that if $A$ is a tilted algebra and $\mathcal{C}$ a regular connecting component of $\Gamma_A$, then $\mathcal{C}$ is the unique regular generalized standard component of $\Gamma_A$ and all but a finite number of modules in $\mathcal{C}$ are faithful $A$-modules (see [**30**, (1.8)]). Moreover, it was recently proved in [**40**] that, if $\mathcal{C}$ is a regular component of $\Gamma_A$ containing a directing module, then $\mathcal{C}$ is generalized standard, has only finitely many $\tau_A$-orbits and consists entirely of directing modules. Further, by [**36**, (2.4)], all sincere directing modules are faithful. Then we have the following consequence of the above corollary.

COROLLARY. *Let $\mathcal{C}$ be a regular component of $\Gamma_A$. Then the following conditions are equivalent.*

(i) *$A$ is a tilted algebra of the form $\mathrm{End}_H(T)$ for some wild hereditary artin algebra $H$ and a regular tilting $H$-module $T$, and $\mathcal{C}$ is the connecting component of $\Gamma_A$.*

(ii) *$\mathcal{C}$ is sincere and contains a directing module.*

(iii) *$\mathcal{C}$ is faithful, generalized standard, and without oriented cycles.*

(iv) *$\mathcal{C}$ is generalized standard, without oriented cycles, and all but a finite number of modules in $\mathcal{C}$ are faithful.*

We may deduce from [**30**, (1.7), (1.9)] and [**29**, (1.6)] the following fact. If $R$ is an algebraically closed field and $\mathcal{C}$ a sincere regular component of $\Gamma_A$ having only finitely many $\tau_A$-orbits and consisting entirely of modules which do not lie on short cycles, then $\mathcal{C}$ is generalized standard without oriented cycles and containing faithful indecomposable modules. Then it is a unique regular generalized standard component of $\Gamma_A$. We shall show in the next section that there are algebras $A$ such that $\Gamma_A$ admits more than one regular generalized standard component without oriented cycles and containing sincere indecomposable $\Lambda$-modules.

Now we shall show that, if $A$ is an arbitrary artin algebra, then $\Gamma_A$ admits at most finitely many generalized standard components without oriented cycles. For this the following result is useful.

LEMMA 3.5. *Let $\mathcal{C}$ be a sincere, regular, generalized standard component of $\Gamma_A$ without oriented cycles. Denote by $t_{\mathcal{C}}$ the ideal of $A$ generated by all images of all maps from modules in $\mathcal{C}$ to $A$. Then* $\operatorname{ann} \mathcal{C}=t_{\mathcal{C}}$.

PROOF. We know from Theorem 2.3 and [48] that $\mathcal{C}=Z\Delta$ for some valued finite quiver $\Delta$ without oriented cycles. As in the proof of Theorem 3.1, let $M$ be the direct sum of all modules corresponding to the vertices of a fixed $\Delta$ in $\mathcal{C}$ and such that $\operatorname{ann} \mathcal{C}=\operatorname{ann} M$. Put $B=A/\operatorname{ann} \mathcal{C}$. Then $M$ is a faithful tilting $B$-module. Hence there is a monomorphism $B \to M^r$ for some $r$. Moreover, $\mathcal{C}$ is a connecting component of $\Gamma_B$, and hence the submodules of $B$ are not successors of $\mathcal{C}$ in $\operatorname{mod} B$. This implies that $t_{\mathcal{C}}$ is contained in $\operatorname{ann} \mathcal{C}$, because $\mathcal{C}$ is generalized standard. Let $D=A/t_{\mathcal{C}}$. Then $\mathcal{C}$ is a generalized standard component of $\Gamma_D$ and $M$ is a sincere $D$-module. By definition of $t_{\mathcal{C}}$ and the fact that $\operatorname{Ext}_A^1(M, t_{\mathcal{C}})=0$, we get $\operatorname{Hom}_D(\tau_D^- M, D)=0$, and consequently $\operatorname{id}_D M \leqq 1$. Then $M$ is a cotilting $D$-module, because $M$ has the correct number of indecomposable direct summands and $\operatorname{Ext}_D^1(M, M) \cong D\,\overline{\operatorname{Hom}}_D(M, \tau_D M)=0$. Since $H=\operatorname{End}_D(M)$ is hereditary, $D$ is a cotilted algebra of the form $\operatorname{End}_H(T)$ for some regular cotilting $H$-module $T$. Hence $M$ is not the middle of a short chain in $\operatorname{mod} D$. Then, by [**29**, (3.2)], $M$ is a faithful $D$-module. But this implies that $\operatorname{ann} \mathcal{C}=t_{\mathcal{C}}$.

THEOREM 3.6. *There is only a finite number of generalized standard components of $\Gamma_A$ which are not stable tubes.*

PROOF. It is enough to show that $\Gamma_A$ admits at most finitely many generalized standard regular components without oriented cycles. Moreover, for a regular generalized standard component $\mathcal{C}$ of $\Gamma_A$, we may write $A=P \oplus Q$ where the simple direct summands of $P/\operatorname{rad} P$ are exactly the simple composition factors of modules in $\mathcal{C}$. Let $t_Q(A)$ be the ideal of $A$ generated by the images of all maps from $Q$ to $A$, and $D=A/t_Q(A)$. Then $t_Q(A)$ is contained in $\operatorname{ann} \mathcal{C}$, and $\mathcal{C}$ is a sincere, regular, generalized standard component of $\Gamma_D$. Observe that $A$ admits only finitely many ideals of the form $t_Q(A)$.

Suppose now that there are pairwise different regular generalized standard components $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \cdots$ of $\Gamma_A$ without oriented cycles. By the above remarks we may assume that all these components $\mathcal{C}_i$ are sincere. We claim that there is a factor algebra $F$ of $A$ such that $\Gamma_F$ admits infinitely many pairwise different faithful generalized standard regular components without oriented cycles. This will give a contradiction to Corollary 3.3.

Let $e_1, \cdots, e_n$ be primitive orthogonal idempotents of $A$ such that $e_1A, \cdots, e_nA$ form a complete list of pairwise nonisomorphic indecomposable projective $A$-modules. For any $r \geqq 1$, let $I_r = \operatorname{ann} C_r$ and $B_r = A/I_r$. Then from Theorem 3.1, each $B_r$ is a tilted algebra of the form $B_r = \operatorname{End}_{H_r}(T_r)$ for some hereditary artin algebra $H_r$ and a regular tilting $H_r$-module $T_r$. In particular, the ordinary quiver of $B_r$ has no oriented cycles. Moreover, each $I_r$ is contained in $\operatorname{rad} A$. To each $C_r$ we associate a partially ordered set $\mathcal{S}_r$ whose set of vertices is equal to $\{1, \cdots, n\}$, and its order $\leqq_r$ is defined as follows: $i \leqq_r j$ if and only if $\operatorname{Hom}_{B_r}(e_iB_r, e_jB_r) \neq 0$. Since the number of partially ordered sets with $n$ elements is finite, we may assume that all partially ordered sets $\mathcal{S}_r$ coincide. Put $\mathcal{S} = \mathcal{S}_1$ and $\leqq = \leqq_1$. We may moreover assume that $\bigcap_{r \geqq 1} I_r = 0$, because otherwise replace $A$ by $A/(\bigcap_{r \geqq 1} I_r)$. This implies that the simple projective $B_r$-modules, corresponding to the minimal elements of $\mathcal{S} = \mathcal{S}_r$, are also simple projective $A$-modules. Indeed, for any minimal element $i$ of $\mathcal{S}$, $e_i \operatorname{rad} A$ is contained in $\bigcap_{r \geqq 1} I_r$, and hence $e_i \operatorname{rad} A = 0$. In particular, $e_iI_r = 0$ for any minimal element $i$ of $\mathcal{S}$ and all $r \geqq 1$. Let $s$ be an elements of $\mathcal{S}$ such that for infinitely many $r \geqq 1$ we have $e_sI_r \neq 0$ and $e_tI_r = 0$ for all elements $t$ of $\mathcal{S}$ with $t < s$. Without loss of generality, we may assume that $e_sI_r \neq 0$ and $e_tI_r = 0$ for all $r \geqq 1$ and all $t$ in $\mathcal{S}$ with $t < s$. Then, for each $r \geqq 1$ and $t$ from $\mathcal{S}$ with $t < s$, the projective $A$-module $e_tA$ is also a projective $B_r$-module. Fix $r \geqq 1$. Then the projective cover of $e_s \operatorname{rad} A/e_sI_r = \operatorname{rad}(e_sA/e_sI_r)$ in $\operatorname{mod} A$ is a direct sum of modules of the form $e_tA$ with $t < s$. Hence, $(e_s \operatorname{rad} A/e_sI_r)I_m = 0$ for all $m \geqq 1$. But then $(e_s \operatorname{rad} A)I_m \subset e_sI_r$ for all $m$ and $r$. Since $\bigcap_{r \geqq 1} I_r = 0$, we then infer that $(e_s \operatorname{rad} A)I_m = 0$ for all $m \geqq 1$. Then, for each $r \geqq 1$, we have $\operatorname{rad}(e_sA) = e_s \operatorname{rad} A = X \oplus Y$, where $X$ is a direct sum of indecomposable $B_r$-modules lying in the torsion-free part $\mathcal{Y}(T_r)$ of $\operatorname{mod} B_r$ and $Y$ is a direct sum of indecomposable $B_r$-modules lying in the torsion part $\mathcal{X}(T_r)$ of $\operatorname{mod} B_r$. Moreover, by Lemma 3.5, we deduce that $Y = e_sI_r$. Since $e_s \operatorname{rad} A$ has only finitely many pairwise nonisomorphic indecomposable direct summands, there is an infinite sequence

$$1 \leqq j_1 < j_2 < \cdots < j_p < \cdots$$

such that $e_sI_{j_1} = e_sI_{j_2} = \cdots = e_sI_{j_p} = \cdots$. Let $J$ be the intersection of all $I_{j_p}$, $p \geqq 1$, and $A' = A/J$, $I'_{j_p} = I_{j_p}/J$. Then $C_{j_p}$, $p \geqq 1$, are generalized standard regular components of $\Gamma_{A'}$, $I'_{j_p}$ are the annihilators of $C_{j_p}$ in $A'$, and $e_tI'_{j_p} = 0$ for all $p \geqq 1$ and elements $t$ of $\mathcal{S}$ with $t \leqq s$. Since $\mathcal{S}$ is a finite partially ordered set, repeating, if necessary, the above arguments finitely many times, we deduce that there is a factor algebra $F$ of $A$ such that infinitely many components from the sequence $C_1, C_2, C_3, \cdots$ are faithful generalized standard components of

$\Gamma_F$. This finishes our proof.

PROPOSITION 3.7. *Assume that $A$ is not strictly wild. Let $\mathcal{C}$ be a generalized standard component of $\Gamma_A$ without projective modules and oriented cycles. Then $\mathcal{C}$ is a preinjective component of Euclidean type.*

PROOF. We know from Theorem 2.3 and [**28**] that there is a finite valued quiver $\Delta$ without oriented cycles such that $\mathcal{C}$ is isomorphic to a full translation subquiver of $\boldsymbol{Z}\Delta$ closed under predecessors. Moreover, $\Delta$ is connected, since $\mathcal{C}$ is connected. Let $I=\operatorname{ann}\mathcal{C}$ and $B=A/I$. Then by Theorem 3.1, $B$ is a tilted algebra of the form $\operatorname{End}_H(T)$ for some connected hereditary artin algebra $H$ of type $\Delta^{op}$ and a tilting $H$-module $T$ without preinjective direct summands, and $\mathcal{C}$ is the connecting component of $\Gamma_B$ corresponding to $T$. We claim that $\Delta$ is a Euclidean quiver and $\mathcal{C}$ is a preinjective component. Clearly, $\Delta$ is not of Dynkin type, since $\mathcal{C}$ has no projective modules. Suppose that $\Delta$ is wild. Then, by [**46**, (7.5)], there exists a decomposition $T=V\oplus W$ such that

(a) $\operatorname{Hom}_H(W, V)=0$, and hence $B\cong\begin{bmatrix} D & M \\ 0 & C\end{bmatrix}$, where $C=\operatorname{End}_H(V)$, $D=\operatorname{End}_H(W)$, and $M=\operatorname{Hom}_H(V, W)$.

(b) There exists a connected hereditary artin algebra $\Lambda$ of wild type and a preprojective tilting $\Lambda$-module $U$ such that $C=\operatorname{End}_\Lambda(U)$.

(c) The preprojective component of $\Gamma_C$ is a full component of $\Gamma_B$. We shall show that $C$ is strictly wild. Then $A$ will be also strictly wild, which is impossible by our assumption on $A$. Since $\Lambda$ is connected of wild type, by [**33**] and [**16**, Section 8], there are indecomposable $\Lambda$-modules $L$ and $N$ satisfying the conditions:

(i) $L$ and $N$ are not preprojective in $\operatorname{mod}\Lambda$,

(ii) $\operatorname{End}_\Lambda(L)$ and $\operatorname{End}_\Lambda(N)$ are division algebras,

(iii) $\operatorname{Hom}_\Lambda(L, N)=0=\operatorname{Hom}_\Lambda(N, L)$,

(iv) $\dim_{\operatorname{End}_\Lambda(N)}\operatorname{Ext}^1_\Lambda(L, N)\cdot\dim\operatorname{Ext}^1_\Lambda(L, N)_{\operatorname{End}_\Lambda(L)}\geqq 5$.

Since $U$ is a preprojective tilting $\Lambda$-module, the modules $L$ and $N$ belong to the torsion part $\mathcal{G}(T)$ of $\operatorname{mod}\Lambda$, and then the $C$-modules $X=\operatorname{Hom}_\Lambda(U, L)$ and $Y=\operatorname{Hom}_\Lambda(U, N)$ satisfy the conditions (ii)—(iv). Hence $C$ is strictly wild. Therefore $\Delta$ is of Euclidean type. Finally, then $\mathcal{C}$ is the connecting component of a representation-infinite tilted algebra $B$ of Euclidean type $\Delta^{op}$ and without projective modules. Then $\mathcal{C}$ is a preinjective component of $\Gamma_A$ and of Euclidean type.

Dually, we get the following fact.

PROPOSITION 3.8. *Assume that $A$ is not strictly wild. Let $\mathcal{C}$ be a generalized standard component of $\Gamma_A$ without injective modules and oriented cycles. Then $\mathcal{C}$ is a preprojective component of Euclidean type.*

Moreover, we have the following consequence of the above propositions and [21], [48], [28].

COROLLARY 3.9. *Assume that $A$ is not strictly wild. Then every generalized standard regular component of $\Gamma_A$ is a stable tube.*

Finally, by (1.4), [28], and the above results, we get the following description of semi-regular generalized standard components over tame algebras.

COROLLARY 3.10. *Assume that $R$ is an algebraically closed field and $A$ is tame. Let $\mathcal{C}$ be a semi-regular generalized standard component of $\Gamma_A$. Then $\mathcal{C}$ is one of the following types:*

(i) *a preprojective component of Euclidean type,*

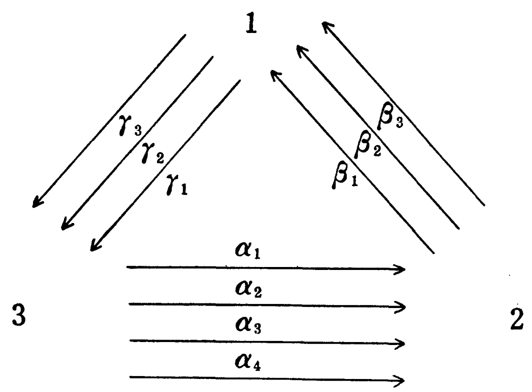
(ii) *a preinjective component of Euclidean type,*

(iii) *a tube.*

COROLLARY 3.11. *Assume that $R$ is an algebraically closed field and $A$ is tame. Then every regular generalized standard component of $\Gamma_A$ is a stable tube.*

## 4. An example.

Let $k$ be an algebraically closed field. Consider the bound quiver algebra $\Lambda=kQ/I$, where $Q$ is the quiver



and $I$ is the ideal in the path algebra $kQ$ of $Q$ generated by the elements $\alpha_1\beta_2$, $\alpha_1\beta_3$, $\alpha_4\beta_1$, $\alpha_4\beta_2$, $\alpha_2\beta_1+\alpha_2\beta_2$, $\alpha_2\beta_2+\alpha_3\beta_1$, $\alpha_3\beta_1+\alpha_4\beta_3$, $\alpha_1\beta_1+\alpha_2\beta_3$, $\alpha_2\beta_3+\alpha_3\beta_2$, $\alpha_3\beta_2+\alpha_3\beta_3$, $\gamma_2\alpha_1$, $\gamma_3\alpha_1$, $\gamma_1\alpha_4$, $\gamma_2\alpha_4$, $\gamma_1\alpha_2+\gamma_2\alpha_2$, $\gamma_2\alpha_2+\gamma_1\alpha_3$, $\gamma_1\alpha_3+\gamma_3\alpha_4$, $\gamma_1\alpha_1+\gamma_3\alpha_2$, $\gamma_3\alpha_2+\gamma_2\alpha_3$, $\gamma_2\alpha_3+\gamma_3\alpha_3$, $\gamma_1\alpha_1\beta_1$, and $\beta_i\gamma_j$, $\beta_i\gamma_i-\beta_j\gamma_j$ for all $i\neq j$, $1\leq i,\ j\leq 3$. We denote by $J$ the ideal of $\Lambda$ generated by the elements $\gamma_1+I$, $\gamma_2+I$, $\gamma_3+I$, by $K$ the ideal of $\Lambda$ generated by $\beta_1+I$, $\beta_2+I$, $\beta_3+I$, by $L$ the ideal of $\Lambda$ generated by $\alpha_1+I$, $\alpha_2+I$, $\alpha_3+I$, $\alpha_4+I$, and put $B=\Lambda/J$, $C=\Lambda/K$ and $D=\Lambda/L$. Moreover, let $E=\Lambda/\Lambda e_1\Lambda$, $\Omega=\Lambda/\Lambda e_2\Lambda$ and $H=\Lambda/\Lambda e_3\Lambda$. Clearly, $E$, $\Omega$, $H$ are hereditary algebras. From [**10**, (3.3)], $B$ is a tilted algebra of the form $\mathrm{End}_{k\Delta}(T)$ where $k\Delta$ is the path algebra of the following quiver $\Delta$

$$\circ \leftleftarrows \circ \longleftarrow \circ$$

and $T$ is a regular tilting $k\Delta$-module. Then, by (1.3), the connecting component $\mathcal{C}$ of $\Gamma_B$ associated to $T$ is regular of type $Z\Delta^{op}=Z\Delta$. Moreover, by [**24**] and [**46**], $\Gamma_B$ admits exactly one preprojective component $\mathcal{P}$ and exactly one preinjective component $\mathcal{I}$. Further, $B$ is not concealed algebra, and then by [**37**] $\mathcal{P}$ (resp. $\mathcal{I}$) has at most two $\tau_B$-orbits. Therefore, $\mathcal{P}$ is the preproinjective component of $\Gamma_H$ and $\mathcal{I}$ is the preinjective component of $\Gamma_E$. Hence, $\Gamma_B$ is of the form

$$\Gamma_B = \mathcal{P} \vee \mathcal{R} \vee \mathcal{C} \vee \mathcal{S} \vee \mathcal{I}$$

where $\mathcal{R}$ is a union of components contained in the torsion-free part $\mathcal{Y}(T)$ and $\mathcal{S}$ is a union of components contained in the torsion part $\mathcal{X}(T)$ of $\mathrm{mod}\, B$. In particular, the projective cover $P_B(3)=P_\Lambda(3)$ of the simple module $S(3)$ at 3 belongs to $\mathcal{R}$ and the injective envelope $I_B(1)=I_\Lambda(1)$ of the simple module $S(1)$ at 1 belongs to $\mathcal{S}$. Then $B$ is a one-point extension $H[X]$ of $H$ by an indecomposable regular $H$-module $X$, being the radical of $P_B(3)$, and is a one-point coextension $[Y]E$ of $E$ by an indecomposable regular $E$-module $Y$, being the socle factor of $I_B(1)$. Moreover, in the above decomposition of $\Gamma_B$, there are only nonzero maps in $\mathrm{mod}\, B$ from any of these classes $\mathcal{P}$, $\mathcal{R}$, $\mathcal{C}$, $\mathcal{S}$, $\mathcal{I}$ to itself or to the classes of its right side. Observe now that $C\cong B^{op}$. Then, as above, $\Gamma_C$ is of the form

$$\Gamma_C = \mathcal{P}' \vee \mathcal{R}' \vee \mathcal{D} \vee \mathcal{S}' \vee \mathcal{I}'$$

where $\mathcal{P}'$ is the preprojective component of $\Gamma_E$, $\mathcal{I}'$ is the preinjective component of $\Gamma_\Omega$, $\mathcal{D}$ is the regular connecting component of type $\boldsymbol{Z}\Delta$, $\mathcal{R}'$ and $\mathcal{S}'$ are unions of components, the projective cover $P_C(1)=P_\Lambda(1)$ of the simple module $S(1)$ belongs to $\mathcal{R}'$, and the injective envelope $I_C(2)=I_\Lambda(2)$ of the simple module $S(2)$ belongs to $\mathcal{S}'$. Moreover, $C$ is a one-point extension $E[Y']$ of $E$ by an indecomposable regular $E$-module $Y'$, being the radical of $P_C(1)$, and is a one-point coextension $[X']\Omega$ of $\Omega$ by an indecomposable regular $\Omega$-module $X'$, being the socle factor of $I_C(2)$. Further, since $\gamma_r\alpha_j\beta_s$ belongs to $I$ for all $1\leqq r,\ s\leqq 3$, $1\leqq j\leqq 4$, the simple module $S(1)$ is not a composition factor of $Y'$. Then using the calculations from [**10**, (3.3)] we deduce that $Y'$, considered as a $B$-module, belongs to one of the components of $\mathcal{S}$. Finally, observe that the projective cover $P_\Lambda(2)$ of $S(2)$ coincides with the injective envelope $I_\Lambda(3)$ of $S(3)$. Hence, $P_D(2)=P_\Lambda(2)=I_\Lambda(2)=I_D(3)$ is a projective-injective $D$-module (and $\Lambda$-module). Therefore $D$ is a one-point extension $\Omega[U]$ of $\Omega$ by the injective envelope $U=I_\Omega(3)$ of $S(3)$ in $\mathrm{mod}\,\Omega$, and is a one-point coextension $[V]H$ of $H$ by the projective cover $V=P_H(2)$ of $S(2)$ in $\operatorname{mod} H$. Hence $\Gamma_D$ is of the form

$$\Gamma_D=\mathcal{P}''\vee\mathcal{R}''\vee\mathcal{X}\vee\mathcal{S}''\vee\mathcal{I}''$$

where $\mathcal{P}''$ is the preprojective component of $\Gamma_\Omega$, $\mathcal{I}''$ is the preinjective component of $\Gamma_H$, $\mathcal{R}''$ is the union of all regular components of $\Gamma_\Omega$, $\mathcal{S}''$ is the union of all regular components of $\Gamma_H$ and $\mathcal{X}$ is a component of the form

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[dr, shift left] \arrow[dr] \arrow[dr, shift right] & & \tau_\Omega S(1) \arrow[dr, shift left] \arrow[dr] \arrow[dr, shift right] & & S(1) \arrow[dr, shift left] \arrow[dr] \arrow[dr, shift right] & & \tau_H^{-}S(1) \arrow[dr, shift left] \arrow[dr] \arrow[dr, shift right] & & \cdots \\
& \tau_\Omega I_\Omega(3) \arrow[ur, shift left] \arrow[ur] \arrow[ur, shift right] & & I_\Omega(3) \arrow[ur, shift left] \arrow[ur] \arrow[ur, shift right] \arrow[dr, shift left] \arrow[dr] \arrow[dr, shift right] & & P_H(2) \arrow[ur, shift left] \arrow[ur] \arrow[ur, shift right] & & \tau_H^{-}P_H(2) \arrow[ur, shift left] \arrow[ur] \arrow[ur, shift right] & \\
& & & & P_D(2)=I_D(3) \arrow[ur, shift left] \arrow[ur] \arrow[ur, shift right] & & & &
\end{tikzcd}$$

obtained by a glueing of the preprojective component of $\Gamma_H$ with the preinjective component of $\Gamma_\Omega$ and adding $P_D(2)=I_D(3)$. Moreover, in the above decomposition of $\Gamma_D$, there are only nonzero maps in $\operatorname{mod} D$ from any of these classes $\mathcal{P}''$, $\mathcal{R}''$, $\mathcal{X}$, $\mathcal{S}''$, $\mathcal{I}''$ to itself or to the classes of its right side. Clearly, $P_D(2)=P_\Lambda(2)$ is a sincere indecomposable $\Lambda$-module. Moreover, by [**30**, (1.8)], all but a finite number of indecomposable modules from $\mathcal{C}$ (resp. $\mathcal{D}$) are sincere $B$-modules (resp. sincere $C$-modules), and hence sincere $\Lambda$-modules. We shall show that $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{X}$ are generalized standard components of $\Gamma_\Lambda$. We apply the Galois covering techniques and for related details we refer to [**12**], [**19**] and [**17**].

Let $\tilde{\Lambda}$ be the locally bounded $k$-category $k\tilde{Q}/\tilde{I}$, where $\tilde{Q}$ is the quiver

$$\begin{tikzcd}[column sep=large]
\cdots & (3,r-1) \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] & (1,r) \arrow[l, shift right=3, "\gamma_{1,r}"'] \arrow[l, "\gamma_{2,r}"] \arrow[l, shift left=3, "\gamma_{3,r}"] & (2,r) \arrow[l, shift right=3, "\beta_{1,r}"'] \arrow[l, "\beta_{2,r}"] \arrow[l, shift left=3, "\beta_{3,r}"] & (3,r) \arrow[l, shift right=4, "\alpha_{1,r}"'] \arrow[l, shift right=1.5, "\alpha_{2,r}"'] \arrow[l, shift left=1.5, "\alpha_{3,r}"] \arrow[l, shift left=4, "\alpha_{4,r}"] & (1,r+1) \arrow[l, shift right=3, "\gamma_{1,r+1}"'] \arrow[l, "\gamma_{2,r+1}"] \arrow[l, shift left=3, "\gamma_{3,r+1}"] & (2,r+1) \arrow[l, shift right=3, "\beta_{1,r+1}"'] \arrow[l, "\beta_{2,r+1}"] \arrow[l, shift left=3, "\beta_{3,r+1}"] & \cdots \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3]
\end{tikzcd}$$

with $r\in\boldsymbol{Z}$, and $\tilde{I}$ is the ideal in the path category $k\tilde{Q}$ of $\tilde{Q}$ generated by all elements $\alpha_{1,r}\beta_{2,r}$, $\alpha_{1,r}\beta_{3,r}$, $\alpha_{4,r}\beta_{1,r}$, $\alpha_{4,r}\beta_{2,r}$, $\alpha_{2,r}\beta_{1,r}+\alpha_{2,r}\beta_{2,r}$, $\alpha_{2,r}\beta_{2,r}+\alpha_{3,r}\beta_{1,r}$, $\alpha_{3,r}\beta_{1,r}+\alpha_{4,r}\beta_{3,r}$, $\alpha_{1,r}\beta_{1,r}+\alpha_{2,r}\beta_{3,r}$, $\alpha_{2,r}\beta_{3,r}+\alpha_{3,r}\beta_{2,r}$, $\alpha_{3,r}\beta_{2,r}+\alpha_{3,r}\beta_{3,r}$, $\gamma_{2,r}\alpha_{1,r-1}$, $\gamma_{3,r}\alpha_{1,r-1}$, $\gamma_{1,r}\alpha_{4,r-1}$, $\gamma_{2,r}\alpha_{4,r-1}$, $\gamma_{1,r}\alpha_{2,r-1}+\gamma_{2,r}\alpha_{2,r-1}$, $\gamma_{2,r}\alpha_{2,r-1}+\gamma_{1,r}\alpha_{3,r-1}$, $\gamma_{1,r}\alpha_{3,r-1}+\gamma_{3,r}\alpha_{4,r-1}$, $\gamma_{1,r}\alpha_{1,r-1}+\gamma_{3,r}\alpha_{2,r-1}$, $\gamma_{3,r}\alpha_{2,r-1}+\gamma_{2,r}\alpha_{3,r-1}$, $\gamma_{2,r}\alpha_{3,r-1}+\gamma_{3,r}\alpha_{3,r-1}$, $\gamma_{1,r+1}\alpha_{1,r}\beta_{1,r}$, $\beta_{i,r}\gamma_{j,r}$, $\beta_{i,r}\gamma_{i,r}-\beta_{j,r}\gamma_{j,r}$ for all $i\neq j$, $1\leqq i$, $j\leqq 3$, $r\in\boldsymbol{Z}$. Let $G$ be the infinite cyclic group of $k$-linear automorphisms of $\tilde{\Lambda}$ generated by the shift

$g:\tilde{\Lambda}\to\tilde{\Lambda}$ such that $g(i, r)=(i, r+1)$, $g(\alpha_{j,r})=\alpha_{j,r+1}$, $g(\beta_{s,r})=\beta_{s,r+1}$ and $g(\gamma_{s,r})=\gamma_{s,r+1}$ for $r\in\boldsymbol{Z}$, $1\leqq i\leqq 3$, $1\leqq j\leqq 4$, $1\leqq s\leqq 3$. Then we can consider the quotient category $\tilde{\Lambda}/G$ (see [**19**]) whose objects are the $G$-orbits of the objects in $\tilde{\Lambda}$, and the Galois covering $F:\tilde{\Lambda}\to\tilde{\Lambda}/G$ which assigns to each object $x=(i, r)$ of $\tilde{\Lambda}$ its $G$-orbit $Gx$. Observe that $\tilde{\Lambda}/G=\Lambda$, if we identify $\Lambda$ with the corresponding $k$-category with the objects 1, 2, 3. Denote by mod $\tilde{\Lambda}$ the category of all finite dimensional $\tilde{\Lambda}$-modules. Then $G$ acts on the category mod $\tilde{\Lambda}$ by the translations ${}^h(-)$, $h\in G$, which assign to each module $M$ from mod $\tilde{\Lambda}$ the $\tilde{\Lambda}$-module ${}^hM=M(h^{-1}(-))$. Moreover, we have the push-down functor $F_\lambda$: mod $\tilde{\Lambda}$ $\to$mod $\tilde{\Lambda}/G=$mod $\Lambda$ (see [**12**]). Since $G$, as a torsion-free group, acts freely on the isoclasses of indecomposable objects in mod $\tilde{\Lambda}$, the functor $F_\lambda$ preserves indecomposable modules and Auslander-Reiten sequences (see [**19**, (3.5), (3.6)]).

We endow the set $Q_0=\{1, 2, 3\}x\boldsymbol{Z}$ of vertices of $Q$ with the partial ordering $\leqq$ such that $(j, s)\leqq(i, r)$ if and only if $s<r$ or $s=r$ and $j\leqq i$. For $(j, s)\leqq(i, r)$ in $Q_0$, we denote by ${}_{(j,s)}\tilde{\Lambda}_{(i,r)}$ the full subcategory of $\tilde{\Lambda}$ formed by all objects $(p, t)$ with $(j, s)\leqq(p, t)\leqq(i, r)$. For each $m\in\boldsymbol{Z}$, we put $B_m={}_{(1,m)}\tilde{\Lambda}_{(3,m)}$, $C_m={}_{(2,m)}\tilde{\Lambda}_{(1,m+1)}$, $D_m={}_{(3,m)}\tilde{\Lambda}_{(2,m+1)}$, $E_m={}_{(2,m)}\tilde{\Lambda}_{(3,m)}$, $\Omega_m={}_{(3,m)}\tilde{\Lambda}_{(1,m+1)}$, and $H_m={}_{(1,m)}\tilde{\Lambda}_{(2,m)}$. Observe that, for each $m\in\boldsymbol{Z}$, there are isomorphisms $B_m\cong B$, $C_m\cong C$, $D_m\cong D$, $E_m\cong E$, $\Omega_m\cong\Omega$, and $H_m\cong H$, if we again identify algebras with the corresponding $k$-categories with finitely many objects. The above isomorphisms induce the corresponding decompositions of the Auslander-Reiten quivers

$$\Gamma_{B_m}=\mathcal{P}_m\vee\mathcal{R}_m\vee\mathcal{C}_m\vee\mathcal{S}_m\vee\mathcal{I}_m\,,$$

$$\Gamma_{C_m}=\mathcal{P}'_m\vee\mathcal{R}'_m\vee\mathcal{D}_m\vee\mathcal{S}'_m\vee\mathcal{I}'_m\,,$$

and

$$\Gamma_{D_m}=\mathcal{P}''_m\vee\mathcal{R}''_m\vee\mathcal{X}_m\vee\mathcal{S}''_m\vee\mathcal{I}''_m\,,$$

$m\in\boldsymbol{Z}$. Moreover, $B_m=H_m[X_m]=[Y_m]E_m$, $C_m=E_m[Y'_m]=[X'_m]\Omega_m$ and $D_m=\Omega_m[U_m]=[V_m]H_{m+1}$ for the corresponding indecomposable modules $X_m$, $Y_m$, $Y'_m$, $X'_m$, $U_m$ and $V_m$. Fix now $m\in\boldsymbol{Z}$. Then $\tilde{\Lambda}$ can be obtained from $B_m$ by successive one-point extensions using the modules $Y'_m$, $U_m$, $X_{m+1}$, $Y'_{m+1}$, $U_{m+1}$, $X_{m+2}$, $\cdots$, and then successive one-point coextensions using the modules $V_{m-1}$, $X'_{m-1}$, $Y_{m-1}$, $V_{m-2}$, $X'_{m-2}$, $Y_{m-2}$, $\cdots$. In this process, the component $\mathcal{C}_m$ of $\Gamma_{B_m}$ remains unchanged. Similarly, $\tilde{\Lambda}$ can be obtained from $C_m$ by successive one-point extensions using the modules $U_m$, $X_{m+1}$, $Y'_{m+1}$, $U_{m+1}$, $X_{m+2}$, $Y'_{m+2}$, $\cdots$, and then by successive one-point coextensions using the modules $Y_m$, $V_{m-1}$, $X'_{m-1}$, $Y_{m-1}$, $V_{m-2}$, $X'_{m-2}$, $\cdots$. In this process, the component $\mathcal{D}_m$ of $\Gamma_{C_m}$ remains unchanged. Finally, $\tilde{\Lambda}$ can be obtained from $D_m$ by successive one-point extensions using the modules $X_{m+1}$, $Y'_{m+1}$, $U_{m+1}$, $X_{m+2}$, $Y'_{m+2}$, $U_{m+2}$, $\cdots$, and then successive one-point coextensions using the modules $X'_m$, $Y_m$, $V_{m-1}$, $X'_{m-1}$, $Y_{m-1}$, $V_{m-2}$, $\cdots$. In this process, the component $\mathcal{X}_m$ of $\Gamma_{D_m}$ remains unchanged.

Therefore, $\mathcal{C}_m$, $\mathcal{D}_m$ and $\mathcal{X}_m$ are full components of $\Gamma_{\tilde{A}}$. Moreover, we have the following decomposition of $\Gamma_{\tilde{A}}$

$$\Gamma_{\tilde{A}} = \bigvee_{m \in \boldsymbol{Z}} (\mathcal{C}_m \vee \mathcal{W}_m \vee \mathcal{D}_m \vee \mathcal{Y}_m \vee \mathcal{X}_m \vee \mathcal{Z}_m)$$

where $\mathcal{W}_m$, $\mathcal{Y}_m$ and $\mathcal{Z}_m$ are unions of components of $\Gamma_{\tilde{A}}$ such that the supports of modules from $\mathcal{W}_m$ are contained ${}_{(1,m)}\tilde{A}_{(1,m+1)}$, the supports of modules from $\mathcal{Y}_m$ are contained in ${}_{(2,m)}\tilde{A}_{(2,m+1)}$, and the supports of modules from $\mathcal{Z}_m$ are contained in ${}_{(3,m)}\tilde{A}_{(3,m+1)}$. In particular, $\tilde{A}$ is a locally support-finite category [17], that is, for each indecomposable projective $\tilde{A}$-module $P$, there are only finitely many nonisomorphic indecomposable projective $\tilde{A}$-modules $P'$ such that $\mathrm{Hom}_{\tilde{A}}(P, M) \neq 0$ and $\mathrm{Hom}_{\tilde{A}}(P', M) \neq 0$ for an indecomposable module $M$ from $\mathrm{mod}\,\tilde{A}$. Then, by [17], the functor $F_\lambda : \mathrm{mod}\,\tilde{A} \to \mathrm{mod}\,A$ is dense, and consequently we have a Galois covering $F_\lambda : \mathrm{mod}\,\tilde{A} \to \mathrm{mod}\,A$ of module categories. In particular, $\Gamma_A$ coincides with the orbit quiver $\Gamma_{\tilde{A}}/G$. Observe also that $g(\mathcal{C}_m) = \mathcal{C}_{m+1}$, $g(\mathcal{W}_m) = \mathcal{W}_{m+1}$, $g(\mathcal{D}_m) = \mathcal{D}_{m+1}$, $g(\mathcal{Y}_m) = \mathcal{Y}_{m+1}$, $g(\mathcal{X}_m) = \mathcal{X}_{m+1}$ and $g(\mathcal{Z}_m) = \mathcal{Z}_{m+1}$. Moreover, the supports of modules from $\mathcal{C}_m$ are contained in $B_m$, the supports of modules from $\mathcal{D}_m$ are contained in $C_m$, and the supports of modules from $\mathcal{X}_m$ are contained in $D_m$. Hence,

(a) $\mathcal{C}_m$, $m \in \boldsymbol{Z}$, are pairwise orthogonal generalized standard components of $\Gamma_{\tilde{A}}$,

(b) $\mathcal{D}_m$, $m \in \boldsymbol{Z}$, are pairwise orthogonal generalized standard components of $\Gamma_{\tilde{A}}$,

(c) $\mathcal{X}_m$, $m \in \boldsymbol{Z}$, are pairwise orthogonal generalized standard components of $\Gamma_{\tilde{A}}$.

Since $F_\lambda : \mathrm{mod}\,\tilde{A} \to \mathrm{mod}\,A$ is a Galois covering, we infer that $\mathcal{C} = F_\lambda(\mathcal{C}_0)$, $\mathcal{D} = F_\lambda(\mathcal{D}_0)$ and $\mathcal{X} = F_\lambda(\mathcal{X}_0)$ are sincere generalized standard components of $\Gamma_A$.

## 5. Generalized standard stable tubes.

We shall first present some characterizations of generalized standard stable tubes of $\Gamma_A$. The following two lemmas will be useful.

LEMMA 5.1. *Let $X$ be an indecomposable $A$-module. Assume that the $\tau_A$-orbit of $X$ is stable and consists of pairwise orthogonal bricks. Then $X$ is periodic.*

PROOF. Suppose that $X$ is not periodic. Consider the modules $M_i = \tau_A^{2i} X$, $1 \leqq i \leqq n+1$. Then $\mathrm{Hom}_A(M_i, \tau_A M_j) = 0$ for all $1 \leqq i$, $j \leqq n+1$, a contradiction with Lemma 2.1. Hence $X$ is periodic.

LEMMA 5.2. *Let $\mathcal{T}$ be a stable tube of $\Gamma_A$. Suppose that* $\mathrm{rad}^\infty(X, Y) \neq 0$ *for some modules $X$ and $Y$ from $\mathcal{T}$. Then there are mouth modules $M$ and $N$ (may* *be* $M=N$) *in* $\mathcal{T}$ *such that* $\mathrm{rad}^{\infty}(M, N)\neq 0$.

PROOF. Let $0\neq f\in\mathrm{rad}^{\infty}(X, Y)$. If $X$ lies on the mouth of $\mathcal{T}$ we put $M=X$. Suppose that $X$ does not lie on the mouth of $\mathcal{T}$. Consider the sectional path

$$X=X_m \xrightarrow{\alpha_{m-1}} X_{m-1} \longrightarrow \cdots \longrightarrow X_2 \xrightarrow{\alpha_1} X_1$$

in $\mathcal{T}$ with $X_1$ lying on its mouth. Let $r$, $1\leqq r\leqq m$, be minimal such that $f=f'\alpha_r\alpha_{r+1}\cdots\alpha_{m-1}$ for some $f': X_r\to Y$. If $r=1$, we put $M=X_1$ and $g=f'$. Suppose that $r>1$. We have then an exact sequence

$$0 \longrightarrow M \xrightarrow{\gamma} X_r \xrightarrow{\alpha_{r-1}} X_{r-1} \longrightarrow 0$$

with $M$ lying on the mouth of $\mathcal{T}$. Since $f'$ does not factor through $X_{r-1}$, the composition $g=f'\gamma$ is nonzero and obviously belongs to $\mathrm{rad}^{\infty}(\mathrm{mod}\, A)$. If $Y$ lies on the mouth of $\mathcal{T}$, we put $N=Y$, and then $0\neq g\in\mathrm{rad}^{\infty}(M, N)$. Suppose that $Y$ does not lie on the mouth of $\mathcal{T}$. Consider the sectional path

$$Y_1 \xrightarrow{\beta_1} Y_2 \xrightarrow{\beta_2} \cdots \longrightarrow Y_{q-1} \xrightarrow{\beta_{q-1}} Y_q=Y$$

in $\mathcal{T}$ with $Y_1$ lying on its mouth. Let $s$, $1\leqq s\leqq q$, be minimal such that $g=\beta_{q-1}\cdots\beta_s g'$ for some $g': M\to Y_s$. If $s=1$, we put $N=Y_1$ and $h=g'$. Suppose that $s>1$. Then we have an exact sequence

$$0 \longrightarrow Y_{s-1} \xrightarrow{\beta_{s-1}} Y_s \xrightarrow{\sigma} N \longrightarrow 0$$

with $N$ lying on the mouth of $\mathcal{T}$. Since $g'$ does not factor through $Y_{s-1}$, the composition $h=\sigma g'$ is nonzero and obviously belongs to $\mathrm{rad}^{\infty}(\mathrm{mod}\, A)$. Therefore, we proved that $0\neq h\in\mathrm{rad}^{\infty}(M, N)$ for some modules $M$ and $N$ lying on the mouth of $\mathcal{T}$.

COROLLARY 5.3. *Let* $\mathcal{T}$ *be a stable tube of* $\Gamma_A$. *Then the following conditions are equivalent:*

(i) $\mathcal{T}$ *is generalized standard.*

(ii) *The mouth modules of* $\mathcal{T}$ *are pairwise orthogonal bricks.*

(iii) *Every loop* $Z\to Z$ *in* mod $A$ *with* $Z$ *from* $\mathcal{T}$ *is finite.*

PROOF. Obviously (i) implies (ii). Suppose that $\mathrm{rad}^{\infty}(Z, Z)\neq 0$ for some module $Z$ from $\mathcal{T}$. Then by the above lemma, $\mathrm{rad}^{\infty}(M, N)\neq 0$ for some modules $M$ and $N$ lying on the mouth of $\mathcal{T}$. Consequently, (ii) implies (iii). Similarly, (iii) implies (i). Indeed, if $\mathrm{rad}^{\infty}(X, Y)\neq 0$ for some modules $X$ and $Y$ from $\mathcal{T}$, then also $\mathrm{rad}^{\infty}(M, N)\neq 0$ for some modules $M$ and $N$ lying on the mouth of $\mathcal{T}$. Let $Z$ be a module in $\mathcal{T}$ lying on the intersection of the sectional path in $\mathcal{T}$

from $M$ to infinity and the sectional path in $\mathcal{T}$ from infinity to $N$. Then, for each $0 \neq h \in \mathrm{rad}^{\infty}(M, N)$, we have $h = ufv$ for some $f \in \mathrm{rad}^{\infty}(Z, Z)$ and $v: M \to Z$, $u: Z \to N$, being the compositions of the corresponding irreducible maps in $\mathcal{T}$. Obviously, $f \neq 0$.

In particular, we have the following facts:

COROLLARY 5.4. *Let $\mathcal{T}$ be a stable tube of $\Gamma_A$ consisting of modules which do not lie on short infinite cycles. Then $\mathcal{T}$ is generalized standard.*

COROLLARY 5.5. *Let $\mathcal{T}$ be a regular component of $\Gamma_A$. Then $\mathcal{T}$ is a generalized standard tube if and only if $\mathcal{T}$ admits a $\tau_A$-orbit consisting of pairwise orthogonal bricks.*

PROOF. It is a direct consequence of Lemma 5.1, Corollary 5.3, and [**21**].

For selfinjective artin algebras we have the following fact.

PROPOSITION 5.6. *Assume that $A$ is selfinjective but not a Nakayama algebra. Let $\mathcal{T}$ be a component of $\Gamma_A$. Then $\mathcal{T}$ is a generalized standard stable tube if and only if $\mathcal{T}$ admits a stable $\tau_A$-orbit consisting of pairwise orthogonal bricks.*

PROOF. One implication is obvious. Assume that $\mathcal{T}$ admits a stable $\tau_A$-orbit, say of a module $X$, consisting of pairwise orthogonal bricks. We claim that $\mathcal{T}$ is a stable tube. Then Corollary 5.5 will imply that $\mathcal{T}$ is also generalized standard. From Lemma 5.1 we know that $X$ is periodic. Let $\mathcal{C}$ be the stable part of $\mathcal{T}$. Observe that $\mathcal{C}$ is connected because $A$ is selfinjective and not Nakayama. Since $\mathcal{C}$ admits the periodic module $X$, by [**21**], $\mathcal{C}$ is either a stable tube, if $\mathcal{C}$ is infinite, or of the form $\boldsymbol{Z}\Delta/G$, where $\Delta$ is a Dynkin quiver and $G$ is an automorphism group of $\boldsymbol{Z}\Delta$. Assume first that $\mathcal{C}$ is a stable tube. We claim that then $\mathcal{C} = \mathcal{T}$. Suppose that $\mathcal{C} \neq \mathcal{T}$. Then there is an Auslander-Reiten sequence in mod $A$

$$0 \longrightarrow \mathrm{rad}\, P \longrightarrow P \oplus \mathrm{rad}\, P/\mathrm{soc}\, P \longrightarrow P/\mathrm{soc}\, P \longrightarrow 0$$

with $P$ projective-injective and $\mathrm{rad}\, P$ in $\mathcal{C}$. Since $\mathcal{C}$ is a stable tube, its mouth is formed by the $\tau_A$-orbit of $X$, and then $\mathrm{rad}\, P/\mathrm{soc}\, P$ is a direct sum of two indecomposable modules. Let $\tau_A^r X$ be the module lying on the sectional path in $\mathcal{C}$ from the mouth to infinity passing through $\mathrm{rad}\, P$, and $\tau_A^s X$ be the module lying on the sectional path in $\mathcal{C}$ from infinity to the mouth passing through $P/\mathrm{soc}\, P$. Then clearly $\mathrm{rad}(\tau_A^r X, \tau_A^s X) \neq 0$, and we have a contradiction to our assumption on the $\tau_A$-orbit of $X$. Hence $\mathcal{T} = \mathcal{C}$ is a stable tube. Now suppose that $\mathcal{C} = \boldsymbol{Z}\Delta/G$ for some Dynkin quiver $\Delta$. From our assumption on the $\tau_A$-orbit of $X$ we deduce that the Auslander-Reiten sequences ending at the modules $\tau_A^r X$ have indecomposable middle terms. Similarly, we infer as above that the Auslander-

Reiten sequences ending at other modules of $\mathcal{C}$ have at most two indecomposable middle terms. Hence $\Delta$ is of type $A_m$. Moreover, $G$ is an infinite cyclic group generated by an element of the form $\tau^s\phi$, where $\tau$ is the translation of $\boldsymbol{Z}\Delta$ and $\phi$ is an automorphism of $\boldsymbol{Z}\Delta$ which fixes a vertex of $\boldsymbol{Z}\Delta$ (see [**31**]). Since we have no sectional path in $\mathcal{C}$ of the form $\tau_A^r X\to\cdots\to\tau_A^s X$, $G$ is generated by a power of the translation $\tau$ in $\boldsymbol{Z}\Delta$, and then $\mathcal{C}$ is a cylinder $\boldsymbol{Z}A_m/(\tau^s)$. Moreover, we conclude as above that each vertex of $\mathcal{T}=\Gamma_A$ is a source (resp. target) of at most two arrows. Then, for any indecomposable projective $A$-module $P$, all its factor modules lie on one sectional path starting at $P$. Hence $A$ is a Nakayama algebra, a contradiction. Therefore $\mathcal{T}$ is a generalized standard stable tube.

A family $\mathcal{T}_i$, $i\in I$, of stable tubes in $\Gamma_A$ is said to be *faithful* if the intersection of the annihilators of all modules from this family is zero. Then we have

Lemma 5.7. *Let $\mathcal{T}_i$, $i\in I$, be a faithful family of stable tubes of $\Gamma_A$. Assume that any short path $X\to Y\to Z$ in* mod $A$ *with $X$ and $Z$ from the family $\mathcal{T}_i$, $i\in I$, is finite. Then gl.* dim. $A\leqq 2$.

Proof. Let $N_1,\cdots,N_t$ be indecomposable modules from the family $\mathcal{T}_i$, $i\in I$, such that $N=N_1\oplus\cdots\oplus N_t$ is a faithful $A$-module. Let $X$ be an indecomposable submodule of an indecomposable projective $A$-module $P$. We claim that $pd_AX\leqq 1$. Suppose that $pd_AX\geqq 2$. Observe first that $X$ does not belong to the family $\mathcal{T}_i$, $i\in I$. Indeed, if this is not the case, we have in mod $A$ a short path $X\xrightarrow{u}P\xrightarrow{v}N_j$, for some $1\leqq j\leqq t$, with $u$ and $v$ from $\mathrm{rad}^\infty(\mathrm{mod}\,A)$, because the tubes $\mathcal{T}_i$ are stable. But this contradicts our assumption on the family $\mathcal{T}_i$, $i\in I$. Further, since $N$ is a faithful $A$-module, there are an epimorphism $N^r\to DA$ and a monomorphism $A\to N^s$, for some $r, s\geqq 1$. Hence, $\mathrm{Hom}_A(X, N)\neq 0$ because $X$ is a submodule of $A$, and $\mathrm{Hom}_A(N, \tau_AX)\neq 0$ because $pd_AX\geqq 2$ implies $\mathrm{Hom}_A(DA, \tau_AX)\neq 0$, by [**36**, (2.4)]. Then $N$ is the middle term of a short chain $X\to N\to\tau_AX$ in mod $A$. Consequently, by [**29**, (1.6)], we have in mod $A$ a short path $N_k\to V\to N_p$, where $1\leqq k$, $p\leqq t$, and $V$ is an indecomposable direct summand of the middle term of an Auslander-Reiten sequence in mod $A$ ending at $X$. Since $X$ does not belong to the family $\mathcal{T}_i$, $i\in I$, also $V$ does not belong to this family, and so the path $N_k\to V\to N_p$ is infinite. We have again a contradiction to our assumption on the family $\mathcal{T}_i$, $i\in I$. Therefore, $pd_AX\leqq 1$, and hence $gl.\,\mathrm{dim.}\,A\leqq 2$.

Corollary 5.8. *Let $\mathcal{T}$ be a faithful stable tube of $\Gamma_A$ consisting entirely of modules which do not lie on short infinite cycles. Then gl.* dim. $A\leqq 2$.

Proof. Suppose that $X\to Y\to Z$ is an infinite short path in mod $A$ with $X$

and $Z$ from $\mathcal{T}$. We know from Corollary 5.4 that $\mathcal{T}$ is generalized standard. Hence $Y$ does not belong to $\mathcal{T}$. Consequently, $\mathrm{rad}^{\infty}(X, Y)\neq 0$ and $\mathrm{rad}^{\infty}(Y, Z)\neq 0$. Then, as in the proof of Lemma 5.2, we deduce that $\mathrm{rad}^{\infty}(M, Y)\neq 0$ and $\mathrm{rad}^{\infty}(Y, N)\neq 0$ for some modules $M$ and $N$ lying on the mouth of $\mathcal{T}$. Let $W$ be the module in $\mathcal{T}$ lying on the intersection of the sectional path in $\mathcal{T}$ from $M$ to infinity and the sectional path in $\mathcal{T}$ from infinity to $N$. Then clearly $\mathrm{rad}^{\infty}(W, Y)\neq 0$ and $\mathrm{rad}^{\infty}(Y, W)\neq 0$, and so we have in $\mathrm{mod}\, A$ a short infinite cycle $W\to Y\to W$ with $W$ from $\mathcal{T}$, a contradiction. Now the corollary is a direct consequence of Lemma 5.7.

We shall need also the following lemma.

LEMMA 5.9. *Let $\mathcal{T}_i$, $i\in I$, be a faithful family of pairwise orthogonal, generalized standard stable tubes of $\Gamma_A$, and let $X$ be a module from this family. Then $pd_A X\leqq 1$ and $\mathrm{id}_A X\leqq 1$.*

PROOF. Let $N_1, \cdots, N_t$ be indecomposable modules from the family $\mathcal{T}_i$, $i\in I$, such that $N=N_1\oplus\cdots\oplus N_t$ is a faithful $A$-module. Then there are an epimorphism $f: N^r\to DA$ and a monomorphism $g: A\to N^s$ for some $r, s\geqq 1$. Since the tubes $\mathcal{T}_i$ are stable, the maps $f$ and $g$ belong to $\mathrm{rad}^{\infty}(\mathrm{mod}\, A)$. Suppose that $pd_A X\geqq 2$. Then, by [**36**, (2.4)], we get that $\mathrm{Hom}_A(DA, \tau_A X)\neq 0$, and so there is a nonzero map $h: DA\to\tau_A X$. But then $hf$ is nonzero and belongs to $\mathrm{rad}^{\infty}(N^r, \tau_A X)$. Hence $\mathrm{rad}^{\infty}(N_j, \tau_A X)\neq 0$ for some $j$. This contradicts our assumption that the tubes $\mathcal{T}_i$ are pairwise orthogonal and generalized standard, because $N_j$ and $\tau_A X$ belong to the family $\mathcal{T}_i$, $i\in I$. Therefore, $pd_A X\leqq 1$. Dual arguments show that also $\mathrm{id}_A X\leqq 1$.

We shall prove now some facts on the ranks of generalized standard stable tubes in $\Gamma_A$.

LEMMA 5.10. *Let $\mathcal{T}_i$, $i\in I$, be a family of pairwise orthogonal, generalized standard stable tubes of $\Gamma_A$. For each $i\in I$, denote by $r_i$ the rank of the tube $\mathcal{T}_i$. Then*

$$\sum_{i\in I}(r_i-1)\leqq n-1\,.$$

PROOF. We know from the Corollaries 2.2 and 5.3 that the number of tubes $\mathcal{T}_i$ with $r_i>1$ is finite. Hence we may assume that $I=\{1, \cdots, m\}$ and $r_i>1$ for all $i\in I$. Let $J$ be the intersection of the annihilators $\mathrm{ann}\,\mathcal{T}_i$, $1\leqq i\leqq m$. Let $B=A/J$. Then $\mathcal{T}_1, \cdots, \mathcal{T}_m$ is a faithful family of pairwise orthogonal, generalized standard stable tubes of $\Gamma_B$. For each $i\in I$, choose a sectional path

$$M_{r_i-1}^{(i)}\longrightarrow M_{r_i-2}^{(i)}\longrightarrow\cdots\longrightarrow M_1^{(i)}$$

in $\mathcal{T}_i$ with $M_1^{(i)}$ lying on the mouth of $\mathcal{T}_i$. Let $M$ be the direct sum of all modules $M_j^{(i)}$, $1 \leqq j < r_i$, $1 \leqq i \leqq m$. Since the tubes $\mathcal{T}_1, \cdots, \mathcal{T}_m$ are generalized standard and pairwise orthogonal, we get that $\mathrm{Hom}_B(M_j^{(i)}, \tau_B M_q^{(p)})=0$ for any $i, j, p, q$, and hence $\mathrm{Hom}_B(M, \tau_B M)=0$. Then $\mathrm{Ext}_B^1(M, M) \cong D\,\overline{\mathrm{Hom}}_B(M, \tau_B M) = 0$. Moreover, by Lemma 5.9, we have $pd_B M \leqq 1$. Consequently, $M$ is a partial tilting $B$-module, and hence $\sum_{i\in I}(r_i-1) \leqq n$. Suppose that $\sum_{i\in I}(r_i-1) = n$. Then $M$ is a tilting $B$-module. Observe that the algebra $H=\mathrm{End}_B(M)$ is a product of hereditary algebras of Dynkin types $A_{r_i-1}$, $1 \leqq i \leqq m$, and hence $B$ is representation-finite. But it is impossible because $\mathcal{T}_1, \cdots, \mathcal{T}_m$ are infinite components of $\Gamma_B$. This proves that $\sum_{i\in I}(r_i-1) \leqq n-1$.

COROLLARY 5.11. *Let $\mathcal{T}$ be a generalized standard stable tube of $\Gamma_A$ and $r$ be the rank of $\mathcal{T}$. Then $r \leqq n$.*

THEOREM 5.12. *Let $\mathcal{T}_i$, $i \in I$, be a family of generalized standard stable tubes of $\Gamma_A$. Assume that any short path $X \to Y \to Z$ in* mod $A$ *with $X$ and $Z$ from the family $\mathcal{T}_i$, $i \in I$, is finite. For each $i \in I$, denote by $r_i$ the rank of the tube $\mathcal{T}_i$. Then*

$$\sum_{i\in I}(r_i-1) \leqq n-2\,.$$

PROOF. Observe that from our assumption the tubes $\mathcal{T}_i$ are pairwise orthogonal. Then from Lemma 5.10 we get that $\sum_{i\in I}(r_i-1) \leqq n-1$. We use the notation from the proof of Lemma 5.10.

Suppose that $\sum_{1\leq i\leq m}(r_i-1)=n-1$. Then, by [**11**, (2.1)], there exists an indecomposable $B$-module $Y$ such that $U=M\oplus Y$ is a tilting $B$-module. Since $pd_B U \leqq 1$, we get by [**36**, (2.4)] that $D\,\mathrm{Hom}_B(U, \tau_B U) = D\,\overline{\mathrm{Hom}}_B(U, \tau_B U) \cong \mathrm{Ext}_B^1(U, U)=0$, and hence $\mathrm{Hom}_B(U, \tau_B U)=0$. We shall prove now that $Y$ does not belong to the family $\mathcal{T}_1, \cdots, \mathcal{T}_m$. Indeed, suppose that $Y$ belongs to some $\mathcal{T}_i$. We know that the irreducible maps in mod $B$ corresponding to the arrows of $\mathcal{T}_i$ pointing to the mouth (respectively, pointing to infinity) are epimorphisms (respectively, monomorphisms). Further, $r_i$ is the rank of $\mathcal{T}_i$ and $M_{r_i-1}^{(i)} \to \cdots \to M_1^{(i)}$ is a sectional path in $\mathcal{T}_i$ with $M_1^{(i)}$ lying on the mouth. Hence we infer that either $\mathrm{Hom}_B(Y, \tau_B Y) \neq 0$ or $\mathrm{Hom}_B(Y, \tau_B M_j^{(i)}) \neq 0$ for some $1 \leqq j < r_i$. This gives a contradiction because $\mathrm{Hom}_B(U, \tau_B U)=0$ and $U=Y\oplus M$. Then our claim follows. Let now $\Lambda=\mathrm{End}_B(U)$. If we show that $\Lambda$ is a hereditary algebra, then the torsion pair $(\mathcal{T}(U), \mathcal{F}(U))$ in mod $B$ induced by $U$ is splitting [**22**, (6.3)], where $\mathcal{T}(U)=\{Z_B \mid \mathrm{Hom}_B(U, Z)=0\}$ and $\mathcal{F}(U)=\{Z_B \mid \mathrm{Ext}_B^1(U, Z)=0\}$. In particular, $\mathcal{F}(U)$ is closed under successors in mod $B$. Hence $\mathcal{T}_i$ is contained in $\mathcal{F}(U)$ for any $i$, $1 \leqq i \leqq m$, because $M_1^{(i)} \in \mathcal{F}(U)$ and every module in $\mathcal{T}_i$ is a successor of $M_1^{(i)}$. On the other hand, $\mathrm{Hom}_B(U, \tau_B U)=0$ implies that $\tau_B M \in \mathcal{T}(U)$, a contradiction. Thus we have only to show that $\Lambda$ is hereditary. Since $\mathcal{T}_1, \cdots, \mathcal{T}_m$ is a faithful family of $\Gamma_B$, it follows from Lemma 5.7 that $gl.\dim. B \leqq 2$, and hence $gl.\dim.\Lambda\leqq 3$ (see [11], [22]). Denote by $P$ the projective $\Lambda$-module $\mathrm{Hom}_B(U, Y)$, by $S$ the top of $P$, and by $Q$ the injective envelope of $S$ in mod $\Lambda$. We have two cases to consider.

Assume that $\mathrm{Hom}_B(Y, M)\neq 0$. We claim that then $\mathrm{Hom}_B(M, Y)=0$. Indeed, if $\mathrm{Hom}_B(M, Y)\neq 0$, then there is a short path $M_j^{(i)}\to Y\to M_q^{(p)}$. Since $Y$ does not belong to the family $\mathcal{T}_1, \cdots, \mathcal{T}_m$, the above path is infinite, a contradiction to our assumption. We know that $\mathrm{End}_B(M)$ is a product of hereditary algebras of Dynkin types $A_{r_i-1}$, $1\leqq i\leqq m$. Moreover, since $\mathrm{Hom}_B(Y, \tau_B M)=0$, every irreducible epimorphism $M_j^{(i)}\to M_{j-1}^{(i)}$ induces an isomorphism $\mathrm{Hom}_B(Y, M_j^{(i)})\xrightarrow{\sim}\mathrm{Hom}_B(Y, M_{j-1}^{(i)})$, $2\leqq j<r_i$, $1\leqq i\leqq m$. Hence, in order to prove that $\Lambda$ is hereditary, it is enough to show that $\mathrm{End}_B(Y)$ is a division algebra. Since $\mathrm{Hom}_B(M, Y)=0$, all simple composition factors of $P$ are isomorphic to $S$. Moreover, $pd_\Lambda S\leqq 3$, because $gl.\dim.\Lambda\leqq 3$. Therefore, $P=S$ and $\mathrm{End}_B(Y)=\mathrm{End}_\Lambda(P)$ is a division algebra.

Assume now that $\mathrm{Hom}_B(Y, M)=0$. Applying Lemma 5.9 to the faithful family $\mathcal{T}_1, \cdots, \mathcal{T}_m$ in $\Gamma_B$ we get that $\mathrm{id}_B M\leqq 1$. Then, by [36, (2.4)], we obtain that $D\,\mathrm{Hom}_B(\tau_B^- M, Y)=D\,\underline{\mathrm{Hom}}_B(\tau_B^- M, Y)\cong\mathrm{Ext}_B^1(Y, M)=0$, and so $\mathrm{Hom}_B(\tau_B^- M, Y)=0$. This implies that $\mathrm{Hom}_B(M_j^{(i)}, Y)=0$ for any $1\leqq j<r_i-1$, $1\leqq i\leqq m$, because $M_j^{(i)}$ lie on the sectional paths $M_{r_i-1}^{(i)}\to\cdots\to M_1^{(i)}$ of the stable tubes $\mathcal{T}_i$ and $M_1^{(i)}$ are on the mouth of $\mathcal{T}_i$. Hence, in order to show that $\Lambda$ is hereditary, it is enough to prove that $\mathrm{End}_B(Y)$ is a division algebra. We claim that every simple composition factor of $Q$ is isomorphic to $S$. Indeed, let $0\neq f\in\mathrm{Hom}_\Lambda(P', Q)$ for some indecomposable projective $\Lambda$-module $P'$. Then $S$ is a submodule of the image of $f$, and so there is a nonzero map $g: P\to\mathrm{Im}\, f$ with $\mathrm{Im}\, g=S$. Since $P$ is projective, we then infer that $\mathrm{Hom}_\Lambda(P, P')\neq 0$. Then $\mathrm{Hom}_B(Y, M)=0$ implies that $P\cong P'$, which proves our claim. We know also that $\mathrm{id}_\Lambda S\leqq 3$ since $gl.\dim.\Lambda\leqq 3$. We then conclude that $Q=S$ and $P$ is of Loewy length 2. Therefore, since $\mathrm{Ext}_\Lambda^1(S, S)=0$, we get that $\mathrm{End}_B(Y)\cong\mathrm{End}_\Lambda(P)$ is a division algebra.

COROLLARY 5.13. *Let $\mathcal{T}_i$, $i\in I$, be a separating tubular family of $\Gamma_A$, in the sense of* [36, (3.1)]. *For each $i\in I$, denote by $r_i$ the rank of the tube $\mathcal{T}_i$. Then*

$$\sum_{i\in I}(r_i-1)\leqq n-2\,.$$

PROOF. It follows from the fact that the separating tubular families satisfy the conditions of Theorem 5.12.

COROLLARY 5.14. *Let $\mathcal{T}$ be a stable tube of $\Gamma_A$ consisting entirely of modules which do not lie on short infinite cycles. Denote by $r$ the rank of $\mathcal{T}$. Then $r\leqq n-1$.*

PROOF. From Corollary 5.4 the tube $\mathcal{T}$ is generalized standard. From the proof of Corollary 5.8 we know also that any short path $X \to Y \to Z$ in $\operatorname{mod} A$ with $X$ and $Z$ from $\mathcal{T}$ is finite. Then the corollary is a direct consequence of Theorem 5.12.

REMARK 5.15. The bounds on the ranks of generalized standard stable tubes stated in (5.12), (5.13) and (5.14) are the best possible. For, if $k$ is an algebraically closed field and $A$ is the path algebra $kQ$ of the following quiver $Q$

$$\begin{tikzcd} & \circ \arrow[dr] & \\ \circ \arrow[ur] \arrow[rr] & & \circ \end{tikzcd}$$

then $\Gamma_A$ consists of a preprojective components $\mathcal{P}$, a preinjective component $\mathcal{Q}$, and one family $\mathcal{T}_\lambda$, $\lambda \in P_1(k)$, of generalized standard stable tubes, separating $\mathcal{P}$ from $\mathcal{Q}$. One of the tubes $\mathcal{T}_\lambda$ has rank $2=3-1$ and the remaining tubes $\mathcal{T}_\lambda$ have rank 1.

The following theorem gives a characterization of generalized standard components of $\Gamma_A$ without rocks.

THEOREM 5.16. *Let $\mathcal{C}$ be a generalized standard component of $\Gamma_A$. Then the following conditions are equivalent.*

(i) *$\mathcal{C}$ is a left stable tube with exactly one maximal ray (in the sense of* [**36**]*) and containing no directing module.*

(ii) $\operatorname{Hom}_A(X, \tau_A X) \neq 0$ *for any indecomposable module $X$ from $\mathcal{C}$.*

PROOF. If (i) holds then, for any module $X$ in $\mathcal{C}$ there is a sectional path from $X$ to $\tau_A X$, and hence $\operatorname{Hom}_A(X, \tau_A X) \neq 0$. Assume now that (ii) holds. Then $\mathcal{C}$ does not contain projective modules. Moreover, if $X$ is an indecomposable module from $\mathcal{C}$, then $\operatorname{Hom}_A(X, \tau_A X) \neq 0$ implies existence of a cycle

$$X \xrightarrow{f} \tau_A X \xrightarrow{g} U \xrightarrow{h} X$$

in $\operatorname{mod} A$, where $U$ is an indecomposable direct summand of the middle term of an Auslander-Reiten sequence ending at $X$, and $g$ and $h$ are irreducible maps. Since $\mathcal{C}$ is generalized standard, $f$ does not belongs to $\operatorname{rad}^\infty(\operatorname{mod} A)$, and consequently $X$ lies on a cycle in $\mathcal{C}$. Hence $\mathcal{C}$ contains oriented cycles. Then, by [**28**, (2.6)], $\mathcal{C}$ is a tube in the sense of [**36**]. Suppose that $\mathcal{C}$ contains an injective module. Then $\mathcal{C}$ is a coray tube, that is, has trivial valuations and its underlying translation quiver is isomorphic to one obtained from a stable tube by coray insertions [**36**, (4.6)]. But then there is an indecomposable injective module $I$ in $\mathcal{C}$ such that the socle factor $I/\operatorname{soc} I$ of $I$ admits an indecomposable noninjective direct summand $Y$ having the property that the middle term of the Auslander-Reiten sequence starting at $Y$ is indecomposable. Since $\mathcal{C}$ is generalized standard, all indecomposable modules $Z$ from $\mathcal{C}$ such that $\mathrm{Hom}_A(Y, Z) \neq 0$ lie on the unique infinite sectional path in $\mathcal{C}$ starting at $Y$. Hence $\tau_A Y$ lies on this sectional path, and consequently $\mathcal{C}$ admits exactly one maximal ray, starting at an injective module and passing through $Y$ and $\tau_A Y$. Moreover, by (ii), $\mathcal{C}$ has no directing module. Assume now that $\mathcal{C}$ is a stable tube. If $\mathcal{C}$ is not of rank 1, then for any module $V$ lying on the mouth of $\mathcal{C}$ we have $\mathrm{Hom}_A(V, \tau_A V)=0$, by (5.3). Therefore $\mathcal{C}$ is of rank 1 and clearly admits exactly one maximal ray. Thus (ii) implies (i), and the proof is completed.

REMARK 5.17. In the forthcoming paper [**42**] we discuss the relations between the indecomposable modules lying in generalized standard stable tubes of $\Gamma_A$ and having the same composition factors.

REMARK 5.18. We are also interested in description of all artin algebras $A$ such that $\Gamma_A$ admits a faithful generalized standard stable tube. In the forthcoming paper [**43**] we solve a related problem for cycle-finite algebras. Recall that following [**2**] an artin algebra $A$ is called cycle-finite if any cycle in mod $A$ is finite. For cycle-finite artin algebras $A$, all stable tubes of $\Gamma_A$ are, by Corollary 5.4, generalized standard. In particular, we prove in [**43**] that, if $A$ is a cycle-finite basic artin algebra such that $\Gamma_A$ admits a sincere stable tube, then $A$ is either tame concealed or tubular (in the sense of [**36**]). In this case, $gl.\dim A \leq 2$, and all sincere stable tubes are faithful. In general, there are algebras $\Lambda$ of infinite global dimension such that $\Gamma_\Lambda$ admits sincere generalized standard stable tubes which are not faithful. Indeed, for the trivial extension $\Lambda = H \propto DH$ of the path algebra $H=kQ$ of the quiver

$$Q : \circ \rightrightarrows \circ$$

by $DH$, over an algebraically closed field $k$, all stable tubes of $\Gamma_\Lambda$ are sincere generalized standard but clearly not faithful.

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## References

[1] I. Assem, Tilting theory- an introduction, in Rings and Representations of Algebras, Banach Center Publications, Vol. 26, Part 1, PWN, Warsaw, 1990, pp. 127-180.

[2] I. Assem and A. Skowroński, Minimal representation-infinite coil algebras, Manuscripta Math., **67** (1990), 305-331.

[3] I. Assem and A. Skowroński, Indecomposable modules over multicoil algebras, Math.

Scand., **71** (1992), 31-61.
[4] I. Assem and A. Skowroński, Multicoil algebras, CMS Conf. Proc., **14** (1993), 29-68.
[5] M. Auslander, Representation theory of artin algebras II, Comm. Algebra, **1** (1974), 269-310.
[6] M. Auslander, Applications of morphisms determined by objects, Lecture Notes in Pure and Appl. Math., **37**, 1976, pp. 245-327.
[7] M. Auslander and I. Reiten, Representation theory of artin algebra III, Comm. Algebra, **3** (1975), 239-294.
[8] M. Auslander and I. Reiten, Representation theory of artin algebra IV, Comm. Algebra, **5** (1977), 443-518.
[9] M. Auslander and I. Reiten, Modules determined by their composition factors, Illinois J. Math., **29** (1985), 280-301.
[10] D. Baer, A note on wild quiver algebras and tilting modules, Comm. Algebra, **17** (1989), 751-757.
[11] K. Bongartz, Tilted algebras, in Representations of Algebras, Lecture Notes in Math., **903**, Springer, Berlin, 1981, pp. 26-38.
[12] K. Bongartz and P. Gabriel, Covering spaces in representation theory, Invent. Math. **65** (1982), 331-378.
[13] O. Bretscher and P. Gabriel, The standard form of a representation-finite algebra, Bull. Soc. Math. France, **111** (1983), 21-40.
[14] W.W. Crawley-Boevey, On tame algebras and bocses, Proc. London Math. Soc., **56** (1988), 451-483.
[15] W.W. Crawley-Boevey, Tame algebras and generic modules, Proc. London Math. Soc., **63** (1991), 241-265.
[16] W.W. Crawley-Boevey, Modules of finite length over their endomorphism rings, in Representations of Algebras and Related Topics, London Math. Soc. Lecture Note Ser., **168**, Cambridge Univ. Press, 1992, pp. 127-184.
[17] P. Dowbor, H. Lenzing and A. Skowroński, Galois coverings of algebras by locally support-finite categories, in Representation Theory I. Finite dimensional algebras, Lecture Notes in Math., **1177**, Springer, Berlin, 1986, pp. 91-93.
[18] Yu. A. Drozd, Tame and wild matrix problems (in Russian), in Representations and quadratic forms, Institute of Mathematics, Academy of Science, Ukrainian SSR, Kiev, 1979, pp. 39-74.
[19] P. Gabriel, The universal cover of a representation-finite algebra, in Representations of Algebras, Lecture Notes in Math., **903**, Springer, Berlin, 1981, pp. 68-105.
[20] D. Happel, U. Preiser and C.M. Ringel, Vinberg's characterization of Dynkin diagrams using subadditive functions with applications to $DTr$-periodic modules, in Representation Theory II, Lecture Notes in Math., **832**, Springer, Berlin, 1980, pp. 280-294.
[21] D. Happel and C.M. Ringel, Tilted algebras, Trans. Amer. Math. Soc., **274** (1982), 399-444.
[22] M. Harada and Y. Sai, On categories of indecomposable modules I, Osaka J. Math., **7** (1970), 323-344.
[23] M. Hoshino, Modules without self-extensions and Nakayama's conjecture, Arch. Math., **43** (1984), 493-500.
[24] O. Kerner, Tilting wild algebras, J. London Math. Soc., **39** (1989), 29-47.
[25] O. Kerner, Stable components of wild tilted algebras, J. Algebra, **142** (1991), 35-57.
[26] O. Kerner and A. Skowroński, On module categories with nilpotent infinite radical,

Compositio Math., **77** (1991), 313-333.

[27] S. Liu, Degrees of irreducible maps and the shapes of the Auslander-Reiten quivers, J. London Math. Soc., **45** (1992), 32-54.

[28] S. Liu, Semistable components of an Auslander-Reiten quiver, J. London Math. Soc., **47** (1993), 405-416.

[29] I. Reiten, A. Skowroński and S.O. Smalø, Short chains and short cycles of modules, Proc. Amer. Math. Soc., **117** (1993), 343-354.

[30] I. Reiten, A. Skowroński and S.O. Smalø, Short chains and regular components, Proc. Amer. Math. Soc., **117** (1993), 601-612.

[31] Ch. Riedtmann, Representation-finite selfinjective algebras of type $A_n$, in Representation Theory II, Lecture Notes in Math., **832**, Springer, Berlin, 1980, pp. 449-520.

[32] Ch. Riedtmann, Many algebras with the same Auslander-Reiten quiver, Bull. London Math. Soc., **15** (1983), 43-47.

[33] C.M. Ringel, Representations of $k$-species and bimodules, J. Algebra, **41** (1976), 269-302.

[34] C.M. Ringel, Finite dimensional hereditary algebras of wild representation type, Math. Z., **161** (1978), 236-255.

[35] C.M. Ringel, Report on the Brauer-Thrall conjectures, in Representation Theory I, Lecture Notes in Math., **831**, Springer, Berlin, 1980, pp. 104-136.

[36] C.M. Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Math., **1099**, Springer, Berlin, 1984.

[37] C.M. Ringel, Representation theory of finite-dimensional algebras, in Representations of Algebras, London Math. Soc. Lecture Note Ser., **116**, Cambridge Univ. Press, 1986, pp. 7-79.

[38] C.M. Ringel, The regular components of the Auslander-Reiten quiver of a tilted algebra, Chinese Ann. Math., **9B**(1) (1988), 1-18.

[39] A. Skowroński, Group algebras of polynomial growth, Manuscripta Math., **59** (1987), 499-516.

[40] A. Skowroński, Regular Auslander-Reiten components containing directing modules, Proc. Amer. Math. Soc., **120** (1994), 19-26,

[41] A. Skowroński, Loop-finite algebras with the separation property, to appear.

[42] A. Skowroński, On the composition factors of periodic modules, J. London Math. Soc., to appear.

[43] A. Skowroński, Cycle-finite algebras, J. Pure Appl. Algebra, to appear.

[44] A. Skowroński and S.O. Smalø, Directing modules, J. Algebra, **147** (1992), 137-146.

[45] A. Skowroński and M. Wenderlich, Artin algebras with directing indecomposable projective modules, J. Algebra, **164** (1994), to appear.

[46] H. Strauss, On the perpendicular category of a partial tilting module, J. Algebra, **144** (1991), 43-66.

[47] L. Unger, Schur modules over wild, finite dimensional path algebras with three simple modules, J. Pure Appl. Algebra, **64** (1990), 205-222.

[48] Y. Zhang, The structure of stable components, Canad. J. Math., **43** (1991), 652-672.

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